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KSB SMOOTHINGS OF SURFACE PAIRS

JÁNOS KOLLÁR

Abstract. We describe KSB smoothings of log canonical surface pairs (S, D), where D is a reduced curve. In sharp contrast with the $D = \emptyset$ case, cyclic quotient pairs always have KSB smoothings, usually forming many irreducible components.

At the boundary of the moduli space of smooth surfaces, we find surfaces with log canonical singularities. Quotient singularities form the largest—and usually most troublesome—subclass.

The semi-universal deformation space $\mathrm{Def}(S)$ of a quotient singularity is quite complicated. The irreducible components were enumerated in [KSB88, Ste91a]; their number grows exponentially with the multiplicity. However, for the moduli of surfaces, we are interested only in those deformations of S for which K_S lifts to a $\mathbb{Q}$-Cartier divisor; see Paragraph 11. These are now called *KSB deformations,* replacing the name qG-*deformations* used in [KSB88].

By [LW86, KSB88], quotient singularities either have no KSB smoothings, or the KSB smoothings form a single, easy to describe, irreducible component of the deformation space; see Paragraph 12.

The aim of this note is to study the analogous question for the moduli theory of simple normal crossing surface pairs. The normal singularities at the boundary are log canonical pairs (S, D), where S itself has quotient singularities whenever $D \neq \emptyset$. For such pairs, we enumerate the irreducible components of $\mathrm{Def}_{\mathrm{KSB}}(S, D)$—the *semi-universal parameter space of KSB deformations*—where we now require $K_S + D$ to lift to a $\mathbb{Q}$-Cartier divisor; see Definitions 15–17.

1. Irreducible components of KSB deformation spaces

Singularities of 2-dimensional log canonical pairs $(0 \in S, D)$ with $D \neq \emptyset$ come in 3 types. If the singularity is a cyclic quotient, then D can have 1 or 2 local branches, and if the singularity is a dihedral quotient, then D has 1 local branch; see Notation 14 for details. The 3 types behave quite differently. We use the dual graph—as in Notation 14—to specify a pair (S, D).

Theorem 1. *The irreducible components of*

$$\mathrm{Def}_{\mathrm{KSB}}\bigl(\bullet\ -\ c_1\ -\ c_2\ -\ \cdots\ -\ c_s\ -\ \bullet\bigr)$$

are in one-to-one correspondence with the irreducible components of

$$\mathrm{Def}\bigl(c_1\ -\ c_2\ -\ \cdots\ -\ c_s\bigr),$$

and, in each of them, a general deformation is smooth.

Theorem 2. *The irreducible components of*

$$\mathrm{Def}_{\mathrm{KSB}}\left(\begin{array}{c} \qquad 2 \\ \qquad | \\ 2 - c_1 - c_2\ -\ \cdots\ -\ c_s\ -\ \bullet \end{array}\right)$$

are in one-to-one correspondence with the irreducible components of

$$\mathrm{Def}\bigl(c_2\ -\ \cdots\ -\ c_s\bigr), \qquad \textit{(note that } c_1 \textit{ is omitted)}$$

and, in each of them, a general deformation has 2 singularities, which are $(2\ -\ \bullet)$.

Theorem 3. *The singularities* $(c_1 \ - \ c_2 \ - \ \cdots \ - \ c_s \ - \ \bullet)$ *are KSB rigid.*

Remarks 4.

(4.1) The dimensions of the irreducible components of $\mathrm{Def}_{\mathrm{KSB}}(S,D)$ are computed in (44) for the cyclic cases, and in (46) for the dihedral cases.

(4.2) For the cyclic quotients in Theorem 1, the image of

$$\mathrm{Def}_{\mathrm{KSB}}\big(\bullet \ - \ c_1 \ - \ \cdots \ - \ c_s \ - \ \bullet\big) \ \to \ \mathrm{Def}\big(c_1 \ - \cdots \ - \ c_s\big)$$

is nowhere dense, with a few exceptions; see (45).

(4.3) For the dihedral pairs (S,D) in Theorem 2, only some irreducible components of $\mathrm{Def}(S)$ contain an irreducible component of $\mathrm{Def}_{\mathrm{KSB}}(S,D)$, see (31).

(4.4) For quotient singularities S, [KSB88, 3.14] gives an algorithm to enumerate the irreducible components of $\mathrm{Def}(S)$, though in practice this can be very cumbersome if there are many exceptional curves. For cyclic quotients S as in Theorem 1, [Ste91a] gives a better method. As a consequence, [Ste91a] shows that if S has multiplicity $m = m(S)$, then $\mathrm{Def}(S)$ has at most $\frac{1}{m-1}\binom{2(m-2)}{m-2}$ irreducible components, and equality holds for 'most' cyclic quotients of a given multiplicity. Note that $m(S) = 2 + \sum(c_i - 2)$.

5 (Sketch of the proofs)**.** Let (S,D) be a pair as in Theorems 1–3. Although we do it in different order, the steps of the proof are the following.

Step 5.1. Fix an irreducible component of $\mathrm{Def}_{\mathrm{KSB}}(S,D)$, and let $g : (\mathbf{S},\mathbf{D}) \to (0,\mathbb{D})$ be a general 1-parameter deformation in it. That is, the fiber over 0 is $(\mathbf{S},\mathbf{D})_0 \cong (S,D)$, and $\mathbb{D}$ denotes either a complex disc or the germ of a smooth curve. We prove in (48) that the general fiber of g is smooth in the cyclic case, and has at worst A_1 singularities in the dihedral case.

Step 5.2. As we recall in (28), by [KSB88, 3.5] there is a proper, birational morphism $\pi : \mathbf{S}_P \to \mathbf{S}$ such that

- $K_{\mathbf{S}_P}$ is $\mathbb{Q}$-Cartier, π-ample, and
- the central fiber $S_P := (g\circ\pi)^{-1}(0)$ has only Du Val and $\hat{\mathbb{A}}^2/\frac{1}{rn^2}(1, arn-1)$ singularities; see Notation 8.

The latter are called T-singularities (12.1), and $S_P \to S$ is called a P-modification, see Definition 29.

Step 5.3. Let $\mathbf{D}_P$ denote the birational transform of $\mathbf{D}$ on $\mathbf{S}_P$, and D_P the birational transform of D on S_P. We show in (36) that $\mathbf{D}_P|_{S_P} = D_P + E_P$, where $E_P \subset S_P$ is the reduced exceptional divisor.

Step 5.4. $K_{\mathbf{S}_P} + \mathbf{D}_P \sim_{\mathbb{Q}} \pi^*\big(K_{\mathbf{S}} + \mathbf{D}\big)$, hence both $K_{\mathbf{S}_P}$ and $K_{\mathbf{S}_P} + \mathbf{D}_P$ are $\mathbb{Q}$-Cartier. Thus $g\circ\pi : (\mathbf{S}_P + \mathbf{D}_P) \to \mathbb{D}$ is a KSB deformation of S_P and also of $(S_P, D_P + E_P)$. We call these *doubly KSB deformations* in Definition 16. Basic results going back to [Gra72, Wah76, Bin87] show that a semi-universal deformation space—denoted by $\mathrm{Def}_{\mathrm{dKSB}}(S_P, D_P + E_P)$—exists; see Paragraph 17. Combining Lemmas 18 and 47 shows that it is smooth.

Step 5.5. A well known argument (34) gives a natural morphism

$$\tau_P : \mathrm{Def}_{\mathrm{dKSB}}(S_P, D_P + E_P) \to \mathrm{Def}_{\mathrm{KSB}}(S,D),$$

which is finite and birational onto our irreducible component; see (36) for details.

Step 5.6. In the cyclic case of Theorem 1, a quick argument shows that every P-modification leads to an irreducible component of $\mathrm{Def}_{\mathrm{KSB}}(S,D)$, see (44). In the dihedral case of Theorem 2, the irreducible components of $\mathrm{Def}(S)$ are enumerated in (31), following [Ste91a, Ste93]. Then in (46) we describe which P-modification leads to an irreducible component of $\mathrm{Def}_{\mathrm{KSB}}(S,D)$. Putting these together gives Theorem 2. Finally Theorem 3 directly follows from [Kol23, 2.23], see (20.1). $\square$

6 (Other results)**.** There are several variants and generalizations.

Fractional coefficients 6.1. If we look at pairs (S, cD) for $\frac{1}{2} < c < 1$, the situation changes dramatically; see Section 8 for details.

For the pairs (S, D) as in Theorem 1, there are either no KSBA deformations (as in Definition 15) for any c, or a single irreducible component for every c; see Corollary 53. For the pairs (S, D) as in Theorem 2, there are no KSBA deformations; see Lemma 54. For almost all of the pairs (S, cD) as in Theorem 3, there are either no KSBA deformations for any c, or a single irreducible component for a unique value of c, see (56).

These describe KSBA deformations of all pairs (S, cD) for $c \in (\frac{5}{6}, 1]$, though the answer is not explicit in terms of the dual graph. Example 56 shows that c can be arbitrarily close to 1.

For $c \in [\frac{1}{2}, 1]$, all lc pairs (S, cD) are listed in [Kol13, Sec.3.3]. The method should give a full answer for all of them. I checked only some examples, but did not find any other pairs (S, cD) with a KSBA smoothing and $\frac{1}{2} < c < 1$. Examples with $c = \frac{1}{2}$ are given in (58).

Higher dimensions 6.2. For an lc pair (X, D) of arbitrary dimension, [KK23] shows that a flat deformation is a KSB deformation iff the KSB condition (15) is satisfied by a general surface section. Thus our results give information not only for surface pairs, but for higher dimensions as well.

Infinitesimal computations 6.3. Usually the space of KSB deformations is not reduced. For cyclic pairs, the method of [AK19] can be used to determine its tangent space, but a complete description is known only in a handful of cases.

Other rational singularities 6.4. Many of the results apply to more general rational singularities S. However, not every irreducible component of $\mathrm{Def}(S)$ is obtained from a P-modification.

The conjectures in [Kol91, Sec.6] ask whether every irreducible component of $\mathrm{Def}(S)$ is obtained in a similar way, using a notion of P-modification that is more general than the one in Definition 29. A positive answer has been known for quotient singularities [KSB88, 3.9] and for points of multiplicity ≤ 4 [dJvS91, Ste91b]. The recent papers [PS22, JS23] develop a method to prove the conjecture in many new cases. In [PS22] this is illustrated by the $W(p, q, r)$ series of singularities, but their method applies more broadly. (The $W(p, q, r)$ series was discovered by Wahl around 1980, see [Wah21] for a recent survey.) In [JS23] the conjecture is proved for most weighted homogeneous singularities.

Especially for the M-modification version as in [BC94] or (34), the singularities with a rational homology disc smoothing—classified by [BS11]—may form the natural class to work with.

2. Quotient singularities

The classification of surface quotient singularities and their dual graphs are given in [Bri68]. We need detailed information about the cyclic (type A) and dihedral (type D) cases; we recall these and fix our notation. The tetrahedral, octahedral and icosahedral quotients (type E) will appear only in examples.

Notation 7. We work over the complex numbers and let $\hat{\mathbb{A}}^2$ denote the germ of $\mathbb{A}^2$ at the origin. For our purposes, we may work with a complex analytic germ, the spectrum of $\mathbb{C}[[x, y]]$, or the spectrum of the Henselisation of $\mathbb{C}[x, y]_{(x,y)}$. For most situations in this paper one can choose global coordinates, so we can even work with $0 \in \mathbb{A}^2$.

$S_{n,q}$ and $S^d_{n,q}$ will denote the cyclic and dihedral quotients as in Notation 8 and 10. The curves $B_{n,q} \subset S_{n,q}$, $D_{n,q} \subset S_{n,q}$ and $D^d_{n,q} \subset S^d_{n,q}$ are defined in (14).

The minimal resolution of a surface S is denoted by $\mu : S^{\mathrm{m}} \to S$. For a curve $D \subset S$, its birational transform on S^{m} is denoted by D^{m}.

Notation 8 (Cyclic quotients)**.** We write $S_{n,q} := \hat{\mathbb{A}}^2/\frac{1}{n}(1,q)$, where the group action is $(x,y) \mapsto (\epsilon x, \epsilon^q y)$ for some primitive nth root of unity ϵ. The action is free outside the origin iff $(n,q)=1$. In general, if $n = n_1(n,q), q = q_1(n,q)$, then

$$\hat{\mathbb{A}}^2/\tfrac{1}{n}(1,q) \cong \hat{\mathbb{A}}^2/\tfrac{1}{n_1}(1,q_1).$$

Let q' denote the multiplicative inverse of q modulo n. Then $S_{n,q} \cong S_{n,q'}$; the isomorphism interchanges the coordinates.

For the dual graph of the minimal resolution, we use $c_i := -(C_i^2)$, the negative of the self-intersection, to denote the vertex corresponding to the exceptional curve $C_i \subset S_{n,q}^{\mathrm{m}}$. For $S_{n,q}$ the dual graph is

$$\mathcal{D}(S_{n,q}) = \quad c_1 \ - \ c_2 \ - \cdots \ - c_s,$$

where the c_i are obtained from the continued fraction expansion

$$\frac{n}{q} = c_1 - \cfrac{1}{c_2 - \cfrac{1}{c_3 - \cdots}}$$

We use $[c_1,\dots,c_s]$ to denote this continued fraction, and $S(c_1,\dots,c_s) \cong S_{n,q}$ the resulting singularity.

9 (Resolution of cyclic quotients)**.** An explicit construction of the minimal resolution is in [Jun08]; see [Rei93] for a very accessible treatment. Here we use an inductive procedure as in [Kol07, 2.31].

For $\mathbb{A}^2_{xy}/\frac{1}{n}(1,q)$, the quotient of the blow-up of the ideal sheaf (x^q,y) gives a proper, birational morphism $\pi_1 : S_1 \to \mathbb{A}^2_{xy}/\frac{1}{n}(1,q)$. It is covered by 2 charts

(9.1) (Singular chart) $\mathbb{A}^2_{x_1y_1}/\frac{1}{q}(1,-n)$, where $x = x_1y_1^{1/n}, y = y_1^{q/n}$.

(9.2) (Smooth chart) $\mathbb{A}^2_{x_2y_2}$, where $x = x_2^{1/n}, y = y_2x_2^{q/n}$.

Iterating this blow-up gives the minimal resolution. The π_1-exceptional curve E_1 is $(y_1=0)$ (resp. $(x_2=0)$). Thus the extended dual graph is

$$(x-\text{axis}) \ - \ c_1 \ - \ c_2 \ - \cdots \ - c_s \ - \ (y-\text{axis}). \tag{9.3}$$

Notation 10 (Dihedral quotients)**.** Let $S^d_{n,q}$ denote the singularity whose minimal resolution dual graph is

$$\begin{array}{ccccccccc} & & 2 & & & & & & \\ & & | & & & & & & \\ 2 & - & c_1 & - & c_2 & - & \cdots & - & c_s \end{array}$$

where $\frac{n}{q} = [c_1,\dots,c_s]$. The curves marked 2 are denoted by C_0', C_0''. We also use the notation

$$S(2^2, c_1, \dots, c_s) := S^d_{n,q}.$$

One usually assumes $s \geq 2$, though later it will be convenient to allow $s=1$.

Claim 10.1. $S^d_{n,q}$ is also the quotient of

$$S_{N,Q} := \hat{\mathbb{A}}^2/\tfrac{1}{2q(n-q)}\bigl(1, 2n'(n-q)+1\bigr), \quad \text{where} \quad qq' = nn'+1,$$

by the involution induced by $(x,y) \mapsto (\epsilon^q y, x)$, where ϵ is any primitive $2q(n-q)$-th root of unity. This gives the correspondence

$$\begin{array}{rcl} (n,q) & \mapsto & \bigl(N = 2q(n-q),\ Q = 2n'(n-q)+1\bigr), \quad \text{and} \\ (N,Q) & \mapsto & \bigl(n = q + \frac{1}{2}(N, Q-1),\ q = N/(N,Q-1)\bigr). \end{array}$$

(So N is even and $Q^2 \equiv 1 \mod N$.)

To see these, contract the curves C_0', C_0'' and $C_2,\dots,C_s$ to get $\bar{S}^d_{n,q} \to S^d_{n,q}$. There is a single exceptional curve $\cong \mathbb{P}^1$ with two A_1 points on it, plus the singularity coming from $S(c_2,\dots,c_s)$.

Thus $\bar S^d_{n,q}$ has a double cover, ramified only at the A_1 points. The corresponding double cover of $S^d_{n,q}$ is a cyclic quotient singularity, whose dual graph is

$$c_s \ - \ \cdots \ - \ c_2 \ - \ \tilde c_1 \ - \ c_2 \ - \ \cdots \ - \ c_s \tag{10.2}$$

where $\tilde c_1 = 2(c_1-1)$. We claim that this is the quotient singularity $S_{N,Q}$. To see this, start with $S_{N,Q}$ and blow up the origin. In the $(x, \frac{y}{x})$-chart we get

$$\mathbb{A}^2/\tfrac{1}{2q(n-q)}\big(1, 2n'(n-q)\big) \cong \mathbb{A}^2/\tfrac{1}{q}\big(1, n'\big).$$

Next, write $\frac{q}{m} := [c_2, \dots, c_s]$, so $n = c_1 - \frac{m}{q}$, and note that $nn' \equiv -1 \mod q$, so $mn' \equiv 1 \mod q$. Therefore

$$\hat{\mathbb{A}}^2/\tfrac{1}{q}\big(1, n'\big) \cong \hat{\mathbb{A}}^2/\tfrac{1}{q}\big(1, m\big)$$

which gives $S(c_2, \dots, c_s)$.

The exceptional curve of the blow-up is point-wise fixed by a subgroup of order $2(n-q)$. Taking the corresponding quotient we have a smooth surface and the self-intersection of the curve is $-2(n-q)$. Then we have a $\frac{1}{q}$-action, which changes the self-intersection to

$$-2(n-q)/q = -2(n/q) + 2.$$

Then we resolve the $\hat{\mathbb{A}}^2/\frac{1}{q}\big(1, m\big)$ singularities. The self-intersection becomes

$$-2\lceil n/q\rceil + 2 = -2c_1 + 2,$$

as claimed.

Finally $(x, y) \mapsto (\epsilon^q y, x)$ descends to the involution on $S_{N,Q}$. □

Definition 11. Let X_0 be a surface with quotient (more generally log canonical) singularities only. A *KSB deformation* of X_0 over a reduced, local scheme $(0, B)$ is a flat morphism $g : X \to (0, B)$ such that $X_0 \cong g^{-1}(0)$ and $K_{X/B}$ is $\mathbb{Q}$-Cartier. The notion was introduced in [KSB88], and called qG-*deformation* there. See Definition 15 for pairs and [Kol23, Sec.6.2] for nonreduced bases.

12 (KSB smoothings)**.** By [LW86, 5.9] and [KSB88, 3.10], quotient singularities that admit a KSB smoothing—frequently called *T-singularities*—are either Du Val or of the form

$$\hat{\mathbb{A}}^2_{uv}/\tfrac{1}{rn^2}(1, arn-1), \quad \text{where } (a, n) = 1 \text{ and } n > 1. \tag{12.1}$$

More generally, these are the singularities that have a KSB deformation with Du Val generic fiber. (There are a few more cases that have nontrivial KSB deformations, but no KSB smoothings; see [Kol23, 2.29].) By [Wah81], the dual resolution graphs are obtained as follows. We start with

$$4 \quad \text{or} \quad 3 \ - \ 2 \ - \ \cdots \ - \ 2 \ - \ 3,$$

and successively apply the operation

$$c_1 \ - \ \cdots \ - \ c_{s-1} \ - \ c_s \quad \mapsto \quad 2 \ - \ c_1 \ - \ \cdots \ - \ c_{s-1} \ - \ (c_s+1),$$

or its symmetric version.

Starting with (4) gives the $\hat{\mathbb{A}}^2_{uv}/\frac{1}{n^2}(1, an-1)$ singularities, and starting with

$$(3 \ - \ 2 \ - \ \cdots \ - \ 2 \ - \ 3)$$

(with $r-2$ curves marked 2) gives the $\hat{\mathbb{A}}^2_{uv}/\frac{1}{rn^2}(1, arn-1)$ singularities.

Taking the quotient of (12.1) by the subgroup of order rn shows that

$$\hat{\mathbb{A}}^2_{uv}/\tfrac{1}{rn^2}(1, arn-1) \cong (xy = z^{rn})/\tfrac{1}{n}(1, -1, a), \tag{12.2}$$

where $x = u^{rn}, y = v^{rn}, z = uv$. Using the second representation, the semi-universal KSB deformation is given by

$$\big(xy = z^{rn} + \textstyle\sum_{i=0}^{r-1} t_i z^{in}\big)/\tfrac{1}{n}(1, -1, a, \mathbf{0}). \tag{12.3}$$

Thus the KSB deformation space is smooth and has dimension r.

For $r=1$ we get a 1-dimensional KSB deformation space. As observed by [BC94], the $r>1$ cases can be reduced to the $r=1$ cases as follows. Change the deformation (12.3) to the factored form

$$\big(xy=\textstyle\prod_{i=1}^{r}(z^n-s_i)\big)/\tfrac{1}{n}(1,-1,a,\mathbf{0}),$$

and repeatedly blow up the divisors $(x=z^n-s_i=0)$ to obtain a small, crepant modification, which is a flat KSB deformation of its central fiber. The central fiber is a modification of $\hat{\mathbb{A}}^2/\frac{1}{rn^2}(1,arn-1)$ that has r singularities of the form $\hat{\mathbb{A}}^2/\frac{1}{n^2}(1,an-1)$, connected by $r-1$ curves whose birational transforms are (-1)-curves on the minimal resolution of the central fiber. Using Notation 13, the dual graph is the following.

$$\boxed{\tfrac{1}{n^2}(1,an-1)}-1-\boxed{\tfrac{1}{n^2}(1,an-1)}-\cdots-\boxed{\tfrac{1}{n^2}(1,an-1)}-1-\boxed{\tfrac{1}{n^2}(1,an-1)} \tag{12.4}$$

This construction leads to the notion of M-modifications in (29).

Note that, as we repeatedly contract (-1)-curves in (12.4), we never contract the curves on the two ends. These show the following.

Claim 12.5. The minimal resolution of $\hat{\mathbb{A}}^2/\frac{1}{rn^2}(1,arn-1)$ and the minimal resolution of (12.4) are isomorphic along the birational transforms of the x and y axes. □

Notation 13. For $\frac{n}{q}=[c_1,\dots,c_s]$ we use

$$a_1-\boxed{\tfrac{n}{q}}-a_2 \quad \text{resp.} \quad a_1-\boxed{\tfrac{n}{q}}$$

to denote 2 (resp. 1) curves through a singular point of type $\hat{\mathbb{A}}^2/\frac{1}{n}(1,q)$, whose resolution dual graph is

$$a_1 - c_1 - \cdots - c_s - a_2 \quad \text{resp.} \quad a_1 - c_1 - \cdots - c_s.$$

3. Surface pairs with reduced boundary

Notation 14. Let (S,D) be an lc surface pair, where D is a reduced curve. There are 3 possible local normal forms at points $p\in D\subset S$; see for example [Kol13, Sec.3.3].

(14.1) *(Cyclic, plt)* $(S_{n,q},B_{n,q}):=\big(\hat{\mathbb{A}}^2_{xy},(x=0)\big)/\frac{1}{n}(1,q)$, where $(n,q)=1$. The minimal resolution dual graph is the following, where $\frac{n}{q}=[c_1,\dots,c_s]$ and $\bullet$ denotes the birational transform of $B_{n,q}$.

$$c_1 - c_2 - \cdots - c_s - \bullet$$

(14.2) *(Cyclic, non-plt)* $(S_{n,q},D_{n,q}):=\big(\hat{\mathbb{A}}^2_{xy},(xy=0)\big)/\frac{1}{n}(1,q)$, where $(n,q)=1$. The minimal resolution dual graph is

$$\bullet - c_1 - c_2 - \cdots - c_s - \bullet,$$

where $\bullet$ denotes a branch of the birational transform of $D_{n,q}$. Note that $S_{n,n-1}$ is the Du Val singularity $(uv=w^n)$. These are the only ones for which $D_{n,q}$ is Cartier; they behave exceptionally in many respects.

(14.3) *(Dihedral)* The pair $(S^d_{n,q},D^d_{n,q})$ has dual graph.

$$\begin{array}{c} \phantom{2-{}}2 \\ \phantom{2-{}}| \\ 2 - c_1 - c_2 - \cdots - c_s - \bullet, \end{array}$$

where $\bullet$ denotes the birational transform of $D^d_{n,q}$. The curve $D^d_{n,q}$ is irreducible and smooth. Note that $s=1$ iff $q=1$. The singularity $S^d_{n,1}\cong S_{4n-4,2n-1}$ is a cyclic quotient, but the curve $D^d_{n,1}$ is different from the curves $B_{4n-4,2n-1}$ and $D_{4n-4,2n-1}$.

As in (10.1), $(S^d_{n,q}, D^d_{n,q})$ is also obtained as the quotient of

$$(S_{N,Q}, D_{N,Q}) := \big(\hat{\mathbb{A}}^2, (xy=0)\big)/\tfrac{1}{2q(n-q)}\big(1, 2n'(n-q)+1\big)$$

by the involution induced by $(x,y) \mapsto (\epsilon^q y, x)$.

Definition 15 (KSBA deformations of pairs)**.** Let $(X_0, \Delta_0 := \sum c_i D_0^i)$ be a normal, log canonical surface pair. Assume that $c_i > \frac{1}{2}$ for every i. A *KSBA deformation* of (X_0, Δ_0) over a reduced base $(0, B)$ is a flat morphism $g: X \to (0, B)$, plus flat divisors $D^i \subset X$ such that $K_{X/B} + \sum c_i D^i$ is $\mathbb{Q}$-Cartier.

Comment on the definition 15.1. The 'correct' definition of KSBA deformations is quite complicated, see [Kol23, Sec.8.2], since the D^i need not be flat if $c \le \frac{1}{2}$. It is a nontrivial result that the general definition is equivalent to the above one if $c_i > \frac{1}{2}$ and the base is reduced; see [Kol23, 2.3 and 4.7] for the key steps.

Since we are concerned with the reduced structure of the deformation spaces, the above naive definition is good enough.

Comment on terminology 15.2. Following [Kol23], I use *KSB deformation* if Δ is a $\mathbb{Z}$-divisor. The usage in the literature is not consistent; this applies to my papers as well.

Definition 16. A flat morphism $g: (X, \Delta) \to (0, B)$ is a *doubly KSBA deformation* of (X_0, Δ_0) if it is a KSBA deformation both for X_0 and for (X_0, Δ_0). Equivalently, if $K_{X/B}$ and Δ are both $\mathbb{Q}$-Cartier.

Thus doubly KSB deformations of $(X_0, c\Delta_0)$ are independent of $c \neq 0$.

17 (Existence of deformation spaces)**.** Let Y be an affine variety (or Stein space) with isolated singularities and $p: X \to Y$ a proper, birational morphism such that $\mathrm{Ex}(p)$ is proper. Then a semi-universal deformation space $\mathrm{Def}(X)$ exists (and is finite dimensional); see [Gra72, Wah76, Bin87]. We get $\mathrm{Def}(Y)$ using the identity map $Y = Y$.

If $Z \subset X$ is a closed subvariety with isolated singularities, then $\mathrm{Def}(X, Z)$ exists, parametrizing deformations where both X and Z are flat over the base.

If $x \in X$ is an isolated singular point of X and Z, then $\mathrm{Def}(x, X, Z)$ denotes the deformation space of a small enough neighborhood of $x \in U \subset X$ (using the identity map $U = U$).

Let (X_0, Δ_0) be a normal, log canonical surface pair. All KSB deformations give a functor, and similarly for doubly KSB deformations. The semi-universal deformation spaces and the universal families over them are denoted by

$$\begin{array}{ccc} \mathrm{Univ}_{\mathrm{KSB}}(X_0, \Delta_0) & & \mathrm{Univ}_{\mathrm{dKSB}}(X_0, \Delta_0) \\ u\downarrow & \text{and} & u\downarrow \\ \mathrm{Def}_{\mathrm{KSB}}(X_0, \Delta_0) & & \mathrm{Def}_{\mathrm{dKSB}}(X_0, \Delta_0). \end{array} \tag{17.1}$$

Our main aim is to describe the reduced structure of $\mathrm{Def}_{\mathrm{KSB}}(X_0, \Delta_0)$, thus, we can as well work with the naive version as in Definition 15.

For the purposes of our proofs, the main example of $\mathrm{Def}_{\mathrm{dKSB}}(X_0, D_0)$ that we need is the following.

Lemma 18. *The semi-universal doubly KSB deformation space of the cyclic pair*

$$\big(\hat{\mathbb{A}}^2_{uv}, (uv=0)\big)/\frac{1}{rn^2}(1, ran-1)$$

is

$$\big((xy = z^{rn} + \textstyle\sum_{i=0}^{r-1} t_i z^{in}), (z=0)\big)/\tfrac{1}{n}(1, -1, a, \mathbf{0}). \tag{18.1}$$

Proof. Using index 1 coverings as in [KSB88, 3.10] or [Kol23, 2.23], all such deformations are quotients of a deformation of $\big((xy - z^{rn} = 0), (z=0)\big)$ by the group action $\frac{1}{n}(1, -1, a, \mathbf{0})$. This is the boundary singularity B_{rn} in the notation of [AGZV85, I.Sec.17.4].

The key point is that we can not eliminate the z^{rn-1} term since we need to keep the divisor equation as $z=0$. The equation must be invariant for $\frac{1}{n}(1,-1,a,\mathbf{0})$, which gives (18.1). $\square$

Example 19. Since we have an explicit description of all KSB deformations, we can use it to describe all doubly KSB deformations in most cases. The basic examples are the following.

(19.1) Every KSB smoothing of $S_{n,q}$ is a doubly KSB deformation of $(S_{n,q}, D_{n,q})$ by Lemma 18.

(19.2) By Theorem 3, $(S_{n,q}, B_{n,q})$ has no doubly KSB deformations.

(19.3) The non-Du Val dihedral singularities have no KSB deformations by Paragraph 12; see (57) for the Du Val cases.

(19.4) If $\frac{5}{6} < c \le 1$ and (S, cD) is lc, then (S, D) is also lc, see [Kol13, 3.44]. Thus the doubly KSB smoothings are fully described by Lemma 18.

(19.5) Other examples with $c = \frac{1}{2}$ are in (58); see also (57).

In order to compute $\mathrm{Def}_{\mathrm{KSB}}(S, D)$ for the pairs in (14.1–3), [KSB88] or [Kol23, 2.23] give the following simplifying step if $K_S + D$ is not Cartier.

20 (The order of $K_S + D$ in the class group)**.** We consider separately the 3 cases in (14.1–3).

(20.1) For (14.1), the group acts faithfully on $\frac{dx\wedge dy}{x}$. Thus $\bigl(\hat{\mathbb{A}}^2_{xy}, (x=0)\bigr)$ is the index 1 cover of $(S_{n,q}, B_{n,q})$; see [Kol13, 2.49]. Therefore, as in [Kol23, 2.23], every KSB deformation of $(S_{n,q}, B_{n,q})$ is induced by a KSB deformation of $\bigl(\hat{\mathbb{A}}^2_{xy}, (x=0)\bigr)$. The latter is rigid, so every KSB deformation of $(S_{n,q}, B_{n,q})$ is trivial.

(20.2) For (14.2), the rational 2 form $\frac{dx\wedge dy}{xy}$ is invariant under the group action, hence descends to a generating section of $\omega_{S_{n,q}}(D_{n,q})$. Thus $\omega_{S_{n,q}}(D_{n,q})$ is locally free and K_S+D is Cartier. Thus the method of [Kol23, 2.23] does not give any information about the KSB deformations.

(20.3) For (14.3), the rational 2 form $\frac{dx\wedge dy}{xy}$ is invariant under the cyclic subgroup of order $2q(n-q)$, but anti-invariant under $(x,y)\mapsto(\epsilon^q y, x)$. Thus only its tensor square descends, so K_S+D is 2-torsion in the local Picard group. The corresponding double cover is $(S_{N,Q}, D_{N,Q})$ as in (14.3).

As in [Kol23, 2.23], every KSB deformation of $(S^d_{n,q}, D^d_{n,q})$ is induced by a KSB deformation of $(S_{N,Q}, D_{N,Q})$. Thus, in principle, this reduces us to the previous case. We give a more direct description in (31).

Pulling $K_S + D$ back to the minimal resolution shows that

(20.4) $K_{S^{\mathrm{m}}} + D^{\mathrm{m}} + \sum_{i=1}^{s} C_i \sim 0$ in case (14.2), and

(20.5) $2\bigl(K_{S^{\mathrm{m}}} + D^{\mathrm{m}} + \sum_{i=1}^{s} C_i\bigr) + (C_0' + C_0'') \sim 0$ in case (14.3).

4. Q-modifications

P-modifications were introduced in [KSB88] (under the name P-resolution) to describe smoothings of quotient singularities. A variant, called M-modification, is developed in [BC94]. To handle arbitrary deformations of pairs we introduce Q-modifications. These are actually very natural objects from the point of view of birational geometry, and many basic results on P- and M-modifications hold for Q-modifications.

Definition 21. Let (s, S) be a rational surface singularity. A proper, birational morphism $\pi: S_Q \to S$ is a *Q-modification* if

(21.1) K_{S_Q} is π-nef, and

(21.2) S_Q has only quotient singularities. (So 'Q' for quotient.)

Both P- and M-modifications are special cases of Q-modifications.

Roughly speaking, studying smoothings of quotient singularities using [KSB88, 3.9] leads to P-modifications, and the [BC94] refinement leads to M-modifications.

Studying deformations of log canonical pairs (S, Δ) using [KSB88, 3.9] leads to Q-modifications with K_{S_Q} π-ample, and the [BC94] refinement leads to general Q-modifications.

Notation 22. Let $(0, S)$ be a rational singularity with minimal resolution $\mu : S^{\mathrm{m}} \to S$. Let $\pi : \bar{S} \to S$ be a proper, birational morphism. Assume that $\bar{S}$ is normal with minimal resolution $\bar{\mu} : \bar{S}^{\mathrm{m}} \to \bar{S}$. Then $\bar{S}^{\mathrm{m}}$ dominates S^{m}, giving a commutative diagram

$$\begin{tikzcd} \bar{S}^{\mathrm{m}} \arrow[r, "\bar{\mu}"] \arrow[d, "\pi^m"'] & \bar{S} \arrow[d, "\pi"] \\ S^{\mathrm{m}} \arrow[r, "\mu"] \arrow[ur, dotted, "\tau"] & S \end{tikzcd} \tag{22.1}$$

We can write

$$K_{\bar{S}^{\mathrm{m}}} \sim_{\mathbb{Q}} \bar{\mu}^* K_{\bar{S}} - \bar{\Delta}, \quad \text{and} \quad K_{\bar{S}^{\mathrm{m}}} \sim (\pi^m)^* K_{S^{\mathrm{m}}} + \bar{F}^m, \tag{22.2}$$

where $\bar{\Delta}$ is effective with $\operatorname{Supp} \bar{\Delta} \subset \operatorname{Ex}(\bar{\mu})$, and $\bar{F}^m$ is effective with $\operatorname{Supp} \bar{F}^m = \operatorname{Ex}(\pi^m)$.

Let $\bar{A}^{\mathrm{m}} \subset \bar{S}^{\mathrm{m}}$ be an irreducible curve that is exceptional over S, with images $A^{\mathrm{m}} \subset S^{\mathrm{m}}$ and $\bar{A} \subset \bar{S}$. (We use $A^{\mathrm{m}} := 0$ if the image is a point.) Then (22.2) gives that

$$(\bar{A} \cdot K_{\bar{S}}) = (A^{\mathrm{m}} \cdot K_{S^{\mathrm{m}}}) + \big(\bar{A}^{\mathrm{m}} \cdot \bar{F}^m\big) + \big(\bar{A}^{\mathrm{m}} \cdot \bar{\Delta}\big). \tag{22.3}$$

From this we conclude the following.

(22.4) If $\bar{A} \neq 0$ and $A^{\mathrm{m}} \neq 0$, then $(\bar{A} \cdot K_{\bar{S}}) \geq 0$, with equality holding iff $(A^{\mathrm{m}} \cdot K_{S^{\mathrm{m}}}) = 0$ and τ is an isomorphism along A^{m}.

(22.5) If $A^{\mathrm{m}} = 0$, then $(\bar{A} \cdot K_{\bar{S}})$ depends only on the intersection numbers of the curves in $\operatorname{Ex}(\bar{\mu}) \cup \operatorname{Ex}(\pi^m)$.

Lemma 23. *Using the notation of (22), assume that $K_{\bar{S}}$ is π-nef. Then $-K_{\bar{S}^{\mathrm{m}}}$ is $\mathbb{Q}$-linearly equivalent to an effective linear combination of curves, that are exceptional for $\bar{S}^{\mathrm{m}} \to S$.*

Proof. By (22.2) $K_{\bar{S}^{\mathrm{m}}} \sim_{\mathbb{Q}} \bar{\mu}^* K_{\bar{S}} - \bar{\Delta}$ where Δ is effective. Since $K_{\bar{S}}$ is π-nef, $-K_{\bar{S}}$ is $\mathbb{Q}$-linearly equivalent to an effective linear combination of π-exceptional curves. □

Remark 24. This Lemma is especially strong if S has quotient singularities. Then, by [KSB88, 3.13], there is a unique, maximal $\pi' : S' \to S$ such that $-K_{S'}$ is $\mathbb{Q}$-linearly equivalent to an effective linear combination of π'-exceptional curves. This gives a procedure to construct all Q-modifications.

Lemma 25. *Let $\pi : \bar{S} \to S$ be a Q-modification and π^m as in (22.1). Then π^m is a composite of a sequence of blow ups*

$$\bar{S}^{\mathrm{m}} =: Y_r \to Y_{r-1} \to \cdots \to Y_1 \to Y_0 := S^{\mathrm{m}},$$

where each $Y_{i+1} \to Y_i$ is the blow-up of a singular point of the image of $\bar{\Delta}$ in Y_i.

Proof. By (22.2) $K_{\bar{S}^{\mathrm{m}}} \sim \bar{\mu}^* K_{\bar{S}} - \bar{\Delta}$, where $\lfloor \bar{\Delta} \rfloor = 0$ since $\bar{S}$ has only quotient singularities.

For any j, let $\bar{\Delta}_j$ denote the push-forward of $\bar{\Delta}$ to Y_j. Then $K_{Y_r} + \bar{\Delta}_r \sim_{\mathbb{Q}} \bar{\mu}^* K_{\bar{S}}$ is nef over S by assumption, hence so is each $K_{Y_j} + \bar{\Delta}_j$. Let now F_{i+1} denote the exceptional curve of $Y_{i+1} \to Y_i$. Then $(K_{Y_{i+1}} \cdot F_{i+1}) = -1$, so $(\bar{\Delta}_{i+1} \cdot F_{i+1}) \geq 1$. Since $\lfloor \bar{\Delta}_j \rfloor = 0$, F_{i+1} must intersect at least 2 irreducible components of $\bar{\Delta}_{i+1}$ (different from F_{i+1}). The images of these in Y_i show that the blow-up center is a singular point of $\bar{\Delta}_i$. □

These show that Q-modifications are combinatorial objects that are determined by the dual graph $\mathcal{D}(S)$. In fact, the following much stronger result holds, which generalizes [Ste91a, 7.2]. To state it, consider a graph with vertices V and edges E. For a subset $V' \subset V$ the *induced subgraph* contains all the edges in E that go between vertices in V'.

Proposition 26. *Let S_1, S_2 be surfaces with rational singularities and $j : \mathcal{D}(S_1) \hookrightarrow \mathcal{D}(S_2)$ an isomorphism onto an induced subgraph that preserves the labeling of vertices. Then there is a one-to-one correspondence between*

(26.1) *Q-modifications $\tau_1 : S_1^{\mathrm{m}} \dashrightarrow \bar{S}_1$ of S_1, and*

(26.2) *Q-modifications $\tau_2 : S_2^{\mathrm{m}} \dashrightarrow \bar{S}_2$ of S_2, such that $\mathrm{Ex}(\tau_2)$ is supported on $j(\mathcal{D}(S_1))$.*

Under this correspondence, $\bar{S}_1$ and $\bar{S}_2$ have the same singularities.

Proof. Given a Q-modification of S_1, we construct a Q-modification of S_2 as follows.

By (25), $\pi_1^{\mathrm{m}} : \bar{S}_1^{\mathrm{m}} \to S_1^{\mathrm{m}}$ is obtained by repeatedly blowing up singular points of the exceptional divisor. We do the same blow ups to get $\bar{S}_2^{\mathrm{m}} \to S_2^{\mathrm{m}}$. Then j lifts to an embedding $\bar{\jmath}$ of the dual graph of $\mathrm{Ex}(\bar{S}_1^{\mathrm{m}} \to S_1)$ into the dual graph of $\mathrm{Ex}(\bar{S}_2^{\mathrm{m}} \to S_2)$.

Next we get $\bar{\mu}_1 : \bar{S}_1^{\mathrm{m}} \to \bar{S}_1$ by contracting certain curves contained in $\mathrm{Ex}(\bar{S}_1^{\mathrm{m}} \to S_1)$. Using $\bar{\jmath}$, we construct $\bar{\mu}_2' : \bar{S}_2^{\mathrm{m}} \to \bar{S}_2'$ by contracting the corresponding curves of $\mathrm{Ex}(\bar{S}_2^{\mathrm{m}} \to S_2)$. By construction, $\bar{S}_2$ has the same singularities as $\bar{S}_1$.

We need to show that $K_{\bar{S}_2}$ has nonnegative intersection number with every exceptional curve of $\bar{S}_2 \to S_2$. For images of curves in $\mathrm{Ex}(\pi_2^{\mathrm{m}})$ this follows from (22.5). For images of curves in $\mathrm{Ex}(\mu_2)$ this follows from (22.4).

Conversely, let $\tau_2 : S_2^{\mathrm{m}} \to S_2 \leftarrow \bar{S}_2$ be a Q-modification of S_2 as in (26.2). Then (25) shows that $\bar{S}_2^{\mathrm{m}} \to S_2^{\mathrm{m}}$ is obtained by repeatedly blowing up singular points of $j(\mathcal{D}(S_1))$. This shows that the above procedure can be reversed to get $\tau_1 : S_1^{\mathrm{m}} \to S_1 \leftarrow \bar{S}_1$ of S_1. □

Definition 27. Let $p : X \to \mathbb{D}$ be a flat morphism such that X_0 has rational singularities. A *simultaneous Q-modification* is a proper, birational morphism $\pi : \bar{X} \to X$ such that

(27.1) $\pi_t : \bar{X}_t \to X_t$ is a Q-modification for every $t \in \mathbb{D}$, and

(27.2) $K_{\bar{X}}$ is $\mathbb{Q}$-Cartier. Equivalently, $p \circ \pi : \bar{X} \to \mathbb{D}$ is a KSB deformation.

Being a Q-modification is an open property for KSB deformations. Thus if $\pi_0 : \bar{X}_0 \to X_0$ is a Q-modification, then the same holds for nearby t.

The next result, proved in [KSB88, 3.5], says that every flat deformation of a surface with quotient singularities has a simultaneous Q-modification; see [Kol23, 5.41] for the version that we use.

Theorem 28. *Let $p : X \to \mathbb{D}$ be a flat morphism such that X_0 has quotient singularities. Then there is a unique simultaneous Q-modification $\pi : \bar{X} \to X$ such that*

(28.1) *π is an isomorphism over $X \setminus X_0$, and*

(28.2) *$K_{\bar{X}}$ is π-ample.*

(28.3) *If X_t has Du Val singularities for $t \neq 0$, then $\pi_0 : \bar{X}_0 \to X_0$ is a P-modification (29).* □

5. P-modifications

By [KSB88, 3.9], for every quotient singularity S, the irreducible components of $\mathrm{Def}(S)$ are in one-to-one correspondence with the P-modifications of S. We define and study these next.

Definition 29. Let (s, S) be a rational surface singularity. A proper, birational morphism $\pi_P : S_P \to S$ is a *P-modification* if

(29.1) K_{S_P} is π_P-ample, and

(29.2) S_P has only Du Val and $\hat{\mathbb{A}}^2/\frac{1}{rn^2}(1, arn{-}1)$ singularities, where $(a, n) = 1$.

Following [BC94], $\pi_M : S_M \to S$ is an *M-modification* if

(29.1) K_{S_M} is π_M-nef, and

(29.2) S_M has only $\hat{\mathbb{A}}^2/\frac{1}{n^2}(1, na{-}1)$ singularities, where $n > 1$ and $(a, n) = 1$.

Given a P-modification $S_P \to S$, resolving its Du Val singularities and applying the construction (12) gives an M-modification $\pi_{MP} : S_M \to S_P$ such that $K_{S_M} \sim_{\mathbb{Q}} \pi^*_{MP} K_{S_P}$. By [BC94] this establishes a one-to-one correspondence between P- and M-modifications of S.

Note on terminology. For P-modifications we follow [KSB88, 3.8]. A more general version is defined in [Kol91, 6.2.13], which allows more singularities on S_P.

For P-modifications we get the following variant of (26).

Corollary 30. *Using the notation and assumptions of (26), there is a one-to-one correspondence between*

(30.1) *P-modifications $\tau_1 : S_1^{\mathrm{m}} \dashrightarrow \bar S_1$ of S_1, and*

(30.2) *P-modifications $\tau_2 : S_2^{\mathrm{m}} \dashrightarrow \bar S_2$ of S_2, such that $\mathrm{Ex}(\tau_2)$ is supported on*

$$j(\mathcal{D}(S_1)) \cup \{(-2) \text{ curves disjoint from } j(\mathcal{D}(S_1))\}.$$

Proof. The only difference is that, as we go from $\bar S_1$ to $\bar S_2$, we have to ensure that $K_{\bar S_2}$ is ample. That is, we have to contract each connected component of all (-2)-curves disjoint from $j(\mathcal{D}(S_1))$ to a Du Val singularity. □

31 (P-modifications of dihedral quotients)**.** By [Ste91a, Ste93] P-modifications

$$\tau_D : S_D^{\mathrm{m}} \to S_D \leftarrow \bar S_D$$

of $S_D := S(2^2, \mathbf{c})$ are obtained from P-modifications $\tau_C : S_C^{\mathrm{m}} \to S_C \leftarrow \bar S_C$ of $S_C := S(2, \mathbf{c})$ as follows.

(31.1) (C_1 is not contracted by τ_C.) Then C_0', C_0'' are contracted to A_1 points by τ_D.

(31.2) ($C_0', C_1, \dots, C_i$ are contracted by τ_C to an A_{i+1} point.) Then C_0'' is also contracted by τ_D, giving a D_{i+2} point.

(31.3) ($C_1, \dots, C_i$ are contracted by τ_C to a non-DV point.) Then C_0', C_0'' are not contracted by τ_D.

(31.4) ($C_0', C_1, \dots, C_i$ are contracted by τ_C to a non-DV point.) Then C_0'' is not contracted by τ_D. There is also a symmetric version, where $C_0'', C_1, \dots, C_i$ are contracted by τ_D but C_0' is not contracted.

The P-modifications that appear in Theorem 2 have several characterizations.

Proposition 32. *Let $\pi_P : S_P \to S$ be a P-modification of a dihedral quotient singularity $(S^d_{n,q}, D^d_{n,q})$ with reduced exceptional curve E_P. The following are equivalent.*

(32.1) *S_P is one of the cases (31.1–2).*

(32.2) *C_0', C_0'' are contracted to Du Val point(s) on S_P.*

(32.3) *$(S_P, D_P + E_P)$ is log canonical.*

(32.4) *$K_{S_P} + D_P + E_P$ is numerically π_P-trivial.*

Proof. The equivalence of (1) and (2) is clear. To see (3) note that if $C_1, \dots, C_i$ are contracted by τ_C to a non-DV point, then there are 3 curves in $D_P + E_P$ through this point, and if $C_0', C_1, \dots, C_i$ are contracted by τ_1 to a non-DV point, then C_0'' meets the resolution at C_1, which is not an end curve of the chain for $i \geq 2$. For $i = 1$ it is an end curve, but then another curve of $D_P + E_P$ meets C_1.

For (4), note that $K_{S^{\mathrm{m}}} + \sum_{i>0} C_i + \frac{1}{2}(C_0' + C_0'')$ is numerically trivial on $S^{\mathrm{m}} \to S$. Thus $K_{S_P} + D_P + E_P - \frac{1}{2}(C_0' + C_0'')$ is numerically π_P-trivial. Thus (4) holds iff both C_0', C_0'' are contracted. This happens only in cases (31.1–2). □

Remark 33. Note that the cases (32) are in bijection with P-modifications of $T := S(c_2, \dots, c_s)$. Indeed, let $\tau_T : T^{\mathrm{m}} \to T \leftarrow T_P$ be a P-modification.

If $c_1 > 2$ then we get τ_D by not contracting C_1 but contracting C_0', C_0''.

If $c_1 = 2$ and C_2 is contracted to a non-DV point by τ_T, then again we contract C_0', C_0'' but not C_1. If $C_2, \dots, C_i$ are contracted to an A_{i-1} point by τ_T, then $C_0', C_0'', C_1, \dots, C_i$ are contracted to a D_{i+2} point by τ_D. Finally if C_2 is not contracted, then C_0', C_0'', C_1 are contracted to an A_3 point by τ_D.

The main reason to study P- and M-modifications is the following, essentially proved in [KSB88, 3.9] and [BC94].

Theorem 34. *Let $(0, S)$ be a rational surface singularity, $\pi_P : S_P \to S$ a P-modification, and $\pi_{MP} : S_M \to S_P$ the corresponding M-modification. Then, using the notation of (17.1), there is a commutative diagram*

$$\begin{array}{ccccc} \mathrm{Univ}_{\mathrm{KSB}}(S_M) & \xrightarrow{c_{MP}} & \mathrm{Univ}_{\mathrm{KSB}}(S_P) & \xrightarrow{c_P} & \mathrm{Univ}(S) \\ u_M \downarrow & & u_P \downarrow & & u_S \downarrow \\ \mathrm{Def}_{\mathrm{KSB}}(S_M) & \xrightarrow{\tau_{MP}} & \mathrm{Def}_{\mathrm{KSB}}(S_P) & \xrightarrow{\tau_P} & \mathrm{Def}(S) \end{array} \tag{34.1}$$

where

(34.2) $\mathrm{Def}_{\mathrm{KSB}}(S_M)$ *and* $\mathrm{Def}_{\mathrm{KSB}}(S_P)$ *are smooth of the same dimension,*

(34.3) $\tau_{MP} : \mathrm{Def}_{\mathrm{KSB}}(S_M) \to \mathrm{Def}_{\mathrm{KSB}}(S_P)$ *is finite and Galois,*

(34.4) $\tau_P : \mathrm{Def}_{\mathrm{KSB}}(S_P) \to \mathrm{Def}(S)$ *is a finite, birational morphism onto an irreducible component of* $\mathrm{Def}(S)$.

Sketch of proof. The existence of the diagram follows from (35).

Since KSB deformations are locally unobstructed, the obstruction to smoothness lies in H^2 of the tangent sheaf. If $X \to U$ is proper of fiber dimension ≤ 1, and U is affine, then H^i of any coherent sheaf on X vanishes for $i \geq 2$. This implies smoothness in (34.2); see (47) for details.

The relation between P- and M-modifications is established in [BC94]. The Galois group in (34.3) is a product of reflection groups, determined by the singularities of S_P; see [BC94].

In order to show (34.4), note first that every fiber of u_P is a P-modification of the corresponding deformation of S. We check in (59) that P-modifications $\pi_P : S_P \to S$ do not have nontrivial 1-parameter deformations fixing S. This shows that τ_P is finite. As in [Art74], τ_P is onto an irreducible component of $\mathrm{Def}(S)$ by openness of versality. Thus $\tau_P : \mathrm{Def}_{\mathrm{KSB}}(S_P) \to \mathrm{Def}(S)$ is a finite cover. In order to show that it has degree 1, let $T \to \mathrm{Def}(S)$ be a morphism from the spectrum of a DVR that maps the closed (resp. generic) point the closed (resp. generic) point. By what we already proved, there is a finite $T' \to T$ such that the pull-back of $\mathrm{Univ}(S)$ to $T' \to T \to \mathrm{Def}(S)$ is obtained from a KSB deformation of S_P. We may assume that T'/T is Galois. As noted in [KSB88, p.312], the action of $\mathrm{Gal}(T'/T)$ lifts to an action on this KSB deformation. Taking the quotient shows that the pull-back of $\mathrm{Univ}(S)$ to $T \to \mathrm{Def}(S)$ is obtained from a KSB deformation of S_P. Thus $T \to \mathrm{Def}(S)$ lifts to $T \to \mathrm{Def}_{\mathrm{KSB}}(S)$, so τ_P is birational. □

Remark 34.5. By [Ste91a], τ_P is an isomorphism onto an irreducible component of $\mathrm{Def}(S)$ for cyclic quotients, but there are dihedral examples where τ_P is not an isomorphism. Also, τ_P is birational but not necessarily finite for the more general P-modifications considered in [Kol91, Sec.6].

The proof of the following is summarized in [KM92, 11.4], which in turn relies mainly on [Wah76].

Lemma 35. *Let $p_X : X \to S$ be a flat morphism of complex analytic spaces. Fix $s \in S$ and let $\pi_s : X_s \to Y_s$ be a proper morphism such that $(\pi_s)_*\mathcal{O}_{X_s} = \mathcal{O}_{Y_s}$, $R^1(\pi_s)_*\mathcal{O}_{X_s} = 0$ and* $\mathrm{Ex}(\pi_s)$ *is* *proper. Then, after shrinking $s\in S$, there is a unique commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{\pi} & Y \\ p_X\downarrow & & \downarrow p_Y \\ S & = & S, \end{array}$$

where $p_Y: Y\to S$ is flat, and $\pi|_{X_s}=\pi_s$. □

Next we show that KSB deformations of a pair can be understood in terms of doubly KSB deformations of P-modifications. This is very similar to [KSB88, 3.9].

Theorem 36. *Let (S,D) be an lc pair such that $\operatorname{Sing} S\subset \operatorname{Supp} D$. There is a bijection between the following sets.*

(36.1) *Irreducible components of $\operatorname{Def}_{\mathrm{KSB}}(S,D)$ (as in Definition 15).*

(36.2) *P-modifications $\pi_P: S_P\to S$ such that*

(36.2.a) *$K_{S_P}+D_P+E_P\sim_{\mathbb{Q}}0$, and*

(36.2.b) *the singularities of $\big(S_P, D_P+E_P\big)$ have doubly KSB deformations with Du Val generic fibers.*

Proof. Assume that $\pi_P: S_P\to S$ satisfies (36.2). We note in (47) that there is a semi-universal doubly KSB deformation

$$u_{S_P}:\big(\mathbf{S}_P,\mathbf{D}_P\big)\to \operatorname{Def}_{\mathrm{dKSB}}\big(S_P,D_P+E_P\big).$$

(The deformation of D_P+E_P is an irreducible divisor, which we denote by $\mathbf{D}_P$.) As in (34), the contraction π_P extends to

$$u_{S_P}:\big(\mathbf{S}_P,\mathbf{D}_P\big)\xrightarrow{\pi}\big(\mathbf{S},\mathbf{D}\big)\xrightarrow{u_S}\operatorname{Def}_{\mathrm{dKSB}}\big(S_P,D_P+E_P\big),$$

where u_S is a flat deformation of S. Since $R^1\pi_*\mathcal{O}_{\mathbf{S}_P}=0$ and $K_{\mathbf{S}_P}+\mathbf{D}_P$ is $\mathbb{Q}$-linearly trivial, its push-forward $K_{\mathbf{S}}+\mathbf{D}$ is $\mathbb{Q}$-Cartier by [KM92, 12.1.4]. Thus u_S is a KSBA deformation of (S,D).

As we noted, $\mathbf{D}_P$ is irreducible, and $-\mathbf{D}_P$ is numerically equivalent to the relative canonical class, hence π-ample. Thus π is an isomorphism on the generic fiber. So, by openness of versality [Art74], $\operatorname{Def}_{\mathrm{dKSB}}\big(S_P,D_P+E_P\big)$ maps onto a whole irreducible component of $\operatorname{Def}_{\mathrm{KSB}}(S,D)$. Thus (36.2) ⇒ (36.1). (Note that this implication holds without any restriction on c.)

Conversely, take a general 1-parameter KSB deformation $p:(\mathbf{S},\mathbf{D})\to\mathbb{D}$ of (S,D). We check in Proposition 48 that the generic fiber has only Du Val singularities. Let $\bar\pi:\bar{\mathbf{S}}\to\mathbf{S}$ be the simultaneous P-modification as in (28.3), with central fiber $\pi_P: S_P\to S$. Then π_P is a isomorphism over the generic fiber, hence the generic fiber of $\bar{\mathbf{S}}\to\mathbb{D}$ also has only Du Val singularities.

Let $\bar{\mathbf{D}}$ be the birational transform of $\mathbf{D}$. Since π is small, $K_{\bar{\mathbf{S}}}+\bar{\mathbf{D}}\sim_{\mathbb{Q}}\bar\pi^*\big(K_{\mathbf{S}}+\mathbf{D}\big)$, and $(\bar{\mathbf{S}},\bar{\mathbf{D}}+S_P)$ is also lc. By adjunction as in [Kol13, 4.9], we get that the pair $\big(S_P,\bar{\mathbf{D}}|_{S_P}\big)$ is also lc.

Since K_{S_P} is π_P-ample, $-\bar{\mathbf{D}}|_{S_P}$ is also π_0-ample, hence its support must contain the hole exceptional divisor. Each divisor in $\bar{\mathbf{D}}|_{S_P}$ appears with integral coefficient. Therefore $\bar{\mathbf{D}}|_{S_P}$ is the sum of D_P (the birational transform of D) plus the π_P-exceptional divisor E_P, so

$$K_{S_P}+(D_P+E_P)=\big(K_{\bar{\mathbf{S}}}+\bar{\mathbf{D}}\big)|_{S_P}\sim_{\mathbb{Q}}\pi^*\big(K_{\mathbf{S}}+\mathbf{D}\big)\sim_{\mathbb{Q}}0. \tag{36.4}$$

Thus (36.1) ⇒ (36.2). □

6. Tangent sheaves

In order to compute the dimensions of the deformation spaces, we need various results on tangent and logarithmic tangent sheaves for modifications of rational singularities. These are mostly taken from [Lau73a, Lau73b] and [LP07, KLP12]. Note that [LP07, KLP12] focus on the global aspects, which tend to be more subtle. See also [PPS23, RU22, DRU22] for similar computations.

37 (Tangent and logarithmic tangent sheaves)**.** Let S be a smooth surface and $D \subset S$ a smooth curve with normal bundle N_D. There is a natural map $T_S \to N_D$. Its kernel is the *logarithmic tangent bundle* of the pair (S, D), denoted by $T_S(-\log D)$. Thus we have an exact sequence

$$0 \to T_S(-\log D) \to T_S \to N_D \to 0. \tag{37.1}$$

The restriction of $T_S(-\log D)$ to D sits in an exact sequence

$$0 \to \mathcal{O}_D \to T_S(-\log D)|_D \to T_D \to 0. \tag{37.2}$$

Locally, if $D = (y = 0)$, then the kernel $\mathcal{O}_D$ is generated by $y\frac{\partial}{\partial y}$ and the quotient T_D by $\frac{\partial}{\partial x}$.

Next assume that S is smooth but $D \subset S$ is a nodal curve with normalization $\tau : \bar{D} \to D$. The *immersed normal sheaf* of D is

$$N_{\bar{D}} := \tau_* \operatorname{coker}\bigl[T_{\bar{D}} \to \tau^* T_S\bigr]. \tag{37.3}$$

Then (37.1) becomes

$$0 \to T_S(-\log D) \to T_S \to N_{\bar{D}} \to 0,$$

where $T_S(-\log D)$ is locally free. More generally, if $D = D_1 + D_2$, then we have

$$0 \to T_S(-\log(D_1 + D_2)) \to T_S(-\log D_1) \to N_{\bar{D}_2} \to 0. \tag{37.4}$$

The sequence (37.2) now gives

$$0 \to \mathcal{O}_{\bar{D}} \to \tau^* T_S(-\log D) \to T_{\bar{D}}(-N) \to 0, \tag{37.5}$$

where $N \subset \bar{D}$ is the preimage of the nodes of D.

If S is a normal surface and $D \subset S$ a reduced curve, then there is a finite subset $P \subset S$ such that $(S \setminus P, D \setminus P)$ is a smooth pair. Let $j : S \setminus P \hookrightarrow S$ be the natural injection. The *tangent sheaf* of S and the *logarithmic tangent sheaf* of (S, D) are defined as

$$T_S := j_* T_{S\setminus P}, \quad \text{and} \quad T_S(-\log D) := j_*\bigl(T_{S\setminus P}(-\log(D \setminus P))\bigr). \tag{37.6}$$

In the log canonical case, we have the following close analog of (37.4).

Lemma 38. *Let $(S, D_1 + D_2)$ be a log canonical pair, $\mu : S^{\mathrm{m}} \to S$ the minimal resolution, and $D_i^{\mathrm{m}} := \mu_*^{-1} D_i$ with normalizations $\tau_i : \bar{D}_i^{\mathrm{m}} \to D_i^{\mathrm{m}}$. Then there is an exact sequence*

$$0 \to T_S(-\log(D_1 + D_2)) \to T_S(-\log(D_1)) \to \mu_* N_{\bar{D}_2^{\mathrm{m}}} \to 0. \tag{38.1}$$

Proof. Over the smooth locus of S, this is (37.4). Thus it remains to check what happens at the singular points that are contained in $D_1 + D_2$. For this we can use local analytic coordinates.

If a finite group G acts with isolated fixed points on a surface X, then the local sections of $T_{X/G}$ are the G-invariant local sections of T_X. Thus the tangent sheaf of $S_{n,q} = \hat{\mathbb{A}}^2_{xy}/\frac{1}{n}(1, q)$ is generated by

$$x\tfrac{\partial}{\partial x}, y^{q'}\tfrac{\partial}{\partial x}, y\tfrac{\partial}{\partial y}, x^q\tfrac{\partial}{\partial y}. \tag{38.2}$$

If $B_x \subset S_{n,q}$ is the image of the x-axis, then we see that $T_S(-\log B_x)$ is generated by $x\frac{\partial}{\partial x}, y^{q'}\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}$, and the quotient $T_S/T_S(-\log B_x)$ is generated by $x^q\frac{\partial}{\partial y}$.

For the minimal resolution of $S_{n,q}$ we use the chart given in (9.2). The chain rule gives that

$$(\pi_1)_*\bigl(\tfrac{\partial}{\partial y_2}\bigr) = x^q\tfrac{\partial}{\partial y}.$$

That is, $T_S/T_S(-\log B_x)$ is naturally isomorphic to the normal bundle of $B_x^{\mathrm{m}} \subset S_{n,q}^{\mathrm{m}}$. This shows (38.1) at the cyclic quotient points.

At a dihedral point $(S^d_{n,q}, B^d_{n,q})$, the minimal resolution can be obtained by first taking a double cover $(S_{N,Q}, D_{N,Q}) \to (S^d_{n,q}, B^d_{n,q})$, resolving $S_{N,Q}$ and then quotienting out by the involution. The latter is fixed point free along the birational transform of $D_{N,Q}$, so the normal bundle computation for the birational transform of $B^d_{n,q}$ is the same as for the birational transform of $D_{N,Q}$. Finally note that $Q^2 \equiv 1 \mod N$, so the quotient $T_S/T_S(-\log B_x)$ is generated by $x^Q \frac{\partial}{\partial y} + y^Q \frac{\partial}{\partial x}$, where x, y are the orbifold coordinates on $S_{N,Q}$. This shows (38.1) at the dihedral quotient points. □

Proposition 39. *Let (X, D) be a log canonical pair with quotient singularities. Let $\pi : X^{\mathrm{m}} \to X$ be either the minimal resolution, or the blow-up of a node of D where X is smooth. Let E^{m} be the exceptional curve and D^{m} the birational transform of D. Then*

(39.1) $\pi_* T_{X^{\mathrm{m}}}\big(-\log(E^{\mathrm{m}} + D^{\mathrm{m}})\big) = T_X(-\log D)$,

(39.2) $R^1\pi_* T_{X^{\mathrm{m}}}\big(-\log(E^{\mathrm{m}} + D^{\mathrm{m}})\big) = R^1\pi_* T_{X^{\mathrm{m}}}\big(-\log(E^{\mathrm{m}})\big) = 0$, *and*

(39.3) $H^i\big(X, T_X(-\log D)\big) = H^i\big(X^{\mathrm{m}}, T_{X^{\mathrm{m}}}(-\log(E^{\mathrm{m}} + D^{\mathrm{m}}))\big)$ *for every i.*

Proof. For (39.1) the blow-up of a node of D is a simple computation. At a singular point of X, it suffices to show the $D = \emptyset$ case. That is, all local generators of T_X lift to sections of $T_{X^{\mathrm{m}}}\big(-\log E^{\mathrm{m}}\big)$. This is done in [BW74, 1.2].

The $D = \emptyset$ case of (39.2) is discussed in (41.1). For general D, we use (37.4) to get

$$0 \to T_{X^{\mathrm{m}}}\big(-\log(E^{\mathrm{m}} + D^{\mathrm{m}})\big) \to T_{X^{\mathrm{m}}}\big(-\log(E^{\mathrm{m}})\big) \to N_{D^{\mathrm{m}}} \to 0.$$

Pushing this forward, we get that

$$R^1\pi_* T_{X^{\mathrm{m}}}\big(-\log(E^{\mathrm{m}} + D^{\mathrm{m}})\big) \cong \operatorname{coker}\big[T_S \to \pi_* N_{\bar{D}^{\mathrm{m}}}\big],$$

and the latter is 0 by (38). The Leray spectral sequence now gives (39.3). □

Remark 39.4. Let $\pi : X' \to X$ be the blow-up of a smooth point with exceptional curve E. Then $h^1(X', T_{X'}(-\log E)) = h^1(X, T_X) + 2$. Deformations of X' are obtained by deforming X and also the point.

Complement 39.5. The computations also show that if all the singularities are $(S_{n,q}, D_{n,q})$, then $T_X(-\log D)$ is locally free and $\pi^* T_X(-\log D) \cong T_{X^{\mathrm{m}}}\big(-\log(E^{\mathrm{m}} + D^{\mathrm{m}})\big)$.

We can now compute the global contribution to the dimension of our deformation spaces.

Proposition 40. *Let S be an affine surface with rational singularities and $D \subset S$ a reduced curve. Let $\mu : S^{\mathrm{m}} \to S$ be the minimal resolution with reduced exceptional set E^{m}, and $\pi : \bar{S} \to S$ a Q-modification with reduced exceptional set $\bar{E}$. Let D^{m} resp. $\bar{D}$ denote the birational transforms of D on S^{m} resp. $\bar{S}$. Assume that $(\bar{S}, \bar{E} + \bar{D})$ is lc and $(S^{\mathrm{m}}, E^{\mathrm{m}} + D^{\mathrm{m}})$ is normal crossing.*

For an irreducible component $\bar{E}_i \subset \bar{E}$, let $e_i \in \mathbb{N}$ be the negative of the self-intersection of its birational transform on the minimal resolution of $\bar{S}$. Then

(40.1) $h^1\big(\bar{S}, T_{\bar{S}}(-\log \bar{D})\big) = h^1\big(\bar{S}, T_{\bar{S}}(-\log(\bar{E} + \bar{D}))\big) + \sum_{i\in I}(e_i - 1)$, *and*

(40.2) $h^1\big(\bar{S}, T_{\bar{S}}(-\log(\bar{E} + \bar{D}))\big) = h^1\big(S^{\mathrm{m}}, T_{S^{\mathrm{m}}}(-\log(E^{\mathrm{m}} + D^{\mathrm{m}}))\big)$.

Proof. (38) gives an exact sequence

$$0 \to T_{\bar{S}}\big(-\log(\bar{E} + \bar{D})\big) \to T_{\bar{S}}(-\log \bar{D}) \to \oplus \mathcal{O}_{\bar{E}_i}(-e_i) \to 0.$$

Here $h^0\big(\bar{E}_i, \mathcal{O}_{\bar{E}_i}(-e_i)\big) = 0$ since $e_i > 0$, so taking cohomologies gives the first claim. For the second, we use the notation of (22), and claim that

$$h^1\big(\bar{S}^{\mathrm{m}}, T_{\bar{S}^{\mathrm{m}}}(-\log(\bar{F}^{\mathrm{m}} + \bar{D}^{\mathrm{m}}))\big)$$

equals both terms in (40.2), where $\bar{F}^{\mathrm{m}}$ is the reduced exceptional set of $\bar{S}^{\mathrm{m}} \to S$. For $\bar{S}^{\mathrm{m}} \to \bar{S}$ we use the minimal resolution case of (39). Next, since $\bar{S}^{\mathrm{m}} \to S^{\mathrm{m}}$ is a composite of node blow-ups by (25), we can repeatedly use the node blow-up case of (39). □

7. Dimension formulas

We compute the dimension of the irreducible components of various deformation spaces using invariants of the singularity and its P-modifications. The formulas are simplest when S is determined by $\mathcal{D}(S)$.

41. Let (s, S) be a rational surface singularity with minimal resolution $\mu : S^{\mathrm{m}} \to S$ and reduced exceptional curve $C^{\mathrm{m}} = \cup_{i\in I} C_i$.

By [Ser06, Sec.3.4.4], $H^1\big(S^{\mathrm{m}}, T_{S^{\mathrm{m}}}(-\log C^{\mathrm{m}})\big)$ is the tangent space to those deformations of S^{m} where every C_i lifts. Equivalently, these are those deformations of S that preserve the dual graph. The dimension of this tangent space is usually hard to compute, but it is known in several important cases.

Quotient singularities 41.1. The dual graph determines the singularity, hence

$$H^1\big(S^{\mathrm{m}}, T_{S^{\mathrm{m}}}(-\log C^{\mathrm{m}})\big) = 0.$$

This has been long known; see for example [Bri68], [BPV84, III.5.1] or [Kol13, 3.32].

Taut singularities 41.2. These are the singularities that are determined by their dual graphs. For these $H^1\big(S^{\mathrm{m}}, T_{S^{\mathrm{m}}}(-\log C^{\mathrm{m}})\big) = 0$. A rather laborious result of [Lau73a, Lau73b] says that this vanishing implies tautness, with a few exceptions; these are enumerated in [Lau73b, 3.2]. The complete list of taut singularities is given in [Lau73b].

A simple case is when $\mathcal{D}(S)$ has only 1 fork, which has degree 3 and self-intersection ≤ -3. This is used for the $W(p, q, r)$ singularities studied in [PS22].

Weighted homogeneous singularities 41.3. For these $\mathcal{D}(S)$ has only 1 fork. Let d_0 denote its degree and $-c_0$ its self-intersection. By [Lau73b, 4.1.III], if $c_0 \geq 2d_0 - 3$ then

$$H^1\big(S^{\mathrm{m}}, T_{S^{\mathrm{m}}}(-\log C^{\mathrm{m}})\big) = d_0 - 3.$$

These are used in [JS23] under the (mostly) weaker assumption $c_0 \geq d_0 + 3$.

For P-modifications, we need only the simplest numerical invariants.

Definition 42. Let $\pi : S_P \to S$ be a P-modification of a normal singularity. Let

$$\{p_j \in S_P : j \in J_P\} \quad \text{resp.} \quad \{E_i \subset S_P : i \in I_P\}$$

denote the singular points of S_P, respectively the irreducible, exceptional curves of $\pi : S_P \to S$. Set $r_j = r$ if p_j is of type A_r, D_r, E_r or $\hat{\mathbb{A}}^2/\frac{1}{rn^2}(1, arn-1)$ for some $n > 1$. Note that $r_j = \dim \mathrm{Def}_{\mathrm{KSB}}(p_j, S_P)$ by (12.2).

Let $e_i \in \mathbb{N}$ denote the negative of the self-intersection of the birational transform of E_i on the minimal resolution of S_P.

We can now state the first dimension formula.

Proposition 43. *Let $\pi : S_P \to S$ be a P-modification of an affine surface with rational singularities, and I_P, J_P as in (42). Let $S^{\mathrm{m}} \to S$ be the minimal resolution, with reduced exceptional curve C^{m}. Then*

$$\dim \mathrm{Def}_{\mathrm{KSB}}(S_P) = \textstyle\sum_{j\in J_P} r_j + \sum_{i\in I_P}(e_i - 1) + h^1\big(S^{\mathrm{m}}, T_{S^{\mathrm{m}}}(-\log C^{\mathrm{m}})\big). \tag{43.1}$$

In particular, if S has taut singularities, then

$$\dim \mathrm{Def}_{\mathrm{KSB}}(S_P) = \textstyle\sum_{j\in J_P} r_j + \sum_{i\in I_P}(e_i - 1). \tag{43.2}$$

Proof. First note that, by (47),

$$\dim \mathrm{Def}_{\mathrm{KSB}}(S_P) = \textstyle\sum_{j\in J_P} \dim \mathrm{Def}_{\mathrm{KSB}}(p_j, S_P) + h^1(S_P, T_{S_P}).$$

Here $r_j = \dim \mathrm{Def}_{\mathrm{KSB}}(p_j, S_P)$ by (12.2), this explains the summand $\sum_{j\in J_P} r_j$. The computation of $h^1(S_P, T_{S_P})$ is the $D = 0$ special case of (40). $\square$

For KSB deformations of pairs we have the following. The dihedral case is discussed in (46).

Theorem 44. *Let $\pi : (S_P, D_P + E_P) \to (S, D) := (S_{n,q}, D_{n.q})$ be a P-modification of a cyclic quotient singularity and J_P as in (42). Then*

$$\dim \operatorname{Def}_{\mathrm{dKSB}}(S_P, D_P + E_P) = \textstyle\sum_{j\in J_P} r_j + |\textit{type } A \textit{ points}|. \tag{44.1}$$

Clarification 44.2. Here |type A points| is the number of points where the pair $(S_P, D_P + E_P)$ is locally of the form $\bigl((xy = z^{r+1}), (xy = 0)\bigr)$ for some $r \geq 0$. For $r = 0$ the surface is smooth but the curve is singular.

Proof. If p_j is a singular point of S_P or of $D_P + E_P$, then $\dim \operatorname{Def}_{\mathrm{dKSB}}(p_j, S_P, D_P + E_P)$ is

(44.3) r for a singularity is of type $\hat{\mathbb{A}}^2/\frac{1}{rn^2}(1, arn{-}1)$ by Lemma 18, but

(44.4) $r + 1$ if the singularity is of type A_r.

(Here we need to count $\bigl(\hat{\mathbb{A}}^2, (xy = 0)\bigr)$ as type A_0.) Thus the right hand side of (44.1) is the sum of the local terms in (47).

It remains to show that the global term $h^1\bigl(S_P, T_{S_P}(-\log(D_P + E_P))\bigr)$ vanishes. This follows from (40.2), (39.2) and (41.1.). □

Remark 45. Comparing (43) and (44) we see that

$$\dim \operatorname{Def}_{\mathrm{dKSB}}(S_P, D_P + E_P) \leq \dim \operatorname{Def}_{\mathrm{KSB}}(S_P),$$

and the inequality is strict with a few exceptions. These are triple points and also P-resolutions of the form

$$\bullet - \boxed{A_{n_1}} - 2 - \boxed{\tfrac{1}{rn^2}(1, arn{-}1)} - 2 - \boxed{A_{n_2}} - \bullet$$

The dihedral version of (44) is the following.

Theorem 46. *Let $\pi : (S_P, D_P + E_P) \to (S, D) = (S^d_{n,q}, D^d_{n.q})$ be a P-modification of a dihedral quotient singularity that satisfies the conditions of (32), and J_P as in (42). Then*

$$\dim \operatorname{Def}_{\mathrm{dKSB}}(S_P, D_P + E_P) = \textstyle\sum_{j\in J_P} r_j + |\textit{type } A \textit{ points}| - 2. \tag{46.1}$$

Proof. Note that here $(S_P, D_P + E_P)$ has a singular point of type

$$\bigl((z^2 = x(y^2 - x^{r-2})), (x = z = 0)\bigr)$$

whose KSB deformation space has dimension $r{-}2$ by (57). (We have 2 singular points if $r = 2$.) This is why we need to subtract 2 on the right hand side of (46.1). The rest of the argument is as for (44). □

We used the following result, whose proof is summarized in [KM92, 11.4].

Lemma 47. *Let S be an affine surface, $\tau : X \to S$ a proper, birational morphism and $D \subset X$ a reduced curve. Assume that X is normal and let $\{p_j : j \in J\}$ be the points of $\operatorname{Sing} X \cup \operatorname{Sing} D$. Then the restriction maps*

(47.1) $\operatorname{Def}(X, D) \to \times_{j\in J} \operatorname{Def}(p_j, X, D)$,

(47.2) $\operatorname{Def}_{\mathrm{KSB}}(X, D) \to \times_{j\in J} \operatorname{Def}_{\mathrm{KSB}}(p_j, X, D)$, *and*

(47.3) $\operatorname{Def}_{\mathrm{dKSB}}(X, D) \to \times_{j\in J} \operatorname{Def}_{\mathrm{dKSB}}(p_j, X, D)$

are smooth of relative dimension $h^1\bigl(X, T_X(-\log D)\bigr)$. □

By (20.1), the pairs $(S_{n,q}, B_{n,q})$ are KSB rigid. By contrast, we show that the pairs $(S_{n,q}, D_{n,q})$ are KSB smoothable, and the pairs $(S^d_{n,q}, D^d_{n,q})$ have KSB deformations whose general fibers have only singularities of type $(S_{2,1}, D_{2,1}) \cong \bigl((xy = z^2), (x = z = 0)\bigr)$; we call these *almost smoothings.* More precisely, we have the following.

Proposition 48. *Every irreducible component of $\operatorname{Def}_{\mathrm{KSB}}(S_{n,q}, D_{n,q})$ contains smoothings, and every irreducible component of $\operatorname{Def}_{\mathrm{KSB}}(S^d_{n,q}, D^d_{n,q})$ contains almost smoothings.*

Proof. By (20.2–3), K_S+D (resp. $2(K_S+D)$) is Cartier, so the same holds for its KSB deformations, hence every singularity of a KSB deformation of our cyclic (resp. dihedral) pair is again cyclic (resp. dihedral or cyclic). By openness of versality [Art74], it is thus enough to show that each cyclic (resp. dihedral) pair is smoothable (resp. almost smoothable).

We find such deformations in the Artin component. For a cyclic pair (S,D), look at the minimal resolution

$$\pi_0 : (S^{\mathrm{m}}, D^{\mathrm{m}} + E^{\mathrm{m}}) \to (S,D).$$

By (47), there is a deformation $p : (\mathbf{S}^{\mathrm{m}}, \mathbf{D}^{\mathrm{m}}) \to \mathbb{D}$ whose special fiber is $(S^{\mathrm{m}}, D^{\mathrm{m}} + E^{\mathrm{m}})$ and whose generic fiber is a smooth pair with $\mathbf{D}^{\mathrm{m}}_{gen}$ irreducible. Next π_0 extends to a contraction

$$\pi : (\mathbf{S}^{\mathrm{m}}, \mathbf{D}^{\mathrm{m}}) \to (\mathbf{S}, \mathbf{D}) \to \mathbb{D},$$

and $(\mathbf{S}, \mathbf{D}) \to \mathbb{D}$ is a KSB smoothing since $K_{S^{\mathrm{m}}} + D^{\mathrm{m}} + E^{\mathrm{m}} \sim 0$ by (20.4).

For a dihedral pair (S,D), we start with $\pi_0 : (S', D'+E') \to (S,D)$ which is obtained from the minimal resolution by contracting the curves C'_0, C''_0 (as in Notation 10). The key point is that $2\big(K_{S'} + D' + E'\big) \sim 0$ by (20.5). The above argument now works for this case too, except that we have 2 singularities of type $\big((xy=z^2),(x=z=0)\big)$. These are KSB rigid, so persist in every deformation. □

8. Fractional coefficients

We consider which of the previous results generalize to pairs $(S, D=\sum c_iD_i)$.

49 (Applying the method of [KSB88, 3.9])**.** Let $(S,\Delta=\sum c_iD_i)$ be a 2-dimensional lc pair, and $p:(\mathbf{S},\mathbf{\Delta})\to\mathbb{D}$ a 1-parameter KSBA deformation of it. Possibly after a base change, [Kol23, 5.41] gives a small modification $\pi:(\mathbf{S}',\mathbf{\Delta}')\to(\mathbf{S},\mathbf{\Delta})$, such that $K_{\mathbf{S}'}$ and $\mathbf{\Delta}'$ are $\mathbb{Q}$-Cartier, with central fiber

$$\pi_0 : \big(S', \Delta' + E' := \mathbf{\Delta}'|_{S'}\big) \to (S,\Delta)$$

satisfying the following.

(49.1) $\pi_0 : S' \to S$ is a Q-modification,

(49.2) $K_{S'}$ is π_0-ample,

(49.3) Δ' is the birational transform of Δ and E' is π_0-exceptional,

(49.4) $K_{S'}+\Delta'+E' \sim_{\mathbb{Q}} 0$, and

(49.5) the coefficients in E' are $\mathbb{N}$-linear combinations of the c_i.

Given (S,Δ), we can now proceed in several steps.

Step 1. Find all Q-modifications $\pi_0 : S' \to S$ such that $K_{S'}$ is π_0-ample.

Step 2. The Hodge index theorem shows that E' is uniquely determined by condition (49.4). Thus we get $\pi_0 : (S', \Delta'+E') \to (S,\Delta)$.

Step 3. In many cases condition (49.5) is not satisfied by E'. Then this $\pi_0 : S' \to S$ is excluded. For some pairs (S,Δ) this always holds, and then there are no KSBA deformations.

Step 4. Unfortunately, a given $\pi_0 : S' \to S$ that satisfies conditions (49.1–5) need not correspond to an irreducible component of $\mathrm{Def}_{\mathrm{KSBA}}(S,\Delta)$. The main difficulty is with irreducible components where a general surface has non-DV singularities. So in practice the hope is to get a short list of the possible $\pi_0 : S' \to S$, and then examine each one using other considerations.

Thus Steps 1–3 lead to the following.

Task 50. Let $(S,\Delta=\sum_i c_iD_i)$ be an lc pair. Find all Q-modifications $\pi : S' \to S$ with reduced exceptional curve $E' = \sum_j E'_j$ and $e_j \in [0,1]$ such that

(50.1) $K_{S'} + \sum_i c_iD'_i + \sum_j e_jE'_j \sim_{\mathbb{Q}} 0$, and

(50.2) the e_j are $\mathbb{N}$-linear combinations of the c_i.

For any given pair (S,Δ), this is an effectively doable linear algebra problem, but there are likely many cases if the c_i are small. However, if $\Delta = cD$ and $c > \frac{1}{2}$, then we must have $e_j = c$ for every j, hence we get the conditions

$$\big((K_{S'} + cD' + cE')\cdot E'_j\big) = 0 \quad \text{for every } j. \tag{50.3}$$

We can always solve for c if there is a single exceptional curve, but we expect no solutions if there are several.

We start by writing down solutions of (50.1), and then use these to prove that (50.3) has very few solutions.

51 (Discrepancy divisors)**.** There are some easy solutions of (50.1).

(51.1) $(S_{n,q}, D_{n,q})$. Let $\pi : S' \to S_{n,q}$ be a Q-modification with reduced exceptional divisor E'. Then $K_{S'} + D' + E' \sim_{\mathbb{Q}} 0$. Indeed, this holds for the minimal resolution by (20.4), and continues to hold since we blow up only nodes.

(51.2) $(S^d_{n,q}, D^d_{n,q})$. On the minimal resolution $S^{\mathrm{m}} \to S := S^d_{n,q}$, we have

$$K_{S^{\mathrm{m}}} + D^{\mathrm{m}}_{n,q} + \tfrac{1}{2}(C'_0 + C''_0) + \textstyle\sum_{i=1}^{s} C_i \sim_{\mathbb{Q}} 0$$

by (20.5). As we blow up nodes, new curves appear with half integer coefficients that are ≤ 1.

(51.3) $(S_{n,q}, B_{n,q})$. The formula is quite complicated already for the minimal resolution, see [Kol13, 3.32–34]. We will not use these. Instead, we will choose a divisor $\bar{B}_{n,q}$ so that $B_{n,q} + \bar{B}_{n,q} = D_{n,q}$ and work with (51.1) instead.

The solution of (50) is especially simple for the pairs $(S_{n,q}, cD_{n,q})$.

Lemma 52. *Let $\pi : S' \to S := S_{n,q}$ be a Q-modification with reduced exceptional curve E' such that $K_{S'}$ is π-ample. Then*

(52.1) *$K_{S'} + D'_{n,q} + E' \sim_{\mathbb{Q}} 0$, and*

(52.2) *if $K_{S'} + c(D' + E') \sim_{\mathbb{Q}} 0$ for some $c < 1$, then π is an isomorphism.*

Proof. The first claim was already noted in (51.1). If π has $r \geq 1$ exceptional curves, then the dual graph of $(S', D' + E')$ is

$$\bullet - \boxed{\tfrac{n_0}{q_0}} - e'_1 - \boxed{\tfrac{n_1}{q_1}} - \cdots - \boxed{\tfrac{n_{s-1}}{q_{s-1}}} - e'_s - \boxed{\tfrac{n_s}{q_s}} - \bullet$$

Since $K_{S'} + D' + E' \sim_{\mathbb{Q}} 0$ and $K_{S'} + c(D' + E') \sim_{\mathbb{Q}} 0$, we get that $D' + E' \sim_{\mathbb{Q}} 0$ for $c < 1$. Then $(D' + E')\cdot E'_1 = 0$ and $K_{S'}\cdot E'_1 = 0$, a contradiction. □

Combining with Lemma 18, we get the following.

Corollary 53. *A pair $(S_{N,Q}, cD_{N,Q})$ is KSBA smoothable iff $S_{N,Q}$ is KSB smoothable, hence $N = rn^2, Q = arn - 1$. If these hold then the semi-universal KSBA deformation is given by*

$$\big((xy = z^{rn} + \textstyle\sum_{i=0}^{r-1} t_i z^{in}), c(z = 0)\big)/\tfrac{1}{n}(1, -1, a, \mathbf{0}). \quad \square$$

For the dihedral pairs we have the following.

Lemma 54. *Let $\pi : S' \to S := S^d_{n,q}$ be a Q-modification such that $K_{S'}$ is π-ample. Assume that $K_{S'} + c(D' + E') \sim_{\mathbb{Q}} 0$ for some $\frac{1}{2} < c < 1$. Then π is an isomorphism.*

Therefore the non-DV pairs $(S^d_{n,q}, cD^d_{n,q})$ have no KSBA deformations with Du Val general fibers for $\frac{1}{2} < c < 1$.

Proof. By (51.2) there are half integers $e'_i \leq 1$ such that $K_{S'} + D' + \sum_i e'_i E'_i \sim_{\mathbb{Q}} 0$. Subtracting gives that

$$(1 - c)D' + \textstyle\sum (e'_i - c)E'_i \sim_{\mathbb{Q}} 0.$$

Since D' is π-nef, $-\sum(e'_i - c)E'_i$ is π-nef, hence $e'_i - c \geq 0$ for every i. Thus in fact $e'_i = 1$ for every i. The rest is now the same as in (52), but $S^d_{n,q}$ itself has no KSB smoothings. □

By contrast, the pairs $(S_{n,q}, cB_{n,q})$ have a much more complicated behavior.

Lemma 55. *Let $\pi : S' \to S := S_{n,q}$ be a Q-modification such that $K_{S'}$ is π-ample. Assume that $K_{S'} + c(B' + E') \sim_{\mathbb{Q}} 0$ for some $c < 1$. Then π has at most 1 exceptional curve.*

Proof. If there are at least 2 exceptional curves, then the dual graph is

$$\bullet - \boxed{\tfrac{n_0}{q_0}} - e_1' - \boxed{\tfrac{n_1}{q_1}} - e_2' - \cdots$$

By (51.3) $K_{S'} + B' + E' + \bar{B}' \sim_{\mathbb{Q}} 0$. Subtracting, we get that $(1-c)(B' + E') + \bar{B}' \sim_{\mathbb{Q}} 0$. Here $\bar{B}'$ is disjoint from E_1', so $(B' + E') \cdot E_1' = 0$. This in turn gives that $K_{S'} \cdot E_1' = 0$, a contradiction. □

Examples 56. Note that S' with dual graph

$$\bullet - \boxed{\tfrac{n_0}{q_0}} - e_1' - \boxed{\tfrac{n_1}{q_1}}$$

has a nontrivial KSB deformation if S_{n_0,q_0} is a T-singularity, and it has a KSB deformation with Du Val general fiber if, in addition, S_{n_1,q_1} is a Du Val singularity. The simplest case is the dual graph

$$\bullet - \boxed{\tfrac{4}{1}} - m.$$

This shows that $\bigl(\hat{\mathbb{A}}^2_{xy}, \frac{2m-3}{2m-1}(y=0)\bigr)/\frac{1}{4m-1}(1,m)$ is KSBA smoothable.

I found only 2 series of pairs with KSBA deformations for different values of c. The first is $\bullet - 4 - 3 - \boxed{A_{n+2}}$ with P-modifications

$$\bullet - \boxed{\tfrac{4}{1}} - 3 - \boxed{A_{n+2}} \quad \text{and} \quad \bullet - \boxed{\tfrac{18}{5}} - 2 - \boxed{A_n}.$$

We get $c = \frac{3n+9}{3n+11}$ and $c = \frac{n+1}{n+4}$.

The second is $\bullet - 2 - 5 - 3 - \boxed{A_{n+1}}$ with P-modifications

$$\bullet - \boxed{\tfrac{9}{5}} - 3 - \boxed{A_{n+1}} \quad \text{and} \quad \bullet - \boxed{\tfrac{25}{14}} - 2 - \boxed{A_n}.$$

We get $c = \frac{5n+10}{5n+13}$ and $c = \frac{3n+3}{3n+8}$.

Example 57. Doubly KSB deformations of D_n singularities are described as follows. For $r \geq 2$ the D_r-type pair $\bigl((z^2 = x(y^2 - x^{r-2})), (x = z = 0)\bigr)$ has an $(r-2)$-parameter deformation

$$z^2 = x\bigl(y^2 - x^{r-2} - \textstyle\sum_{i=0}^{r-3} t_i x^i\bigr).$$

(For $r = 2$ we mean $(z^2 = x(y^2 - 1))$, which has 2 singular points of type $\bigl((z^2 = xy), (x = z = 0)\bigr)$ at $(0, \pm 1, 0)$.)

The divisor $D := (x = z = 0)$ is $\mathbb{Q}$-Cartier, since $2D = (x = 0)$. So these are doubly KSB deformations. The generic deformation has 2 singular points of type $\bigl((z^2 = xy), (x = z = 0)\bigr)$ at $(0, \pm\sqrt{-t_0}, 0)$. By (20.1) the latter has no KSB smoothings.

Example 58. There are a several KSBA smoothable examples with $c = \frac{1}{2}$. These are given by the dual graph, where $*$ can be either $\bullet$ or 2

$$\begin{array}{ccccc} & & * & & \\ & & | & & \\ * & - & 4 & - & * \\ & & | & & \\ & & * & & \end{array} \quad \text{where } D^{\mathrm{m}} + \Delta^{\mathrm{m}} \text{ is } \quad \begin{array}{ccccc} & & \frac{1}{2} & & \\ & & | & & \\ \frac{1}{2} & - & 1 & - & \frac{1}{2} \\ & & | & & \\ & & \frac{1}{2} & & \end{array}$$

Contracting the curve (4) we get doubly KSBA smoothable examples.

9. Deformations of modifications

We prove that Q-modifications $X \to S$ have no (small or large) KSB deformations over reduced bases that keep S fixed. (See (61) for other deformations and non-reduced bases.) More generally, we have the following.

Theorem 59. *Let S be a normal surface, B a smooth, irreducible curve, and $g : Y \to S \times B$ a proper birational morphism, Y normal. Assume that*

(59.1) *$g_b : Y_b \to S$ is birational for every $b \in B$, and*

(59.2) *there is a dense, open subset $B^\circ \subset B$ such that, for every $b \in B^\circ$, $Y_b \to S$ is a Q-modification.*

Then, $Y \to B$ is trivial. That is, for any $b \in B$,

$$\big(Y \to S \times B \to B\big) \cong \big(Y_b \times B \to S \times B \to B\big). \tag{59.3}$$

Proof. Over a dense, open subset $B^* \subset B^\circ$, the minimal resolutions $\{Y_b^{\mathrm{m}} \to Y_b : b \in B^*\}$ form a flat family. By (25), we have only finitely many choices for the centers of the blow-ups giving $Y_b^{\mathrm{m}} \to S^{\mathrm{m}}$, so the $\{Y_b^{\mathrm{m}} : b \in B^*\}$, and hence also $\{Y_b : b \in B^*\}$, form locally trivial families.

Thus the family is also trivial over B by (60). ☐

Lemma 60. *Let $g : Y \to X \times B$ be a projective morphism such that Y is normal and $g_b : \mathrm{red}(Y_b) \to X_b$ is birational for every b. Assume that there is a dense, open $B^\circ \subset B$ such that $(Y^\circ \to B^\circ) \cong ((Y' \times B^\circ) \to B^\circ)$ for some normal Y'. Then $(Y \to B) \cong ((Y' \times B) \to B)$.*

Proof. Let H be relatively ample on $Y \to X \times B$ and let H' be its birational transform on $(Y' \times B) \to B$. H' is Cartier over B°, so Cartier by [Ram63, Sam62]; see also [Kol23, 4.21]. Thus H' is relatively ample on $(Y' \times B) \to B$. Thus $(Y \to B) \cong ((Y' \times B) \to B)$ by [MM64]; see [Kol23, 11.39] for the form that we use. ☐

Example 61. M-modifications have nontrivial deformations, as the next examples show.

(61.1) Let $S_P \to S$ be any M-modification whose construction involves blowing up a point $p \in S^{\mathrm{m}}$. We can move the point p along the exceptional curve and contract in the family to get a nontrivial flat deformation of $S_P \to S$ (keeping S fixed). Here the canonical class of the generic fiber is not relatively nef. The simplest example is the singularity $3 - 3$. After one blow-up, the central and generic fibers are

$$4 - 1 - 4 \quad \text{resp.} \quad 3 - 4 - 1.$$

All curves marked 4 are then contracted as in (35). The central fiber is an M-modification. In the generic fiber, the canonical class has degree $-\frac{1}{2}$ on the image of the curve marked 1; so it is not even a Q-modification.

(61.2) The semi-universal deformation of $A_n := (xy = z^{n+1})$ is

$$\big(xy = z^{n+1} + \textstyle\sum_{i=0}^{n-1} t_i z^i\big),$$

while the semi-universal deformation of its minimal resolution is obtained from

$$\big(xy = \textstyle\prod_{i=0}^{n}(z - s_i)\big), \quad \text{(subject to } \textstyle\sum s_i = 0)$$

by repeatedly blowing up the divisors $(x = z - s_i = 0)$ as in (12.3).

Let σ_j be the jth elementary symmetric polynomial in the s_i. Then, over the Artin ring $k[s_0, \dots, s_n]/(\sigma_0, \dots, \sigma_n)$, we get a nontrivial deformation of the minimal resolution, which contracts to the trivial deformation of A_n.

A similar result holds for all Du Val singularities by [Art74], and for M-resolutions by [BC94].

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János Kollár, Princeton University, Princeton NJ 08544-1000
Email address: `kollar@math.princeton.edu`