ON THE NUMBER OF POPULATIONS OF CRITICAL POINTS OF MASTER FUNCTIONS

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ABSTRACT. We consider the master functions associated with one irreducible integrable highest weight representation of a Kac-Moody algebra. We study the generation procedure of new critical points from a given critical point of one of these master functions. We show that all critical points of all of these master functions can be generated from the critical point of the master function with no variables. This means that the set of all critical points of all of these master functions form a single population of critical points.

We formulate a conjecture that the number of populations of critical points of master functions associated with a tensor product of irreducible integrable highest weight representations of a Kac-Moody algebra are labeled by homomorphisms to $\mathbb{C}$ of the Bethe algebra of the Gaudin model associated with this tensor product.

1. INTRODUCTION

Master functions form an important class of functions. They were introduced more than twenty years ago in [SV] to solve KZ equations in CFT and to find eigen-states in quantum spin models [BF, RV], although examples of master functions can be found in works by Heine and Stieltjes in 19th century. Master functions are associated with tensor products of representations of Kac-Moody algebras.

A quantum integrable Gaudin model is assigned to a tensor product of representations of a Kac-Moody algebra, see [G, FFR]. This model is an algebra of commuting linear operators on the tensor product. The algebra is called the Bethe algebra. Eigen-states of the model are common eigenvectors of the Bethe algebra. The eigenvectors are constructed from critical points of the associated master functions. This construction is called the Bethe ansatz method, see [BF, RV].

In [MV1], it was discovered that the critical points of master functions come in families called the populations. Starting from a critical point one can generate a one-parameter family of new critical points. The set of all critical points obtained by the iteration of this procedure is called a population.

It turned out that for the Lie algebra $\mathfrak{gl}_N$ the number of populations equals the number of irreducible components in the decomposition of the tensor product into irreducibles [MTV]. This fact showed an unexpected connection between the nonlinear analytic object (critical points of master functions) and the linear algebraic object (the number of direct irreducible summands in a tensor product of representations).

In [VW, VWW] it was observed that a certain population of the critical points of the master functions associated with one irreducible representation with highest weight zero of an affine Kac-Moody algebra is related to the mKdV integrable hierarchy associated with that Kac-Moody algebra. Namely, one can construct rational solutions of the hierarchy from the critical points of the population.

These facts show that populations of critical points of master functions are an interesting object.

In this paper we consider the master functions associated with one irreducible integrable highest weight representation of a Kac-Moody algebra. We study the generation procedure of new critical points from a given critical point of one of these master functions. We show that all critical points of all of these master functions can be generated from the critical point of the master function with no variables. In particular this means that the set of all critical points of all of these master functions form a single population of critical points.

We formulate a conjecture that the number of populations of critical points of master functions associated with a tensor product of irreducible integrable highest weight representations of a Kac-Moody algebra are labeled by homomorphisms to $\mathbb{C}$ of the Bethe algebra of the Gaudin model associated with this tensor product.

In Section 2 we introduce master functions and describe different ways to characterize critical points of master functions. In Section 3, we introduce populations of critical points and formulate the conjecture. The main results are Theorems 3.8 and 3.9.

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2. Master functions and critical points, [MV1]

2.1. **Kac-Moody algebras.** Let $A = (a_{i,j})_{i,j=1}^r$ be a generalized Cartan matrix, $a_{i,i} = 2$, $a_{i,j} = 0$ if and only $a_{j,i} = 0$, $a_{i,j} \in \mathbb{Z}_{\leqslant 0}$ if $i \neq j$. We assume that A is symmetrizable, i.e. there exists a diagonal matrix $B = \operatorname{diag}(b_1, \ldots, b_r)$ with positive integers b_i such that BA is symmetric.

Let $\mathfrak{g} = \mathfrak{g}(A)$ be the corresponding complex Kac-Moody Lie algebra (see [K, Section 1.2]), $\mathfrak{h} \subset \mathfrak{g}$ the Cartan subalgebra. The associated symmetric bilinear form $(\ ,\ )$ is nondegenerate on $\mathfrak{h}^*$ and $\dim \mathfrak{h} = r + 2d$, where d is the dimension of the kernel of the Cartan matrix A, see [K, Chapter 2].

Let $\alpha_i \in \mathfrak{h}^*$, $\alpha_i^\vee \in \mathfrak{h}$, $i = 1, \ldots, r$, be the sets of simple roots, coroots, respectively. We have $(\alpha_i, \alpha_j) = b_i a_{i,j}$, $\langle \lambda, \alpha_i^\vee \rangle = 2(\lambda, \alpha_i)/(\alpha_i, \alpha_i)$, $\lambda \in \mathfrak{h}^*$. In particular, $\langle \alpha_j, \alpha_i^\vee \rangle = a_{i,j}$. Let $P = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}\}$ and $P^+ = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geqslant 0}\}$ be the sets of integral and dominant integral weights, respectively.

Fix $\rho \in \mathfrak{h}^*$ such that $\langle \rho, \alpha_i^\vee \rangle = 1$, $i = 1, \ldots, r$. We have $(\rho, \alpha_i) = (\alpha_i, \alpha_i)/2$. The Weyl group $\mathcal{W} \in \operatorname{End}(\mathfrak{h}^*)$ is generated by reflections s_i, $i = 1, \ldots, r$, where $s_i(\lambda) = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i$, $\lambda \in \mathfrak{h}^*$. We use the notation

$$(2.1) \qquad w \cdot \lambda = w(\lambda + \rho) - \rho, \qquad w \in \mathcal{W}, \ \lambda \in \mathfrak{h}^*,$$

for the *shifted action* of the Weyl group.

2.2. **Master functions,** [SV]. Let $\boldsymbol{\Lambda} = (\Lambda_1, \ldots, \Lambda_n)$, $\Lambda_a \in P^+$ be a collection of dominant integral weights. Let $\boldsymbol{k} = (k_1, \ldots, k_r) \in \mathbb{Z}_{\geqslant 0}^r$ be a collection of nonnegative integers. Denote $k = k_1 + \cdots + k_r$. Denote $\Lambda_\infty(\boldsymbol{\Lambda}, \boldsymbol{k}) = \sum_{a=1}^n \Lambda_a - \sum_{j=1}^r k_j \alpha_j \in P$. The weight $\Lambda_\infty(\boldsymbol{\Lambda}, \boldsymbol{k})$ will be called the *weight at infinity*.

Consider $\mathbb{C}^n$ with coordinates $\boldsymbol{z} = (z_1, \ldots, z_n)$. Consider $\mathbb{C}^k$ with coordinates $\boldsymbol{u}$ collected into r groups, the j-th group consisting of k_j variables, $\boldsymbol{u} = (u^{(1)}, \ldots, u^{(r)})$, $u^{(j)} = (u_1^{(j)}, \ldots, u_{k_j}^{(j)})$. The *master function* is the multivalued function on $\mathbb{C}^k \times \mathbb{C}^n$ defined by the formula

$$(2.2) \qquad \Phi(\boldsymbol{u}, \boldsymbol{z}, \boldsymbol{\Lambda}, \boldsymbol{k}) = \sum_{a<b} (\Lambda_a, \Lambda_b) \ln(z_a - z_b) - \sum_{a,i,j} (\alpha_j, \Lambda_a) \ln(u_i^{(j)} - z_a) +$$

$$+ \sum_{j<j'} \sum_{i,i'} (\alpha_j, \alpha_{j'}) \ln(u_i^{(j)} - u_{i'}^{(j')}) + \sum_j \sum_{i<i'} (\alpha_j, \alpha_j) \ln(u_i^{(j)} - u_{i'}^{(j)}),$$

with singularities at the places where the arguments of the logarithms are equal to zero. The product of symmetric groups $\Sigma_{\boldsymbol{k}} = \Sigma_{k_1} \times \cdots \times \Sigma_{k_r}$ acts on the set of variables $u_i^{(j)}$ by permuting the coordinates with the same upper index. The function Φ is symmetric with respect to the $\Sigma_{\boldsymbol{k}}$-action.

For $q \in \mathbb{C}^n$, a point $p \in \mathbb{C}^k$ is a *critical point* if $\Phi(\cdot, \boldsymbol{z}(q), \boldsymbol{\Lambda}, \boldsymbol{k})$ is defined at p and

$$d_u \Phi(\cdot, \boldsymbol{z}(q), \boldsymbol{\Lambda}, \boldsymbol{k}) = 0$$

at p. In other words, p is a critical point if

$$(2.3) \qquad \sum_{i', \, i' \neq i} \frac{(\alpha_j, \alpha_j)}{u_i^{(j)}(p) - u_{i'}^{(j)}(p)} + \sum_{j', i'} \frac{(\alpha_j, \alpha_{j'})}{u_i^{(j)}(p) - u_{i'}^{(j')}(p)} - \sum_a \frac{(\alpha_j, \Lambda_a)}{u_i^{(j)}(p) - z_a(q)} = 0,$$

$j = 1, \ldots, r$, $i = 1, \ldots, k_j$. The critical set of the function $\Phi(\cdot, \boldsymbol{z}(q), \boldsymbol{\Lambda}, \boldsymbol{k})$ is $\Sigma_{\boldsymbol{k}}$-invariant. All orbits in the critical set have the same cardinality $k_1! \ldots k_r!$. We do not make distinction between critical points in the same orbit.

2.3. Polynomials representing critical points.

Let $q \in \mathbb{C}^n$. Let $p \in \mathbb{C}^k$ be a critical point of the master function $\Phi(\cdot, \boldsymbol{z}(q), \boldsymbol{\Lambda}, \boldsymbol{k})$. Introduce an r-tuple of polynomials $\boldsymbol{y} = (y_1(x), \ldots, y_r(x))$, $y_j(x) = \prod_{i=1}^{k_j} (x - u_i^{(j)}(p))$. Each polynomial is considered up to multiplication by a nonzero number. The tuple defines a point in the direct product $\boldsymbol{P}(\mathbb{C}[x])^r$ of r copies of the projective space associated with the vector space of polynomials in x. We say that the tuple $\boldsymbol{y}$ *represents the critical point* p. The vector $\boldsymbol{k} = (k_1, \ldots, k_r)$ will be called the *degree vector* of the tuple $\boldsymbol{y}$.

It is convenient to think that the tuple $(1, \ldots, 1)$ of constant polynomials represents in $\boldsymbol{P}(\mathbb{C}[x])^r$ the critical point of the master function with no variables. This corresponds to the degree vector $\boldsymbol{k} = (0, \ldots, 0)$.

Introduce polynomials

$$(2.4) \qquad T_j(x) = \prod_{a=1}^{n} (x - z_a(q))^{\langle \Lambda_a, \alpha_j^\vee \rangle}, \qquad j = 1, \ldots, r.$$

Denote $\tau_j = \deg T_j(x)$. We say that a given tuple $\boldsymbol{y} \in \boldsymbol{P}(\mathbb{C}[x])^r$ is *generic* with respect to a point $q \in \mathbb{C}^n$ and weights $\boldsymbol{\Lambda}$ if: (i) each polynomial $y_j(x)$ has no multiple roots; (ii) all roots of $y_j(x)$ are different from roots of the polynomial T_j; (iii) any two polynomials $y_i(x)$, $y_j(x)$ have no common roots if $i \neq j$ and $a_{i,j} \neq 0$. It is clear that if a tuple represents a critical point, then it is generic, see equations (2.3).

Denote $W(f, g) = fg' - f'g$ the Wronskian determinant of functions f, g in x. A tuple $\boldsymbol{y}$ is called *fertile* with respect to weights $\boldsymbol{\Lambda}$ and a point $q \in \mathbb{C}^n$, if there exist polynomials $\tilde{y}_1(x), \ldots, \tilde{y}_r(x)$ satisfying the equations

$$(2.5) \qquad W(y_j, \tilde{y}_j) = T_j \prod_{i, \, i \neq j} y_i^{-a_{j,i}}, \qquad j = 1, \ldots, r,$$

see [MV1]. Equation (2.5) is a first order linear inhomogeneous differential equation with respect to $\tilde{y}_j$. Its solutions are

$$(2.6) \qquad \tilde{y}_j = y_j \int T_j \prod_{i=1}^{r} y_i^{-a_{j,i}} dx + c y_j,$$

where c is any number. The tuples

$$(2.7) \qquad \boldsymbol{y}^{(j)}(x, c) = (y_1(x), \ldots, y_{j-1}(x), \tilde{y}_j(x, c), y_{j+1}(x), \ldots, y_r(x)) \in \boldsymbol{P}(\mathbb{C}[x])^r$$

form a one-parameter family. This family is called the *generation of tuples from $\boldsymbol{y}$ in the j-th direction*. A tuple of this family is called an *immediate descendant* of $\boldsymbol{y}$ in the j-th direction.

Theorem 2.1 ([MV1])**.**

(i) *A generic tuple* $\boldsymbol{y} = (y_1, \ldots, y_r)$ *represents a critical point of the master function* $\Phi(\cdot, \boldsymbol{z}(q), \boldsymbol{\Lambda}, \boldsymbol{k})$, *where* $k_j = \deg y_j$, *if and only if* $\boldsymbol{y}$ *is fertile with respect to weights* $\boldsymbol{\Lambda}$ *and a point* $q \in \mathbb{C}^n$

(ii) *If* $\boldsymbol{y}$ *represents a critical point, then for any* $c \in \mathbb{C}$ *the tuples* $\boldsymbol{y}^{(j)}(x, c)$, $j = 1, \ldots, r$, *are fertile.*

(iii) *If* $\boldsymbol{y}$ *is generic and fertile, then for almost all values of the parameter* $c \in \mathbb{C}$ *the tuples* $\boldsymbol{y}^{(j)}(x, c)$, $j = 1, \ldots, r$, *are generic. The exceptions form a finite set in* $\mathbb{C}$.

(iv) *Assume that a sequence* $\boldsymbol{y}_i, i = 1, 2, \ldots,$ *of fertile tuples has a limit* $\boldsymbol{y}_\infty$ *in* $\boldsymbol{P}(\mathbb{C}[x])^r$ *as* i *tends to infinity.*

   (a) *Then the limiting tuple* $\boldsymbol{y}_\infty$ *is fertile.*

   (b) *For* $j = 1, \ldots, r$, *let* $\boldsymbol{y}_\infty^{(j)}$ *be an immediate descendant of* $\boldsymbol{y}_\infty$. *Then for* $j = 1, \ldots, r$, *there exist immediate descendants* $\boldsymbol{y}_i^{(j)}$ *of* y_i *such that* $\boldsymbol{y}_\infty^{(j)}$ *is the limit of* $\boldsymbol{y}_i^{(j)}$ *as* i *tends to infinity.*

Consider a generic fertile tuple $\boldsymbol{y} = (y_1(x), \ldots, y_r(x))$ as in Theorem 2.1. Let $k_j = \deg y_j$ for $j = 1, \ldots, r$. Consider a generic fertile tuple $\boldsymbol{y}^{(j)}(x, c) = (y_1(x), \ldots, y_{j-1}(x), \tilde{y}_j(x, c), y_{j+1}(x), \ldots, y_r(x))$ for some $c \in \mathbb{C}$ and some $j, 1 \leqslant j \leqslant r$, as in part (iii) of Theorem 2.1. It is easy to see that the polynomial $\tilde{y}_j(x, c)$ is of degree k_j or $\tilde{k}_j = \tau_j + 1 - k_j - \sum_{i \neq j} a_{j,i} k_i$. Denote

$$\boldsymbol{k}^{(j)} = (k_1, \ldots, k_{j-1}, \tilde{k}_j, k_{j+1} \ldots, k_r).$$

By Theorem 2.1, if $\deg \tilde{y}_j(x, c) = k_j$, then the generic fertile tuple $\boldsymbol{y}^{(j)}(x, c)$ represents a critical point of the master function $\Phi(\cdot, \boldsymbol{z}(q), \boldsymbol{\Lambda}, \boldsymbol{k})$. By Theorem 2.1, if $\deg \tilde{y}_j(x, c) = \tilde{k}_j$, then the generic fertile tuple $\boldsymbol{y}^{(j)}(x, c)$ represents a critical point of the master function $\Phi(\cdot, \boldsymbol{z}(q), \boldsymbol{\Lambda}, \boldsymbol{k}^{(j)})$.

The transformation from $\boldsymbol{k}$ to $\boldsymbol{k}^{(j)}$ can be described in terms of the shifted Weyl group action.

Lemma 2.2 ([MV1])**.** *We have* $\Lambda_\infty(\boldsymbol{\Lambda}, \boldsymbol{k}^{(j)}) = s_j \cdot \Lambda_\infty(\boldsymbol{\Lambda}, \boldsymbol{k})$.

2.4. **Another way to characterize critical points.**

Theorem 2.3 ([MSTV])**.** *Let* $\boldsymbol{y} = (y_1, \ldots, y_r)$ *be generic with respect to a point* $q \in \mathbb{C}^n$ *and weights* $\boldsymbol{\Lambda}$. *Then* $\boldsymbol{y}$ *represents a critical point of the master function* $\Phi(\cdot, \boldsymbol{z}(q); \boldsymbol{\Lambda}, \boldsymbol{k})$ *if and only if there exist numbers* $\mu_1, \ldots, \mu_n$, $\mu_1 + \cdots + \mu_n = 0$, *such that*

$$(2.8) \qquad \sum_{j=1}^{r} (\alpha_j, \alpha_j) \frac{y_j''}{y_j} + \sum_{i \neq j} (\alpha_i, \alpha_j) \frac{y_i' y_j'}{y_i y_j} - \sum_{j=1}^{r} (\alpha_j, \alpha_j) \frac{T_j' y_j'}{T_j y_j} +$$

$$+ \sum_{a=1}^{n} \frac{1}{x - z_a(q)} \left(\mu_a - \sum_{b \neq a} \frac{(\Lambda_a, \Lambda_b)}{z_a(q) - z_b(q)} \right) = 0.$$

Lemma 2.4 ([MSTV])**.** *Let* $\boldsymbol{y} = (y_1, \ldots, y_r)$ *be generic and fertile with respect to a point* $q \in \mathbb{C}^n$ *and weights* $\boldsymbol{\Lambda}$. *Let* $j \in \{1, \ldots, r\}$. *Let* $\boldsymbol{y}^{(j)}(x, c) = (y_1(x), \ldots, \tilde{y}_j(x, c), \ldots, y_r(x))$ *be a generic fertile descendant in the* j-*th direction. Then the tuple* $\boldsymbol{y}^{(j)}(x, c)$ *satisfies equation* (2.8) *with the same numbers* $\mu_1, \ldots, \mu_n$ *as the tuple* $\boldsymbol{y}$.

3. Populations of critical points

3.1. **Populations.** Let $\boldsymbol{y} = (y_1(x), \ldots, y_r(x)) \in \boldsymbol{P}(\mathbb{C}[x])^r$ be the tuple representing a critical point $p \in \mathbb{C}^k$ of the master function $\Phi(\cdot, \boldsymbol{z}(q), \boldsymbol{\Lambda}, \boldsymbol{k})$. By Theorem 2.1, we can construct r one-parameter families $\boldsymbol{y}^{(j)}(x, c) \in \boldsymbol{P}(\mathbb{C}[x])^r$ of fertile tuples almost all of which are generic

with respect to the point $q \in \mathbb{C}^n$ and weights $\boldsymbol{\Lambda}$ and hence represent critical points of master functions. After that we can start with any tuple $\boldsymbol{y}^{(j)}(x, c)$ of these r families and generate new r one-parameter families of fertile tuples by using Theorem 2.1. This two-step procedure gives us r^2 two-parameter families of fertile tuples in $\boldsymbol{P}(\mathbb{C}[x])^r$ almost all of which are generic with respect to the point $q \in \mathbb{C}^n$ and weights $\boldsymbol{\Lambda}$ and hence represent critical points of master functions. We may repeat this procedure any number of times and after, say, m repetitions we will obtain r^m families of fertile tuples almost all of which are generic with respect to the point $q \in \mathbb{C}^n$ and weights $\boldsymbol{\Lambda}$ and hence represent critical points of master functions. The union in $\boldsymbol{P}(\mathbb{C}[x])^r$ of all tuples obtained after all possible repetitions is called the *population of tuples generated from the tuple $\boldsymbol{y}$* with given additional data $q, \boldsymbol{\Lambda}$ (or called the *population of critical points generated from the critical point p of the master function* $\Phi(\cdot, \boldsymbol{z}(q); \boldsymbol{\Lambda}, \boldsymbol{k})$). All tuples in the population are fertile with respect to the data $q, \boldsymbol{\Lambda}$. Almost all tuples are generic with respect to the data $q, \boldsymbol{\Lambda}$.

Lemma 3.1 ([MV1]). *For given data $q, \boldsymbol{\Lambda}$, if two populations intersect, then they coincide.*

Lemma 3.2 ([MV1]). *Let $\mathcal{P} \subset \boldsymbol{P}(\mathbb{C}[x])^r$ be the population generated from the tuple $\boldsymbol{y}$ representing a critical point $p \in \mathbb{C}^k$ of the master function $\Phi(\cdot, \boldsymbol{z}(q), \boldsymbol{\Lambda}, \boldsymbol{k})$. Let $\tilde{\boldsymbol{y}} = (\tilde{y}_1, \ldots, \tilde{y}_r)$ be a point of $\mathcal{P}$ and $\tilde{\boldsymbol{k}} = (\tilde{k}_1, \ldots, \tilde{k}_r)$, where $\tilde{k}_j = \deg \tilde{y}_j$. Then the vector $\Lambda_\infty(\boldsymbol{\Lambda}, \tilde{\boldsymbol{k}})$ lies in the orbit of the vector $\Lambda_\infty(\boldsymbol{\Lambda}, \boldsymbol{k})$ under the shifted action of the Weyl group. Conversely, if a vector $\sum_{a=1}^n \Lambda_a - \sum_{j=1}^r \tilde{k}_j \alpha_j$ lies in the orbit of $\Lambda_\infty(\boldsymbol{\Lambda}, \boldsymbol{k})$, then there exists a tuple $\tilde{\boldsymbol{y}} \in \mathcal{P}$ with degree vector $(\tilde{k}_1, \ldots, \tilde{k}_r)$.*

Lemma 3.3. *Let $\mathcal{P}$ be a population associated with data $q, \boldsymbol{\Lambda}$. Let $\boldsymbol{y} \in \mathcal{P}$ be a generic fertile tuple. Then the numbers $\mu_1, \ldots, \mu_n$ in equation (2.8) satisfied by $\boldsymbol{y}$ do not depend on the choice of $\boldsymbol{y}$ in $\mathcal{P}$.*

Proof. The lemma follows from Lemma 2.4. $\square$

Remark. For the Lie algebra $\mathfrak{sl}_2$, different populations with the same polynomial $T_1(x)$ have different sets of numbers $\mu_1, \ldots, \mu_n$. For the Lie algebra $\mathfrak{sl}_3$, a population is not uniquely determined by polynomials T_1, T_2 and numbers $\mu_1, \ldots, \mu_n$. For example, let

$$(\Lambda_1, \Lambda_2) = (\alpha_1 + \alpha_2, \alpha_1 + \alpha_2),$$

$(k_1, k_2) = (1, 1)$, $(z_1, z_2) = (1, -1)$. Then $T_1 = T_2 = x^2 - 1$. The critical point equations $1/(t - 1) + 1/(t + 1) + 1/(t - s) = 0$, $1/(s - 1) + 1/(s + 1) + 1/(s - t) = 0$ have two solutions $t = 1/\sqrt{5}, s = -1/\sqrt{5}$ and $t = -1/\sqrt{5}, s = 1/\sqrt{5}$. These solutions generate different populations, see [MV1, Section 5]. For the first of them we have $(y_1, y_2) = (x - 1/\sqrt{5}, x + 1/\sqrt{5})$ and for the second $(y_1, y_2) = (x + 1/\sqrt{5}, x - 1/\sqrt{5})$. For both of them we have

$$2\frac{y_1''}{y_1} + 2\frac{y_2''}{y_2} - 2\frac{y_1'y_2'}{y_1y_2} - 2\frac{T_1'y_1}{T_1y_1} - 2\frac{T_2'y_2'}{T_2y_2} + \frac{5}{x - 1} - \frac{5}{x + 1} = 0$$

and for both of them the numbers μ_1, μ_2 are the same.

Given data $q, \boldsymbol{\Lambda}$, let $T_j(x)$ be polynomials defined by (2.4). Introduce a quadratic polynomial in variables $k_1, \ldots, k_r$,

$$(3.1) \qquad B(k_1, \ldots, k_r) = \sum_{j=1}^r (\alpha_j, \alpha_j) k_j (k_j - 1 - \tau_j) + \sum_{i \neq j} (\alpha_i, \alpha_j) k_i k_j.$$

We have

$$(3.2) \quad B(k_1, \ldots, k_r) =$$

$$= (\rho + \sum_{a=1}^{n} \Lambda_a - \sum_{j=1}^{r} k_j \alpha_j, \ \rho + \sum_{a=1}^{n} \Lambda_a - \sum_{j=1}^{r} k_j \alpha_j) - (\rho + \sum_{a=1}^{n} \Lambda_a, \ \rho + \sum_{a=1}^{n} \Lambda_a).$$

Lemma 3.4. *Let $\mathcal{P}$ be a population generated from a tuple, which is generic and fertile with respect to the data q, Λ. Then there exists an integer $c(\mathcal{P})$ such that for any*

$$\boldsymbol{y} = (y_1(x), \ldots, y_r(x)) \in \mathcal{P}$$

with $\deg y_j = k_j$, we have

$$(3.3) \qquad\qquad\qquad\qquad B(k_1, \ldots, k_r) = c(\mathcal{P}).$$

Proof. By Lemma 3.3, there exists numbers $\mu_1, \ldots, \mu_n, \mu_1 + \cdots + \mu_n = 0$, such that all generic fertile tuples $\boldsymbol{y} \in \mathcal{P}$ satisfy equation (2.8) with numbers $\mu_1, \ldots, \mu_n$. Equating to zero the coefficient of x^{-2} in the Laurent expansion at infinity of the left hand side of (2.8) we prove (3.3) for $c(\mathcal{P})$ equal to the coefficient of x^{-2} in the Laurent expansion at infinity of

$$-\sum_{a=1}^{n} \frac{1}{x - z_a(q)} \Big(\mu_a - \sum_{b \neq a} \frac{(\Lambda_a, \Lambda_b)}{z_a(q) - z_b(q)} \Big).$$

$\square$

The integer $c(\mathcal{P})$ will be called the *charge* of the population $\mathcal{P}$.

Let $\mathcal{P}$ be a population associated with data q, Λ. A tuple $\boldsymbol{y} \in \mathcal{P}$ with degree vector

$$\boldsymbol{k} = (k_1, \ldots, k_r)$$

will be called *minimal* if $\tau_j + 1 - k_j - \sum_{i \neq j} a_{j,i} k_i > k_j$ for $j = 1, \ldots, r$. These inequalities can be rewritten as

$$(3.4) \qquad\qquad\qquad \tau_j + 1 - \sum_{i=1}^{r} a_{j,i} k_i > 0, \qquad j = 1, \ldots, r.$$

Lemma 3.5. *Every population has a minimal tuple.*

Lemma 3.6. *A tuple $\boldsymbol{y} \in \mathcal{P}$ is minimal if and only if the vector $\Lambda_\infty(\Lambda, \boldsymbol{k})$ is integral dominant.*

Lemma 3.7. *Let $\mathcal{P}$ be a population associated with data q, Λ. Let $\boldsymbol{y} \in \mathcal{P}$ be a minimal tuple with degree vector $\boldsymbol{k} = (k_1, \ldots, k_r)$. Then either $\boldsymbol{k} = 0$ and $c(\mathcal{P}) = 0$ or $\boldsymbol{k} \neq 0$, $c(\mathcal{P}) < 0$ and*

$$(3.5) \qquad\qquad\qquad c(\mathcal{P}) + \sum_{j=1}^{r} b_j(\tau_j + 1)k_j < 0.$$

Proof. If $\boldsymbol{k} = 0$, then $c(\mathcal{P}) = 0$. If $\boldsymbol{k} \neq 0$, then $\sum_{j=1}^{r} b_j k_j (\tau_j + 1 - \sum_{i=1}^{r} a_{j,i} k_i) > 0$. Hence

$$0 \ < \ \sum_{j=1}^{r} b_j k_j (\tau_j + 1 - \sum_{i=1}^{r} a_{j,i} k_i) =$$

$$= \ \sum_{j=1}^{r} b_j k_j (\tau_j + 1 - \sum_{i=1}^{r} a_{j,i} k_i) + B(k_1, \ldots, k_r) - c(\mathcal{P}) = -c(\mathcal{P}) - \sum_{j=1}^{r} b_j k_j (\tau_j + 1).$$

$\square$

The tuple $\boldsymbol{y}^{\emptyset} = (1, \ldots, 1)$ is generic and fertile with respect to any data q, Λ. Denote by $\mathcal{P}_{\boldsymbol{y}^{\emptyset}, q, \Lambda} \subset \boldsymbol{P}(\mathbb{C}[x])^r$ the population of tuples generated from $\boldsymbol{y}^{\emptyset}$. Clearly we have $c(\mathcal{P}_{\boldsymbol{y}^{\emptyset}, q, \Lambda}) = 0$.

Theorem 3.8. *The population* $\mathcal{P}_{\boldsymbol{y}^\emptyset,q,\boldsymbol{\Lambda}}$ *is the only population* $\mathcal{P}$ *associated with the data* $q,\boldsymbol{\Lambda}$ *and such that* $c(\mathcal{P}) = 0$.

Proof. Let $\mathcal{P}$ be any such population. Let $\boldsymbol{y} \in \mathcal{P}$ be a minimal tuple and $\boldsymbol{k}$ its degree vector. By Lemma 3.7, we get $\boldsymbol{k} = 0$. Hence $\mathcal{P}$ contains $\boldsymbol{y}^\emptyset$ and therefore $\mathcal{P} = \mathcal{P}_{\boldsymbol{y}^\emptyset,q,\boldsymbol{\Lambda}}$. $\square$

If $n = 1$, then $q \in \mathbb{C}$ and $\boldsymbol{\Lambda} = (\Lambda_1)$.

Theorem 3.9. *If* $n = 1$, *then* $\mathcal{P}_{\boldsymbol{y}^\emptyset,q,\boldsymbol{\Lambda}}$ *is the only populations associated with data* $q,\boldsymbol{\Lambda}$.

Proof. Let $n = 1$. Let $\mathcal{P}$ be a population associated with data $q,\boldsymbol{\Lambda}$ and $\boldsymbol{y} \in \mathcal{P}$ a generic fertile tuple. Then the numbers $\mu_1,\ldots,\mu_n$ in formula (2.8) are equal to zero. Hence $c(\mathcal{P}) = 0$. By Theorem 3.8 we get $\mathcal{P} = \mathcal{P}_{\boldsymbol{y}^\emptyset,q,\boldsymbol{\Lambda}}$. $\square$

Remark. In [ScV, MV1, MV2, F], it is shown that a population $\mathcal{P} \subset \boldsymbol{P}(\mathbb{C}[x])^r$ is a variety isomorphic to the flag variety of the Kac-Moody algebra $\mathfrak{g}(A^t)$ Langlands dual to the Kac-Moody algebra $\mathfrak{g}(A)$.

Remark. If $\mathfrak{g}(A)$ is an affine Lie algebra of type A_r or $A_2^{(2)}$, the population $\mathcal{P}_{\boldsymbol{y}^\emptyset,q,\boldsymbol{\Lambda}=0}$ for $n = 1$ was used in [VW, VWW] to construct rational solutions of the corresponding integrable hierarchy.

3.2. **Conjecture.**

Conjecture 3.10. *Given a Kac-Moody algebra* $\mathfrak{g}(A)$ *and data* $q,\boldsymbol{\Lambda}$, *the populations* $\mathcal{P}$ *are in one-to-one correspondence with characters* $\chi : \mathcal{B} \to \mathbb{C}$ *of the Bethe algebra* $\mathcal{B}$ *of the Gaudin model associated with the tensor product* $\otimes_{a=1}^n L_{\Lambda_a}$ *of the irreducible* $\mathfrak{g}(A)$-*modules* L_{Λ_a} *with highest weights* Λ_a.

The definition of the Bethe algebra of the Gaudin model see in [FFR, T, MTV].

The statement of the conjecture for $\mathfrak{g}(A) = \mathfrak{sl}_{r+1}$ is a corollary of the main theorem in [MTV] and the description of $\mathfrak{sl}_{r+1}$-populations $\mathcal{P} \subset \boldsymbol{P}(\mathbb{C}[x])^r$ in [MV1, Section 5].

Conjecture also holds for $n = 1$. Indeed, in this case, the Bethe algebra $\mathcal{B}$ associated with an irreducible highest weight representation L_{Λ_1} consists of scalar operators on L_{Λ_1} and is isomorphic to $\mathbb{C}$. The Bethe algebra has the single character $id : \mathbb{C} \to \mathbb{C}$ and the conjecture says that there is only one population associated with $\mathfrak{g}(A)$ and the data $q \in \mathbb{C}$, $\boldsymbol{\Lambda} = (\Lambda_1)$. Our Theorem 3.9 says that indeed there is exactly one population and this is the population $\mathcal{P}_{\boldsymbol{y}^\emptyset,q,\boldsymbol{\Lambda}} \subset \boldsymbol{P}(\mathbb{C}[x])^r$ generated from the fertile generic tuple $\boldsymbol{y}^\emptyset = (1,\ldots,1)$ representing the critical point of the master function with no variable and associated with $\boldsymbol{k} = (0,\ldots,0)$.

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