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ON THE GEOMETRY AND ARITHMETIC OF INFINITE TRANSLATION SURFACES

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In honorem professoris Xavier Gómez-Mont sexagesimum annum complentis.

Abstract. Precompact translation surfaces, *i.e.* closed surfaces which carry a translation atlas outside of finitely many finite angle cone points, have been intensively studied for about 25 years now. About 5 years ago the attention was also drawn to general translation surfaces. In this case the underlying surface can have infinite genus, the number of finite angle cone points of the translation structure can be infinite, and there can be singularities which are not finite angle cone points. There are only a few invariants one classically associates with precompact translation surfaces, among them certain number fields, *i.e.* fields which are finite extensions of $\mathbb{Q}$. These fields are closely related to each other; they are often even equal. We prove by constructing explicit examples that most of the classical results for the fields associated with precompact translation surfaces fail in the realm of general translation surfaces and investigate the relations among these fields. A very special class of translation surfaces are so called square-tiled surfaces or origamis. We give a characterisation for infinite origamis.

1. Introduction

Let S be a translation surface, in the sense of Thurston [Thu97], and denote by $\overline{S}$ the metric completion with respect to its natural translation invariant flat metric. S is called *precompact* if $\overline{S}$ is homeomorphic to a compact surface. We call translation surfaces *origamis*, if they are obtained from gluing copies of the Euclidean unit square along parallel edges by translations; see Definition 2.6. They are precompact translation surfaces if and only if the number of copies is finite. An important invariant associated with a translation surface S is the *Veech group* $\Gamma(S)$ formed by the differentials of affine diffeomorphisms of S that preserve orientation; as further invariants one considers the *trace field* $K_{\mathrm{tr}}(S)$, the *holonomy field* $K_{\mathrm{hol}}(S)$, the *field of cross ratios of saddle connections* $K_{\mathrm{cr}}(S)$ and the *field of saddle connections* $K_{\mathrm{sc}}(S)$; compare Definition 2.9 and Definition 3.2. For precompact surfaces we have the following characterisation:

Theorem A. [GJ00, Theorem 5.5] *Let S be a precompact translation surface, and let $\Gamma(S)$ be its Veech group. The following statements are equivalent.*

- *(i) The groups $\Gamma(S)$ and $\mathrm{SL}(2,\mathbb{Z})$ are commensurable.*
- *(ii) Every cross ratio of saddle connections is rational. Equivalently the field $K_{cr}(S)$ is equal to $\mathbb{Q}$.*
- *(iii) There exists a translation covering from a puncturing of S to a once-punctured flat torus.*

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(iv) *S is an origami up to an affine homeomorphism,* i.e. *there is a Euclidean parallelogram that tiles S by translations.*

The first result of this article explores what remains of the preceding characterisation if S is a general *tame* translation surface. Tame translation surfaces are the translation surfaces all of whose singularities are cone angle singularities (possibly of infinite angle). This includes surfaces like $\mathbb{R}^2$, but also surfaces whose fundamental group is not finitely generated. We define tameness and the different types of singularities in Section 2. Furthermore, we call S *maximal*, if it has no finite singularities of total angle 2π; compare Definition 2.6.

Theorem 1. *Let S be a maximal tame translation surface. Then,*

(i) *S is affine equivalent to an origami if and only if the set of* developed cone points *is contained in $L + x$, where $L \subset \mathbb{R}^2$ is a lattice and $x \in \mathbb{R}^2$ is fixed.*

(ii) *If S is an origami the following statements (b)-(d) hold. In (a) and (e) we require in addition that there are at least two nonparallel saddle connections on S:*
   (a) *The Veech group of S is commensurable to a subgroup of* $\mathrm{SL}(2,\mathbb{Z})$.
   (b) *The field of cross ratios $K_{\mathrm{cr}}(S)$ is isomorphic to $\mathbb{Q}$.*
   (c) *The holonomy field $K_{\mathrm{hol}}(S)$ is isomorphic to $\mathbb{Q}$.*
   (d) *The saddle connection field $K_{\mathrm{sc}}(S)$ is isomorphic to $\mathbb{Q}$.*
   (e) *The trace field $K_{\mathrm{tr}}(S)$ is isomorphic to $\mathbb{Q}$.*

   However, none of (a)-(e) implies that S is an origami.

In the proof of Theorem 1 we will show that even if we require that in (a) the Veech group of S is equal to $\mathrm{SL}(2,\mathbb{Z})$, this condition does not imply that S is an origami.

If S is precompact, then the four fields $K_{\mathrm{tr}}(S)$, $K_{\mathrm{hol}}(S)$, $K_{\mathrm{cr}}(S)$ and $K_{\mathrm{sc}}(S)$ are number fields and we have the following hierarchy:

$$\mathbb{Q} \subseteq K_{\mathrm{tr}}(S) \subseteq K_{\mathrm{hol}}(S) \subseteq K_{\mathrm{cr}}(S) = K_{\mathrm{sc}}(S) \tag{1.1}$$

Thus by Theorem Theorem A the conditions (a), (b) and (d) in (ii) of Theorem 1 are, for precompact surfaces, equivalent to being an origami. Conditions (c) and (e), however, are even for precompact translation surfaces not equivalent to being an origami. Indeed, recall that the "general" precompact translation surface has trivial Veech group, *i.e.* Veech group $\{I, -I\}$, where I is the identity matrix (see [Möl09, Thm. 2.1]). This implies that (e) is not equivalent to being an origami. Furthermore, in Example 4.5 we construct an explicit example of a precompact surface S that is not an origami and such that $K_{\mathrm{hol}}(S) = \mathbb{Q}$. This shows that (c) is not equivalent to being an origami.

In the case of general tame translation surfaces, the fields $K_{\mathrm{tr}}(S)$, $K_{\mathrm{hol}}(S)$, $K_{\mathrm{cr}}(S)$ and $K_{\mathrm{sc}}(S)$ are not necessarily number fields anymore; compare Proposition 3.6. Furthermore from the hierarchy in (1.1) it just remains true in general that $K_{\mathrm{hol}}(S)$ and $K_{\mathrm{cr}}(S)$ are both subfields of $K_{\mathrm{sc}}(S)$. Some of the other relations in (1.1) hold under extra assumptions on S; compare Corollary 4.7. It follows that, in general, if $K_{\mathrm{sc}}(S)$ is isomorphic to $\mathbb{Q}$, then both $K_{\mathrm{hol}}(S)$ and $K_{\mathrm{cr}}(S)$ are isomorphic to $\mathbb{Q}$. In terms of Theorem 1, part (ii), this is equivalent to say that (d) implies both (b) and (c). Note that furthermore trivially (a) implies (e). We treat the remaining of these implications in the next theorem.

Theorem 2. *There are examples of tame translation surfaces S for which*

(i) *The Veech group $\Gamma(S)$ is equal to $\mathrm{SL}(2,\mathbb{Z})$ and K is not equal to $\mathbb{Q}$, where K can be chosen from $K_{\mathrm{cr}}(S)$, $K_{\mathrm{hol}}(S)$ and $K_{\mathrm{sc}}(S)$.*

(ii) *The fields $K_{\mathrm{sc}}(S)$ (hence also $K_{\mathrm{cr}}(S)$ and $K_{\mathrm{hol}}(S)$) and $K_{\mathrm{tr}}(S)$ are equal to $\mathbb{Q}$, but $\Gamma(S)$ is not commensurable to a subgroup of $\mathrm{SL}(2,\mathbb{Z})$.*

(iii) *$K_{\mathrm{cr}}(S)$ or $K_{\mathrm{hol}}(S)$ is equal to $\mathbb{Q}$, but $K_{\mathrm{sc}}(S)$ is not.*

(iv) *The field $K_{\mathrm{cr}}(S)$ is equal to $\mathbb{Q}$, but $K_{\mathrm{hol}}(S)$ is not or* vice versa*: $K_{\mathrm{hol}}(S)$ is equal to $\mathbb{Q}$, but $K_{\mathrm{cr}}(S)$ is not.*

(v) *The field $K_{\mathrm{tr}}(S)$ is equal to $\mathbb{Q}$, but none of the conditions (a), (b), (c) or (d) in Theorem 1 hold. Moreover, none of the conditions (b), (c) or (d) imply that $K_{\mathrm{tr}}(S)$ is isomorphic to $\mathbb{Q}$.*

The proofs of the preceding two theorems heavily rely on modifications of the construction in [PSV11, Construction 4.9] which was there used to determine all possible Veech groups of tame translation surfaces. We summarise this construction in Section 2.3. One can furthermore modify the construction to prove that any subgroup of $\mathrm{SL}(2,\mathbb{Z})$ is the Veech group of an origami. From this we will deduce the following statement about the oriented outer automorphism group $\mathrm{Out}^+(\mathrm{F}_2)$ of the free group F_2 in two generators:

Corollary 1.1. *Every subgroup of $\mathrm{Out}^+(\mathrm{F}_2)$ is the stabiliser of a conjugacy class of some (possibly infinite index) subgroup of F_2.*

If S is a precompact translation surface, the existence of hyperbolic elements, *i.e.* matrices whose trace is bigger than 2, in $\Gamma(S)$ has consequences for the image of $H_1(S,\mathbb{Z})$ in $\mathbb{R}^2$ under the developing map (also called *holonomy* map; see Section 2) and for the nature of some of the fields associated with S. To be more precise, if S is precompact, the following is known:

(A) If there exists $M \in \Gamma(S)$ hyperbolic, then the holonomy field of S is equal to $\mathbb{Q}[tr(M)]$. In particular, the traces of any two hyperbolic elements in $\Gamma(S)$ generate the same field over $\mathbb{Q}$; see [KS00, Theorem 28].

(B) If there exists $M \in \Gamma(S)$ hyperbolic and $tr(M) \in \mathbb{Q}$, then S is an origami; see [McM03b, Theorem 9.8].

(C) If S is a "bouillabaisse surface" (*i.e.* if $\Gamma(S)$ contains two transverse parabolic elements), then $K_{\mathrm{tr}}(S)$ is totally real; compare [HL06a, Theorem 1.1]. This implies that if there exists an hyperbolic M in $\Gamma(S)$ such that $\mathbb{Q}[tr(M)]$ is not totally real then $\Gamma(S)$ does not contain any parabolic elements; see Theorem 1.2 in *ibid.*

(D) Let Λ and Λ_0 be the subgroups of $\mathbb{R}^2$ generated by the image under the holonomy map of $H_1(\overline{S},\mathbb{Z})$ and $H_1(\overline{S},\Sigma;\mathbb{Z})$, respectively. Here Σ is the set of cone angle singularities of S. If the affine group of S contains a pseudo-Anosov element, then Λ has finite index in Λ_0; see [KS00, Theorem 30].

The third main result of this paper shows that when passing to general tame translation surfaces there are no such consequences. For such surfaces, an element of $\Gamma(S) < \mathrm{GL}_+(2,\mathbb{R})$ will be called hyperbolic, parabolic or elliptic if its image in $\mathrm{PSL}(2,\mathbb{R})$ is hyperbolic, parabolic or elliptic respectively.

Theorem 3. *There are examples of tame translation surfaces S for which (A), (B), (C) or (D) from above do not hold.*

We remark that all tame translation surfaces S that we construct in the proof of the preceding theorem have the same topological type: one end and infinite genus. Such topological surfaces are called Loch Ness monster; see Section 2.

This paper is organised as follows. In Section 2 we review the basics about general translation surfaces, tame translation surfaces and origamis, their singularities and possible Veech groups.

In Section 3 we present the definitions of the fields listed in Theorem 1 for general tame translation surfaces. We prove that the main algebraic properties of these fields which are true for precompact translation surfaces no longer hold for general translation surfaces. For example, we construct examples of tame translation surfaces for which the trace field is not a number field. We furthermore show those inclusions from (1.1) which still are valid for tame translation surfaces. Section 4 deals with the proofs of the three theorems stated in this section. We refer the reader to [HS10], [HLT11] or [HHW13] for recent developments concerning tame translation surfaces.

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2. Preliminaries

2.1. **General translation surfaces and their singularities.** In this section we review some basic notions needed for the rest of the article. For a detailed exposition, we refer to [GJ00] and [Thu97].

A *translation surface* S will be a 2-dimensional real G-manifold with $G = \mathbb{R}^2 = Trans(\mathbb{R}^2)$; that is, a surface on which coordinate changes are translations of the real plane $\mathbb{R}^2$. We can pull back to S the standard translation invariant flat metric of the plane and obtain this way a flat metric on the surface. We denote by $\overline{S}$ the metric completion of S with respect to this natural flat metric. A *translation map* is a G-map between translation surfaces. Every translation map $f : S_1 \longrightarrow S_2$ has a unique continuous extension $\overline{f} : \overline{S_1} \longrightarrow \overline{S_2}$.

Definition 2.1. If $\overline{S}$ is homeomorphic to an orientable compact surface, we say that S is a *precompact translation surface*. Else we say that S is non precompact. Observe that a not precompact translation surface is not necessarily of infinite type. The union of all precompact and not precompact translation surfaces form the set of *general translation surfaces*.

Definition 2.2. Let S be a translation surface. We call the points of $\overline{S}\backslash S$ *singularities* of the translation surface S. A point $x \in \overline{S} \setminus S$ is called a *finite angle singularity* or *finite angle cone point* of total angle $2\pi m$, where $m \geq 1$ is a natural number, if there exists a neighbourhood of x which is isometric to a neighbourhood of the origin in $\mathbb{R}^2$ with a metric that, in polar coordinates (r, θ), has the form $ds^2 = dr^2 + (mrd\theta)$. The set of finite angle singularities of $\overline{S}$ is denoted by Σ_{fin}.

Precompact translation surfaces are obtained by glueing finitely many polygons (deprived of their vertices) along parallel edges by translations. One even obtains all precompact translation surfaces in this way; see [Mas06]. Thus if S is a precompact translation surface, all of its singularities are finite angle singularities. If furthermore $\overline{S}$ has genus at least 2, then, by a simple Euler characteristic calculation, $\overline{S}$ always has singularities. For non precompact translation surfaces, new kinds of singularities will occur. We illustrate this in the following example.

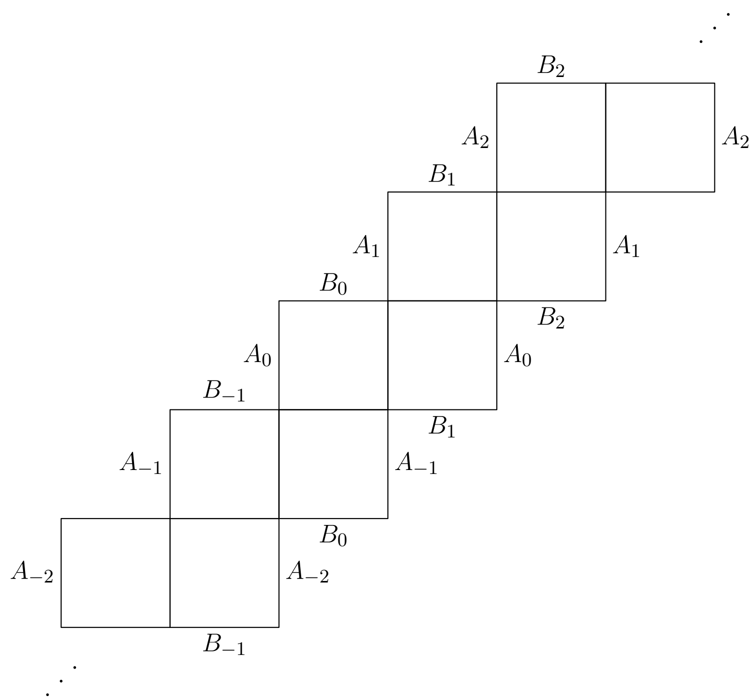

FIGURE 1. An infinite-type translation surface.

Example 2.3. In Figure 1 we depict a translation surface obtained from infinitely many copies of the Euclidean unit square. More precisely, we remove the vertices from all the squares in the figure. Some pairs of edges are already identified; among the remaining edges we identify opposite ones which are labelled by the same letter by translations. The result is a translation surface S which is not precompact. It is called *infinite staircase* because of its shape. This and similar shaped surfaces have been intensively studied in the literature; see *e.g.* [HS10], [HHW13], [HW12] and [CG12]. S is a prototype for what we will call in this text an infinite origami or infinite square-tiled surface; compare Definition 2.6. The translation surface S comes with a natural cover p to the once punctured torus obtained from glueing parallel edges of the Euclidean unit square again with its vertices removed.
Observe furthermore that the metric completion of the infinite staircase S has four singularities x_1, x_2, x_3 and x_4. Restricted to a punctured neighbourhood of them p is infinite cyclic and the universal cover of a once punctured disk. In this sense the singularities $x_1, \ldots, x_4$ generalise finite angle singularities of angle $2\pi m$. They are prototypes for what we call infinite angle singularities; compare Definition 2.4.

Definition 2.4. Let S be a translation surface. A point $x \in \overline{S}$ is called an *infinite angle singularity* or *infinite angle cone point* if there exists a neighbourhood of x isometric to the neighbourhood of the branching point of the infinite cyclic flat branched covering of $\mathbb{R}^2$. The set of infinite angle singularities of $\overline{S}$ is denoted by Σ_{inf}. Points in the set $\Sigma = \Sigma_{\text{fin}} \cup \Sigma_{\text{inf}}$ will be called *cone angle singularities* of S or just *cone points*.

Definition 2.5. A translation surface S is called *tame* if all points in $\overline{S} \setminus S$ are cone angle singularities (of finite or infinite total angle). A tame translation surface S is said to be of *infinite-type* if the fundamental group of S is not finitely generated.

Every precompact translation surface is tame. There are tame translation surfaces with infinite angle singularities which are not of infinite type. For example consider the infinite cyclic covering of the once punctured plane. Explicit examples of infinite-type translation surfaces arise naturally when studying the billiard on a polygon whose interior angles are not rational multiples of π (see [Val09]). Nevertheless, not all translation surfaces are tame. If one allows infinitely many polygons, *wild* types of singularities may occur. Simple examples of not tame translation surfaces can be found in [Cha04] and [BV13].

In the following we define the very special class of translation surfaces called origamis or square-tiled surfaces. With Example 2.3 we have already seen a specific instance of them.

Definition 2.6. A translation surface is called *origami* or *square-tiled surface*, if it fulfils one of the two following equivalent conditions:

(i) S is a translation surface obtained from glueing (possibly infinitely many) copies of the Euclidean square along edges by translations according to the following rules:
  - each left edge is glued to precisely one right edge,
  - each upper edge to precisely one lower edge and
  - the resulting surface is connected;

  and removing all singularities.

(ii) S allows an unramified covering $p : S^* \to T_0$ of the once-punctured unit torus

$$T_0 = (\mathbb{R}^2 \backslash L_0)/L_0,$$

such that p is a translation map. Here S^* is a subset of S such that the complement $S \backslash S^*$ is a discrete set of points on S. L_0 is the lattice in $\mathbb{R}^2$ spanned by the two standard basis $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Furthermore, S is maximal in the sense that $\overline{S} \backslash S$ contains no finite angle singularities of angle 2π.

An origami will be called *finite* if the number of squares needed to construct it is finite or, equivalently, if the unramified covering $p : S \to T_0$ is finite. Else, the origami will be called *infinite*. See [Sch06, Section 1] for a detailed introduction to finite origamis. Infinite origamis were studied *e.g.* in [HS10] and [Gut10].

2.2. **Developed cone points and the Veech group.** In the following we introduce the *set of developed cone points* for tame translation surfaces, which will play an important role in the proof of Theorem 1. Let $\pi_S : \widetilde{S} \longrightarrow S$ be a universal cover of a translation surface S and $\mathrm{Aut}(\pi_S)$ the group of its deck transformations. From now on, $\widetilde{S}$ is endowed with the translation structure obtained as pull-back from the one on S via π_S. Recall from [Thu97, Section 4.3] that for every deck transformation γ, there is a unique translation $\mathrm{hol}(\gamma)$ satisfying

$$\mathrm{dev} \circ \gamma = \mathrm{hol}(\gamma) \circ \mathrm{dev}, \tag{2.2}$$

where $\mathrm{dev} : \widetilde{\mathrm{S}} \longrightarrow \mathbb{R}^2$ denotes the developing map. The map hol: $\mathrm{Aut}(\pi_S) \to \mathrm{Trans}(\mathbb{R}^2) \cong \mathbb{R}^2$ is a group homomorphism. By considering the continuous extension of each map in Equation (2.2) to the metric completion of its domain, we obtain

$$\overline{\mathrm{dev}} \circ \overline{\gamma} = \mathrm{hol}(\gamma) \circ \overline{\mathrm{dev}}. \tag{2.3}$$

Overall we have the following commutative diagram:

$$
\begin{tikzcd}
& \overline{\widetilde{S}} \arrow[rrr, "\overline{\gamma}"] \arrow[dddl, bend right, "\overline{\mathrm{dev}}"'] & & & \overline{\widetilde{S}} \arrow[dddr, bend left, "\overline{\mathrm{dev}}"] & \\
& & \widetilde{S} \arrow[ul, hook] \arrow[r, "\gamma"] \arrow[d, "\pi_S"] \arrow[ddll, "\mathrm{dev}"'] & \widetilde{S} \arrow[ur, hook] \arrow[d, "\pi_S"] \arrow[ddrr, "\mathrm{dev}"] & & \\
& & S & S & & \\
\mathbb{R}^2 \arrow[rrrrr, "\mathrm{hol}(\gamma)"] & & & & & \mathbb{R}^2
\end{tikzcd}
\quad . \tag{2.4}
$$

Definition 2.7. *The set of developed singularities* of S is the subset of the plane $\mathbb{R}^2$ given by $\overline{\mathrm{dev}}(\overline{\widetilde{S}} \setminus \widetilde{S})$. We denote it by $\widetilde{\Sigma}(S)$. If S is a tame translation surface, we also call $\widetilde{\Sigma}(S)$ the *set of developed cone points*.

Definition 2.8. A *singular geodesic* of a translation surface S is an open geodesic segment in the flat metric of S whose image under the natural embedding $S \hookrightarrow \overline{S}$ issues from a singularity of $\overline{S}$, contains no singularity in its interior and is not properly contained in some other geodesic segment. A *saddle connection* is a finite length singular geodesic.

To each saddle connection we can associate a *holonomy vector*: we 'develop' the saddle connection in the plane by using local coordinates of the flat structure. The difference vector defined by the planar line segment is the holonomy vector. Two saddle connections are *parallel*, if their corresponding holonomy vectors are linearly dependent.

Next, we introduce the *Veech group*, which since Veech's article [Vee89] from 1989 has been studied for precompact translation surfaces as the natural object associated with the surface. Let $\mathrm{Aff}_+(S)$ be the group of affine orientation preserving homeomorphisms of a translation surface S. Consider the map

$$\mathrm{Aff}_+(S) \xrightarrow{D} \mathbf{GL}_+(2, \mathbb{R}) \tag{2.5}$$

that associates to every $\phi \in \mathrm{Aff}_+(S)$ its (constant) Jacobian derivative $D\phi$.

Definition 2.9. Let S be a translation surface. We call $\Gamma(S) = D(\mathrm{Aff}_+(S))$ the *Veech group* of S.

Remark 2.10. The group $\mathbf{GL}_+(2, \mathbb{R})$ naturally acts on the set of translation surfaces: We define $A \cdot S$ to be the translation surface obtained from S by composing each chart in the translation atlas with the linear map $\binom{x}{y} \mapsto A \cdot \binom{x}{y}$. Since the map $\mathrm{id}_A : S \to A \cdot S$ which topologically is the identity map has derivative A, we have that $\Gamma(A \cdot S) = A \cdot \Gamma(S) \cdot A^{-1}$.

2.3. **Constructing tame surfaces with prescribed Veech groups.** The proofs of our main results heavily rely on slight modifications of the construction in the proof of [PSV11, Proposition 4.1]. In this section we review this construction. We will mainly use the notation of [PSV11]. The construction we are about to review proves the following:

Proposition 2.11 ([PSV11, Proposition 4.1])**.** *For any countable subgroup G of* $\mathrm{GL}_+(2, \mathbb{R})$ *disjoint from* $\mathcal{U} = \{g \in \mathbf{GL}_+(2, \mathbb{R}) \ : \ ||g|| < 1\}$ *there exists a tame translation surface* $S = S(G)$*, which is homeomorphic to the* Loch Ness monster*, with Veech group* G*.*

The *Loch Ness monster* is the unique topological surface S (up to homeomorphism) of infinite genus and one end. By one end we mean that for every compact set $K \subset S$ there exists a compact set $K \subset K' \subset S$ such that $S \setminus K'$ is connected. We refer the reader to [Ric63] for a more detailed discussion on surfaces of infinite genus and ends.

First we have to recall a basic geometric operation which will play an important role in the construction: *glueing translation surfaces along marks.*

Definition 2.12. Let S be a tame translation surface. A *mark* on S is an oriented finite length geodesic (with endpoints) on S. The *vector* of a mark is its holonomy vector, which lies in $\mathbb{R}^2$. If m, m' are two disjoint marks on S with equal vectors, we can perform the following operation. We cut S along m and m', which turns S into a surface with boundary consisting of four straight segments. Then we reglue these segments to obtain a tame translation surface S' different from the one we started from. We say that S' is obtained from S by *reglueing along m and m'*. Let $S_0 = S \setminus (m \cup m')$. Then S' admits a natural embedding i of S_0. If $A \subset S_0$, then we say that $i(A)$ is *inherited* by S' from A.

Remark 2.13. If S' is obtained from S by reglueing, then the number of singularities of S' of a fixed angle equals the one of S, except for 4π–angle singularities, whose number in S' is greater by 2 to that in S (we put $\infty + 2 = \infty$). The Euler characteristic of S is greater by 2 than the Euler characteristic of S'.
We can extend the notion of reglueing to ordered families $\mathcal{M} = (m_n)_{n=1}^{\infty}$ and $\mathcal{M}' = (m'_n)_{n=1}^{\infty}$ of disjoint marks, which do not accumulate in $\overline{S}$, and such that the vector of m_n equals the vector of m'_n, for each n.

Outline of the construction. Let $\{a_i\}_{i \in I}$ (with $I \subseteq \mathbb{N}$) be a (possibly infinite) set of generators for G. We make use of the fact that any group G acts on its Cayley graph Γ and turn the graph Γ in a G-equivariant way into a translation surface. In the following we describe the general idea of the construction; below we give the explicit construction for the case that G is generated by two elements. The construction then works just in the same way for general groups; compare [PSV11, Construction 4.9].

- With each vertex g of Γ we associate a translation surface V_g. More precisely we start from some translation surface V_{Id} and define V_g to be its translate $g \cdot V_{\mathrm{Id}}$ by the action of $\mathbf{GL}_+(2, \mathbb{R})$ on the set of translation surfaces described in Remark 2.10. Observe that the linear group G naturally acts via affine homeomorphisms on the disjoint union of the V_g's; an element $h \in G$ maps V_g to $V_{h \cdot g}$. In the next step we will choose disjoint marks on the translation surfaces V_g. Reglueing the disjoint union of the surfaces V_g along these marks will give us a connected surface on which G acts by affine homeomorphisms. At the moment, we can assume V_{Id} just to be the real plane $\mathbb{R}^2$ equipped with an origin and a coordinate system.
- We choose marks on the starting surface V_{Id} in the following way:
  - For each i in I we choose a family $\mathcal{C}^i = \{m_j^i\}_{j \in J}$ (with $J \subseteq \mathbb{N}$) of horizontal marks m_j^i of length 1, *i.e.* the vector of each mark m_j^i is the first standard basis vector e_1.
  - For each i we choose a family $\mathcal{C}^{-i} = \{m_j^{-i}\}_{j \in J}$ of marks with vector $a_i^{-1}(e_1)$, *i.e.* the vector of m_j^{-i} is equal to $a_i^{-1} \cdot e_1$.
  - All marks are disjoint.
- On each V_g we take the corresponding marks $g(m_j^i)$ and $g(m_j^{-i})$ with $i \in I$ and $j \in \mathbb{N}$. The mark $g(m_j^i)$ has the vector $g \cdot e_1$ and $g(m_j^{-i})$ has the vector $g a_i^{-1} \cdot e_1$.

- We pair the mark $g(m_j^i)$ on the surface V_g with the mark $ga_i(m_j^{-i})$ on V_{ga_i}. Observe that for both the vector is $g \cdot e_1$.
  We now reglue the disjoint union of the V_g's along these pairs of marks.

This gives us a translation surface S_1 on which the elements of G act via affine homeomorphisms, *i.e.* $\Gamma(S_1)$ contains G. However we are not yet done, but still have the following problems:

(i) The Veech group $\Gamma(S_1)$ can be bigger than G.
(ii) The singularities can accumulate. In this case S_1 is not tame.
(iii) We want the translation surface to have one end.

We resolve the problems in the following way: To enforce that all elements in the Veech group are in G, we will modify the starting surface V_{Id}. We will replace it by a surface obtained from glueing a *decorated surface* $\widetilde{L}'_{\mathrm{Id}}$ (described below) to a plane $A_{\mathrm{Id}} = \mathbb{R}^2$. The surface $\widetilde{L}'_{\mathrm{Id}}$ will be decorated with special singularities. This will guarantee that every orientation preserving affine homeomorphism permutes the set of the singularities on the $\widetilde{L}'_g$'s and with some more care we will establish that it actually acts as one of the elements of G. To avoid accumulation of singularities, we will associate with each edge in the Cayley graph between two vertices g and g' (let us say that $g^{-1}g' = a_i$ is the i-th generator) a *buffer surface* $\hat{E}_g^i$ which connects V_g to $V_{g'}$, but separates them by a definite distance. Finally, we keep track of the end by providing that each V_g and $\hat{E}_g^i$ is one-ended and that after glueing all V_g and $\hat{E}_g^i$, their ends actually merge into one end. This actually is the reason why we have to choose infinite families of marks. If we do not require the surface to be a Loch Ness monster, then it suffices to take one mark from each infinite family.

An illustrative example. In the following paragraphs we carry out the construction for the case where G is generated by two matrices a_1 and a_2. The general case works in the same way; compare [PSV11, Construction 4.9].

Constructing the translation surface V_g: We first construct the surface V_{Id}. We will obtain it by glueing two surfaces A_{Id} and $\widetilde{L}'_{Id}$ along an infinite family of marks. Let A_{Id} be an oriented flat plane, equipped with an origin and the standard basis $e_1 = (1, 0)$ and $e_2 = (0, 1)$. We define the families of marks as follows:

- For $i = 0, 1, 2$ let $\mathcal{C}^i$ be the family of marks on A_{Id} with endpoints $ie_2 + (2n-1)e_1,\ ie_2 + 2ne_1$, for $n \geq 1$. All these marks are pairwise disjoint.
- Given $x_1, y_1 \in \mathbb{R}$, consider the family $\mathcal{C}^{-1}$ of marks on A_{Id} with endpoints
  $$(nx_1, y_1),\ (nx_1, y_1) + a_1^{-1}(e_1),$$
  for $n \geq 1$. We can choose $x_1 > 0$ sufficiently large and $y_1 < 0$ sufficiently small so that all these marks are pairwise disjoint and disjoint from the ones in $\mathcal{C}^i$ for $i = 0, 1, 2$.
- Observe that a translate of the lower half-plane in A_{Id} is avoided by all already constructed marks. In this way we can choose $x_2, -y_2 \in \mathbb{R}$ sufficiently large so that the marks with endpoints $(nx_2, y_2),\ (nx_2, y_2) + a_2^{-1}(e_1)$, for $n \geq 1$, are pairwise disjoint and disjoint with the previously constructed marks. We denote this family by $\mathcal{C}^{-2}$.

Let L_{Id} be an oriented flat plane, equipped with an origin O_{Id}. Let $\widetilde{L}_{\mathrm{Id}}$ be the threefold cyclic branched covering of L_{Id}, which is branched over the origin. Denote the projection map from $\widetilde{L}_{\mathrm{Id}}$ onto L_{Id} by π. Denote by R the closure in $\widetilde{L}_{\mathrm{Id}}$ of one connected component of the preimage under π of the open right half-plane in L_{Id}. On R consider coordinates induced from L_{Id} via π. We define the following family of marks on $\widetilde{L}_{\mathrm{Id}}$:

- Let $\mathcal{C}'$ be the family of marks in R with endpoints $(2n-1)e_1, 2ne_1$, for $n \geq 1$.
- Let t and b be the two marks in $\widetilde{L}_{\mathrm{Id}}$ with endpoints in R with coordinates $e_2, 2e_2$ and $-2e_2, -e_2$, respectively.

Let $\widetilde{L}'_{\mathrm{Id}}$ be the tame flat surface obtained from $\widetilde{L}_{\mathrm{Id}}$ by regluing along t and b. We call $\widetilde{L}'_{\mathrm{Id}}$ the *decorated surface*. Finally, we obtain V_{Id} by glueing A_{Id} with $\widetilde{L}'_{\mathrm{Id}}$ along the families of marks $\mathcal{C}^0$ and $\mathcal{C}'$. For each $g \in G$ we define V_g as the translation surface $g \cdot V_{\mathrm{Id}}$.

Observe that if we denote by $\widetilde{O}_g$ the unique preimage on $\widetilde{L}_g$ of the origin O_g of L_g via the three-fold covering, then $\widetilde{O}_g$ is a singularity of total angle 6π and there are precisely three saddle connections starting in $\widetilde{O}_g$.

Constructing the buffer surface $\hat{E}^i_g$: Let $E_{\mathrm{Id}}, E'_{\mathrm{Id}}$ be two oriented flat planes, equipped with origins that allow us to identify them with $\mathbb{R}^2$. We define the following families of vector e_1 marks on $E_{\mathrm{Id}} \cup E'_{\mathrm{Id}}$.

- Let $\mathcal{S}$ be the family of marks on E_{Id} with endpoints $4ne_1, (4n+1)e_1$, for $n \geq 1$.
- Let $\mathcal{S}_{\mathrm{glue}}$ be the family of marks on E_{Id} with endpoints $(4n+2)e_1, (4n+3)e_1$, for $n \geq 1$.
- Let $\mathcal{S}'$ be the family of marks on E'_{Id} with endpoints $2ne_2, 2ne_2 + e_1$, for $n \geq 1$.
- Finally, let $\mathcal{S}'_{\mathrm{glue}}$ be the family of marks on E'_{Id} with endpoints $(2n+1)e_2, (2n+1)e_2+e_1$, for $n \geq 1$.

Let $\hat{E}_{\mathrm{Id}}$ be the tame flat surface obtained from E_{Id} and E'_{Id} by regluing along $\mathcal{S}_{\mathrm{glue}}$ and $\mathcal{S}'_{\mathrm{glue}}$. We call $\hat{E}_{\mathrm{Id}}$ the *buffer surface*. The surface $\hat{E}_{\mathrm{Id}}$ comes with the distinguished families of marks inherited from $\mathcal{S}$ and $\mathcal{S}'$, for which we retain the same notation. Let $\hat{E}^1_{\mathrm{Id}}$ and $\hat{E}^2_{\mathrm{Id}}$ be two copies of $\hat{E}_{\mathrm{Id}}$ and for each $g \in G$ let $\hat{E}^i_g$ to be the translation surface $g \cdot \hat{E}^i_g$ ($i \in \{1,2\}$). It is endowed with the two family of marks $\mathcal{S}^i_g$ and $\mathcal{S}'^i_g$.

Construction of the surface S: We finally obtain the desired surface S from the disjoint union of all V_g's and $\hat{E}^i_g$ in the following way:

- Reglue each mark $\mathcal{C}^1_g$ on V_g with $\mathcal{S}^1_g$ on $\hat{E}^1_g$, and each mark $\mathcal{S}'^1_g$ on $\hat{E}^1_g$ with $\mathcal{C}^{-1}_{ga_1}$ on V_{ga_1}.
- Reglue each mark $\mathcal{C}^2_g$ on V_g with $\mathcal{S}^2_g$ on $\hat{E}^2_g$, and each mark $\mathcal{S}'^2_g$ on $\hat{E}^2_g$ with $\mathcal{C}^{-2}_{ga_2}$ on V_{ga_2}.

In [PSV11, Section 4] it is carefully carried out that the construction is well defined and gives the desired result from Proposition 2.11.

3. Fields associated with translation surfaces

There are four subfields of $\mathbb{R}$ in the literature which are naturally associated with a translation surface S. They are called the *holonomy field* $K_{\mathrm{hol}}(S)$, the *segment field* or *field of saddle connections* $K_{\mathrm{sc}}(S)$, the *field of cross ratios of saddle connections* $K_{\mathrm{cr}}(S)$, and the *trace field* $K_{\mathrm{tr}}(S)$; compare [KS00] and [GJ00]. In the following, we extend their definitions to (possibly non precompact) tame translation surfaces.

Remark 3.1. It follows from [PSV11, Lemma 3.2] that there are only three types of tame translation surfaces such that $\overline{S}$ has no singularity: $\mathbb{R}^2$, $\mathbb{R}^2/\mathbb{Z}$ and flat tori. Furthermore, tame translation surfaces with only one singularity are cyclic coverings of $\mathbb{R}^2$ ramified over the origin. Finally, if $\overline{S}$ has at least two singularities, then there exists at least one saddle connection.

Definition 3.2. Let S be a tame translation surface and $\overline{S}$ the metric completion of S.

(i) (Following [KS00, Section 7].) Let Λ be the image of $H_1(\overline{S}, \mathbb{Z})$ in $\mathbb{R}^2$ under the holonomy map h and let n be the dimension of the smallest $\mathbb{R}$-subspace of $\mathbb{R}^2$ containing Λ; in particular n is 0, 1 or 2. The *holonomy field* $K_{\mathrm{hol}}(S)$ is the smallest subfield k of $\mathbb{R}$ such that
$$\Lambda \otimes_{\mathbb{Z}} k \cong k^n.$$

(ii) Let Σ denote the set of all singularities of $\overline{S}$. Using in (i) $H_1(\overline{S}, \Sigma_{\mathrm{fin}}; \mathbb{Z})$, the homology relative to the set of finite angle singularities, instead of the absolute homology $H_1(\overline{S}, \mathbb{Z})$, we obtain the *segment field* or *field of saddle connections* $K_{\mathrm{sc}}(S)$.

(iii) (Following [GJ00, Section 5].) The *field of cross ratios of saddle connections* $K_{\text{cr}}(S)$ is the field generated by the set of all cross ratios $(v_1, v_2; v_3, v_4)$, where the v_i's are four pairwise nonparallel holonomy vectors of saddle connections of S; compare Remark 3.3 iii).

(iv) Finally, the *trace field* $K_{\text{tr}}(S)$ is the field generated by the traces of elements in the Veech group: $K_{\text{tr}}(S) = \mathbb{Q}[\text{tr}(A) | A \in \Gamma(S)]$.

In the rest of this section we mean by a *holonomy vector* always the holonomy vector of a saddle connection.

Remark 3.3. (i) Definition 3.2 (i) is equivalent to the following: If $n = 2$, take any two nonparallel vectors $\{e_1, e_2\} \subset \Lambda$, then $K_{\text{hol}}(S)$ is the smallest subfield k of $\mathbb{R}$ such that every element v of Λ can be written in the form $a \cdot e_1 + b \cdot e_2$, with $a, b \in k$. If $n = 1$, any element v of Λ can be written as $a \cdot e_1$, with $a \in K_{\text{hol}}(S)$ and e_1 any nonzero (fixed) vector in Λ. If $n = 0$, $K_{\text{hol}}(S) = \mathbb{Q}$.
The same is true for $K_{\text{sc}}(S)$, if Λ is the image of $H_1(\overline{S}, \Sigma_{\text{fin}}; \mathbb{Z})$ in $\mathbb{R}^2$.

(ii) Recall that $\overline{S}$ is a topological surface if and only if all of its singularities have finite cone angles. However, if Σ_{inf} (resp. Σ_{fin}) is the set of infinite (resp. finite) angle singularities, then $\widehat{S} = \overline{S} \backslash \Sigma_{\text{inf}} = S \cup \Sigma_{\text{fin}}$ is a surface, possibly of infinite genus. We furthermore have that the fundamental group $\pi_1(\overline{S})$ equals $\pi_1(\widehat{S})$ and thus

$$H_1(\overline{S}, \mathbb{Z}) \cong H_1(\widehat{S}, \mathbb{Z}).$$

Indeed, for every infinite angle singularity $p_0 \in \overline{S}$, there exists by definition a neighbourhood U of p_0 in $\overline{S}$ which is isometric to a neighbourhood of the branching point z_0 of the infinite flat cyclic covering X_0 of $\mathbb{R}^2$ branched over 0. Without loss of generality we may choose the neighbourhood of z_0 as an open ball of radius ε in X_0. We then have that U is homeomorphic to $\{(x, y) \in \mathbb{R}^2 | x > 0\} \cup \{(0, 0)\} \subset \mathbb{R}^2$. In particular, U and $U \backslash \{p_0\}$ are both contractible, and by the Seifert-van Kampen theorem we have $\pi_1(\overline{S} \backslash \{p_0\}) \cong \pi_1(\overline{S})$.

(iii) Recall that the cross ratio r of four vectors $v_1, \ldots, v_4$ with $v_i = (x_i, y_i)$ is equal to the cross ratio of the real numbers $r_1 = y_1/x_1, \ldots, r_4 = y_4/x_4$, *i.e.*

$$(v_1, v_2; v_3, v_4) = \frac{(r_1 - r_3) \cdot (r_2 - r_4)}{(r_2 - r_3) \cdot (r_1 - r_4)}. \tag{3.6}$$

If $r_i = \infty$ for some $i = 1, \ldots, 4$, one eliminates the factors on which it appears in Equation (3.6). For example, if $r_1 = \infty$, then $(v_1, v_2; v_3, v_4) = \frac{r_2 - r_4}{r_2 - r_3}$. If there are no four non parallel holonomy vectors, $K_{\text{cr}}(S)$ is equal to $\mathbb{Q}$.

(iv) The four fields from Definition 3.2 are invariant under the action of $\mathbf{GL}(2, \mathbb{R})$ described in Remark 2.10, *i.e.* for $A \in \mathbf{GL}(2, \mathbb{R})$ we have

$$\begin{array}{rclrcl} K_{\text{hol}}(S) & = & K_{\text{hol}}(A \cdot S), & K_{\text{sc}}(S) & = & K_{\text{sc}}(A \cdot S), \\ K_{\text{cr}}(S) & = & K_{\text{cr}}(A \cdot S), & K_{\text{tr}}(S) & = & K_{\text{tr}}(A \cdot S). \end{array}$$

For $K_{\text{hol}}(S)$ and $K_{\text{sc}}(S)$ this follows from (i). Recall that the cross ratio is invariant under linear transformation. Thus the claim is true for the field $K_{\text{cr}}(S)$. Finally, we have that $\Gamma(A \cdot S)$ is conjugated to $\Gamma(S)$; compare Remark 2.10. Since the trace of a matrix is invariant under conjugation, the claim also holds for $K_{\text{tr}}(S)$.

It follows directly from the definitions that $K_{\text{hol}}(S) \subseteq K_{\text{sc}}(S)$. Furthermore, we see from Remark 3.3 that $K_{\text{cr}}(S) \subseteq K_{\text{sc}}(S)$: Suppose S has two linearly independent holonomy vectors w_1 and w_2. By (iv) in the preceding remark we may assume that $w_1 = e_1$, $w_2 = e_2$ is the

standard basis. Let v_1, v_2, v_3, v_4 be four arbitrary pairwise nonparallel holonomy vectors with $v_i = (x_i, y_i)$. By (i) we have that all the coordinates x_i and y_i are in $K_{\mathrm{sc}}(S)$. Thus in particular the cross ratio $(v_1, v_2; v_3, v_4)$ is in $K_{\mathrm{sc}}(S)$. If there is no pair (w_1, w_2) of linearly independent holonomy vectors, then $K_{\mathrm{cr}}(S) = \mathbb{Q}$ and the inclusion $K_{\mathrm{cr}}(S) \subseteq K_{\mathrm{sc}}(S)$ trivially holds.
Since the Veech group preserves the set of holonomy vectors, we furthermore have that if there are at least two linearly independent holonomy vectors, then $K_{\mathrm{tr}}(S) \subseteq K_{\mathrm{hol}}(S)$. However, if all holonomy vectors are parallel, it is not in general true that $K_{\mathrm{tr}}(S) \subseteq K_{\mathrm{hol}}(S)$. An example of a surface S showing this is given in [PSV11, Lemma 3.7]: The surface S is obtained from glueing two copies of $\mathbb{R}^2$ along horizontal slits l_n of the plane with end points $(4n+1, 0)$ and $(4n+3, 0)$. In particular all saddle connections are horizontal and the fields $K_{\mathrm{hol}}(S)$, $K_{\mathrm{cr}}(S)$ and $K_{\mathrm{sc}}(S)$ are all $\mathbb{Q}$. But the Veech group is very big. It consists of all matrices in $\mathbf{GL}_+(2, \mathbb{R})$ which fix the first standard basis vector e_1; compare [PSV11, Lemma 3.7].

Remark 3.4. The translation surface S from [PSV11, Lemma 3.7] has the following properties:

$$\Gamma(S) = \left\{ \begin{pmatrix} 1 & t \\ 0 & s \end{pmatrix} | t \in \mathbb{R}, s \in \mathbb{R}_+ \right\}$$

and $K_{\mathrm{hol}}(S) = K_{\mathrm{cr}}(S) = K_{\mathrm{sc}}(S) = \mathbb{Q}$. In particular, we have $K_{\mathrm{tr}}(S) = \mathbb{R}$.

Finally, in Proposition 3.5 we see that for a large class of translation surfaces we have that $K_{\mathrm{cr}}(S) = K_{\mathrm{sc}}(S)$. The main argument of the proof was given in [GJ00] for precompact surfaces.

Proposition 3.5. *Let S be a (possibly non precompact) tame translation surface, $\overline{S}$ its metric completion and $\Sigma \subset \overline{S}$ its set of singularities. Suppose that $\overline{S}$ has a geodesic triangulation by countably many triangles Δ_k ($k \in I$ for some index set I) such that the set of vertices equals Σ. We then have $K_{\mathrm{cr}}(S) = K_{\mathrm{sc}}(S)$.*

Proof. The inclusion "$\subset$" was shown in general in the paragraph below Remark 3.3. The inclusion "$\supset$" follows from [GJ00, Proposition 5.2]. The statement there is for precompact surfaces, but the proof works in the same way if there exists a triangulation as required in this proposition. More precisely, in [GJ00] it is shown that the $K_{\mathrm{cr}}(S)$-vector space $V(S)$ spanned by the image of $H_1(\overline{S}, \Sigma; \mathbb{Z})$ under the holonomy map is 2-dimensional over $K_{\mathrm{cr}}(S)$. Hence $K_{\mathrm{sc}}(S) \subseteq K_{\mathrm{cr}}(S)$. ☐

It follows from Theorem 2 that in general no further inclusions between the four fields from Definition 3.2 hold than those stated above; see Corollary 4.7 for a subsumption of the relations between the fields.

If S is a precompact translation surface of genus g, then $[K_{\mathrm{tr}}(S) : \mathbb{Q}] \leq g$. Moreover, the traces of elements in $\Gamma(S)$ are algebraic integers (see [McM03a]). When dealing with tame translation surfaces, such algebraic properties do not hold in general.

Proposition 3.6. *For each $n \in \mathbb{N} \cup \{\infty\}$ there exists a tame translation surface S_n of infinite genus such that the transcendence degree of the field extension $K_{\mathrm{tr}}(S_n)/\mathbb{Q}$ is n. S_n can be chosen to be a Loch Ness monster.*

Proof. Let $\{\lambda_1, \ldots, \lambda_n\}$ be $\mathbb{Q}$-algebraically independent real numbers with $|\lambda_i| > 2$. Define

$$G_n := \left\langle \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} \mid \mu + \mu^{-1} = \lambda_i \text{ with } i \in \{1, \ldots, n\} \right\rangle.$$

G_n is countable and a subgroup of the diagonal group. In particular, G_n is disjoint from the set $\mathcal{U}$ of contraction matrices; recall Proposition 2.11 for the definition of $\mathcal{U}$. Thus, we can apply Proposition 2.11 and obtain a surface S_n with Veech group G_n. We have

$$\mathbb{Q} \subset \mathbb{Q}(\lambda_1, \dots, \lambda_n) \subseteq K_{\mathrm{tr}}(S_n) \subseteq L = \mathbb{Q}(\mu | \mu + \mu^{-1} = \lambda_i \text{ with } i \in \{1, \dots, n\}).$$

Since the generators μ of L are algebraic over $\mathbb{Q}(\lambda_1, \dots, \lambda_n)$, it follows that $L/\mathbb{Q}(\lambda_1, \dots, \lambda_n)$ and thus also $K_{\mathrm{tr}}(S_n)/\mathbb{Q}(\lambda_1, \dots, \lambda_n)$ is algebraic and we obtain the claim. □

If μ_1 is one of the two solutions of $\mu + \mu^{-1} = \pi$ and

$$G := \left\langle \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_1^{-1} \end{pmatrix} \right\rangle,$$

then we obtain in the same way the following corollary.

Corollary 3.7. *There are examples of tame translation surfaces S of infinite genus with a cyclic hyperbolic Veech group such that $K_{\mathrm{tr}}(S)$ is not a number field. Again the translation surface can be chosen to be as a Loch Ness monster.*

Transcendental numbers naturally appear also in fields associated with Veech groups arising from a generic triangular billiard. Indeed, let $\mathcal{T} \subset \mathbb{R}^2$ denote the space of triangles parametrised by two angles (θ_1, θ_2). Remark that $\mathcal{T}$ is a simplex. For every $T = T_{(\theta_1, \theta_2)} \in \mathcal{T}$, a classical construction due to Katok and Zemljakov produces a tame flat surface S_T from T [ZK75]. If T has an interior angle which is not commensurable with π, S_T is a Loch Ness monster; compare [Val09].

Proposition 3.8. *The set $\mathcal{T}' \subset \mathcal{T}$ formed by those triangles such that $K_{\mathrm{sc}}(S_T)$, $K_{\mathrm{cr}}(S_T)$ and $K_{\mathrm{tr}}(S_T)$ are not number fields, is of total (Lebesgue) measure in $\mathcal{T}$.*

Proof. Since S_T has a triangulation with countably many triangles satisfying the hypotheses of Proposition 3.5, the fields $K_{\mathrm{sc}}(S_T)$ and $K_{\mathrm{cr}}(S_T)$ coincide. Without loss of generality we can assume that the triangle $T = T_{(\theta_1, \theta_2)}$ has the vertices 0, 1 and $\rho e^{i\theta_1}$ (with $\rho > 0$) in the complex plane $\mathbb{C}$. When doing the Katok-Zemljakov construction we start by reflecting T at its edges. Thus in particular S_T contains the geodesic quadrangle shown in Figure 2.

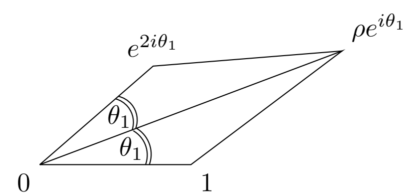

FIGURE 2. Geodesic quadrangle in the surface S_T with T the triangle $T_{(\theta_1, \theta_2)}$

Thus the vectors $v_1 = (1, 0)$, $v_2 = (\rho \cos \theta_1, \rho \sin \theta_1)$ and $v_3 = (\cos 2\theta_1, \sin 2\theta_1)$ are holonomy vectors. Choose $\{v_1, v_2\}$ as basis of $\mathbb{R}^2$. We then have $v_3 = a \cdot v_1 + b \cdot v_2$ with

$$a = -1 \text{ and } b = \frac{2 \cos \theta_1}{\rho}.$$

Therefore $\frac{2 \cos \theta_1}{\rho}$ is an element of $K_{\mathrm{sc}}(S) = K_{\mathrm{cr}}(S)$. Furthermore, from [Val12] we know that the matrix representing the rotation by θ_1 is in $\Gamma(S_T)$. Hence $2 \cos \theta_1$ is in $K_{\mathrm{tr}}(S_T)$. Thus if we choose the values $\frac{\cos\theta_1}{\rho}$, respectively $\cos\theta_1$, to be non algebraic numbers, then $K_{\mathrm{sc}}(S) = K_{\mathrm{cr}}(S)$, respectively $K_{\mathrm{tr}}(S_T)$, are not number fields. $\square$

4. Proof of main results

In this section we prove the results stated in the introduction.

Proof Theorem 1. We begin proving part (i). Let $T_0 = T \setminus \{\infty\}$ be the once punctured torus with $T = \mathbb{R}^2/L$, where L is a lattice in $\mathbb{R}^2$, and with $\infty \in T$ the image of the origin removed. Let $p : S \longrightarrow T_0$ be an unramified translation covering. The existence of such a covering is equivalent to S being affine equivalent to an origami. We use the notation from Section 2. In particular $\pi_S : \widetilde{S} \longrightarrow S$ is a universal cover, $\overline{\widetilde{S}}$ and $\overline{S}$ are the metric completions of $\widetilde{S}$ and S, respectively, and $\widetilde{\Sigma}(S)$ is the set of developed cone points. We then have that the following diagram commutes, since p and π_S are translation maps:

$$\begin{tikzcd}
\widetilde{S} \arrow[r, hook] \arrow[d, "\pi_S"] & \overline{\widetilde{S}} \arrow[d, "\overline{\pi_S}"] \arrow[dr, "\overline{dev}"] & \\
S \arrow[r, hook] \arrow[d, "p"] & \overline{S} \arrow[d, "\overline{p}"] & \mathbb{R}^2 \arrow[dl, "\pi_T"] \\
T_0 \arrow[r, hook] & T &
\end{tikzcd} \quad . \tag{4.7}$$

Given that $T \setminus T_0 = \infty = \overline{p} \circ \overline{\pi_S}(\overline{\widetilde{S}} \setminus \widetilde{S})$, the projection of $\widetilde{\Sigma}(S)$ to T is just a point. This proves sufficiency.

Equation (2.3) implies that if $\widetilde{\Sigma}(S)$ is contained in $L + x$ then every $\mathrm{hol}(\gamma)$ is a translation of the plane of the form $z \to z + \lambda_\gamma$, where $\lambda_\gamma \in L$. Puncture $\widetilde{S}$ and S at $\mathrm{dev}^{-1}(L + x)$ and $\pi_S(\mathrm{dev}^{-1}(L + x))$ respectively to obtain $\widetilde{S}_0$ and S_0 and denote $\mathbb{R}^2_0 = \mathbb{R}^2 \setminus (L + x)$. Let $\pi_{S|} : \widetilde{S}_0 \longrightarrow S_0$ and $\pi_{T|} : \mathbb{R}^2_0 \longrightarrow T_0$ be the restrictions of the universal covers π_S and π_T. Given that $\widetilde{S}_0$ has the translation structure induced by pull-back of $\pi_{S|}$, the map $\mathrm{dev}_| : \widetilde{S}_0 \longrightarrow \mathbb{R}^2_0$ is a flat surjective map; compare [Thu97, §3.4]. Equation (2.2) implies that

$$\begin{tikzcd}
\widetilde{S}_0 \arrow[r, "\mathrm{dev}_|"] \arrow[d, "\pi_{S|}"] & \mathbb{R}^2_0 \arrow[d, "\pi_{T|}"] \\
S_0 & T_0
\end{tikzcd} \tag{4.8}$$

descends to a flat covering map $p : S_0 \longrightarrow T_0$. Hence $\overline{S} = \overline{S_0}$ defines a covering over a flat torus ramified at most over one point. This proves necessity.

Now we prove part (ii). First we prove that every origami satisfies conditions (a), (b), (c), (d) and (e).

Let $\overline{p} : \overline{S} \to T$ be an origami ramified at most over $\infty \in T$. All saddle connections of S are preimages of closed simple curves on T with a base point at ∞. This implies that all holonomy vectors have integer coordinates. Thus $K_{\mathrm{sc}}(S) = K_{\mathrm{hol}}(S) = K_{\mathrm{cr}}(S) = \mathbb{Q}$. Hence every origami fulfils conditions (b), (c) and (d) in part (ii). If S furthermore has at least two linearly independent holonomy vectors, then the Veech group must preserve the lattice spanned by them. Thus it is commensurable to a (possible infinite index) subgroup of $\mathrm{SL}(2,\mathbb{Z})$ and S fulfils in addition (a) and (e).

We finally prove that none of the conditions in theorem (a) to (e) imply that S is an origami. Example 4.1 shows that neither (a) nor (e) imply that S is an origami. Example 4.2 shows that neither (b), nor (c), nor (d) imply that S is an origami.

Example 4.1. In this example we construct a tame translation surface S whose Veech group $\Gamma(S)$ is $\mathrm{SL}(2,\mathbb{Z})$, hence $K_{\mathrm{tr}}(S)=\mathbb{Q}$, but which is not an origami. We achieve this making a slight modification of the construction presented in Section 2.3. Let $G=\mathrm{SL}(2,\mathbb{Z})$. Apply the construction in Section 2.3 to G but choose the family of marks $\mathcal{C}^{-1}$ in such a way that the there exists $N\in\mathbb{Z}$ and irrational $\alpha>0$ so that (α,N) is a holonomy vector. This is possible since in the cited construction the choice of the point (x_1,y_1) is free. Observe that $v_1=(-1,1)$, $v_2=(0,1)$, $v_3=(1,0)$ and $v_4=(\alpha,N)$ are holonomy vectors of $\overline{S}$. Let l_i be lines in $\mathbb{P}^1(\mathbb{R})$ containing v_i, $i=1,\dots,4$ respectively. A direct calculation shows that the cross ratio of these four lines is $\frac{\alpha}{\alpha+N}$, which lies in $K_{\mathrm{cr}}(S)$. Hence $K_{\mathrm{cr}}(S)$ is not isomorphic to $\mathbb{Q}$ and therefore S cannot be an origami.

Example 4.2. In this example we construct a surface S whose Veech group is not a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$ (hence S cannot be an origami, since in addition S has two non parallel saddle connections) but such that

$$K_{\mathrm{cr}}(S)=K_{\mathrm{hol}}(S)=K_{\mathrm{sc}}(S)=K_{\mathrm{tr}}(S)=\mathbb{Q}. \tag{4.9}$$

Consider $G=\mathrm{SL}(2,\mathbb{Q})$ or $G=\mathrm{SO}(2,\mathbb{Q})$. These are non-discrete countable subgroups of $\mathrm{SL}(2,\mathbb{R})$ with no contracting elements. Hence we can apply the construction from Proposition 2.11 to G but choosing the points (x_i,y_i) that define the families of marks $\mathcal{C}^i$ in $\mathbb{Q}\times\mathbb{Q}$ for all $i\geq 1$ indexing a countable set of generators of G. The result is a tame translation surface S whose Veech group is isomorphic to G and whose holonomy vectors S have all coordinates in $\mathbb{Q}\times\mathbb{Q}$. This implies (4.9).

$\square$

Proof Theorem 2. Let us first show that (i) holds. The tame translation surface S in Example 4.1 is such that $\Gamma(S)=\mathrm{SL}(2,\mathbb{Z})$ and $K_{\mathrm{cr}}(S)$ is not isomorphic to $\mathbb{Q}$. Since in general $K_{\mathrm{cr}}(S)$ is a subfield of $K_{\mathrm{sc}}(S)$ this surface also satisfies that $\Gamma(S)=\mathrm{SL}(2,\mathbb{Z})$ and $K_{\mathrm{sc}}(S)$ is not isomorphic to $\mathbb{Q}$. To finish the proof of (i) we consider the following example.

Example 4.3. In this example we construct a tame translation surface such that $\Gamma(S)=\mathrm{SL}(2,\mathbb{Z})$ and $K_{\mathrm{hol}}(S)$ is not isomorphic to $\mathbb{Q}$. Apply the construction described in Section 2.3 to $G=\mathrm{SL}(2,\mathbb{Z})$ but consider the following modification. Let $\{e_1,e_2\}$ be the standard basis of $\mathbb{R}^2$. There exists a natural number $n>0$ such that the mark M in A_{Id} whose end points are $-ne_1$ and $-(n-1)e_1$ does not intersect all other marks used in the construction. On a $[0,\pi]\times[0,e]$ rectangle R, where e is Euler's number, identify opposite sides to obtain a flat torus T. Consider on T a horizontal mark M' of length 1 and glue A_{Id} with T along M and M'. Then proceed with the construction in a $\mathrm{SL}(2,\mathbb{Z})$-equivariant way. This produces a tame translation surface S whose Veech group is $\mathrm{SL}(2,\mathbb{Z})$. The image of $H_1(\overline{S},\mathbb{Z})$ under the holonomy map contains the vectors e_1, $e\cdot e_1$ and $\pi\cdot e_2$. Hence $K_{\mathrm{hol}}(S)$ is not isomorphic to $\mathbb{Q}$.

Part (ii) follows from Example 4.2. We now prove (iii). First we construct S such that $K_{\mathrm{cr}}(S)=\mathbb{Q}$ but $K_{\mathrm{sc}}(S)$ is not. Consider the following example.

Example 4.4. Let P_1, P_2 and P_3 be three copies of $\mathbb{R}^2$; choose on each copy an origin, and let $\{e_1,e_2\}$ be the standard basis. Consider the following:

(i) Let v_n be the mark on the plane P_1 along segments whose end points are $n \cdot e_2$ and $n \cdot e_2 + e_1$ with $n = 0, 1$.
(ii) Marks on P_2 and P_3 along the segments w_0, w_1 whose end points are $(0, 0)$ and $(1, 0)$, and then along the segments z_0 and z_1 whose end points are $(2, 0)$ and $(2 + \sqrt{p}, 0)$, for some prime p.

Glue the three planes along slits as follows: v_i to w_i, for $i = 0, 1$ and z_0 to z_1. The result is a surface S for which $\{0, 1, -1, \infty\}$ parametrizes all possible slopes of lines through the origin in $\mathbb{R}^2$ containing holonomy vectors of saddle connections. Hence $K_{\mathrm{cr}}(S) = \mathbb{Q}$. On the other hand, the set of holonomy vectors contains $(1, 0)$, $(0, 1)$, $(1, 1)$ and $(\sqrt{p}, 0)$. Therefore $K_{\mathrm{sc}}(S)$ contains $\mathbb{Q}(\sqrt{p})$ as a subfield.

We finish the proof of (iii) by constructing a precompact tame translation surface such that $K_{\mathrm{hol}}(S) = \mathbb{Q}$, but $K_{\mathrm{sc}}(S)$ is not. Consider the following example.

Example 4.5. Consider two copies L_1 and L_2 of the L-shaped origami tiled by three unit squares; see *e.g.* [HL06b, Example on p. 293]. Consider a point $p_i \in L_i$ at distance $0 < \varepsilon << 1$ from the 6π-angle singularity s_i, $i = 1, 2$. Let m_i be a marking of length ε on L_i defined by a geodesic of length ε joining p_i to s_i, $i = 1, 2$. We can choose p_i so that both markings are parallel and the vector defined by them has irrational coordinates. Glue then L_1 and L_2 along m_1 and m_2 to obtain S. By construction $h(H_1(\overline{S}, \mathbb{Z})) = \mathbb{Z} \times \mathbb{Z}$, hence $K_{\mathrm{hol}}(S) = \mathbb{Q}$, but $h(H_1(\overline{S}, \Sigma; \mathbb{Z}))$ contains an orthonormal basis $\{e_1, e_2\}$ and a vector $h(m_1)$ with irrational coordinates. This implies that $K_{\mathrm{sc}}(S)$ is not isomorphic to $\mathbb{Q}$.

We address (iv) now. Observe that the surface S constructed in Example 4.5 satisfies that $K_{\mathrm{hol}}(S) = \mathbb{Q}$ but $K_{\mathrm{cr}}(S)$ is not equal to $\mathbb{Q}$. Indeed, we have saddle connections of slope 0, 1 and ∞. Since the slope of $h(m_1)$ is irrational, we are done. We now construct S such that $K_{\mathrm{cr}}(S) = \mathbb{Q}$, but $K_{\mathrm{hol}}(S)$ is not. We will furthermore have that S has four pairwise nonparallel holonomy vectors thus $K_{\mathrm{cr}}(S)$ is not trivially $\mathbb{Q}$.

Example 4.6. Take two copies of the real plane P_1 and P_2. Choose an origin and let e_1, e_2 be the standard basis. Let $\mu_i > 1$, $i = 1, 2, 3$ be three distinct irrational numbers and define $\lambda_0 = 0$ and $\lambda_n = \sum_{i=1}^{n} \mu_i$ for $n = 1, 2, 3$. On P_1 consider the markings m_n whose end points are ne_2 and $ne_2 + e_1$ for $n = 0, \ldots, 3$. On P_2 consider the markings m'_n whose end points are $(n + \lambda_n)e_1$ and $(n + \lambda_n + 1)e_1$ for $n = 0, \ldots, 3$. Glue P_1 and P_2 along the markings m_n and m'_n. The result is a tame flat surface S with eight 4π-angle singularities. These singularities lie on P_2 on a horizontal line, and hence we can naturally order them from, say, left to right. Let us denote these ordered singularities by a_j, $j = 1, \ldots, 8$. Let $g_{e_1}(a_i, a_j)$ (respectively $g_{e_2}(a_i, a_j)$) be the directed geodesic in S parallel to e_1 (respectively e_2) joining a_i with a_j. Define in $H_1(\overline{S}, \mathbb{Z})$

- the cycle c_1 as $g_{e_1}(a_3, a_4) g_{e_1}(a_4, a_5) g_{e_2}(a_5, a_3)$,
- the cycle c_2 as $g_{e_1}(a_4, a_3) g_{e_1}(a_3, a_2) g_{e_2}(a_2, a_4)$,
- the cycle c_3 as $g_{e_2}(a_6, a_8) g_{e_1}(a_8, a_7) g_{e_1}(a_7, a_6)$.

Where the product is defined to be the composition of geodesics on S, *i.e.* following one after the other. Note that $h(c_1) = (1 + \mu_2, -1)$, $h(c_2) = (-(1 + \mu_1), 1)$ and $h(c_3) = (-(1 + \mu_3), 1)$. We can choose parameters μ_i, $i = 1, 2, 3$ so that the $\mathbb{Z}$-module generated by these 3 vectors has rank 3. Therefore $K_{\mathrm{hol}}(S)$ cannot be isomorphic to $\mathbb{Q}$.

We address now (v). We construct first a flat surface S for which $K_{\mathrm{tr}}(S) = \mathbb{Q}$ but none of the conditions (a), (b), (c) or (d) in Theorem 1 hold. We achieve this by making a slight modification

on the construction of the surface in Example 4.3 in the following way. First, change $\mathrm{SL}(2,\mathbb{Z})$ for $\mathrm{SL}(2,\mathbb{Q})$. Second, let the added mark M be of unit length and such that the vector defined by developing it along the flat structure neither lies in the lattice $\pi\mathbb{Z}\times e\mathbb{Z}$ nor has rational slope. The result of this modification is a tame translation surface S homeomorphic to the Loch Ness monster for which $\Gamma(S) = \mathrm{SL}(2,\mathbb{Q})$ and such that both $K_{\mathrm{cr}}(S)$ and $K_{\mathrm{hol}}(S)$ (hence $K_{\mathrm{sc}}(S)$ as well) have transcendence degree at least 1 over $\mathbb{Q}$.
Finally, an example of a surface S which satisfies (c), (d) and (b), but with $K_{\mathrm{tr}}(S) \neq \mathbb{Q}$, is given in [PSV11, Lemma 3.7]; see Remark 3.4. We underline that all holonomy vectors in this surface S are parallel and hence $K_{\mathrm{cr}}(S)$ is by definition isomorphic to $\mathbb{Q}$. For the sake of completeness we construct a tame translation surface S where not all holonomy vectors are parallel, such $K_{\mathrm{cr}}(S) = \mathbb{Q}$, but where $K_{\mathrm{tr}}(S)$ is not equal to $\mathbb{Q}$. Let E_0 be a copy of the affine plane $\mathbb{R}^2$ with a chosen origin and (x,y)-coordinates. Slit E_0 along the rays $R_v := (0, y \geq 1)$ and $R_h := (x \geq 1, 0)$ to obtain $\hat{E}_0$. Choose an irrational $0 < \lambda < 1$ and $n \in \mathbb{N}$ so that $1 < n\lambda$. Define

$$M := \begin{pmatrix} \lambda & 0 \\ 0 & n\lambda \end{pmatrix} \qquad R_v^k := M^k R_v \qquad R_h^k := M^k R_h \quad k \in \mathbb{Z}.$$

Here M^k acts linearly on E_0. For $k \neq 0$, slit a copy of E_0 along the rays R_v^k and R_h^k to obtain $\hat{E}_k$. We glue the family of slitted planes $\{\hat{E}_k\}_{k\in\mathbb{Z}}$ to obtain the desired tame flat surface as follows. Each $\hat{E}_k$ has a "vertical boundary" formed by two vertical rays issuing from the point of coordinates $(0,(n\lambda)^k)$. Denote by $R_{v,l}^k$ and $R_{v,r}^k$ the boundary ray to the left and right respectively. Identify by a translation the rays $R_{v,r}^k$ with $R_{v,l}^{k+1}$, for each $k \in \mathbb{Z}$. Denote by $R_{h,b}^k$ and $R_{h,t}^k$ the horizontal boundary rays in $\hat{E}_k$ to the bottom and top respectively. Identify by a translation $R_{h,b}^k$ with $R_{h,t}^{k+1}$ for each $k \in \mathbb{Z}$.
By construction, $\{(-\lambda^k,(n\lambda)^k)\}_{k\in\mathbb{Z}}$ is the set of all holonomy vectors of S. Clearly, all slopes involved are rational; hence $K_{\mathrm{cr}}(S) = \mathbb{Q}$. On the other hand, $M \in \Gamma(S)$ and $tr(M) = (n+1)\lambda$. Note that the surface S constructed in this last paragraph admits no triangulation satisfying the hypotheses of Proposition 3.5. □

Corollary 4.7. *The four fields $K_{\mathrm{tr}}(S)$, $K_{\mathrm{hol}}(S)$, $K_{\mathrm{cr}}(S)$ and $K_{\mathrm{sc}}(S)$ satisfy the following relations:*

- *(i)* $K_{\mathrm{hol}}(S) \subseteq K_{\mathrm{sc}}(S)$ *and* $K_{\mathrm{cr}}(S) \subseteq K_{\mathrm{sc}}(S)$.
- *(ii) For each other pair (i,j), with $i,j \in \{\mathrm{tr},\mathrm{hol},\mathrm{cr},\mathrm{sc}\}$, $(i,j) \neq (\mathrm{hol},\mathrm{sc})$ and $(i,j) \neq (\mathrm{cr},\mathrm{sc})$, we can find surfaces S such that $K_i(S) \not\subseteq K_j(S)$. In these examples we can always choose $K_j(S)$ to be $\mathbb{Q}$.*
- *(iii) If S has two non parallel holonomy vectors, then $K_{\mathrm{tr}}(S) \subseteq K_{\mathrm{hol}}(S)$.*
- *(iv) If $\overline{S}$ has a geodesic triangulation by countably many triangles whose vertices form the set Σ of singularities of S, then $K_{\mathrm{cr}}(S) = K_{\mathrm{sc}}(S)$.*

Proof. (i) is shown in Section 3 before Remark 3.4; (ii) is shown in Theorem 2 and in Remark 3.4; (iii) is shown before Remark 3.4 and (iv) is the result of Proposition 3.5. □

Proof Corollary 1.1:
Let Γ be a subgroup of $\mathrm{SL}_2(\mathbb{Z})$. By Proposition 2.11 we know that there is a translation surface S with Veech group Γ. Furthermore in the construction all slits can be chosen such that their end points are integer points in the corresponding plane; thus S is an origami by Theorem 1, part (i). Hence it allows for a subset S^* of S, whose complement is a discrete set of points, an unramified covering $p : S^* \to T_0$ to the once puncture unit torus T_0. Recall that p defines the conjugacy class $[U]$ of a subgroup U of F_2 as follows. Let U be the fundamental group of S^*. It is

embedded into $F_2 = \pi_1(T_0)$ via the homomorphism p_* between fundamental groups induced by p. The embedding depends on the choices of the base points up to conjugation. In [Sch04] this is used to give the description of the Veech group completely in terms of $[U]$; compare Theorem B below. Recall for this that the outer automorphism group $\mathrm{Out}(F_2)$ is isomorphic to $\mathrm{GL}_2(\mathbb{Z})$. Furthermore it naturally acts on the set of the conjugacy classes of subgroups U of F_2.

Theorem B. *The Veech group $\Gamma(S^*)$ equals the stabiliser of the conjugacy class $[U]$ in $SL_2(\mathbb{Z})$ under the action described above.*

This theorem, that can be found in [*Ibid.*], considers only finite origamis, but the proof works in the same way for infinite origamis. Recall furthermore that $\Gamma(S^*) = \Gamma(S) \cap \mathrm{SL}_2(\mathbb{Z})$ and $\Gamma(S) \subseteq \mathrm{SL}_2(\mathbb{Z})$ if and only if the $\mathbb{Z}$-module spanned by the holonomy vectors of the saddle connections equals $\mathbb{Z}^2$. We can easily choose the marks in the construction in [PSV11] such that this condition is fulfilled.

□

Proof Theorem 3: First notice that the translation surfaces constructed in the proof of Theorem 2 parts (i) and (v) are both counterexamples for statements (A) and (B). Furthermore, Proposition 3.6 shows that two hyperbolic elements in $\Gamma(S)$ do not have to generate the same trace field. To disprove (C) we let μ be a solution to the equation $\mu + \mu^{-1} = \sqrt[3]{11}$ and G is the group generated by the matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}, \tag{4.10}$$

then Proposition 2.11 produces a tame translation surface S with Veech group G for which $K_{\mathrm{tr}}(S) = \mathbb{Q}(\sqrt[3]{11})$ is not totally real and thus is a counterexample for (C).
Finally for disproving (D) we construct a tame translation surface S with a hyperbolic element in its Veech group for which Λ has infinite index in Λ_0. The construction has two steps.
Step 1: Let M be the matrix given by

$$\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}. \tag{4.11}$$

Let S' be the tame translation surface obtained from Proposition 2.11 for the group G' generated by M. Let Λ' be the image in $\mathbb{R}^2$ under the holonomy map of $H_1(\overline{S'}, \mathbb{Z})$, $\{e_1, e_2\}$ be the standard basis of $\mathbb{R}^2$ and $\beta := G' \cdot \{e_1, e_2\}$. We suppose without loss of generality that e_1 and e_2 lie in Λ'.
Step 2: Let $\alpha = \{v_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^2 \setminus \Lambda'$ be a sequence of $\mathbb{Q}$-linearly independent vectors. We modify the construction in Proposition 2.11 (applied to G') in the following way. We add to the page A_{Id} a family of marks parallel to vectors in α. We can suppose that the new marks lie in the left-half plane $Re(z) < 0$ in A_{Id} and are disjoint by pairs and do not intersect any of the marks in $\mathcal{C}_1$ used in the construction from Step 1. For each $j \in \mathbb{N}$ there exists a natural number k_j such that $2k_j > |v_j|$. Let T_j be the torus obtained from a $2k_j \times 2k_j$ square by identifying opposite sides. Slit each T_j along a vector parallel to v_j and glue it to A_{Id} along the mark parallel to v_j. Denote by A'_{Id} the result of performing this operation for every $j \in \mathbb{N}$, then proceed just the same construction as in Proposition 2.11. Let S be the resulting translation surface. Observe that glueing in the tori T_j produces new elements in $H_1(\overline{S}, \mathbb{Z})$ whose image under the holonomy map lie in $\mathbb{Z} \times \mathbb{Z}$. Thus the subgroups of $\mathbb{R}^2$ generated by the image under the holonomy map of $H_1(\overline{S}', \mathbb{Z})$ and $H_1(\overline{S}, \mathbb{Z})$ are the same. Let Λ be the image in $\mathbb{R}^2$ under the holonomy map of $H_1(S, \mathbb{Z})$. By construction, the index of Λ in Λ_0 is at least the cardinality of α, which is infinite.

□

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