# Journal *of* Singularities

Volume 7

www.journalofsing.org

# Journal

# *of*

# Singularities

Volume 7
2013

ISSN: # 1949-2006

Journal *of* Singularities

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**Journal *of* Singularities**
**Volume 7 (2013), 1-20**

received 18 July 2012
in revised form 1 January 2013
DOI: 10.5427/jsing.2013.7a

# EQUIVARIANT AND INVARIANT THEORY OF NETS OF CONICS WITH AN APPLICATION TO THOM POLYNOMIALS

M. DOMOKOS, L. M. FEHÉR, AND R. RIMÁNYI

Abstract. Two parameter families of plane conics are called nets of conics. There is a natural group action on the vector space of nets of conics, namely the product of the group reparametrizing the underlying plane, and the group reparametrizing the parameter space of the family. We calculate equivariant fundamental classes of orbit closures. Based on this calculation we develop the invariant theory of nets of conics. As an application we determine Thom polynomials of contact singularities of type (3,3). We also show how enumerative problems, in particular the intersection multiplicities of the determinant map from nets of conics to plane cubics, can be solved studying equivariant classes of orbit closures.

## 1. Introduction

The Thom polynomial is a multivariable polynomial associated with a singularity, governing its global behavior. Thom polynomials of certain singularities are known, mostly of those with small codimension, or with some other simple structure. The Thom polynomials of the simplest so-called $\Sigma^3$ singularities are not known. The first objective of this paper is to calculate the Thom polynomials of these simplest $\Sigma^3$ singularities. These singularities are closely related with nets (i.e., two-parameter families) of (plane) conics. Recent work of the second and the third authors [FR12] reduces the calculations of these Thom polynomials to the calculation of the equivariant cohomology classes of the corresponding classes of nets of conics.

While calculating the equivariant cohomology classes of nets of conics we realized that most of the classical results on the invariant theory of nets of conics can be reproduced using our equivariant techniques. We included the invariant theory of nets of conics and its relation with the equivariant theory for two reasons. First, this makes the paper more self-contained, and second, we find the relation between equivariant and invariant theory interesting, and possibly applicable in the future for more general representations.

Another byproduct of our calculations is a list of intersection multiplicities for the determinant map (assigning a plane cubic to a net of conics, its discriminant). Intersection multiplicities are difficult to calculate in general. We will define and compute equivariant intersection multiplicities, and show that they agree with the non-equivariant ones.

## 2. Structure of the paper

Two parameter families of plane conics are called nets of conics. There is a natural group action on the vector space of nets of conics, namely the product of the group of linear reparametrizations of the underlying plane, and the group of linear reparametrizations of the parameter space of the family.

In Section 3 we develop the equivariant cohomology theory of nets of conics with respect to this action. The results of the section are summarized in Theorems 3.1 and 3.3.

The first named author is supported by OTKA grants NK 81203 and K 101515. The second author is supported by the OTKA grants 72537 and 81203. The third author is supported by NSA grant CON:H98230-10-1-0171.

In Section 4, using results of [FR12], we determine Thom polynomials of contact singularities of type (3,3). The main results of the paper are Theorem 4.1 and 4.4.

In Section 5 we develop the invariant theory of nets of conics. While this theory is mostly known, we emphasize that the equivariant point of view gives a conceptional way to approach invariant theory. This section however is relatively independent of the rest of the paper, it does not use cohomology theory. It is known that the corresponding ring of invariants is generated by two algebraically independent invariants having degree 6 and 12, see Section 5 for references. We give a formula for the degree 6 invariant (already known to Salmon and Sylvester) in terms of the Plücker coordinates as a pull-back of a degree 2 invariant (6) of the Plücker space. The expression (6) for this degree 2 invariant appears to be new.

In Section 6 we describe the hierarchy (see Figure 2) of the orbits of nets of conics. This hierarchy was first obtained by different techniques in [Wal77]. We show how the cohomological data obtained in Section 3 through the notion of *incidence class* can recover the result on hierarchy. This method of reducing hierarchy to equivariant cohomology calculations may be applied in the future to other representations with small GIT quotients. The technique is particularly promising in the emerging field of Geometric Complexity Theory, see [Mul09].

In Section 7 we show how enumerative problems, in particular the intersection multiplicities of the determinant map from nets of conics to plane cubics, can be solved studying the equivariant cohomology classes of the orbits. The main result of the section is Theorem 7.1. We also explain that in some cases the intersection multiplicities agree with the algebraic multiplicities. We hope that these calculations may lead to the determination of which orbit-closures of the plane cubics are Cohen-Macaulay (see Remark 7.2). For completeness we included the multiplicities of the induced map between the corresponding GIT quotients though these results are probably known.

Throughout the paper we work in the complex algebraic category, hence in particular, $\mathrm{GL}(U)$ means the group of complex linear transformations of the complex vector space $U$, and $\mathrm{GL}_n$ denotes $\mathrm{GL}(\mathbb{C}^n)$. Cohomology will be considered with integer coefficients.

The authors are grateful to M. Kazarian for several useful discussions on Thom polynomials and to C. T. C. Wall for very valuable comments on nets of conics. Additionally the authors would like to thank I. Dolgachev, J. Chipalkatti, L. Oeding and P. Frenkel for useful conversations on the topics in this paper.

## 3. Classification of orbits of nets and their equivariant classes

3.1. **Orbits of nets of conics.** Let $S^2U$ denote the second symmetric power of the vector space (or representation) $U$. Consider the vector spaces $U = \mathbb{C}^3$ and $V = \mathbb{C}^3$. The main object of this paper is the $\mathrm{GL}(U) \times \mathrm{GL}(V)$ representation $\mathbf{Noc} = \mathrm{Hom}(S^2U, V)$.

Through the natural isomorphism $\mathrm{Hom}(S^2U, V) = \mathrm{Hom}(V^*, S^2U^*)$, elements of this vector space are families of homogeneous degree 2 polynomials on $U$ parameterized by the vector space $V^*$. Lines in $S^2U^*$ determine conics in the projective plane $\mathbf{P}(U)$, hence elements of $\mathrm{Hom}(V^*, S^2U^*)$ are 2-parameter families (*nets*) of plane conics (**Noc** stands for *nets of conics*). The $\mathrm{GL}(U)$ action reparametrizes the underlying plane $\mathbf{P}(U)$, and the $\mathrm{GL}(V)$ action reparametrizes the parameter space $V^*$.

There is a natural stratification $\Sigma^2 \cup \Sigma^1 \cup \Sigma^0$ of $\mathbf{Noc} - \{0\}$, according to corank. Geometrically the strata correspond to conics, pencils of conics (i.e., 1-parameter families of conics), and (proper) nets of conics, respectively. The orbit structure of conics and pencils of conics is widely known. The classification of orbits of proper nets of conics is given in [Wal77] for the codimension$>1$ cases and in [WdP95] for the family of codimension 1 orbits. We will use their notations.

TABLE 1. Codimension> 1 orbits and symmetries

| | cd | representative | symmetry | Poincaré | $\delta$ |
|---|---|---|---|---|---|
| $\Sigma^0$ | | | | | |
| $C$ | 2 | $y^2+2xz, 2yz, -x^2$ | $(2\alpha, \alpha+\beta, 2\beta), (2\alpha+2\beta, \alpha+3\beta, 4\alpha)$ | 1,1 | $\nu$ |
| $D$ | 2 | $x^2, y^2, z^2+2xy$ | $(2\alpha, 2\beta, \alpha+\beta), (4\alpha, 4\beta, 2\alpha+2\beta)$ | 1,2 | $\theta$ |
| $D^*$ | 2 | $2xz, 2yz, z^2+2xy$ | $(2\alpha, 2\beta, \alpha+\beta), (3\alpha+\beta, \alpha+3\beta, 2\alpha+2\beta)$ | 1,2 | $\theta$ |
| $E$ | 3 | $x^2, y^2, z^2$ | $(\alpha, \beta, \gamma), (2\alpha, 2\beta, 2\gamma)$ | 1,2,3 | А |
| $E^*$ | 3 | $2xy, 2yz, 2zx$ | $(\alpha, \beta, \gamma), (\alpha+\beta, \beta+\gamma, \gamma+\alpha)$ | 1,2,3 | А |
| $F$ | 3 | $x^2+y^2, 2xy, 2yz$ | $(\alpha, \alpha, \beta), (2\alpha, 2\alpha, \alpha+\beta)$ | 1,1 | $\neq$ |
| $F^*$ | 3 | $x^2+y^2, xz, z^2$ | $(\alpha, \alpha, \beta), (2\alpha, \alpha+\beta, 2\beta)$ | 1,1 | $\Omega$ |
| $G$ | 4 | $x^2, y^2, yz$ | $(\alpha, \beta, \gamma), (2\alpha, 2\beta, \beta+\gamma)$ | 1,1,1 | $\neq$ |
| $G^*$ | 4 | $xy, xz, z^2$ | $(\alpha, \beta, \gamma), (\alpha+\beta, \alpha+\gamma, 2\gamma)$ | 1,1,1 | $\neq$ |
| $H$ | 5 | $x^2, 2xy, y^2+2xz$ | $(2\alpha, \alpha+\beta, 2\beta), (4\alpha, 3\alpha+\beta, 2\alpha+2\beta)$ | 1,1 | $\Xi$ |
| $I$ | 7 | $x^2, xy, y^2$ | $(\alpha, \beta, \gamma), (2\alpha, \alpha+\beta, 2\beta)$ | 1,1,2 | 0 |
| $I^*$ | 7 | $xz, yz, z^2$ | $(\alpha, \beta, \gamma), (\alpha+\gamma, \beta+\gamma, 2\gamma)$ | 1,1,2 | 0 |
| $\Sigma^1$ | | | | | |
| $(1^4)$ | 4 | $x^2-xz, y^2-yz, 0$ | $(\alpha, \alpha, \alpha), (2\alpha, 2\alpha, \beta)$ | 1,1 | Ж |
| $(21^2)$ | 5 | $xy, xz+yz, 0$ | $(\alpha, \alpha, \beta), (2\alpha, \alpha+\beta, \gamma)$ | 1,1,1 | $\neq$ |
| $(31)$ | 6 | $xz, x^2-yz, 0$ | $(\alpha+\beta, 2\alpha, 2\beta), (\alpha+3\beta, 2\alpha+2\beta, \gamma)$ | 1,1,1 | $\Xi$ |
| $(22)$ | 6 | $x^2, yz, 0$ | $(\alpha, \beta, \gamma), (2\alpha, \beta+\gamma, \delta)$ | 1,1,2 | $\neq$ |
| $(4)$ | 7 | $xz+y^2, x^2, 0$ | $(2\alpha, \alpha+\beta, 2\beta), (2\alpha+2\beta, 4\alpha, \gamma)$ | 1,1,1 | $\Xi$ |
| $K$ | 8 | $y^2, z^2, 0$ | $(\beta, \alpha, \gamma), (2\alpha, 2\gamma, \delta)$ | 1,1,1,2 | 0 |
| $L$ | 8 | $xy, xz, 0$ | $(\alpha, \beta, \gamma), (\alpha+\beta, \alpha+\gamma, \delta)$ | 1,1,1,2 | 0 |
| $M$ | 9 | $yz, y^2, 0$ | $(\alpha, \beta, \gamma), (\beta+\gamma, 2\beta, \delta)$ | 1,1,1,1 | 0 |
| $\Sigma^2$ | | | | | |
| $S$ | 10 | $xy-z^2, 0, 0$ | $(2\alpha, 2\beta, \alpha+\beta), (2\alpha+2\beta, \gamma, \delta)$ | 1,1,2,2 | 0 |
| $PL$ | 11 | $xy, 0, 0$ | $(\alpha, \beta, \gamma), (\alpha+\beta, \delta, \epsilon)$ | 1,1,1,2,2 | 0 |
| $DL$ | 13 | $x^2, 0, 0$ | $(\alpha, \beta, \gamma), (2\alpha, \delta, \epsilon)$ | 1,1,1,2,2 | 0 |
| 0 | 18 | $0, 0, 0$ | $(\alpha, \beta, \gamma), (\delta, \epsilon, \kappa)$ | 1,1,2,2,3,3 | 0 |

The list of codimension> 1 orbits is given in the first 3 columns of Table 1 with the following conventions. Column 1 is the name of the orbit, column 2 is its codimension, and column 3 names three plane conics that span the image of $\phi \in \mathrm{Hom}(V^*, S^2U^*)$. Here we used the letters $x, y, z$ for the coordinates on $U$.

3.2. **Equivariant classes.** $G$-invariant subvarieties (e.g., orbit closures) represent cohomology classes in the equivariant cohomology ring of a $G$-representation. We want to determine the equivariant classes $[\overline{\eta}] \in H^*_{\mathrm{GL}(U)\times\mathrm{GL}(V)}(\mathbf{Noc}) \cong \mathbb{Z}[u_1, u_2, u_3, v_1, v_2, v_3]$ for the orbits $\eta \subset \mathbf{Noc}$. Here $u_1, u_2, u_3$ and $v_1, v_2, v_3$ denote the Chern classes of the groups $\mathrm{GL}(U)$ and $\mathrm{GL}(V)$ respectively. These classes contain a lot of geometric information as we will show later.

For the codimension> 1 orbits we use the method of *restriction equations* of [Rim01, Thm.2.4], see also [FR04, Sect.3], so we need the symmetries of these orbits. More precisely, we need only a maximal torus of their stabilizer subgroups. These elementary calculations can be reduced to the level of Lie algebras. The results are summarized in the "symmetry" column of Table 1.

To explain the notation, consider the orbit $C$ represented by the net of conics

$$(y^2 + 2xz, 2yz, -x^2).$$

The pair of matrices

$$(1)\qquad \left( \begin{pmatrix} a^2 & 0 & 0 \\ 0 & ab & 0 \\ 0 & 0 & b^2 \end{pmatrix}, \begin{pmatrix} a^2b^2 & 0 & 0 \\ 0 & ab^3 & 0 \\ 0 & 0 & a^4 \end{pmatrix} \right) \in \mathrm{GL}(U) \times \mathrm{GL}(V), \qquad a, b \in \mathrm{GL}_1$$

stabilize this net of conic. Since the codimension of $C$ is 2, the dimension of the stabilizer subgroup has to be 2 as well ($\dim G = \dim V = 18$), so we determined the maximal torus. This is the data that is encoded as $(2\alpha, \alpha+\beta, 2\beta), (2\alpha+2\beta, \alpha+3\beta, 4\alpha)$ in Table 1.

**Theorem 3.1.** *Consider the* $\mathrm{GL}(U) \times \mathrm{GL}(V)$ *representation* **Noc**. *The Theorem of Restriction Equations* [FR04, Thm. 3.5] *determines all the* $\mathrm{GL}(U) \times \mathrm{GL}(V)$ *equivariant classes of the codimension*$>1$ *orbit closures, e.g., we have*

- $[\overline{C}] = 8(v_1 - 2u_1)^2$,
- $[\overline{D}] = -3u_2 + 3v_2 - 16u_1v_1 + 3v_1^2 + 17u_1^2$,
- $[\overline{D^*}] = 12u_2 - 3v_2 - 20u_1v_1 + 6v_1^2 + 16u_1^2$,
- $[\overline{E}] = 3u_3 + 3v_3 - 3u_1u_2 + u_2v_1 - 6u_1v_1^2 + 13u_1^2v_1 - 2u_1v_2 - 8u_1^3 + v_1^3$
- $[\overline{E^*}] = -24u_3 + 3v_3 - 24u_1u_2 + 16u_2v_1 - 16u_1v_1^2 + 20u_1^2v_1 - 6v_1v_2 + 10u_1v_2 - 8u_1^3 + 4v_1^3$
- $[\overline{F}] = 2(v_1 - 2u_1)(6u_1^2 - 4u_1v_1 - 6u_2 + 3v_2)$,
- $[\overline{F^*}] = 2(v_1 - 2u_1)(5u_1^2 - 8u_1v_1 + 9u_2 - 3v_2 + 3v_1^2)$.

*Proof.* The proof does not follow from any general principle we are aware of, it is just an experimental fact. (The condition under which the Restriction Equations determine all equivariant classes in [FR04, Thm. 3.5] is that there are finitely many orbits, and each of them satisfy an Euler-class condition. For our representation there is a moduli of orbits.) The symmetry data of the table put constraints on the classes $[\overline{\eta}]$. One can write down all these constraints for each codimension$>1$ orbit $\eta$. A computer program shows that for each codimension$>1$ orbit there is only one equivariant class in $H^*\big(B(\mathrm{GL}(U) \times \mathrm{GL}(V))\big)$ satisfying the constraints. □

For the family of codimension 1 orbits we look at the Wall-DuPlessis classification [WdP95] from an equivariant point of view.

The affine plane $N_C = \{\nu_{c,g} : c, g \in \mathbb{C}\}$, where

$$\nu_{c,g} = (y^2 + 2xz, 2yz, -x^2 + 2g(xz - y^2) + cz^2),$$

is normal to the orbit $C$ at the point $(y^2 + 2xz, 2yz, -x^2)$. This plane is invariant under the action of the complex 2-torus $T_C$ of (1). The $T_C$ action on $N_C$ has weights $2\alpha - 2\beta$ and $4\alpha - 4\beta$, corresponding to the weight vectors $(0, 0, xz - y^2)$ and $(0, 0, z^2)$. Hence, the orbits of $T_C$ on $N_C$ correspond to the parabolas with $\mu = (c : g^2) \in \mathbf{P}^1$ constant.

According to [WdP95] these parabolas are exactly the intersections of the codimension 1 **Noc**-orbits with the normal slice $N_C$. We will refer to the orbit of $\nu_{c,g}$ with $\mu = (c : g^2)$ as $A_\mu$. In [Wal77] $A_{-9}$ is called $B$ and $A_0$ is called $B^*$. We will refer to a **Noc**-orbit representative lying in $N_C$ as a *c-g-form*. Recall the following *Incidence Theorem*.

**Theorem 3.2.** [FP09] *Consider a Lie group $G$ acting on a vector space $V$ complex linearly. For $v \in V$ let $G_v$ denote the stabilizer subgroup of $v$. Let $S$ be a subgroup of $G_v$ and $N_v$ an $S$-invariant normal slice to the orbit $Gv$ at $v$. Suppose that $\eta \subset V$ is a $G$-invariant subvariety. Then*

$$[\eta \subset V]_S = [(\eta \cap N_v) \subset N_v]_S.$$

**Theorem 3.3.** *The equivariant classes of the $A_\mu$ orbits are $4(v_1 - 2u_1)$ for $\mu \neq \infty$ and $2(v_1 - 2u_1)$ for $\mu = \infty$.*

*Proof.* If $\mu \neq \infty$, then the class $[(\mu g^2 = c) \subset N_C]$ is equal to the weight of the $g = 0$ direction. Hence we have $[(\mu g^2 = c) \subset N_C] = 4\alpha - 4\beta$. For the curve $g = 0$ we have

$$[(g = 0) \subset N_C] = 2\alpha - 2\beta.$$

For the restriction homomorphism $r : H^*_{\mathrm{GL}(U)\times\mathrm{GL}(V)} \to H^*_{T_C} = \mathbb{Z}[\alpha, \beta]$, we have

$$r(u_1) = 2\alpha + (\alpha + \beta) + 2\beta = 3\alpha + 3\beta$$

and

$$r(v_1) = (2\alpha + 2\beta) + (\alpha + 3\beta) + 4\alpha = 7\alpha + 5\beta$$

by (1). Hence, if $[A_\mu] = Au_1 + Bv_1$ ($\mu \neq \infty$), then according to Theorem 3.2, we have

$$A(3\alpha + 3\beta) + B(7\alpha + 5\beta) = 4\alpha - 4\beta.$$

The only solution is $A = -8, B = 4$. For $\mu = \infty$ the calculation is similar. □

**Remark 3.4.** From the equivariant classes of (cone) varieties one can calculate their degrees, see, e.g., [FNR05, Sec. 6] and Section 7. Therefore, Theorem 3.3 implies that the degree of the hypersurfaces given by the closures of the $A_\mu$ orbits is $4(1+1+1-2(0+0+0)) = 12$ in general, and 6 for $\mu = \infty$. This implies that the ring of invariants $R(\mathbf{Noc})$ on $\mathbf{Noc}$ is generated by a degree 6 and a degree 12 polynomial. We will give a full description of $R(\mathbf{Noc})$ in Section 5.

## 4. Thom polynomials of contact singularities corresponding to nets of conics

Consider a polynomial map $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$. In global singularity theory one studies its Thom polynomial $\mathrm{Tp}_g \in \mathbb{Z}[\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_p]^{S_n \times S_p}$. Here $S_n$ permutes the $\alpha_i$ variables (the so-called "source Chern roots"), and $S_p$ permutes the $\beta_j$ variables (the so-called "target Chern roots").

To recall the definition of $\mathrm{Tp}_g$ we need to define contact equivalence. Consider the maximal ideal $\mathfrak{m}$ of the ring of formal power series $\mathbb{C}[[x_1, \dots, x_n]]$. To any polynomial map $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ (or, equivalently a jet in $J^k(n, p)$) we assign the ideal $I_g = (g_1, \dots, g_p) \lhd \mathfrak{m}$ generated by the coordinate functions of $g$. We say that two jets are contact equivalent if their ideals are equivalent under a local holomorphic reparametrization of $\mathbb{C}^n$. The Thom polynomial $\mathrm{Tp}_g$ of the map $g$ is the $T = U(1)^n \times U(1)^p$-equivariant cohomology class represented by the set of jets contact equivalent to $g$, in the jet space $J^k(n, p)$ (where $k$ is high enough). Sets of contact equivalent jets will be called contact classes. They can either be defined by $g$ or by the ideal $I_g$.

We will assign two contact classes to a net of conics $f \in \mathrm{Hom}(S^2U, V)$. Let $f$ be represented by three degree-2 polynomials $f_1, f_2, f_3$ in $x, y, z$ (see examples in Table 1). Consider the ideals $I_f = (f_1, f_2, f_3)$ and $I_{\tilde f} = (f_1, f_2, f_3) + (x, y, z)^3$, they define contact classes, $I_f$ for $p \geq 3$ and $I_{\tilde f}$ only for higher values of $p$.

We are mainly interested in the Thom polynomial defined by $I_f$ for $p = 3$, since among these are the the smallest (lowest degree) Thom polynomials that can not be computed by previous methods ([Rim01]). However we need to make a detour to calculate Thom polynomials corresponding to $I_{\tilde f}$ first. The reason is explained in Section 4.1.

**Theorem 4.1.** *Let $f \in \mathrm{Hom}(S^2U, V)$ be a $\Sigma^0$ net of conics, and let $[f] \in \mathbb{Z}[u_1, u_2, u_3, v_1, v_2, v_3]$ be the equivariant class of its $\mathrm{GL}(U) \times \mathrm{GL}(V)$-orbit considered in Section 3. Let $\sigma_f : (\mathbb{C}^3, 0) \to (\mathbb{C}^p, 0)$ be a polynomial map with $I_{\tilde f} = I_{\sigma_f}$. Let $W = \{2\alpha_1, 2\alpha_2, 2\alpha_3, \alpha_1 + \alpha_2, \alpha_1 + \alpha_3, \alpha_2 + \alpha_3\}$. Then*

$$\mathrm{Tp}_{\sigma_f} = \prod_{i=1}^{3} \prod_{j=1}^{p} (\beta_j - \alpha_i) \cdot \sum_{H \subset W, |H|=3} \frac{D_H \cdot [f]|_H}{e_H}, \tag{2}$$

*where*

$$D_H=\prod_{w\notin H}\prod_{j=1}^{p}(\beta_j-w),\qquad e_H=\prod_{w_1\notin H}\prod_{w_2\in H}(w_2-w_1),\qquad [f]|_H=[f]|_{v_i=\sigma_i(H),u_i=\sigma_i(\alpha_1,\alpha_2,\alpha_3)}.$$

*In the last substitution $\sigma_i$ means the i'th elementary symmetric polynomial.*

In Section 4.1 we will show how this Theorem follows from the "Localization Formula" [FR12, Thm 6.1].

**Remark 4.2.** Similar, and in some sense simpler formulas can be given for the non-$\Sigma^0$ nets. The corresponding polynomial maps belong to very large codimensional singularities, consequently their Thom polynomials are of very large degree, so we omit the description.

Theorem 4.1 gives the Thom polynomial in the Chern root variables $\alpha_i$ and $\beta_j$. A more compact way to write down these polynomials is in *quotient variables*, using the basis of Schur polynomials. To indicate that we use quotient variables we use the notation $\mathrm{tp}_{\sigma_f}$. The Schur polynomials $\Delta$ are defined by $\Delta_{(\lambda_1,\dots,\lambda_r)}=\det(c_{\lambda_i+j-i})_{r\times r}$; for example,

$$\Delta_{(31)}=\det\begin{pmatrix}c_3 & c_4\\ c_0 & c_1\end{pmatrix}=c_3c_1-c_4,$$

and $c_i$ denotes the degree $i$ homogeneous part of $\frac{\prod_{j=1}^{p}(1+\beta_j)}{\prod_{i=1}^{3}(1+\alpha_i)}$. Note that this convention for Schur polynomials is that of, e.g., [FP98], which slightly differs from that of, e.g., [Mac98].

Theorem 4.1, and some calculation, gives that for example for $\sigma_f:\mathbb{C}^3\to\mathbb{C}^4$,

$$\sigma_f(x,y,z)=(y^2+2xz,2yz,-x^2+2g(xz-y^2)+cz^2,z^3)$$

we have

$$\mathrm{tp}_{\sigma_f}=8\Delta_{(544111)}+4\Delta_{(444211)}+16\Delta_{(844)}+20\Delta_{(6442)}+32\Delta_{(64411)}+120\Delta_{(6541)}+160\Delta_{(655)}+ \\ 16\Delta_{(54421)}+32\Delta_{(55411)}+40\Delta_{(5542)}+80\Delta_{(5551)}+80\Delta_{(664)}+40\Delta_{(7441)}+112\Delta_{(754)}, \tag{3}$$

if $g\neq 0$ and half of it for $g=0$.

4.1. **Fibered localization.** Now we show how the arguments in [FR12] can be applied to prove Theorem 4.1, some familiarity of Sections 5, 6 of [FR12] are assumed. The localization formula of [FR12, Thm 6.1] is based on a generalization of the Atiyah-Bott-Berline-Vergne localization: the fibered localization formula of Vergne-Rossmann-Bérczi-Szenes ([BS12]). This formula calculates, from some fixed point data, the torus equivariant cohomology of a variety which is the birational image of a vector bundle $E\to Y$ such that the base $Y$ is a subvariety of a smooth manifold (everything is equipped with compatible torus actions).

For the proof of Theorem 4.1 we consider the correspondence variety

$$C(Y)=\{(I,g)\in \mathrm{Gr}^m\left(J^k(n)\right)\times J^k(n,p)\mid I\in Y, I_g\subset I\},\tag{4}$$

where $g\in J^k(n,p)$ is a jet, $m$ is the codimension of $I_g\lhd J^k(n)$, $\mathrm{Gr}^m\left(J^k(n)\right)$ denotes the $m$-codimensional subspaces of the jet space $J^k(n)$ and $Y$ is the closure of the set of ideals equivalent to $I_g$. The first projection $C(Y)\to Y$ is a vector bundle, and it can be shown that the second projection is a birational map to the closure of the contact class of $g$. The variety $Y$ is contained in the manifold $\mathrm{Gr}_3(S^2\mathbb{C}^3)$ (for $f\in\Sigma^i$ it would be $\mathrm{Gr}_{3-i}(S^2\mathbb{C}^3)$). Hence, the fibered localization formula ([FR12, Prop 5.1 or Thm 6.1]) can be applied.

This formula expresses $\mathrm{Tp}_{\sigma_f}$ as a sum whose terms correspond to torus fixed points of $\mathrm{Gr}_3(S^2\mathbb{C}^3)$, that is, to the set $\{H\subset W,|H|=3\}$. The term corresponding to $H$ has a "fiber" component and a "horizontal" component. The fiber component is explicitly $D_H$, and the "horizontal" component, expressing the class of $Y$ at the fixed point $H$ is

$$[Y]|_H = [f]|_{v_i=\sigma_i(H), u_i=\sigma_i(\alpha_1,\alpha_2,\alpha_3)}.$$

**Remark 4.3.** What made this calculation very simple is the fact that $Y$ is contained in the Grassmannian $\mathrm{Gr}_3(S^2\mathbb{C}^3)$. For the ideals $I_f$ we are not aware of any similar simplification. However for $p=3$ we will calculate the Thom polynomials for some nets of conics in the next section.

4.2. **Thom polynomials of equidimensional maps.** Notice that for any $\sigma_f$ satisfying the condition of Theorem 4.1, $p$ is at least 4. It is possible to substitute however $p=3$ formally into the formula, and in some cases we can interpret the result as a Thom polynomial. This phenomenon was called the "small $p$ case" in [FR12, §12]. According to the previous section we need to check that the second projection of the correspondence variety (4) for $Y$ being the closure of the set of ideals equivalent to $I_{\tilde{f}}$ is a birational map to the closure of the contact class of $f \in J^2(3,3)$. Obviously $f$, therefore its contact class, is in the image, so the following two conditions will imply birationality:

(1) The second projection is generically one-to-one.
(2) The dimension of the contact class of $f$ equals to that of the correspondence variety $C(Y)$.

Condition (1) is equivalent to the condition that there is only one ideal equivalent to $I_{\tilde{f}}$ containing $I_f$: the ideal $I_{\tilde{f}}$ itself. This is satisfied for all $f$, since any ideal equivalent to $I_{\tilde{f}}$ should contain the invariant ideal $(x,y,z)^3$. By calculating the dimension of the correspondence variety we can see that the second condition is equivalent to the condition that the codimension of the contact class of $f$ in $J^2(3,3)$ is $\deg[f]+9$.

We give 3 examples when the second condition is also satisfied (see [WdP95, p. 315]).

1. The Thom polynomial of the germ $h_{c,g}:\mathbb{C}^3\to\mathbb{C}^3$,

$$h_{c,g}(x,y,z) = (y^2+2xz, 2yz, -x^2+2g(xz-y^2)+cz^2)$$

is

$$\mathrm{tp}_{h_{c,g}} = 8\Delta_{(433)} + 4\Delta_{(3331)},$$

if $g\neq 0$ and half of it for $g=0$. The singularities $h_{c,g}$ form the smallest codimensional example of a family of non-equivalent contact singularities $(\mathbb{C}^n,0)\to(\mathbb{C}^n,0)$ ([Mat71]). Because of the presence of this continuous modulus in the classification of singularities, the method of [Rim01] calculating Thom polynomials broke down at codimension 9. (In [FR12] a different representation of $h_{c,g}$ is used, namely $(x^2-\lambda yz, y^2-\lambda xz, z^2-\lambda xy)$. This representation is less adapted to our purposes, in general 12 different values of $\lambda$ correspond to the same orbit, and the singularities corresponding to $B$ and $B^*$ cannot be written in this form.)

2 and 3: The singularities corresponding to nets from the orbits $D$ and $E$ are denoted by $KD$ and $KE$ in [WdP95, p. 315]. Hence [FR12, Section 7] implies:

**Theorem 4.4.** *The Thom polynomials of maps $(\mathbb{C}^n,0)\to(\mathbb{C}^n,0)$ of type $KD$ and $KE$ are*

$$\mathrm{tp}_{KD} = 3\Delta_{(33311)} + 6\Delta_{(3332)} + 14\Delta_{(443)} + 16\Delta_{(4331)} + 17\Delta_{(533)},$$

$$\mathrm{tp}_{KE} = \Delta_{(333111)} + 2\Delta_{(33321)} + 4\Delta_{(3333)} + 6\Delta_{(43311)} + 8\Delta_{(4332)} + 14\Delta_{(4431)} + 8\Delta_{(444)} + 13\Delta_{(5331)} + 19\Delta_{(543)} + 8\Delta_{(633)}.$$

**Remark 4.5.** For some pair of contact singularities only condition (2) is satisfied. Examples are given in [FR12, §12]. In such cases the localization formula gives the Thom polynomial multiplied by the number of sheets of the second projection. (The simplest example we are aware of is the singularities $\tilde{f} = (x^3, xy, y^2)$ and $f = (xy)$. In this case we have two sheets: the ideal $(x^2, xy, y^3)$ also contains $(xy)$.)

**Remark 4.6.** A polynomial map is called equidimensional, if the dimension of its source is the same as the dimension of its target. Thom polynomials of equidimensional polynomial maps were calculated in [Rim01] up to degree 8. The results above on the Thom polynomials of $h_{c,g}$, $KD$, and $KE$ are essentially the first examples beyond degree 8. Notice that we calculated the Thom polynomials of these singularities only for $p = 3$, as opposed to the singularities corresponding to $I_{\tilde{f}}$ where we obtained the Thom polynomial for all $p$.

4.3. **Duality and the Kazarian theory.** Recently M. Kazarian developed a method ([Kaz]) to calculate Thom polynomials, in particular Section 8.8 of [Kaz] deals with nets of conics. His approach is dual to the method of [FR12] in the sense that it uses the quotient algebra as opposed to the ideal of the singularity. For a singularity corresponding to a $\Sigma^0$ net $f$ in the above sense the quotient algebra is $Q_f = J^2(U, V)/(f_1, f_2, f_3)$. This is a graded algebra of graded degree $(3, 3)$, so the multiplication is determined by a linear map $m \in \mathrm{Hom}(S^2U^*, \ker f)$. The map $m$ is the *apolar* of $f$ in the terminology of [Wal77]. The input of the Kazarian method is the equivariant class of $m$. The apolar of a net lives in the dual representation, however, by fixing a basis of $U$ we can identify these spaces. The identification depends on the basis but a basis change does not change the orbit, so we get a well defined duality on the $\Sigma^0$ orbits. Wall denotes this duality by $*$. It is also true (and easy to see) that the equivariant classes of the orbits are the same up to sign in the dual representation, however in general the dual orbit has different equivariant class, see, e.g., $[D] \neq [D^*]$ and consequently the Thom polynomials of the corresponding singularities are also different. Using the Kazarian theory we were able to double check our results.

## 5. Invariant theory of nets of conics

Most of the results in this section are known (see [Wal77], [Wal10], [WdP95] [Vin76] and [AN02]), but we also would like to show how equivariant theory leads to these results with the hope that it can be applied in a more general context.

5.1. **The ring of (semi) invariants.** Suppose that the Lie group $G$ acts on the vector space $W$ and $\hat{G}$ is the character group of $G$. We say that $f \in \mathbb{C}[W]$ is a *relative invariant corresponding to the character* $\chi \in \hat{G}$ (i.e., $f \in R_\chi(W)$) if for all $v \in W$ and $g \in G$

$$f(gv) = \chi(g) f(v). \tag{5}$$

The ring of *semi-invariants* is $R(W) := \bigoplus_{\chi \in \hat{G}} R_\chi(W)$.

Note that an element of $R(W)$ is not necessarily a relative invariant for any $\chi \in \hat{G}$. Semi-invariants of $G$ are always invariants of the commutator subgroup $G'$, but in general the ring of invariants of $G'$ can be bigger. In the following two examples, however, they coincide.

(a) The $\mathrm{GL}(U) \times \mathrm{GL}(V)$-action on **Noc**: Any character of $\mathrm{GL}(U) \times \mathrm{GL}(V)$ is of the form $\chi_{a,b}(g, h) := \det^a(g) \det^b(h)$. If $f \in R(\mathbf{Noc})$ is homogeneous of degree $l = 3d$, then

$$f\big((\lambda I, \mu I)v\big) = f(\lambda^{-2}\mu v) = \lambda^{-2l}\mu^l f(v) = \det{}^{-2d}(\lambda I)\det{}^d(\mu I) f(v),$$

therefore $f$ is a relative invariant corresponding to the character $\chi_{-2d,d}$. (In GIT language we see that there is a unique linearization for GIT-quotient.) In other words

$$R(\mathbf{Noc}) = \bigoplus_{d\in\mathbb{N}} R_{\chi_{-2d,d}}(\mathbf{Noc}).$$

In this case all characters are determinants, so $R(\mathbf{Noc})$ coincides with the ring of absolute invariant polynomials for the $\mathrm{SL}_3 \times \mathrm{SL}_3$-action on $\mathbf{Noc}$.

(b) As we discussed before, the maximal torus of the stabilizer of the net $(y^2 + 2xz, 2yz, -x^2)$ is $\mathrm{GL}_1 \times \mathrm{GL}_1$. It acts on the normal space $N_C$ to its orbit with weights $4\alpha - 4\beta$ and $2\alpha - 2\beta$. With respect to this action we have

$$R(N_C) = \bigoplus_{d\in\mathbb{N}} R_{\chi_{2d,-2d}}(N_C) = \mathbb{C}[N_C] = \mathbb{C}[c, g],$$

where $c \in R_{\chi_{4,-4}}(N_C)$ and $g \in R_{\chi_{2,-2}}(N_C)$. Consider the restriction map $i^* : R(\mathbf{Noc}) \to R(N_C)$. From the first line of Table 1 (equivalently, from (1)) one sees that $i^*$ maps $R_{\chi_{-2d,d}}(\mathbf{Noc})$ into $R_{\chi_{d,-d}}(N_C)$.

We claim that $i^* : R(\mathbf{Noc}) \to R(N_C)$ is injective. Indeed, according to the splitting above it is enough to show that it is injective on $R_{\chi_{-2d,d}}(\mathbf{Noc})$ for any given $d$, which follows from the fact that the values of a relative invariant $f$ on $N_C$ determine its values on $(\mathrm{GL}(U)\times\mathrm{GL}(V))\cdot N_C$ via (5) and $(\mathrm{GL}(U) \times \mathrm{GL}(V)) \cdot N_C$ is dense in $\mathbf{Noc}$. Now we prove that the homomorphism $i^*$ is also surjective, by finding relative invariants of $\mathbf{Noc}$ mapped to $c$ and $g$.

5.2. **The determinant map.** Composing nets with the determinant $S^2U^* \to \mathbb{C}$ (well defined up to a scalar factor) we get a degree 3 polynomial map $\delta : \mathbf{Noc} \to S^3V$. This map is $\mathrm{GL}(U) \times \mathrm{GL}(V)$-equivariant if we let $\mathrm{GL}(U)$ act on $S^3V$ as scalars by the second tensor power of the determinant representation. Consequently we have a homomorphism $\delta^* : R(S^3V) \to R(\mathbf{Noc})$. Let us review now the invariant theory of the plane cubics $S^3\mathbb{C}^3$ which was one of the first achievements of the early invariant theory.

**Theorem 5.1.** [Aro50] *There are invariants $a, b$ of $S^3\mathbb{C}^3$ of degree 4 and 6, respectively such that $R(S^3\mathbb{C}^3) \cong \mathbb{C}[a, b]$ and every smooth plane cubic $\gamma$ can be transformed (using the $\mathrm{GL}_3$-action) into the Weierstrass-form:*

$$y^2z + x^3 + a(\gamma)xz^2 + b(\gamma)z^3.$$

**Remark 5.2.** The Weierstrass-form is analogous to the $c$-$g$-form of nets of conics from Section 3. The subset $\{y^2z + x^3 + axz^2 + bz^3 : a, b \in \mathbb{C}\}$ is a normal slice to the orbit of the cuspidal cubic $y^2z + x^3$. This observation was used in [Kőm03] to calculate the equivariant classes for plane cubics. The choice of these orbits is not accidental. Their closure is the nullcone, so the normal slice intersects all invariant hypersurfaces.

Consider now the determinant of the $c$-$g$-form $\nu_{c,g} = (y^2 + 2xz, 2yz, -x^2 + 2g(xz - y^2) + cz^2)$. Considering the three $3 \times 3$ matrices of the three components of $\nu_{c,g}$ we obtain

$$\begin{aligned}\delta(\nu_{c,g}) &= \det\left((-x)\begin{pmatrix} 0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix} + (-y)\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix} + z\begin{pmatrix} -1 & 0 & g\\ 0 & -2g & 0\\ g & 0 & c\end{pmatrix}\right)\\ &= y^2z + x^3 + (c - 3g^2)xz^2 + 2g(c + g^2)z^3.\end{aligned}$$

Here we parameterized $\mathbb{C}^3$ with $-x, -y, z$ to obtain our result, the Weierstrass form, without sign changes. Therefore $i^*\delta^*a = c - 3g^2$ and $i^*\delta^*b = 2g(c + g^2)$. We denote the degree 12 invariant $-48\delta^*a$ by $J_{12}$.

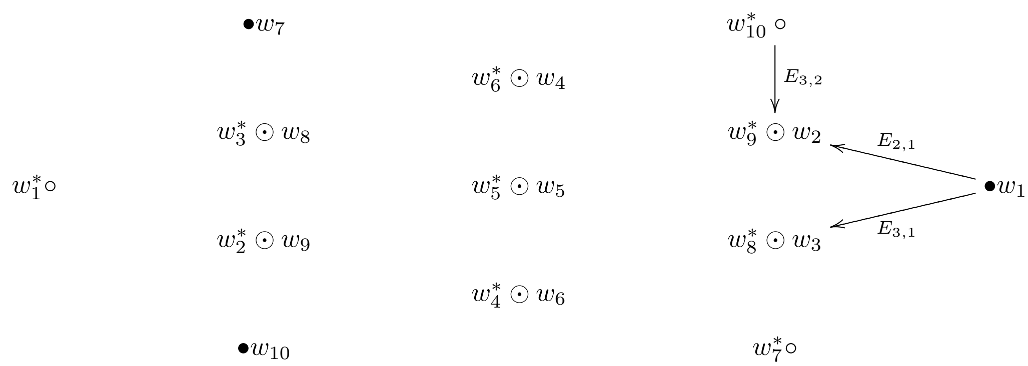


FIGURE 1. The 20 weights of **Pl**

To complete the calculation of $R(\mathbf{Noc})$ we need to find the degree 6 invariant which restricts to $g$ on the normal slice $N_C$. This is a straightforward job with a computer, but using a geometric idea it can be done by hand.

5.3. **The Plücker map and the invariant $I_2$.** We have a degree 3 $\mathrm{GL}(U)\times\mathrm{GL}(V)$-equivariant map

$$\psi : \mathbf{Noc} = \mathrm{Hom}(S^2U, V) \overset{\wedge^3}{\to} \mathrm{Hom}(\bigwedge^3 S^2U, \bigwedge^3 V) \cong \bigwedge^3 S^2U^* \otimes \bigwedge^3 V.$$

We will call $\mathbf{Pl} := \bigwedge^3 S^2U^* \otimes \bigwedge^3 V$ the *Plücker space.*

Picking a basis for $V^*$, to each net we can associate a triple $M = (M_1, M_2, M_3)$ of conics, and $\psi$ sends $M$ to $M_1 \wedge M_2 \wedge M_3 \in \bigwedge^3 S^2U^*$. The image of $\psi$ is the cone of the Grassmannian $\mathrm{Gr}_3(S^2U^*)$. Our next goal is to show that the representation **Pl** has a degree 2 invariant $I_2$ which pulls back to a degree 6 invariant $J_6$ of **Noc**.

Let $\xi, \eta, \nu$ be a basis in $U^*$ and $x, y, z$ the corresponding dual basis in $U$. Write $E_{ij}$ for the element of $\mathfrak{sl}(U)$ mapping the $i$th basis vector of $U$ to the $j$th basis vector and annihilating the other two basis vectors.

Consider the basis $e_1 := \xi^2$, $e_2 := \xi\eta$, $e_3 := \xi\nu$, $e_4 := \eta^2$, $e_5 := \eta\nu$, $e_6 := \nu^2$ in $S^2U^*$, and set $e_{ijk} := e_i \wedge e_2 \wedge e_3 \in \bigwedge^3 S^2(U^*)$. Under the natural identification of $S^2U$ with the dual of $S^2(U^\star)$, the basis in $S^2(U)$ dual to $e_1, \dots, e_6$ is $t_1 := x^2$, $t_2 := 2xy$, $t_3 := 2xz$, $t_4 := y^2$, $t_5 := 2yz$, $t_6 := z^2$. Then $t_{ijk} := t_i \wedge t_j \wedge t_k \in \bigwedge^3 S^2(U)$ is the basis dual to $e_{ijk}$. In particular, $t_{ijk}(\pi(M))$ is the $3\times 3$ minor corresponding to the $(i, j, k)$ columns of the $3 \times 6$ matrix of the net $M$ viewed as a linear map from $V^*$ to $S^2U^*$, with respect to the chosen bases.

As an $\mathrm{SL}(U)$-representation $\mathbf{Pl} \cong S^3(U) \oplus S^3(U^*)$. This follows for example from the calculations below or by calculating the weights of **Pl**. (See Figure 1, picturing the weights of **Pl**, denoting the weights of $S^3(U)$ by dots and the weights of $S^3(U^*)$ by circles. We also indicated the weight vectors spanning the corresponding weight spaces.)

Denoting by $W$ the 10-dimensional $\mathrm{SL}(U)$-module $S^3(U)$ we have

$$S^2\,\mathbf{Pl}^* \cong S^2W \oplus S^2W^* \oplus W \otimes W^*.$$

The first two summands do not contain the trivial $\mathrm{SL}(U)$-module (say by the theorem of Aronhold on the invariants of ternary cubic forms), and the third summand contains one copy of the trivial representation by Schur's Lemma, spanned by $w_1w_1^* + \dots + w_{10}w_{10}^*$, where $w_1, \dots, w_{10}$ is a basis of $W$ and $w_1^*, \dots, w_{10}^*$ is the corresponding dual basis in $W^*$. Table 2 contains explicit elements $w_1, \dots, w_{10} \in \bigwedge^3 S^2(U)$ spanning an $\mathfrak{sl}(U)$-summand isomorphic to $W$. Clearly $x^2 \wedge xy \wedge xz$

is a highest weight vector generating an $\mathfrak{sl}(U)$-module isomorphic to $W$. The $w_2, \dots, w_{10}$ are obtained by applying successively the operators $E_{21}, E_{31} \in \mathfrak{sl}(U)$, as indicated in Table 2.

Recall that the action of $E_{i,j}$ is the sum of the replacements of each occurrence of the $j$th basis vector to the $i$th one, e.g.,

$$E_{3,1}(x^2 \wedge xy \wedge xz) = 2xz \wedge xy \wedge xz + x^2 \wedge yz \wedge xz + x^2 \wedge xy \wedge z^2 = x^2 \wedge xy \wedge z^2 - x^2 \wedge xz \wedge yz.$$

Up to non-zero scalars $w_{10}^*$ in Table 2 is the only weight vector whose weight is opposite to the weight of the lowest weight element $w_{10}$ in $W$, therefore it must be a highest weight vector generating a submodule isomorphic to $W^*$ in $\bigwedge^3 S^2U^*$. Applying successively appropriate elements of $\mathfrak{sl}(U)$ to $w_{10}^*$ one computes $w_2^*, \dots, w_{10}^* \in \bigwedge^3 S^2U^*$. For example, by the left column of the table we have $w_{10} = E_{31}w_6$, hence $w_6^* = -E_{31}w_{10}^*$. In addition to the information in the left column of the table we need also the relations

$$w_8 = \frac{1}{3}E_{32}w_7, \quad w_9 = \frac{1}{2}E_{32}w_8, \quad \text{and} \quad w_{10} = E_{32}w_9.$$

With the $w_i, w_i^*$ given in Table 2 (for example, $w_2 = -x^2 \wedge xz \wedge y^2 + x^2 \wedge xy \wedge yz = -\frac{1}{2}t_{134} + \frac{1}{4}t_{125}$), we have the equality

$$\begin{aligned}
-8I_2 := 8\sum_{i=1}^{10} w_i w_i^* = t_{235}^2 - 8t_{146}^2 & \\
& - 8t_{134}t_{346} + 8t_{126}t_{246} + 8t_{145}t_{156} \\
& + 6t_{123}t_{456} - 6t_{136}t_{245} + 6t_{124}t_{356} \\
& - 4t_{125}t_{256} + 4t_{135}t_{345} - 4t_{234}t_{236} \\
& + 2t_{134}t_{256} - 2t_{125}t_{346} + 2t_{135}t_{246} - 2t_{126}t_{345} \\
& + 2t_{145}t_{236} + 2t_{156}t_{234} - 2t_{146}t_{235}.
\end{aligned} \tag{6}$$

**Remark 5.3.** (i) Set $u_{ijk} := t_{ijk} \circ \psi$, so $u_{ijk}$ is an element of the coordinate ring of **Noc**. Recall a classical result of Sylvester (see page 365 in [Sal79] or [Giz07]), asserting (after a change to our coordinate system) that

$$\begin{aligned}
-8\theta := & \, u_{235}^2 - 8u_{146}^2 \\
& + 4u_{146}u_{235} + 4u_{135}u_{345} - 4u_{125}u_{256} - 4u_{234}u_{236} \\
& + 8u_{145}u_{156} - 8u_{134}u_{346} + 8u_{126}u_{246} \\
& + 8u_{123}u_{456} - 8u_{136}u_{245} + 8u_{124}u_{356}
\end{aligned}$$

is an $\mathrm{SL}(U)$-invariant on **Noc**. We thank I. Dolgachev for bringing this reference to our attention. One can easily verify using the straightening algorithm (cf. Section 13.2.2 in [Pro07]) that $J_6 := I_2 \circ \psi$ coincides with Salmon's $\theta$. Notice that one cannot reconstruct $I_2$ from $\theta$ since $\psi^*$ has a kernel, generated by the *Plücker relations*. To the best of our knowledge formula (6) for $I_2$ is new.

(ii) It is proved in [Vin76] that the ring of $\mathrm{SL}(\mathbb{C}^3) \times \mathrm{SL}(\mathbb{C}^3) \times \mathrm{SL}(\mathbb{C}^3)$-invariants in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ is generated by three algebraically independent elements of degree 6, 9, and 12. An alternative way to construct the invariant $J_6$ (not as a pullback from **Pl**) is to restrict the degree 6 generator to the subspace of $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ corresponding to symmetric matrix triples. Explicit formulae for the three generators can be found in [BLT04] or [DD12].

One can calculate that

$$\psi(\nu_{c,g}) = (y^2 + 2xz) \wedge 2yz \wedge (x^2 + 2g(xz - y^2) + cz^2) = 12ge_{345} + 2c(e_{456} + 2e_{356}) - 2e_{145} - 4e_{135},$$

TABLE 2. Un-normalized generators

| | | | |
|---|---|---|---|
| $w_1$ | $= x^2 \wedge xy \wedge xz$ | $w_{10}^*$ | $= x^2 \wedge xy \wedge y^2$ |
| $w_2 = E_{21}w_1$ | $= -x^2 \wedge xz \wedge y^2 + x^2 \wedge xy \wedge yz$ | $w_9^* = -E_{32}w_{10}^*$ | $= -x^2 \wedge xz \wedge y^2 - 2x^2 \wedge xy \wedge yz$ |
| $w_3 = E_{31}w_1$ | $= x^2 \wedge yz \wedge xz + x^2 \wedge xy \wedge z^2$ | $w_8^* = -\frac{1}{2}E_{32}w_9^*$ | $= 2x^2 \wedge xz \wedge yz + x^2 \wedge xy \wedge z^2$ |
| $w_4 = E_{21}w_2$ | $= 2xy \wedge y^2 \wedge xz + 2x^2 \wedge y^2 \wedge yz$ | $w_7^* = -\frac{1}{3}E_{32}w_8^*$ | $= -x^2 \wedge xz \wedge z^2$ |
| $w_5 = E_{31}w_2$ | $= x^2 \wedge y^2 \wedge z^2 + 2xz \wedge xy \wedge yz$ | $w_6^* = -E_{31}w_{10}^*$ | $= 2xy \wedge xz \wedge y^2 + x^2 \wedge y^2 \wedge yz$ |
| $w_6 = E_{31}w_3$ | $= 2x^2 \wedge yz \wedge z^2 + 2xz \wedge xy \wedge z^2$ | $w_5^* = -E_{31}w_9^*$ | $= -4xy \wedge xz \wedge yz - x^2 \wedge y^2 \wedge z^2$ |
| $w_7 = E_{21}w_4$ | $= 6xy \wedge y^2 \wedge yz$ | $w_4^* = -E_{21}w_7^*$ | $= 2xy \wedge xz \wedge z^2 + x^2 \wedge yz \wedge z^2$ |
| $w_8 = E_{31}w_4$ | $= 2xy \wedge y^2 \wedge z^2 + 2xz \wedge y^2 \wedge yz$ | $w_3^* = -E_{31}w_6^*$ | $= 2xy \wedge y^2 \wedge z^2 - 4xz \wedge y^2 \wedge yz$ |
| $w_9 = E_{31}w_5$ | $= 2xy \wedge yz \wedge z^2 + 2xz \wedge y^2 \wedge z^2$ | $w_2^* = -E_{21}w_4^*$ | $= 2xz \wedge y^2 \wedge z^2 - 4xy \wedge yz \wedge z^2$ |
| $w_{10} = E_{31}w_6$ | $= 6xz \wedge yz \wedge z^2$ | $w_1^* = -E_{21}w_2^*$ | $= 6y^2 \wedge yz \wedge z^2$ |

and hence, for $J_6 = I_2 \circ \psi$ we have $J_6(\nu_{c,g}) = I_2(\psi(\nu_{c,g})) = 24g$. This concludes our proof of the following known theorem (see [AN02], [Vin76], where the ring of invariants of **Noc** is identified with the ring of invariants of a finite complex pseudo-reflection group):

**Theorem 5.4.** *The ring of invariants of* **Noc** *is freely generated by $J_6$ and $J_{12}$.*

5.4. **A geometric interpretation of the splitting of Pl.** Following C.T.C. Wall [Wal10] we can interpret the projection maps from **Pl** to its irreducible factors.

5.4.1. *The Jacobi map.* A net $\varphi$ is a linear map from $S^2U$ to $V$, alternatively a quadratic map from $U$ to $V$. Its derivative at $u \in U$ is a linear map $d_u\varphi : T_uU \to T_uV$. Since tangent spaces of a vector space can be canonically identified with the vector space itself and $d_u\varphi$ is linear in $u$, the derivative $d\varphi$ defines a linear map from $U$ to $\mathrm{Hom}(U, V)$. We also have the degree 3 determinant map

$$\det : \mathrm{Hom}(U, V) \overset{\bigwedge^3}{\to} \mathrm{Hom}(\bigwedge^3 U, \bigwedge^3 V) \cong \bigwedge^3 U^* \otimes \bigwedge^3 V,$$

which can be composed with $d\varphi$ to obtain a degree 3 map $\mathrm{Jac}(\varphi)$ from $U$ to $\bigwedge^3 U^* \otimes \bigwedge^3 V$. We can also consider Jac as a map

$$\mathrm{Jac} : \mathbf{Noc} \to S^3U^* \otimes \bigwedge^3 U^* \otimes \bigwedge^3 V.$$

The map Jac factors through the Plücker map providing a linear projection

$$\pi_1 : \mathbf{Pl} \to S^3U^* \otimes \bigwedge^3 U^* \otimes \bigwedge^3 V.$$

Picking a basis in $V^*$ we can identify Jac with the Jacobian covariant $\mathrm{Jac} : \bigoplus^3 S^2(U^*) \to S^3(U^*)$, which is a joint covariant of triples of conics defined as

$$\mathrm{Jac}(M) := \det \begin{pmatrix} \partial_\xi M_1 & \partial_\xi M_2 & \partial_\xi M_3 \\ \partial_\eta M_1 & \partial_\eta M_2 & \partial_\eta M_3 \\ \partial_\nu M_1 & \partial_\nu M_2 & \partial_\nu M_3 \end{pmatrix}$$

(recall that $\xi, \nu, \eta$ is our basis in $U^*$, and the $M_i$ are homogeneous quadratic polynomials in $\xi, \eta, \nu$). Now Jac is an alternating trilinear function in $M_1, M_2, M_3$, hence it factors through an $\mathrm{SL}(U)$-equivariant linear map

$$\pi_1 : \mathbf{Pl} \to S^3(U^*).$$

It maps the basis vector $e_{ijk} \in \mathbf{Pl}$ to the $3 \times 3$ minor corresponding to the $i, j, k$ columns of the matrix

$$\begin{pmatrix} 2\xi & \eta & \nu & 0 & 0 & 0 \\ 0 & \xi & 0 & 2\eta & \nu & 0 \\ 0 & 0 & \xi & 0 & \eta & 2\nu \end{pmatrix}$$

(the columns contain the partial derivatives for each of $\xi^2, \xi\eta, \xi\nu, \nu^2, \eta\nu, \nu^2$). Recall that $x^3, 3x^2y, \ldots, 6xyz, \ldots$ is the basis in $S^3(U)$ dual to the basis $\xi^3, \xi^2\eta, \ldots, \xi\eta\nu, \ldots$ of $S^3(U^*)$. Now $\pi_1^*$ embeds $S^3(U^*)^* \cong S^3(U)$ into $\mathbf{Pl}^*$. For example, $\pi_1(e_{123}) = 2\xi^3$, and no other $\pi_1(e_{ijk})$ contains the monomial $\xi^3$. This shows that $\pi_1^*(x^3) = 2t_{123} = 8w_1$ (where $w_1$ is the element given in Table 2). Similarly $\xi^2\eta$ is contained only in $\pi_1(e_{125}) = 2\xi^2\eta$ and $\pi_1(e_{134}) = -4\xi^2\eta$. It follows that $\pi_1^*(3x^2y) = 2t_{125} - 4t_{134} = 8w_2$. One checks in the same way that the basis $x^3, 3x^2y, \ldots 6xyz, \ldots, z^3$ of $S^3(U)$ is mapped under $\pi_1^*$ onto $8w_1, \ldots, 8w_{10}$ (cf. Table 2).

5.4.2. *The dual Jacobi map.* We have a degree 2 map

$$\mathrm{Hom}(\mathbb{C}^2, U) \overset{S^2}{\to} \mathrm{Hom}(S^2\mathbb{C}^2, S^2U).$$

Composing with a net $\varphi$ and choosing an element in $\mathrm{Hom}(\mathbb{C}^2, U)$ we get a linear map in $\mathrm{Hom}(S^2\mathbb{C}^2, V)$. Taking its determinant we get a degree 6 map from $U \oplus U \cong \mathrm{Hom}(\mathbb{C}^2, U)$ to the one-dimensional vector space $L = \mathrm{Hom}(\bigwedge^3 S^2\mathbb{C}^2, \bigwedge^3 V) \cong \bigwedge^3 V$. (As a representation of $\mathrm{GL}(U)$ the line $L$ is isomorphic to the trivial one-dimensional representation.) Notice that this map factors through $\bigwedge^2 U$, providing a degree 3 map from $\bigwedge^2 U$ to $L$. Varying $\varphi$ we end up with a degree 3 map from $\mathbf{Noc} \to S^3 \bigwedge^2 U^* \otimes L$ which factors through the Plücker map $\psi$. Now notice that $\bigwedge^2 U^* \cong U \otimes \bigwedge^3 U^*$. Hence we defined a linear map

$$\pi_2 : \mathbf{Pl} \to S^3U \otimes \left(\bigwedge^3 U^*\right)^3 \otimes \bigwedge^3 V.$$

Notice that the $\mathrm{GL}(V)$-action played no active role in the projections $\pi_1$ and $\pi_2$ as it was expected from the abstract splitting of the representation $\bigwedge^3 S^2U^*$.

As an $\mathrm{SL}(U)$-equivariant linear map $\pi_2 : \mathbf{Pl} \to S^3(U)$ can be constructed as follows: Picking a basis in $V^*$ we can identify a net with a triple $M = (M_1, M_2, M_3)$ where $M_i \in S^2U^*$. We may think of $S^3(U)$ as the space of cubic polynomial functions on $U^*$. Now $\pi_2(M_1 \wedge M_2 \wedge M_3)$ vanishes on a linear form $f \in U^*$ if the net $M$ restricted to the zero locus of $f$ does not have full rank.

More explicitly, eliminate the variable $\nu$ from the ternary quadratic forms $M_i$ using the relation $x\xi + y\eta + z\nu = 0$; we obtain three binary quadratic forms in the variables $\xi, \eta$. Now $\pi_2(M_1 \wedge M_2 \wedge M_3)$ is the determinant of the $3 \times 3$ matrix whose columns contain the coefficients of these three binary quadratic forms. In particular, $\pi_2(e_{ijk})$ is the $(i, j, k)$ minor of

$$\begin{pmatrix} z & 0 & -x & 0 & 0 & x^2/z \\ 0 & 0 & 0 & z & -y & y^2/z \\ 0 & z & -y & 0 & -x & 2xy/z \end{pmatrix}$$

(showing also that we end up with a cubic polynomial in $x, y, z$). The dual $\pi_2^*$ embeds $S^3U^*$ into $\mathbf{Pl}^*$, and in the same way as in the case of the Jacobi map one may check that the basis vectors $\xi^3, \xi^2\eta, \ldots, \nu^3$ are mapped to $\frac{-1}{3}w_1^*, \ldots, \frac{-1}{3}w_{10}^*$ from Table 2.

5.5. **Stability.** A net of conics is in the nullcone if both $J_6$ and $J_{12}$ are zero on it. The G.I.T. quotient of $\mathbf{Noc}$ is $\mathbf{P}^1$ and the quotient map on the complement of the nullcone is given by $k := J_6^2/J_{12}$. An orbit $\eta$ is strictly semistable if $k^{-1}(k(\eta)) \supsetneq \eta$. We can use formula (6) to calculate $J_6$ and the explicit form of the degree 4 invariant of the plane cubics to calculate $J_{12}$.

Notice that Theorem 5.1 is not sufficient, since non-smooth cubics do not admit a Weierstrass form. Nevertheless these are simple calculations which show that the only codimension$>1$ orbits outside the nullcone are $D, D^*, E, E^*$ with

| | $J_6$ | $J_{12}$ | $k$ |
|---|---|---|---|
| $D$ | 1 | 1 | 1 |
| $D^*$ | $-8$ | 16 | 4 |
| $E$ | 1 | 1 | 1 |
| $E^*$ | $-8$ | 16 | 4 |

Since $k$ is a bijection on the codimension 1 orbits it is enough to find values of $c$ and $g$ with the given $k$. Since $k(\nu_{c,g})=\frac{12g^2}{3g^2-c}$, it is immediate that $k(B)=1$ for $B:=\nu_{-9,1}$ and $k(B^*)=4$ for $B^*:=\nu_{0,1}$ ($B$ and $B^*$ are notations from [Wal77]). Consequently the complete list of strictly semistable orbits are $B, B^*, D, D^*, E, E^*$.

5.5.1. *The discriminant.* For the representations **Noc** and $S^3\mathbb{C}^3$ the nonstable variety is a hypersurface and we call their defining equation the *discriminant* of the representation. It is a classical result that the discriminant of the plane cubics is $\Delta=4a^3+27b^2$. Using the $c$-$g$-form we can quickly check that $-2^8 3^3\delta^*\Delta=(J_6^2-J_{12})^2(J_6^2-4J_{12})$ (see Section 7.1 for the details), consequently a net is unstable if and only if its determinant cubic is unstable, but the $\delta^*$-image of the discriminant is not the discriminant: the component $(J_6^2-J_{12})=\overline{B}$ is counted with multiplicity 2.

## 6. Hierarchy of the nets of conics

C. T. C. Wall's result on the classification of the **Noc**-orbits can be verified using the results of Sections 3 and 5. The codimension 1 orbits are classified by their $k$-invariant. The fact stated in Theorem 3.1, namely that the restriction equations determine the equivariant classes imply that no orbit is missed in Table 1. Any missing orbit would cause an indeterminacy in the solution of the restriction equations, hence would contradict to Theorem 3.1.

To determine the hierarchy we use that a cohomologically defined incidence class determines adjacency for *positive* orbits: Consider a Lie group $G$ acting on a vector space $V$ complex linearly. For $v\in V$ let $T_v$ denote the maximal torus of the stabilizer subgroup of $G$.

**Definition 6.1.** *The orbit $Gv$ is positive if there is a linear functional $\varphi$ on the weight lattice of $T_v$ such that for all weights $w_i$ of the $T_v$-action on the normal space of the orbit $Gv$ at $v$ we have $\varphi(w_i)>0$.*

**Theorem 6.2.** [FP09] *Let $\eta\subset V$ be a $G$-invariant subvariety and suppose that the orbit $Gv$ is positive for some $v\in V$. Then $v\in\eta$ if and only if $[\eta\subset V]_{T_v}\neq 0$.*

Table 3 contains the description of normal slices to orbits, and their weights. The last column contains the values of the functional $\varphi$ (that is, the values $\varphi(\alpha),\varphi(\beta),\dots$) if such a functional, proving the positivity of the given orbit, exists. By inspection we obtain the following fact.

**Proposition 6.3.** *All unstable orbits of* **Noc** *are positive.*

Thus, Theorem 6.2 determines almost all adjacencies of the orbits, namely the ones involving unstable orbits. The missing adjacencies of the semistable orbits can be determined by calculating the $k$-invariant. As a result we obtain the complete hierarchy, depicted on Figure 2.

**Example 6.4.** Consider the orbits $F$ and $F^*$, and their adjacency with the orbit $(1^4)$. Let $v$ be the point in the $(1^4)$ orbit given in the Table, and let $j_{(1^4)}$ be the restriction homomorphism

TABLE 3. Normal weights and positivity

| $\Sigma^0$ | | | normal weights | $\varphi(\alpha),\varphi(\beta),\ldots$ |
|---|---|---|---|---|
| $C$ | 2 | $y^2+2xz, 2yz, -x^2$ | $4\alpha-4\beta, 2\alpha-2\beta$ | 1,0 |
| $D$ | 2 | $x^2, y^2, z^2+2xy$ | $3\alpha-3\beta, 3\beta-3\alpha$ | - |
| $D^*$ | 2 | $2xz, 2yz, z^2+2xy$ | $3\alpha-3\beta, 3\beta-3\alpha$ | - |
| $E$ | 3 | $x^2, y^2, z^2$ | $2\alpha-\beta-\gamma, 2\beta-\alpha-\gamma, 2\gamma-\alpha-\beta$ | - |
| $E^*$ | 3 | $2xy, 2yz, 2zx$ | $-2\alpha+\beta+\gamma, -2\beta+\alpha+\gamma, -2\gamma+\alpha+\beta$ | - |
| $F$ | 3 | $x^2+y^2, 2xy, 2yz$ | $2\alpha-2\beta, 2\alpha-2\beta, \alpha-\beta$ | 1,0 |
| $F^*$ | 3 | $x^2+y^2, xz, z^2$ | $2\beta-2\alpha, 2\beta-2\alpha, \beta-\alpha,$ | -1,0 |
| $G$ | 4 | $x^2, y^2, yz$ | $2\alpha-2\gamma, \beta-\alpha, 2\beta-\alpha-\gamma, 2\beta-2\gamma$ | 1,2,0 |
| $G^*$ | 4 | $xy, xz, z^2$ | $2\gamma-2\alpha, \alpha-\beta, -2\beta+\alpha+\gamma, 2\gamma-2\beta$ | -1,-2,0 |
| $H$ | 5 | $x^2, 2xy, y^2+2xz$ | $2\alpha-2\beta, 2\alpha-2\beta, 3\alpha-3\beta, 3\alpha-3\beta, 4\alpha-4\beta$ | 1,0 |
| $I$ | 7 | $x^2, xy, y^2$ | $\alpha-\gamma, 2\alpha-\beta-\gamma, 2\alpha-2\gamma, \beta-\gamma, -2\gamma+\alpha+\beta, 2\beta-2\gamma$ | 0,0,-1 |
| $I^*$ | 7 | $xz, yz, z^2$ | $-\alpha+\gamma, -2\alpha+\beta+\gamma, -2\alpha+2\gamma, -\beta+\gamma, 2\gamma-\alpha-\beta, -2\beta+2\gamma$ | 0,0,1 |
| $\Sigma^1$ | | | | |
| $(1^4)$ | 4 | $x^2-xz, y^2-yz, 0$ | $\beta-2\gamma, \beta-2\gamma, \beta-2\gamma, \beta-2\gamma$ | 1,0 |
| $(21^2)$ | 5 | $xy, xz+yz, 0$ | $2\alpha-2\beta, \gamma-2\alpha, \gamma-2\alpha, \gamma-\alpha-\beta, \gamma-2\beta$ | 1,0,3 |
| $(31)$ | 6 | $xz, x^2-yz, 0$ | $3\beta-3\alpha, 2\beta-2\alpha, \gamma-3\alpha-\beta, \gamma-4\alpha, \gamma-2\alpha-2\beta, \gamma-4\beta$ | 0,1,5 |
| $(22)$ | 6 | $x^2, yz, 0$ | $2\alpha-2\beta, 2\alpha-2\gamma, \delta-\alpha-\beta, \delta-\alpha-\gamma, \delta-2\beta, \delta-2\gamma$ | 1,0,0,2 |
| $(4)$ | 7 | $xz+y^2, x^2, 0$ | $2\alpha-2\beta, 3\alpha-3\beta, 4\alpha-4\beta, \gamma-3\alpha-\beta, \gamma-2\alpha-2\beta, \gamma-4\beta$ | 1,0,5 |
| $K$ | 8 | $y^2, z^2, 0$ | $2\alpha-2\beta, 2\alpha-\beta-\gamma, 2\gamma-2\beta, 2\gamma-\alpha-\beta, \delta-2\beta,$ $\delta-\alpha-\beta, \delta-\beta-\gamma, \delta-\alpha-\gamma$ | 0,-1,0,0 |
| $L$ | 8 | $xy, xz, 0$ | $\alpha-\beta, \alpha-\gamma, \alpha+\beta-2\gamma, \alpha+\gamma-2\beta, \delta-2\alpha, \delta-2\beta,$ $\delta-2\gamma, \delta-\beta-\gamma$ | 1,0,0,3 |
| $M$ | 9 | $yz, y^2, 0$ | $\beta+\gamma-2\alpha, \beta-\alpha, 2\beta-2\alpha, 2\beta-\alpha-\gamma, 2\beta-2\gamma, \delta-2\alpha,$ $\delta-\alpha-\beta, \delta-\alpha-\gamma, \delta-2\gamma$ | 0,1,0,2 |
| $\Sigma^2$ | | | | |
| $S$ | 10 | $xy-z^2, 0, 0$ | $\gamma-2\alpha-2\beta, \gamma-3\alpha-\beta, \gamma-4\alpha, \gamma-\alpha-3\beta, \gamma-4\beta,$ $\delta-2\alpha-2\beta, \delta-3\alpha-\beta, \delta-4\alpha, \delta-\alpha-3\beta, \delta-4\beta$ | 0,0,1,1 |
| $PL$ | 11 | $xy, 0, 0$ | $\delta-2\alpha, \delta-2\beta, \delta-\alpha-\gamma, \delta-\beta-\gamma, \delta-2\gamma$ $\epsilon-2\alpha, \epsilon-2\beta, \epsilon-\alpha-\gamma, \epsilon-\beta-\gamma, \epsilon-2\gamma, \alpha+\beta-2\gamma$ | 1,0,0,3,3 |
| $DL$ | 13 | $x^2, 0, 0$ | $\delta-\alpha-\beta, \delta-2\beta, \delta-\alpha-\gamma, \delta-2\gamma, \delta-\beta-\gamma$ $\epsilon-\alpha-\beta, \epsilon-2\beta, \epsilon-\alpha-\gamma, \epsilon-2\gamma, \epsilon-\beta-\gamma$ $2\alpha-2\beta, 2\alpha-\beta-\gamma, 2\alpha-2\gamma$ | 1,0,0,3,3 |

$H^*_{\mathrm{GL}(U)\times\mathrm{GL}(V)}(\mathbf{Noc}) \to H^*_{T_v}(\mathbf{Noc})$. One can read from the table above that the homomorphism $j_{(1^4)}$ is the substitution

$$c_i = \sigma_i(\alpha,\alpha,\alpha), \qquad d_i = \sigma_i(2\alpha, 2\alpha, \beta),$$

where $\sigma_i$ denotes the $i$th elementary symmetric polynomial. For the equivariant classes given in Theorem 3.1 we have

$$j_{(1^4)}([\overline{F}]) = 0, \qquad j_{(1^4)}([\overline{F^*}]) = -6(2\alpha-\beta)(4\alpha^2-4\alpha\beta+\beta^2) \neq 0.$$

Hence, we have that $(1^4)$ is contained in the orbit closure of $F^*$, but is not contained in the orbit closure of $F$.

6.1. **The equivariant cohomology rings of the orbits.** With a little extra sudoku type calculations one can determine the equivariant cohomology rings $H^*(BG_x)$ of the orbits $Gx$. In column 5 of Table 1 we listed the degrees of a free generating set for these rings. This information can be used, e.g., to define certain "higher" Thom polynomials. Since the equivariant cohomology spectral sequence of the codimension filtration (the Kazarian spectral sequence; see

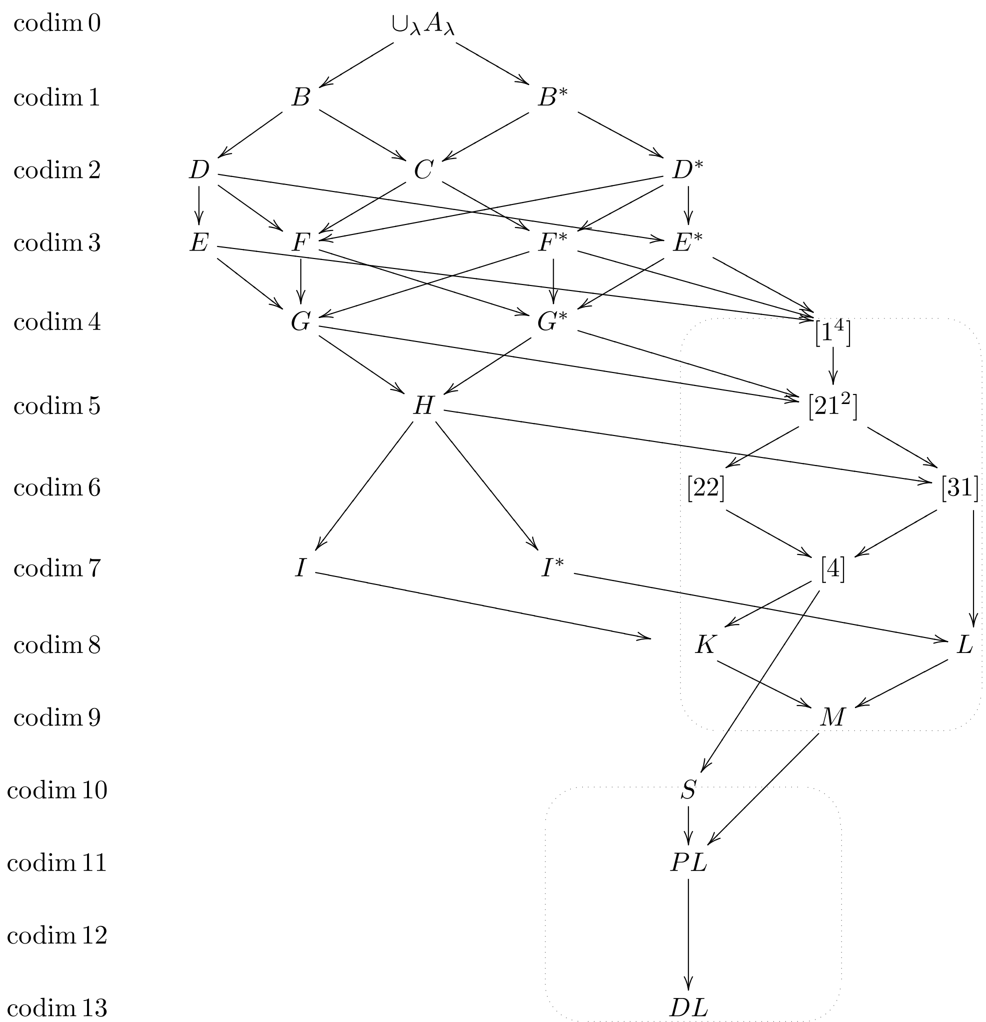


FIGURE 2. The hierarchy of **Noc** orbits

[Kaz97], [FR04, Sect.10]) degenerates, the Poincaré series of the rings $H^*(BG_x)$ shifted by the codimension add up to the Poincaré series of $H^*(BG)$. For the open stratum $O = \bigcup_{\mu \in \mathbf{P}^1} A_\mu$ we have $H^*_G(O) = H^*_{\mathrm{GL}_1}(\mathbf{P}^1)$ for the trivial $\mathrm{GL}_1$-action, so the Poincaré series of $H^*_G(O)$ is $\frac{1+t}{1-t}$, and we get that

$$\frac{1+t}{1-t} + \frac{t^2}{(1-t)^2} + \frac{2t^2}{(1-t)(1-t^2)} + \ldots + \frac{t^{18}}{(1-t)^2(1-t^2)^2(1-t^3)^2} = \frac{1}{(1-t)^2(1-t^2)^2(1-t^3)^2}.$$

Here the $t$-exponents of the numerator are the codimensions (column 2) and the $t$-exponents of the denominators are degrees of the generators of $H^*(BG_x)$ (column 5).

## 7. Enumerative questions, multiplicities of the determinant map

Equivariant classes contain enumerative data in a compressed form. One of the simplest enumerative invariants is the degree. Using [FNR05, Section 6] we see that the degree of a $G$-invariant subvariety $\eta \subset \mathbf{Noc}$ can be obtained by substituting 1 to the Chern roots of $\mathrm{GL}(V)$ and 0 to the Chern roots of $\mathrm{GL}(U)$ in the polynomial $[\eta] \in H^*(B(\mathrm{GL}(U) \times \mathrm{GL}(V)))$. For example, we obtain

$$\deg(\overline{C}) = 72, \qquad \deg(\overline{D}) = 36, \qquad \deg(\overline{D^*}) = 45.$$

Consequently in a generic two-parameter linear family of nets of conics there are 72 of type $C$, 36 of type $D$ and 45 of type $D^*$, etc.

A more subtle enumerative invariant is the *intersection multiplicity*. Suppose that $f : X \to Y$ is a map of smooth varieties and $Z \subset Y$ a codimension $d$ subvariety. Suppose also that the preimage $f^{-1}Z$ is pure $d$-codimensional. Then we can assign positive integers $\mu_i$ to every component $Y_i$ of $f^{-1}Z$ called intersection multiplicities (for more details see [Ful84]). If $X = Y = \mathbb{C}$ and $Z = \{0\}$, then the intersection multiplicities are the usual multiplicities of the roots of $f$.

An important property of the intersection multiplicity ([Ful84, Sect. 7]) is that if all components $Y_i$ of $f^{-1}Z$ are of codimension $d$, then

$$f^*([X]) = \sum \mu_i [Y_i]. \tag{7}$$

In general this equation does not determine the intersection multiplicities $\mu_i$. However (7) generalizes to the equivariant setting where there is more chance that the classes $[Y_i]$ will be linearly independent.

Consider the determinant map $\delta : \mathbf{Noc} \to S^3V$ studied in Section 5.2. From the normal forms in Table 1 it is easy to calculate the image under the determinant map. E.g., for $C$ the normal form is $y^2 + 2xz, 2yz, -x^2$ which means that the generic net in matrix form is

$$\begin{pmatrix} -\kappa & \cdot & \lambda \\ \cdot & \lambda & \mu \\ \lambda & \mu & \cdot \end{pmatrix},$$

with determinant $\mu^2\kappa - \lambda^3$ corresponding to the cuspidal curve $\nu$. A table can be found in [Wal77]. For the readers' convenience we included the images of the determinant map in the last column of Table 1. The notation tries to imitate the shape of the various classes of plane cubics, i. e. $\theta$ is the orbit (closure) of conic intersected by line, $\Omega$ the conic and tangent line, $\mathsf{A}$ the three nonconcurrent lines, $\neq$ is the double line intersected by a third line, $\Xi$ is the triple line and Ж is the three concurrent lines. The codimension 1 orbits will be treated in the next section.

The equivariant classes of the $\mathrm{GL}_3$-representation $S^3\mathbb{C}^3$ (i.e., plane cubics) were calculated by B. Kőműves [Kőm03]. The cases we need are

$$[\nu] = 24e_1^2,\ [\theta] = 18e_1^2 + 9e_2,\ [\Omega] = 36e_1^3 + 18e_1e_2,\ [\mathsf{A}] = 12e_1^3 + 6e_1e_2 + 27e_3,\ [Ж] = e_1[\mathsf{A}], \tag{8}$$

where $e_i$ denote the $\mathrm{GL}_3$-Chern classes.

The map $\delta$ is $\mathrm{GL}_3 \times \mathrm{GL}_3$-equivariant, if we replace the $\mathrm{GL}_3$-action on the plane cubics by the $\mathrm{GL}(U) \times \mathrm{GL}(V)$-action as in Section 5.2. The effect of this change on (8) is replacing the $\mathrm{GL}_3$-Chern roots $\epsilon_i$ ($\epsilon_1 + \epsilon_2 + \epsilon_3 = e_1$ etc) by some linear combination of the $\mathrm{GL}(U) \times \mathrm{GL}(V)$-Chern roots. A comparison of the actions of the maximal tori of $\mathrm{GL}_3$ and $\mathrm{GL}(U) \times GL(V)$ gives the substitution $\epsilon_i \mapsto \delta_i - 2/3u_1$ (where $\delta_i$ denote the $\mathrm{GL}(V)$-Chern roots). Consequently for the Chern classes we obtain the substitution $e_1 \mapsto v_1 - 2u_1$, $e_2 \mapsto v_2 - 4/3u_1v_1 + 4/3u_1^2$ and $e_3 \mapsto v_3 - 2/3v_2u_1 + 4/9v_1u_1^2 - 8/27u_1^3$.

After this substitution we have all the ingredients of (7) for the $\mathrm{GL}(U) \times \mathrm{GL}(V)$-equivariant map $\delta$. Then explicit calculation implies the following theorem.

**Theorem 7.1.** *In $\mathrm{GL}(U) \times \mathrm{GL}(V)$-equivariant cohomology we have*

$$\delta^*([\nu]) = 3[C],$$
$$\delta^*([\theta]) = 4[D] + [D^*],$$
$$\delta^*([\Omega]) = 9[F] + 6[F^*],$$
$$\delta^*([\mathsf{A}]) = 8[E] + [E^*] + 2[F],$$

*and the coefficients on the right hand side are uniquely determined, therefore they are the intersection multiplicities.*

For the three concurrent lines the codimension condition is *not* satisfied, but a similar calculation gives the equality

$$\delta^*([Ж]) = 12[(1^4)] + \big(4[G] + 1/2[G^*] + 2(d_1 - 2c_1)[F]\big),$$

and, like above, the coefficients are uniquely determined. The class in the bracket is supported on $\overline{F}$, so we can call 12 the intersection multiplicity of $(1^4)$.

**Remark 7.2.** The intersection multiplicities are always at most the algebraic multiplicities by [Ful84, Pr. 8.2] and they agree if the image is (locally) Cohen-Macaulay and the preimage has the same codimension by [FP98, p.108]. In [Chi02] J. Chipalkatti determines which orbit closures are arithmetically Cohen-Macaulay for the plane cubics. Among the orbits of Theorem 7.1 $\nu$ is Cohen-Macaulay, since it is a complete intersection, $\Omega$ is arithmetically Cohen-Macaulay consequently Cohen-Macaulay, but $\theta$ and $\mathsf{A}$ are not arithmetically Cohen-Macaulay. Therefore the intersection multiplicities for $\nu$ and $\Omega$ are algebraic multiplicities as well. For $\theta$ and $\mathsf{A}$ we do not know if they are Cohen-Macaulay. If the algebraic multiplicities for $\theta$ and $\mathsf{A}$ differ from the intersection multiplicities, then they cannot be Cohen-Macaulay. Unfortunately we were not able to calculate these algebraic multiplicities.

For the codimension 1 orbits we study the induced map on the GIT quotients.

7.1. **The induced map on the GIT quotients of nets of conics to plane cubics.** To see that the $G = \mathrm{GL}(U) \times \mathrm{GL}(V)$-equivariant determinant map $\delta$ induces a map $d$ of the corresponding GIT quotients we need to check that semistable orbits are mapped to semistable orbits. It follows from general principles but here is a direct verification. For codimension $> 1$ orbits of **Noc** we see this fact from Table 1. (For the plane cubics the semistable orbits are the smooth curves which are also stable, together with the nodal curve and $\theta$ and $\mathsf{A}$.) The codimension 1 orbits $A_\mu$ have a $\nu_{c,g}$ representative with $(c, g) \neq (0, 0)$, so for $\delta(\nu_{c,g})$ either $a = c - 3g^2$ or $b = 2g(c + g^2)$ is not zero, therefore the image is semistable.

The traditional parametrization of the GIT quotient $S^3\mathbb{C}^3//G$ is the $j$-invariant $j = \frac{4a^3}{\Delta} : S^3\mathbb{C}^3//G \to \mathbf{P}^1$, where $\Delta = 4a^3 + 27b^2$ is the discriminant. On the other hand we saw in Section 5.5 that the invariant $k = \frac{J_6^2}{J_{12}} : \mathbf{Noc}//G \to \mathbf{P}^1$ parametrizes the GIT quotient $\mathbf{Noc}//G$. Using these parametrizations we can calculate $\tilde{d} := j \circ d \circ k^{-1} : \mathbf{P}^1 \to \mathbf{P}^1$:

Using the $c$-$g$-form we get (by some abuse of notation)

$$\Delta = \Delta(\nu_{c,g}) = 4a^3 + 27b^2 = 4a^3 + 27(4g^2)(a + 4g^2)^2 = 4(a + 3g^2)(a + 12g^2)^2,$$

and we have $J_6 = 24g$ and $J_{12} = -48a$, so $48^3\Delta = (-4J_{12} + J_6^2)(-J_{12} + J_6^2)^2$. Therefore $j \circ d = \frac{J_{12}^3}{(J_6^2 - 4J_{12})(J_6^2 - J_{12})^2}$ and

$$\tilde{d}(x : y) = (4y^3 : (x - 4y)(x - y)^2).$$

The branching points are at the singular points (take the affine chart $y = 1$, and find the zeroes of $3(x-1)(x-3)$, the derivative of $(x-4)(x-1)^2$, and an extra branching at $x = \infty$.):

$x = 1$ ($\Delta = 0$, i.e., $j$-invariant is $\infty$) corresponds to the semistable point (class of the nodal curve). It has preimages $B$ with $k(B) = 1$ and $B^*$ with $k(B^*) = 4$. The point $B$ has multiplicity two. Notice that $B$ and $B^*$ represent the two semistable points in $\mathbf{Noc}//G$ (in the GIT quotient $B \sim D \sim E$ and $B^* \sim D^* \sim E^*$).

$x = \infty$ ($j$-invariant is 0) corresponds to the degree 4 orbit of elliptic curves defined by $a = 0$. It has one preimage (of multiplicity 3): the orbit of nets of conics defined by $J_{12} = 0$.

$x = 3$ ($j$-invariant is 1) corresponds to the degree 6 orbit of elliptic curves defined by $b = 0$. It has one preimage with $k = 3$ and another one with multiplicity 2, the degree 6 orbit of nets of conics defined by $J_6 = 0$.

**Remark 7.3.** For hypersurfaces the intersection multiplicities agree with the algebraic (or scheme theoretic) multiplicities. To determine these multiplicities in our case it is enough to compare degrees. One obtains that all algebraic multiplicities are 1, except for $B$ which has multiplicity 2.

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Rényi Institute of Mathematics, Hungarian Academy of Sciences, Budapest, Hungary
*E-mail address*: `domokos.matyas@renyi.mta.hu`

Department of Analysis, Eotvos University, Budapest, Hungary
*E-mail address*: `lfeher@renyi.mta.hu`

Department of Mathematics, University of North Carolina, Chapel Hill, NC, USA
*E-mail address*: `rimanyi@email.unc.edu`

**Journal *of* Singularities**
**Volume 7 (2013), 21-22**

received 8 February 2013
in revised form 15 February 2013
DOI: 10.5427/jsing.2013.7b

# ERRATUM: MILNOR FIBRATIONS AND THE THOM PROPERTY FOR MAPS $f\bar{g}$

ANNE PICHON AND JOSÉ SEADE

The main theorem in our article [2] is not correct as stated. Presumably, there exist stronger hypotheses under which it does hold. This is the case, for instance, when $n = 2$ and the germs $f, g$ do not have a common branch (see [1, Proposition 1.4]). We thank Adam Parusiński for having pointed out to us this error by sending us two counter-examples. We also thank Mutsuo Oka for having located the mistake in our proof : It comes from the fact that the equation we give in page 147 line -5 is not sufficient to define the tangent space $T_{x_k}G$. In fact, the normal space to it is defined by the two real vectors $\operatorname{grad} u$ and $\operatorname{grad} v$ where $f\bar{g} = u + iv$, while in our calculation we considered only the vector $w_k = 2 \operatorname{grad} u + 2i \operatorname{grad} v$.

Here are the two counter-examples sent to us by A. Parusiński. The first of these was suggested by comments of M. Tibăr. In both examples, the map $f\bar{g}$ has an isolated critical value at $0 \in \mathbb{C}$, but $f\bar{g}$ does not possess the Thom $a_{f\bar{g}}$-property. We reproduce below the arguments given to us by A. Parusiński.

**Example 1.** Let $f\bar{g}\colon \mathbb{C}^2 \to \mathbb{C}$ be given by $f(z_1, z_2) = z_1 z_2$, $g(z_1, z_2) = z_2$. We have

$$f\bar{g}(z_1, z_2) = z_1 \|z_2\|^2 = x_1(x_2^2 + y_2^2) + iy_1(x_2^2 + y_2^2) = u + iv,$$

$$\operatorname{grad} u = (x_2^2 + y_2^2, 0, 2x_1x_2, 2x_1y_2) \qquad \text{and} \qquad \operatorname{grad} v = (0, x_2^2 + y_2^2, 2y_1x_2, 2y_1y_2).$$

Thus, the critical locus of $f\bar{g}$ is $Y = \{z_2 = 0\}$ and 0 is the only critical value of $f\bar{g}$. We show that, for the stratification $\{\mathbb{C}^2 \setminus Y, Y\}$, the Thom condition $a_{f\bar{g}}$ fails at every point of $Y$.

Fix $P = (p, q, 0, 0) \in Y$ and $(a, b) \in \mathbb{R}^2 \setminus 0$ such that $ap + bq = 0$. Let $z = (z_1, z_2)$ tend to $P$ and satisfy $ax_1 + by_1 = 0$. Then, at these points,

$$a \operatorname{grad} u + b \operatorname{grad} v = (x_2^2 + y_2^2)(a, b, 0, 0)$$

and, hence,

$$\frac{a \operatorname{grad} u + b \operatorname{grad} v}{\|a \operatorname{grad} u + b \operatorname{grad} v\|} = \frac{(a, b, 0, 0)}{\|(a, b)\|}\ ,$$

which contradicts the Thom condition.

In fact, we can deduce from the above arguments that there is no stratification of $f\bar{g}$ satisfying the Thom condition. Indeed, $Y$, as it is the critical locus, has to be a union of strata for any stratification of $f\bar{g}$. If $P$ is in a stratum open in $Y$, we may choose points $z = (z_1, z_2)$ that tend to $P$, are in a stratum open in $\mathbb{C}^2$, and are close to the points considered above. It suffices to suppose that they satisfy $|ax_1 + by_1| \le x_2^2 + y_2^2$, since then

$$a \operatorname{grad} u + b \operatorname{grad} v = (x_2^2 + y_2^2)(a, b, 0, 0) + (ax_1 + by_1)(0, 0, 2x_2, 2y_2),$$

and the second term tends faster to 0 than the first one if $z_2 \to 0$.

**Example 2.** Consider $f\bar{g}\colon \mathbb{C}^3 \to \mathbb{C}$ given by $f(z_1, z_2, z_3) = z_1(z_2 + z_3^2)$, $g(z_1, z_2, z_3) = z_2$. Write as before $f\bar{g} = u + iv$.

First we determine the critical locus of $f$. Since $f\overline{g}$ is holomorphic with respect to $z_1$ and $z_3$, then for $i = 1, 3$ we have $\frac{\partial(f\overline{g})}{\partial\overline{z}_i} = 0$ and the vectors $(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial y_1})$, $(\frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial y_1})$ are independent if and only if $\frac{\partial(f\overline{g})}{\partial z_1} \neq 0$. The critical locus is then contained in the set with equations

$$\frac{\partial(f\overline{g})}{\partial z_1} = (z_2 + z_3^2)\overline{z}_2 = 0 \quad ; \quad \frac{\partial(f\overline{g})}{\partial z_3} = 2z_1\overline{z}_2 z_3 = 0. \tag{1}$$

The solution set of (1) consists of two components: $\{z_2 = 0\}$ and $\{z_1 = z_2 + z_3^2 = 0\}$. On the second one we have $\frac{\partial u}{\partial x_2} = \frac{\partial u}{\partial y_2} = \frac{\partial v}{\partial x_2} = \frac{\partial v}{\partial y_2} = 0$, and hence the entire component is included in the critical set. We write

$$f\overline{g} = f_1\overline{g} + f_2\overline{g},$$

where $f_1(z_1, z_2, z_3) = z_1 z_2$, $f_2(z_1, z_2, z_3) = z_1 z_3^2$. We write $f_1\overline{g} = u_1 + iv_1$, $f_2\overline{g} = u_2 + iv_2$. On the set $\{z_2 = 0\}$ we have $\frac{\partial u_1}{\partial x_2} = \frac{\partial u_1}{\partial y_2} = \frac{\partial v_1}{\partial x_2} = \frac{\partial v_1}{\partial y_2} = 0$ and hence on this set we consider only the partial derivatives of $f_2\overline{g}$ with respect to $x_2, y_2$. Since $f_2\overline{g}$ is antiholomorphic with respect to $z_2$ we get a new set of equations

$$z_2 = 0 \quad \text{and} \quad \frac{\partial(f_2\overline{g})}{\partial\overline{z}_2} = z_1 z_3^2 = 0.$$

This allows us to conclude that

$$Crit(f\overline{g}) = \{z_1 = z_2 + z_3^2 = 0\} \cup \{z_1 = z_2 = 0\} \cup \{z_2 = z_3 = 0\}.$$

Note that 0 is the only critical value of $f\overline{g}$.

Denote $Y = \{z_2 = z_3 = 0\}$. We show that for any stratification of $\mathbb{C}^3$ the Thom condition $a_{f\overline{g}}$ fails at a generic point of $Y$. Fix $P = (p, q, 0, 0, 0, 0) \in Y$ and $(a, b) \in \mathbb{R}^2 \setminus 0$ such that $ap + bq = 0$. Let $z = (z_1, z_2, z_3)$ tend to $P$ and satisfy

$$|ax_1 + by_1| \leq \|z_2\|^2, \quad \|z_3\| \leq \|z_2\|^4.$$

Then at these points

$$a \operatorname{grad} u_1 + b \operatorname{grad} v_1 = \|z_2\|^2(a, b, 0, 0, 0, 0) + o(\|z_2\|^2)$$

and

$$\|\operatorname{grad} u_2, \operatorname{grad} v_2\| \leq \|z_3\| = o(\|z_2\|^2).$$

Thus we may conclude as in Example 1.

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**Journal *of* Singularities**
**Volume 7 (2013), 23-38**

received 25 February 2012
in revised form 26 February 2013
DOI: 10.5427/jsing.2013.7c

# Exchange between perverse and weight filtration for the Hilbert schemes of points of two surfaces

Mark Andrea A. de Cataldo*, Tamás Hausel†, Luca Migliorini‡

**Abstract**

We show that a natural isomorphism between the rational cohomology groups of the two zero-dimensional Hilbert schemes of $n$-points of two surfaces, the affine plane minus the axes and the cotangent bundle of an elliptic curve, exchanges the weight filtration on the first set of cohomology groups with the perverse Leray filtration associated with a natural fibration on the second set of cohomology groups. We discuss some associated hard Lefschetz phenomena.

# 1 Introduction

## 1.1 The main result

The theory of mixed Hodge structures endows the rational cohomology groups $H^*(Z, \mathbb{Q})$ of a complex algebraic variety $Z$ with the increasing weight filtration $\mathscr{W}_Z$. On the other hand, given a map $f : Z \longrightarrow Z'$ of algebraic varieties, the theory of perverse sheaves (with middle perversity) endows the rational cohomology groups $H^*(Z, \mathbb{Q})$ with the increasing perverse Leray filtration $\mathscr{P}_Z$ (see [12], for example).

In [4], it was proved that the non-Abelian Hodge theory diffeomorphism between the twisted character variety $\mathcal{M}_{\mathrm{B}}$ of representations of a compact Riemann surface $C$ into $\mathrm{GL}_2(\mathbb{C})$ and the moduli space $\mathcal{M}_{\mathrm{Dol}}$ of rank 2 degree 1 stable Higgs bundles on $C$ identifies the weight filtration $\mathscr{W}_{\mathcal{M}_{\mathrm{B}}}$ with the perverse filtration $\mathscr{P}_{\mathcal{M}_{\mathrm{Dol}}}$ induced by the projective Hitchin map $\chi : \mathcal{M}_{\mathrm{Dol}} \to \mathcal{A}$. The so-called $P = W$ conjecture, i.e. that this phenomenon should be valid for other groups, such as $\mathrm{GL}_n(\mathbb{C})$, has received some evidence in [2] in a string-theoretic framework.

However, at present, we are unable to use the approach of [4] to attack this conjecture in the case of $\mathrm{GL}(n, \mathbb{C})$ for $n > 2$. One exception is the case of the moduli space of rank $n$ stable mirabolic Higgs bundles on an elliptic curve $E$. Thanks to [18], in this case we have a global understanding of this moduli space as the Hilbert scheme of $n$ points $X^{[n]}$ on the complex surface $X := T^\vee(E) \simeq E \times \mathbb{C}$ the total space of the cotangent bundle of $E$. Additionally, the Hitchin map becomes a proper flat map $h_n : X^{[n]} \to \mathbb{C}^{(n)} \simeq \mathbb{C}^n$ of relative dimension $n$ onto the $n$-th symmetric product $\mathbb{C}^{(n)}$ of $\mathbb{C}$, which can be understood explicitly. In a similar

*Partially supported by N.S.A. and N.S.F.
†Supported by a Royal Society University Research Fellowship
‡Partially supported by PRIN 2007 project "Spazi di moduli e teoria di Lie"

vein one can prove that a corresponding character variety is isomorphic to $Y^{[n]}$ the Hilbert scheme of $n$ points of the surface $Y := \mathbb{C}^* \times \mathbb{C}^*$, whose Hodge theory is well understood. The expectation from the $P = W$ conjecture is that the we again have an exchange of filtration phenomenon.

Indeed the main result of this paper is Theorem 4.1.1 which establishes that there is a natural isomorphism of graded vector spaces $\phi^{[n]} : H^*(X^{[n]}, \mathbb{Q}) \simeq H^*(Y^{[n]}, \mathbb{Q})$ that exchanges the perverse Leray filtration for the map $h_n$ with the halved weight filtration :

$$\phi^{[n]}\left(\mathscr{P}_{X^{[n]}}\right) = {}_{\frac{1}{2}}\mathscr{W}_{Y^{[n]}}.$$

In words, the inverse of the isomorphism $\phi^{[n]}$ sends a class of type $(p,p)$ for $\mathscr{W}_{Y^{[n]}}$ to a class in in $\mathscr{P}_{X^{[n]},p}$. Theorem 4.3.2 relates the hard Lefschetz theorem on the products of symmetric products of the curve $E$ with the relative hard Lefschetz theorem for the map $h_n$ and with a "curious" hard Lefschetz theorem on the cohomology of $Y^{[n]}$.

The example dealt with in this paper presents a striking difference with respect to the one treated in [4]. In the latter case, due to Ngô's support theorem, most of the perverse sheaves showing up in the decomposition theorem are supported on the whole target space of the Hitchin map. On the other hand, in the case treated here, every stratum in $\mathbb{C}^{(n)}$ of the map $h_n : X^{[n]} \to \mathbb{C}^{(n)}$ contributes several perverse sheaves showing up in the decomposition theorem.

At the moment, we cannot explain the exchange of filtration phenomena described above, beyond the fact that we can observe them. In §4.4 we discuss some properties shared by the example considered in this paper and the one treated in [4], and we speculate on the possibility of a more general statement regarding the phenomenon of exchange of filtrations.

## 1.2 Notation

We work over the field of complex numbers $\mathbb{C}$ and with singular cohomology with rational coefficients $\mathbb{Q}$. The results hold with no essential changes over any algebraically closed field and with $\mathbb{Q}_\ell$-adic cohomology. A variety is a separated scheme of finite type over $\mathbb{C}$.

We employ freely the language of derived categories, perverse sheaves and the decomposition theorem as well as the language of Deligne's mixed Hodge structures (MHS); the reader may consult [1], the survey [11] and the textbooks [15, 21, 23, 26]. For the convenience of the reader we summarize our notation and terminology below.

Given a variety $Z$, we work with the full subcategory $D_Z$ of the derived category of the category of sheaves of rational vector spaces on $Z$ given by those bounded complexes $K$ on $Z$ whose cohomology sheaves $\mathcal{H}^i(K)$ on $Z$ are constructible; a sheaf on $Z$ is constructible if there is a partition $Z = \coprod Z_a$ of $Y$ given by locally closed subvarieties such that the restriction $F_{|Z_a}$ is locally constant for every $a$. We denote the $i$-th perverse cohomology sheaf of a complex $K$ on $Z$ by ${}^{p}\mathcal{H}^i(K)$; it is a perverse sheaf on $Z$. Given a map $f : Z \to Z'$ of algebraic varieties, we denote the derived direct image functor $Rf_*$ simply by $f_*$ and the $i$-th direct image functor by $R^i f_*$.

A filtration $F$ on a vector space is a finite increasing filtration

$$\ldots \subseteq F_i V \subseteq F_{i+1} V \subseteq \ldots;$$

finite means that $F_i V = \{0\}$ for $i \ll 0$ and $F_i = V$ for $i \gg 0$. A filtration $F$ on $V$ has type $[a, b]$ if $F_{a-1} V = \{0\}$ and $F_b V = V$.

Given a variety $Z$, the weight filtration on the cohomology groups $H^d(Z,\mathbb{Q})$ is denoted by $\mathscr{W}_Z$. A map $f: Z \to Z'$ endows the cohomology groups $H^d(Z,\mathbb{Q})$ with two distinct filtrations, the Leray filtration $\mathscr{L}_Z$ and the perverse Leray filtration $\mathscr{P}_Z$.

In this paper, we are concerned with the Hilbert schemes of $n$ points $X^{[n]}$ and $Y^{[n]}$ associated with the two complex surfaces $X := T^\vee E \simeq E \times \mathbb{C}$, the total space of the cotangent bundle of an elliptic curve $E$, and $Y := \mathbb{C}^* \times \mathbb{C}^*$. We shall consider a certain natural proper map $h_n : X^{[n]} \to \mathbb{C}^{(n)}$.

# 2 The Hilbert scheme of a surface and its cohomology groups

## 2.1 The decomposition theorem for the Hilbert-Chow map $\pi_n : S^{[n]} \to S^{(n)}$

Let $S$ be a nonsingular connected complex analytic surface $S$ and $n \geq 0$ be a non-negative integer. We refer the reader to [5, 6, 8, 16, 25] for background and references on Hilbert schemes of surfaces.

We denote by $S^{(n)} := S^n/\mathfrak{S}_n$ the $n$-th symmetric product of $S$, i.e. the quotient of $S^n$ by the obvious action of the $n$-th symmetric group. A partition of $\nu = \{\nu_1, \ldots, \nu_l\}$ of $n$ is an unordered collection of positive integers such that $\nu_1 + \ldots + \nu_l = n$; the integer $l = l(\nu)$ is called the length of $\nu$. A point $x \in S^{(n)}$ gives rise to a partition $\nu = \nu(x)$, for $x$ admits a unique representation as a formal sum $\nu_1 s_1 + \ldots + \nu_l s_l$, with $\nu_i$ positive integers adding up to $n$, and $s_i \in S$ distinct. The subset $S^{(n)}_\nu \subseteq S^{(n)}$ of points yielding the same partition $\nu$ is a locally closed, irreducible, nonsingular subvariety of $S^{(n)}$ and we have that the symmetric product $S^{(n)}$ is the disjoint union over the set of partitions on $n$ of these subvarieties: $S^{(n)} = \coprod_\nu S^{(n)}_\nu$. A partition $\nu$ gives rise to a new variety $S^{(\nu)}$ as follows: represent the partition $\nu$ as a symbol $1^{a_1} 2^{a_2} \cdots n^{a_n}$, where $a_i$ is the number of times $i$ appears in $\nu$; the $a_i \geq 0$, the length $l(\nu) = \sum a_i$ and $n = \sum_i i\, a_i$; finally, define $S^{(\nu)} := \prod_i S^{(a_i)}$ to be the indicated product of symmetric product of $S$. If we define, $\mathfrak{S}_\nu := \prod \mathfrak{S}_{a_i}$, then $S^{(\nu)} = S^{l(\nu)}/\mathfrak{S}_\nu$. There is a natural finite map $r^{(\nu)} : S^{(\nu)} \to S^{(n)}$ with image the closure $\overline{S^{(n)}_\nu}$ and the resulting map $S^{(\nu)} \to \overline{S^{(n)}_\nu}$ is the normalization of the image.

The Hilbert scheme $S^{[n]}$ of zero-dimensional length $n$ subschemes of $S$ is a connected complex manifold of dimension $2n$ and, if $S$ is algebraic, then so is $S^{[n]}$. There is the $n$-th Hilbert-Chow map $\pi_n : S^{[n]} \to S^{(n)}$ sending a scheme to its support, counting multiplicities; this map is proper and it is a resolution of singularities of the symmetric product.

In view of [8], §2.5, by using the correspondences in $S^{(\nu)} \times_{S^{(n)}} S^{[n]}$ inside $S^{(\nu)} \times S^{[n]}$ the decomposition theorem for the map $\pi_n$ yields a canonical isomorphism in the category $D_{S^{(n)}}$:

$$\gamma_S^{[n]} := \textstyle\sum_\nu \gamma_S^{(\nu)} : \bigoplus_\nu r_*^{(\nu)} \mathbb{Q}_{S^{(\nu)}}[2l(\nu)] \xrightarrow{\simeq} \pi_* \mathbb{Q}_{S^{[n]}}[2n]. \tag{1}$$

## 2.2 The MHS on $H^*(S^{[n]}, \mathbb{Q})$

If $S$ is algebraic, then, by using the compatibility (see [9]) with MHS of the constructions leading to the isomorphism (1), we obtain a canonical isomorphism of MHS (recall that a

Tate twist in cohomology $(-i)$ increases the weights by $2i$):

$$\gamma_S^{[n]} = \sum \gamma_S^{(\nu)} : \bigoplus_\nu \left(H^{*-2[n-l(\nu)]}\left(S^{(\nu)}, \mathbb{Q}\right)(l(\nu)-n)\right) \xrightarrow{\simeq} H^*\left(S^{[n]}, \mathbb{Q}\right). \tag{2}$$

The fact that the two sides of (2) are isomorphic has been first proved in [17] by using the theory of mixed Hodge modules.

Given a partition $\nu$ of $n$, consider the mixed Hodge substructure

$$H^*_\nu\left(S^{[n]}, \mathbb{Q}\right) := \operatorname{Im} \gamma_S^{(\nu)} \simeq H^{*-2[n-l(\nu)]}\left(S^{(\nu)}, \mathbb{Q}\right)(l(\nu)-n) \tag{3}$$

so that the isomorphism of MHS (2) now reads as the internal direct sum decomposition

$$H^*\left(S^{[n]}, \mathbb{Q}\right) = \bigoplus_\nu H^*_\nu\left(S^{[n]}, \mathbb{Q}\right). \tag{4}$$

### 2.3 The map $\phi^{[n]}$ induced by a diffeomorphism $S_2 \simeq S_1$

The canonical isomorphism (2) has the following simple consequence. Let $S_1$ and $S_2$ be two nonsingular surfaces and

$$\phi : H^*(S_1, \mathbb{Q}) \simeq H^*(S_2, \mathbb{Q}) \tag{5}$$

be an isomorphism of graded vector spaces. By taking tensor products and invariants, the map $\phi$ induces, for every partition $\nu$, an isomorphism of graded vector spaces

$$\phi^{(\nu)} : H^*(S_1^{(\nu)}, \mathbb{Q}) \simeq H^*(S_2^{(\nu)}, \mathbb{Q}).$$

By using the isomorphisms (1), we define the map

$$\phi^{[n]} := \left(\gamma_{S_2}^{[n]}\right) \circ \left(\sum_\nu \phi^{(\nu)}\right) \circ \left(\gamma_{S_1}^{[n]}\right)^{-1} : H^*\left(S_1^{[n]}, \mathbb{Q}\right) \simeq H^*\left(S_2^{[n]}, \mathbb{Q}\right) \tag{6}$$

which is an isomorphisms of graded vector spaces.

If the surfaces $S_i$ are algebraic and $\phi$ is an isomorphism of MHS, then so is (6). However, in this paper we use this set-up in the case: $S_1 = E \times \mathbb{C}$ ($E$ an elliptic curve) and $S_2 = \mathbb{C}^* \times \mathbb{C}^*$ and $\phi = \Phi^*$, where $\Phi : S_2 \simeq S_1$ is a diffeomorphism. In this case, due to the incompatibility of the weights, $\phi$ and $\phi^{[n]}$ cannot be isomorphisms of MHS.

It is likely that the results in [27] imply that if we have a diffeomorphism $S_2 \simeq S_1$ of nonsingular algebraic surfaces, then there is a diffeomorphism $S_2^{[n]} \simeq S_1^{[n]}$. At present, we do not know this and we do not need it here.

## 3 The surfaces $X$ and $Y$ and the filtrations ${}_{\frac{1}{2}}\mathscr{W}_{Y^{[n]}}$ and $\mathscr{P}_{Y^{[n]}}$

For the remainder of the paper, we fix $n \geq 0$, an elliptic curve $E$ and we set

$$Y := \mathbb{C}^* \times \mathbb{C}^*, \qquad X := T^\vee E \simeq E \times \mathbb{C},$$

i.e. $X$ is the total space of the cotangent (canonical) bundle of $E$. The isomorphism above is well-defined up to multiplication by a non-zero scalar.

The two surfaces $X$ and $Y$ are noncanonically diffeomorphic: choose $E$ to be $\mathbb{C}/\Gamma$ where $\Gamma$ is the lattice of Gaussian integers; then use polar coordinates to identify $X$ and $Y$. Let $\Phi : Y \simeq X$ be any diffeomorphism and set $\phi := \Phi^* : H^*(X,\mathbb{Q}) \simeq H^*(Y,\mathbb{Q})$. We are in the situation of §2.3.(5) so that, for every $n \geq 0$, we obtain the linear isomorphism (6) of graded vector spaces

$$\phi^{[n]} : H^*\left(X^{[n]},\mathbb{Q}\right) \xrightarrow{\simeq} H^*\left(Y^{[n]},\mathbb{Q}\right). \tag{7}$$

As it was observed in §2.3, for $n \geq 1$, the two sides are never isomorphic as MHS. In particular, (7) does not preserve the weight filtrations.

Let us remark that each $H^d(X^{[n]},\mathbb{Q})$ is a pure Hodge structure of weight $d$. Since $H^*(X,\mathbb{Q}) \simeq H^*(E,\mathbb{Q})$ is an isomorphism of MHS, we have that the same is true for $H^*(X^{(\nu)},\mathbb{Q}) \simeq H^*(E^{(\nu)},\mathbb{Q})$ for every partition $\nu$ of $n$. In view of the splitting of MHS (4), we have the following canonical isomorphism of MHS

$$H^*(X^{[n]},\mathbb{Q}) \overset{(4)}{=} \bigoplus_{\nu} H^*_{\nu}(X^{[n]},\mathbb{Q}) \simeq \bigoplus_{\nu} H^{*-2[n-l(\nu)]}(E^{(\nu)},\mathbb{Q})(l(\nu)-n).$$

Since each $H^d(E^{(\nu)},\mathbb{Q})$ is pure of weight $d$, we conclude that each $H^d(X^{[n]},\mathbb{Q})$ is pure of weight $d$ as well. In particular, the weight filtration $\mathscr{W}_{X^{[n]}}$ on $H^*(X^{[n]},\mathbb{Q})$ is simply the filtration by cohomological degree and this should be contrasted with Proposition 3.1.2.

## 3.1 The halved weight filtration $\frac{1}{2}\mathscr{W}_{Y^{[n]}}$ on $H^*((\mathbb{C}^* \times \mathbb{C}^*)^{[n]},\mathbb{Q})$

In this section, we first compute the MHS on $H^*(Y^{[n]},\mathbb{Q})$ and determine the weight filtration $\mathscr{W}_{Y^{[n]}}$ on $H^*(Y^{[n]},\mathbb{Q})$. We then observe that $\mathscr{W}_{Y^{[n]}}$ has no odd weights so that we can define the halved weight filtration $\frac{1}{2}\mathscr{W}_{Y^{[n]},k} := \mathscr{W}_{2k}$ on $H^*(Y^{[n]},\mathbb{Q})$ by simply halfing the weights.

Recall that: an MHS is of Hodge-Tate type if the odd graded pieces of the weight filtration are zero and every even graded piece $\mathrm{Gr}^{\mathscr{W}}_{2p}$ is of pure type $(p,p)$; an MHS is split of Hodge-Tate type if it is isomorphic to a direct sum of pure MHS of Hodge-Tate type.

**Lemma 3.1.1** *For every partition $\nu$ of $n$, the natural MHS on $H^*(Y^{(\nu)},\mathbb{Q})$ is split of Hodge-Tate type and, more precisely,*

$$H^d\left(Y^{(\nu)},\mathbb{Q}\right) \textit{ is pure of weight } 2d \textit{ and Hodge-type } (d,d),$$

$$0 = \mathscr{W}_{2d-1} \subseteq \mathscr{W}_{2d} = H^d(Y^{(\nu)},\mathbb{Q}).$$

*Proof.* Since $H^d(\mathbb{C}^*,\mathbb{Q})$ has type $(d,d)$, for $d=0,1$, and it is trivial otherwise, the statement follows from the Künneth isomorphism and the naturality of the mixed Hodge structure for the inclusion $H^d(Y^{(\nu)},\mathbb{Q}) \subseteq H^d(Y^{l(\nu)},\mathbb{Q})$ coming from the quotient map

$$Y^{l(\nu)} \longrightarrow Y^{l(\nu)}/\mathfrak{S}_{\nu} = Y^{(\nu)}.$$

□

The following proposition is an immediate consequence of Lemma 3.1.1 and of the equality of MHS (4).

**Proposition 3.1.2** *The natural mixed Hodge structure on $H^*(Y^{[n]}, \mathbb{Q})$ is split of Hodge-Tate type. More precisely, in terms of the decomposition (4), we have:*

$$\mathscr{W}_{2k}\, H^d\left(Y^{[n]}, \mathbb{Q}\right) = \bigoplus_{d-(n-l(\nu)) \leq k} H^d_\nu\left(Y^{[n]}, \mathbb{Q}\right), \qquad \mathscr{W}_{2k} = \mathscr{W}_{2k+1}.$$

Proposition 3.1.2 allows us to define the halved weight filtration $\frac{1}{2}\mathscr{W}_{Y^{[n]}}$ by setting

$$\tfrac{1}{2}\mathscr{W}_{Y^{[n]},k} := \mathscr{W}_{Y^{[n]},2k}.$$

The halved weight filtration $\frac{1}{2}\mathscr{W}_{Y^{[n]}}$ on $H^*(Y^{[n]}, \mathbb{Q})$ has type $[0, 2n]$.

## 3.2 Decomposition theorem for the Hitchin-like fibration $h_n : X^{[n]} \to \mathbb{C}^{(n)}$

Let $p : X \to \mathbb{C}$ be the induced projection. Recall the notation in §2.2. We have the commutative diagram

$$\begin{tikzcd}
& & X^{[n]} \arrow[d, "\pi_n"] \arrow[dd, bend left, "h_n"] \\
X^{l(\nu)} = E^{l(\nu)} \times \mathbb{C}^{l(\nu)} \arrow[r, "/\mathfrak{S}_\nu"] \arrow[d, "p^{l(\nu)}"] & X^{(\nu)} \arrow[r, "r_X^{(\nu)}"] \arrow[d, "p^{(\nu)}"] & X^{(n)} \arrow[d, "p^{(n)}"] \\
\mathbb{C}^{l(\nu)} \arrow[r, "/\mathfrak{S}_\nu"] & \mathbb{C}^{(\nu)} \arrow[r, "r_{\mathbb{C}}^{(\nu)}"] & \mathbb{C}^{(n)}.
\end{tikzcd} \tag{8}$$

The maps $p^{l(\nu)}$ and $p^{(\nu)}$ are of relative dimension $l(\nu)$ and the map $p^{(n)}$ is of relative dimension $n$. In particular, note that

$$\dim \left\{ {p^{(n)}}^{-1}\left(\mathbb{C}^{(n)}_\nu\right)\right\} = l(\nu) + n, \qquad \dim \left\{ {p^{(\nu)}}^{-1}\left(\mathbb{C}^{(\nu)}_\nu\right)\right\} = 2l(\nu).$$

The fiber of $p^{(\nu)}$ over the general point of $\mathbb{C}^{(\nu)}$ is isomorphic to $E^{l(\nu)}$. All the other fibers are isomorphic to quotients of $E^{l(\nu)}$ under the action of suitable, not necessarily normal, subgroups groups of the finite group $\mathfrak{S}_\nu$. The fibers over the points in the small diagonal in $\mathbb{C}^{(\nu)}$ are all isomorphic to $E^{(\nu)} = E^{l(\nu)}/\mathfrak{S}_\nu$ so that, by the compatibility with MHS of Grothendieck's theorem on the rational cohomology of quotient varieties, we have a canonical isomorphism of MHS

$$H^*\left(E^{(\nu)}, \mathbb{Q}\right) = H^*\left(E^{l(\nu)}, \mathbb{Q}\right)^{\mathfrak{S}_\nu}. \tag{9}$$

The map $h_n : X^{[n]} \to \mathbb{C}^{(n)}$ is projective of relative dimension $n = \frac{1}{2}\dim X^{[n]} = \dim \mathbb{C}^{(n)}$, flat by [24], Corollary to Theorem 23.1, and, as it is observed above, it has general fiber the Abelian variety $E^n$.

**Remark 3.2.1** We say that $h_n$ is a Hitchin-type map because of the analogy it presents with the Hitchin map associated with the moduli of Higgs bundles on a curve, where the dimensions of domain $M$, target $A$ and fibers $F$ are related as above: $\dim M = 2\dim A = 2\dim F$ and also because our main result Theorem 4.1.1 is analogous to the main result of [4], which deals with rank two Higgs bundles of odd degree on a curve.

Due to the commutativity of the diagram (8) and to the functoriality of derived push-forwards applied to $h_n = p^{(n)} \circ \pi_n$, the decomposition theorem (1) for the map $\pi_n$ implies that we have natural isomorphisms

$$\bigoplus_\nu r^{(\nu)}_{\mathbb{C}*} p^{(\nu)}_* \mathbb{Q}_{X^{(\nu)}}[2l(\nu)] \longrightarrow \bigoplus_\nu p^{(n)}_* r^{(\nu)}_{X*} \mathbb{Q}_{X^{(\nu)}}[2l(\nu)] \xrightarrow{\simeq} h_{n*}\mathbb{Q}_{X^{[n]}}[2n].$$

By applying Grothendieck's theorem on the invariant part of push-forwards under a quotient map under a finite group action, and by recalling that $p^{l(\nu)}$ is a projection map, we get a canonical isomorphism

$$p^{(\nu)}_* \mathbb{Q}_{X^{(\nu)}} = \left(p^{l(\nu)}_* \mathbb{Q}_{X^{(\nu)}}\right)^{\mathfrak{S}_\nu} = \bigoplus_{i=0}^{2l(\nu)} \left(R^i p^{l(\nu)}_* \mathbb{Q}\right)^{\mathfrak{S}_\nu}[-i].$$

We thus get the distinguished splitting isomorphism in the category $D_{\mathbb{C}^{(n)}}$

$$\Gamma^{[n]}_X : \bigoplus_\nu \bigoplus_{i=0}^{2l(\nu)} \left\{ \left[ r^{(\nu)}_{\mathbb{C}*} \left(R^i p^{l(\nu)}_* \mathbb{Q}\right)^{\mathfrak{S}_\nu} \right] [l(\nu)] \right\} [-(i-l(\nu))] \xrightarrow{\quad\simeq\quad} h_* \mathbb{Q}_{X^{[n]}}[2n]. \tag{10}$$

Since every $r^{(\nu)}$ is finite, every direct summand in square brackets is an ordinary sheaf (not just a complex). Moreover, since the functors $r^{(\nu)}_*$ are $t$-exact, every summand in curly brackets is a perverse sheaf, in fact an intersection cohomology complex with twisted coefficients supported on $\overline{\mathbb{C}^{(n)}_\nu} \subseteq \mathbb{C}^{(n)}$.

It follows that (10) "is" the decomposition theorem for the map $h_n$ in the sense that we decomposed the right-hand-side as direct sum of shifted intersection cohomology complexes supported on $\mathbb{C}^{(n)}$. We note that, unlike the general statement of the decomposition theorem, we have obtained (10) as a distinguished isomorphism.

In order to simplify the notation, we set

$$R^i_\nu := r^{(\nu)}_{\mathbb{C}*} \left(R^i p^{l(\nu)}_* \mathbb{Q}\right)^{\mathfrak{S}_\nu}.$$

For our purposes, it is convenient to re-write (10) in the following two different ways, where the former emphasizes the perverse-sheaf-nature of the summands, and the latter emphasizes the ordinary-sheaf-nature of the summands. One merely needs to apply the appropriate shift and re-organize the terms. By abuse of notation, we denote the resulting maps with the same symbol $\Gamma^{[n]}_{X^{[n]}}$:

$$\Gamma^{[n]}_X : \bigoplus_{t=0}^{2n} \left(\bigoplus_{i+(n-l(\nu))=t} R^i_\nu[l(\nu)]\right)[n-t] \xrightarrow{\simeq} h_{n*}\mathbb{Q}_{X^{[n]}}[n]; \tag{11}$$

$$\Gamma^{[n]}_X : \bigoplus_{k=0}^{2n} \left(\bigoplus_{i+2(n-l(\nu))=k} R^i_\nu\right)[-k] \xrightarrow{\simeq} h_{n*}\mathbb{Q}_{X^{[n]}}. \tag{12}$$

We now turn to the decompositions in cohomology stemming from the isomorphism(s) $\Gamma^{[n]}_{X^{[n]}}$. By taking components in (10), we have the equality of maps in the derived category

$$\Gamma^{[n]}_X = \sum_\nu \Gamma^{(\nu)}_X = \sum_\nu \sum_{i=0}^{2l(\nu)} \Gamma^{(\nu),i}_X$$

and, by taking the images in cohomology, we set

$$G_\nu^*\left(X^{[n]},\mathbb{Q}\right) := \operatorname{Im}\Gamma_X^{(\nu)} \subseteq H^*\left(X^{[n]},\mathbb{Q}\right). \tag{13}$$

By the very construction of the splitting (10), i.e. the fact that is it obtained by pushing forward (1), we have that

$$G_\nu\left(X^{[n]},\mathbb{Q}\right) :\overset{(13)}{=} \operatorname{Im}\Gamma_X^{(\nu)} = \operatorname{Im}\gamma_X^{(\nu)} \overset{(3)}{=}: H_\nu^*\left(X^{[n]},\mathbb{Q}\right) \subseteq H^*\left(X^{[n]},\mathbb{Q}\right),$$

or, in words, the two distinguished splittings of $H^*(X^{[n]},\mathbb{Q})$ into $\nu$-components arising from the decomposition theorem for the Hilbert-Chow map $\pi_n$ and for the Hitchin-like map $h_n$ coincide.

For every fixed partition $\nu$ of $n$ and for every $0\leq i\leq 2l(\nu)$, we set

$$H_{\nu,i}^*\left(X^{[n]},\mathbb{Q}\right) := \operatorname{Im}\Gamma_X^{(\nu),i} \subseteq H_\nu^*\left(X^{[n]},\mathbb{Q}\right),$$

so that

$$H_\nu^*\left(X^{[n]},\mathbb{Q}\right) = \bigoplus_{i=0}^{2l(\nu)} H_{\nu,i}^*\left(X^{[n]},\mathbb{Q}\right). \tag{14}$$

The following lemma shows that in each cohomological degree $d$, there is at most one non-zero summand $H_{\nu,i}^d(X^{[n]},\mathbb{Q})$ in (14).

**Lemma 3.2.2** *We have the following*

$$H^q\left(\mathbb{C}^{(n)},R_\nu^i\right) \simeq \begin{cases} 0 & \textit{if } q\neq 0,\\ H^i\left(E^{(\nu)},\mathbb{Q}\right) & \textit{if } q=0.\end{cases}$$

*In particular, for every $d$, we have that*

$$H_\nu^d\left(X^{[n]},\mathbb{Q}\right) = H_{\nu,i=d-2(n-l(\nu))}^d\left(X^{[n]},\mathbb{Q}\right) \simeq H^{d-2(n-l(\nu))}\left(E^{(\nu)},\mathbb{Q}\right).$$

*Proof.* We have

$$H^q\left(\mathbb{C}^{(n)}, r_{\mathbb{C}*}^{(\nu)}\left(R^i p_*^{l(\nu)}\mathbb{Q}\right)^{\mathfrak{S}_\nu}\right) = H^q\left(\mathbb{C}^{(\nu)},\left(R^i p_*^{l(\nu)}\mathbb{Q}\right)^{\mathfrak{S}_\nu}\right) = H^q\left(\mathbb{C}^{l(\nu)}, R^i p_*^{l(\nu)}\mathbb{Q}\right)^{\mathfrak{S}_\nu}.$$

Since $\mathbb{C}^{l(\nu)}$ is contractible, the groups above are zero whenever $q\neq 0$. In view of (9), for $q=0$ we have:

$$H^0\left(\mathbb{C}^{l(\nu)},\left(R^i p_*^{l(\nu)}\mathbb{Q}\right)\right)^{\mathfrak{S}_\nu} = H^i\left(E^{l(\nu)},\mathbb{Q}\right)^{\mathfrak{S}_\nu} = H^i\left(E^{(\nu)},\mathbb{Q}\right).$$

This proves the first statement.

According to (10) and the diagram (8), each summand $H_{\nu,i}^d(X^{[n]},\mathbb{Q})$ is the subspace of $H^d(X^{[n]},\mathbb{Q})$ injective image of

$$H^{d-2(n-l(\nu))-i}\left(\mathbb{C}^{(n)}, r_{\mathbb{C}*}^{(\nu)}\left(R^i p_*^{l(\nu)}\mathbb{Q}\right)^{\mathfrak{S}_\nu}\right) \;=\; H^{d-2(n-l(\nu))-i}\left(\mathbb{C}^{(\nu)},\left(R^i p_*^{l(\nu)}\mathbb{Q}\right)^{\mathfrak{S}_\nu}\right) \;=$$

$$\left(H^{d-2(n-l(\nu))-i}\left(\mathbb{C}^{l(\nu)},\left(R^i p_*^{l(\nu)}\mathbb{Q}\right)\right)\right)^{\mathfrak{S}_\nu}.$$

The second statement now follows from (14) and from the first statement. □

Summarizing: we have that for every $d$:

$$H^d\left(X^{[n]},\mathbb{Q}\right)=\bigoplus_\nu H^*_\nu\left(X^{[n]},\mathbb{Q}\right)=$$

$$\bigoplus_\nu H^d_{\nu,d-2(n-l(\nu))}\left(X^{[n]},\mathbb{Q}\right)\simeq\bigoplus_\nu H^{d-2(n-l(\nu))}\left(E^{(\nu)},\mathbb{Q}\right). \tag{15}$$

### 3.3 The perverse Leray filtration $\mathscr{P}_{X^{[n]}}$ on $H^*(X^{[n]},\mathbb{Q})$

The theory of perverse sheaves endows $H^*(X^{[n]},\mathbb{Q})$ with the perverse Leray filtration $\mathscr{P}_{X^{[n]}}$, i.e. with the perverse filtration associated with the complex $h_{n*}\mathbb{Q}_{X^{[n]}}[n]$; see [12]. Note that if we replace $h_{n*}\mathbb{Q}_{X^{[n]}}[n]$ with another shift $h_{n*}\mathbb{Q}_{X^{[n]}}[m]$, the resulting filtrations gets translated. We have made the choice $m=n$ so that, in view of (11), the result has the same type $[0,2n]$ as the one of $_{\frac{1}{2}}\mathscr{W}_{Y^{[n]}}$.

While, in general, the perverse (Leray) filtration is canonically defined, there is no natural splitting of it. In our situation, in view of (11) and of (15), we have that the perverse Leray filtration is naturally split:

$$\mathscr{P}_{X^{[n]},p}H^d\left(X^{[n]},\mathbb{Q}\right)=\bigoplus_{t\le p}\bigoplus_{d-(n-l(\nu))=t}H^d_\nu\left(X^{[n]},\mathbb{Q}\right)=\bigoplus_{d-(n-l(\nu))\le p}H^d_\nu\left(X^{[n]},\mathbb{Q}\right). \tag{16}$$

**Remark 3.3.1** In view of the expression (12), it is straightforward to verify with the aid of Lemma 3.2.2 that the ordinary Leray filtration $\mathscr{L}_{X^{[n]}}$ on $H^*(X^{[n]},\mathbb{Q})$ for the map $h_n$ is the filtration by cohomological degree. In particular, by comparing with (16), it is clear that the Leray filtration is strictly included in the perverse Leray filtration.

## 4 The main result, relation with hard Lefschetz, and a speculation

### 4.1 "$\mathscr{P}_{X^{[n]}}={}_{\frac{1}{2}}\mathscr{W}_{Y^{[n]}}$"

We are now ready to state and prove the main result of this paper.

**Theorem 4.1.1** *For every $n\ge 0$, the map $\phi^{[n]}$ (7) is a filtered isomorphism, i.e.*

$$\phi^{[n]}(\mathscr{P}_{X^{[n]}})={}_{\frac{1}{2}}\mathscr{W}_{Y^{[n]}}.$$

*Proof.* By its very definition, the map $\phi^{[n]}$ is a direct sum map with respect to the $\nu$ decompositions (4) for $S=X$ and $S=Y$, respectively It remains to apply Proposition 3.1.2 and (16). □

We would like to remark on the exceptional circumstance highlighted by Theorem 4.1.1. In view of the canonical splitting (16), we say that a class $a \in H^d(X^{[n]}, \mathbb{Q})$ has perversity $p$ if $a \in \oplus_{d-(n-l(\nu))=p} H^d(X^{[n]}, \mathbb{Q})$. Theorem 4.1.1 shows that, regardless of the $(r,s)$ type of $a$ with respect to the pure Hodge structure $H^d(X^{[n]}, \mathbb{Q})$, we have that $\phi^{[n]}(a) \in H^d(Y^{[n]}, \mathbb{Q})$ has type $(p,p)$ and, more precisely, lives in the $(p,p)$ part of the split Hodge-Tate type structure.

The proof of Theorem 4.1.1 is heavily based on the fact that we have constructed the explicit splitting (16) of the perverse Leray filtration. There is a different approach which is based on the following geometric description [12] of the perverse Leray filtration: let $s \geq 0$ and let $\Lambda^s \subseteq \mathbb{C}^{(n)} \simeq \mathbb{C}^n$ be a general $s$-dimensional linear section of $\mathbb{C}^n$; then

$$\mathscr{P}_{X^{[n]},p} H^d\left(X^{[n]}, \mathbb{Q}\right) = \operatorname{Ker}\left\{H^d\left(X^{[n]}, \mathbb{Q}\right) \longrightarrow H^d\left(h_n^{-1}(\Lambda^{d-p-1}), \mathbb{Q}\right)\right\}.$$

While we omit the details of this approach, we do point out the basic fact leading to the identification of the kernel above with the right-hand-side of (16): a general linear section $\Lambda^{d-p-1}$ avoids the closure of a stratum $\overline{\mathbb{C}^{(n)}_\nu}$, which has dimension $l(\nu)$, if and only if

$$d - (n - l(\nu)) \leq p.$$

## 4.2 The curious hard Lefschetz (CHL) for $H^*((\mathbb{C}^* \times \mathbb{C}^*)^{[n]}, \mathbb{Q})$

Let $(z,w)$ be coordinates on $Y = \mathbb{C}^* \times \mathbb{C}^*$. The 2-form

$$\alpha_Y := \frac{1}{(2i\pi)^2} \frac{dz \wedge dw}{zw}$$

is closed and defines an integral cohomology class which we denote with the same symbol. We have $\alpha_Y \in H^2(Y, \mathbb{Q}) \cap H^{2,2}(Y)$. Let $p_i : Y^n \to Y$ be the $i$-th projection. Set

$$\alpha_{Y^n} = \sum_{i=1}^n p_i^* \alpha_Y \in H^2(Y^n, \mathbb{Q}) \cap H^{2,2}(Y^n).$$

Let $\alpha_{Y^{(n)}} \in H^2(Y^{(n)}, \mathbb{Q}) \cap H^{2,2}(Y^{(n)})$ and $\alpha_{Y^{(\nu)}} \in H^2(Y^{(\nu)}, \mathbb{Q}) \cap H^{2,2}(Y^{(\nu)})$ be the naturally induced classes. Let

$$\alpha_{Y^{[n]}} := \pi_n^* \alpha_{Y^{(n)}} \in H^2(Y^{(n)}, \mathbb{Q}) \cap H^{2,2}(Y^{(n)})$$

be the pullback via the Hilbert-Chow map $\pi_n : Y^{[n]} \to Y^{(n)}$.

Note that because of Hodge type, none of the $\alpha$-type classes above is the first Chern class of a holomorphic line bundle on $Y^{[n]}$. Nonetheless, a simple explicit computation based on Proposition 3.1.2 shows that cupping with the powers of $\alpha_{Y^{[n]}}$, gives rise to isomorphisms

$$\mathrm{Gr}^{\mathscr{W}_{Y^{[n]}}}_{2n-2k} H^*(Y^{[n]}, \mathbb{Q}) \xrightarrow[\simeq]{\alpha^k_{Y^{[n]}}} \mathrm{Gr}^{\mathscr{W}_{Y^{[n]}}}_{2n+2k} H^{*+2k}(Y^{[n]}, \mathbb{Q}). \tag{17}$$

These isomorphisms are analogous to the "curious hard Lefschetz" theorem of [19]. Its curiosity consists of the fact that it is a statement concerning a $(2,2)$ class on a noncompact variety, instead of a $(1,1)$-class on a projective variety. This apparently mysterious fact receives an explanation from the coincidence of the halved weight filtration with the perverse Leray filtration proved in the main Theorem 4.1.1.

**Question 4.2.1** What corresponds to the CHL (17) under the identification

$$H^*(Y^{[n]}, \mathbb{Q}) \simeq H^*(X^{[n]}, \mathbb{Q})$$

given by (7)? We answer this question in Theorem 4.3.2.

## 4.3 CHL on $Y^{[n]}$ $\Leftrightarrow$ the HL on $E^{(\nu)}$ $\Leftrightarrow$ RHL for $h_n$

In this section, we say that a rational cohomology class of degree two on a variety $Z$ is good (resp. ample) if it is a non-zero (resp. positive) rational multiple of the Chern class of an ample line bundle on $Z$. The point of this definition is that the hard Lefschetz theorem holds for a good class on a nonsingular projective manifold as well as on its quotients by a finite group acting by algebraic isomorphisms.

Fix any diffeomorphism $\Phi : Y = \mathbb{C}^* \times \mathbb{C}^* \simeq X = E \times \mathbb{C}$. We obtain the linear isomorphism (7) of graded vector spaces: $\phi_{[n]} : H^*(X^{[n]}, \mathbb{Q}) \simeq H^*(Y^{[n]}, \mathbb{Q})$.

Let $\alpha_X, \alpha_{X^n}, \alpha_{X^{(n)}}, \alpha_{X^{(\nu)}}, \alpha_{X^{[n]}}$ be the classes obtained by transplanting the $\alpha$-classes defined starting from $Y$ in section 4.2 via $\phi_{[n]}^{-1}$.

Note that by construction, for every surface $S$, the inclusion $H^*(S^{(n)}, \mathbb{Q}) \subseteq H^*(S^{[n]}, \mathbb{Q})$ is given by the pull-back $\pi_n^*$ via the Hilbert-Chow map $\pi_n : S^{[n]} \to S^{(n)}$. In particular, we have that $\alpha_{X^{[n]}} = \pi_n^* \alpha_{X^{(n)}}$. This has to be verified in view of the fact that $\phi_{[n]}$ has not been defined using a diffeomorphism $Y^{[n]} \simeq X^{[n]}$ between the Hilbert schemes.

Note that $\phi_{[n]}$ is not a map of MHS (this is already apparent for $n = 1$). On the other hand, since $H^2(X, \mathbb{C}) = H^2(E \times \mathbb{C}) \simeq H^2(E, \mathbb{C}) = H^{1,1}(E)$, we see that all the $\alpha$-classes $\alpha_X, \ldots, \alpha_{X^{[n]}}$ are in fact in $H^2(-, \mathbb{Q}) \cap H^{1,1}(-)$.

Moreover, the class $\alpha_X \in H^2(X, \mathbb{Q}) \simeq \mathbb{Q}$, being non-zero, is automatically good. In fact, it is ample if and only if the diffeomorphism $\Phi : Y \simeq X$ preserves the canonical orientations of the complex analytic surfaces.

It follows that the $\alpha$-classes $\alpha_X, \alpha_{X^n}, \alpha_{X^{(n)}}$ and $\alpha_{X^{(\nu)}}$ are good. Since $\alpha_{X^{(\nu)}}$ is good, so is its restriction to the fibers of $X^{(\nu)} \to \mathbb{C}^{(\nu)}$. The fibers of this map over points in the dense open stratum of $\mathbb{C}^{(\nu)}$ consisting of multiplicity-free cycles are isomorphic to the product $E^{l(\nu)}$. Over the remaining points, the fibers are isomorphic to finite quotients $E^{l(\nu)}/G$, where the $G$ are suitable subgroups of $\mathfrak{S}_\nu$ (see section 2.2).

On the other hand, if $n \geq 2$, then $\alpha_{X^{[n]}}$ is not good: being a pull-back from $X^{(n)}$, it is trivial on the positive dimensional projective fibers of the Hilbert Chow birational map $\pi_n : X^{[n]} \to X^{(n)}$, a fact that prohibits goodness.

In view of the identifications of Lemma 3.2.2 and of the fact that $\alpha_{X^{(\nu)}}$ and its restriction to $E^{(\nu)}$ are good, we have that the classical hard Lefschetz isomorphisms for the nonsingular projective $E^{(\nu)}$ of dimension $l(\nu)$ reads as follows

$$\alpha^j_{X^{(\nu)}} : H^{l(\nu)-j}_\nu\left(X^{[n]}, \mathbb{Q}\right) = H^{l(\nu)-j}\left(E^{(\nu)}, \mathbb{Q}\right) \xrightarrow{\simeq} H^{l(\nu)+j}\left(E^{(\nu)}, \mathbb{Q}\right) = H^{l(\nu)+j}_\nu\left(X^{[n]}, \mathbb{Q}\right). \tag{18}$$

**Remark 4.3.1** Since $\alpha_{X^{[n]}}$ is a pull-back from $X^{(n)}$, its action via cup product on

$$\pi_{n*}\mathbb{Q}_{X^{[n]}}[2n]$$

is diagonal with respect to the decomposition into $\nu$-summands (1). Moreover, the induced action on each $\nu$-summand is the action via cup product with $\alpha_{X^{(\nu)}}$. The same holds after taking cohomology.

The hard Lefschetz isomorphisms (18) express a property of this cup product action with $\alpha_{X^{[n]}}$ in cohomology. In fact, (18) is the reflection in cohomology of the fact that the conclusion of the relative hard Lefschetz theorem ([1], Theorem 5.4.10; see also [10]) holds for the map $h_n : X^{[n]} \to \mathbb{C}^{(n)}$ and for the cup-product action with $\alpha_{X^{[n]}}$, i.e. that we have isomorphisms

$$\alpha^j_{X^{[n]}} : {}^p\mathcal{H}^{-j}(h_{n*}\mathbb{Q}[2n]) \xrightarrow{\simeq} {}^p\mathcal{H}^j(h_{n*}\mathbb{Q}[2n]), \tag{19}$$

where, in view of (10), the perverse cohomology sheaves are

$${}^p\mathcal{H}^j(h_{n*}\mathbb{Q}[2n]) = \bigoplus_{i-l(\nu)=j} R^i_\nu[l(\nu)].$$

In fact, the map of perverse sheaves (19) is defined simply because $\alpha_{X^{[n]}} \in H^2(X^{[n]}, \mathbb{Q})$; see [10]), §4.4. By using the identifications of Lemma 3.2.2, we deduce that the map (19) is an isomorphism: in fact, in view of the isomorphisms (18), it is an isomorphism on the stalks of the respective cohomology sheaves.

Recall that $\alpha_{X^{[n]}}$ is not good for $n \geq 2$, i.e. it is neither "positive", nor "negative" on the fibers of $h_n$, so that the relative hard Lefschetz theorem does not apply in this context, yet we have (19). This situation is similar to the one of the paper [7], where the notion of lef line bundles has been introduced and where it is proved that it is strongly linked to the hard Lefschetz theorem. The relation with the present situation is that, up to sign, $\alpha_{X^{[n]}}$ is not ample on the fibers of $h_n$, but it is lef.

Recalling the expression (16) for the perverse Leray filtration and Remark 4.3.1, a direct calculation using the hard Lefschetz isomorphisms (18) and Theorem 4.1.1 implies the following result, which answers Question 4.2.1.

**Theorem 4.3.2** *Under the identification $\phi_{[n]} : H^*(X^{[n]}, \mathbb{Q}) = H^*(Y^{[n]}, \mathbb{Q})$, the CHL (17) becomes the (relative) hard Lefschetz (19).*

We conclude this section by remarking that the splitting (10) of $h_{n*}\mathbb{Q}_{X^{[n]}}$ has a remarkable property. Deligne's paper [14] implies that once we have the relative hard Lefschetz-type isomorphisms (19), we can construct three a priori distinct isomorphisms between the l.h.s and the r.h.s of (10). Each one of these three splittings is characterized by a certain property of the matrices that express the action of the cup product operations

$$\alpha^k_{X^{[n]}} : h_{n*}\mathbb{Q}_{X^{[n]}} \to h_{n*}\mathbb{Q}_{X^{[n]}}[2k]$$

with respect to the splitting; see [14], p.118 for the definition of this matrix, Proposition 2.7 for the first splitting, section 3.1 for the second, and Proposition 3.5 for the third. In general, these three splittings differ from each other, e.g. in the case of the projectivization of a vector bundle with non trivial Chern classes, projecting over the base.

In our situation, there is the fourth splitting (10). The remarkable fact is that, in view of Remark 4.3.1, it is a matter of routine to verify that the four splittings coincide.

## 4.4 Speculating on where to find the exchange of filtrations

The example treated in this paper and the one considered in [4] have some properties in common which lead us to conjecture that the exchange of filtration occur for a certain class of varieties and maps. Let us recall the main theorem of [4]:

Consider the moduli space of semistable Higgs bundles $\mathcal{M}_{\mathrm{Dol}}$ parametrizing stable rank 2 Higgs bundles $(E,\phi)$ of degree 1 on a fixed nonsingular projective curve $C$ of genus $g\geq 2$. There is the Hitchin proper and flat map $h:\mathcal{M}_{\mathrm{Dol}}\longrightarrow\mathbb{C}^{4g-3}$, which gives rise to the perverse Leray filtration $\mathscr{P}_{\mathcal{M}_{\mathrm{Dol}}}$. By the *non-Abelian Hodge theorem*, $\mathcal{M}_{\mathrm{Dol}}$ is naturally diffeomorphic to the twisted character variety

$$\mathcal{M}_{\mathrm{B}}:=\left\{A_1,B_1,\dots,A_g,B_g\in\mathrm{GL}_2(\mathbb{C})\mid A_1^{-1}B_1^{-1}A_1B_1\dots A_g^{-1}B_g^{-1}A_gB_g=-\mathrm{I}\right\}/\mathrm{GL}_2(\mathbb{C})$$

where the quotient is taken in the sense of invariant theory. The twisted character variety $\mathcal{M}_{\mathrm{B}}$ carries a natural structure of nonsingular complex affine variety, with Hodge structure of Hodge-Tate type, with a natural splitting.

In complete analogy with Theorem 4.1.1, we have the main result in [4], Theorem 4.2.9

**Theorem 4.4.1** *In terms of the isomorphism $H^*(\mathcal{M}_{\mathrm{B}})\xrightarrow{\simeq}H^*(\mathcal{M}_{\mathrm{Dol}})$ induced by the diffeomorphism $\mathcal{M}_{\mathrm{B}}\xrightarrow{\simeq}\mathcal{M}_{\mathrm{Dol}}$ stemming from the non-Abelian Hodge theorem, we have*

$$\mathscr{W}_{\mathcal{M}_B,2k}\,H^*(\mathcal{M}_{\mathrm{B}})=\mathscr{W}_{\mathcal{M}_B,2k+1}\,H^*(\mathcal{M}_{\mathrm{B}})=\mathscr{P}_{\mathcal{M}_{\mathrm{Dol}},k}\,H^*(\mathcal{M}_{\mathrm{Dol}}).$$

The varieties $\mathcal{M}_{\mathrm{Dol}}$ and $X^{[n]}$ belong to the following class of varieties $Z$:

1. $Z$ is a quasi-projective nonsingular variety of even dimension $2m$ endowed with a holomorphic symplectic structure $\omega\in H^0(Z;\Lambda^2T^*Z)$ and with a $\mathbb{C}^*$-action $\phi:\mathbb{C}^*\times Z\to Z$, such that for $\phi_\lambda^*\omega=\lambda\omega$ for $\lambda\in\mathbb{C}^*$ .

2. The ring $\Gamma(Z,\mathcal{O}_Z)$ is finitely generated and the affine reduction map

   $$h_Z:Z\longrightarrow A=\operatorname{Spec}\Gamma(Z,\mathcal{O}_Z)$$

   is proper with fibres of dimension $m$.

3. The induced action on $A$ has a unique fixed point $o$ such that $\lim_{t\to 0}t\,y=o$ for all $y\in A$.

Let us note that, under these hypotheses, the Hodge structure on the cohomology groups $H^d(Z,\mathbb{Q})$ is pure of weight $d$: the inclusion $h^{-1}(o)\subset Z$ induces an isomorphism

$$H^d(Z,\mathbb{Q})\simeq H^d(h^{-1}(o),\mathbb{Q})$$

of MHS; since $Z$ is nonsingular, the weight inequalities ([13] Theorem 8.2.4, iii. and iv.) imply the purity of $H^*(Z,\mathbb{Q})$.

Additionally we see that if $f$ and $g$ are functions in $\Gamma(A,\mathcal{O}_A)\cong\Gamma(Z,\mathcal{O}_Z)$ then we can write them as $f=\sum_{i>0}f_i$ and $g=\sum_{i>0}f_i$ and $g=\sum_{i>0}g_i$ such that $\phi_\lambda^*(f_i)=\lambda^if_i$ and $\phi_\lambda^*(g_i)=\lambda^ig_i$. Then the Poisson bracket satisfies

$$\{f,g\}=\sum_{i,j>0}\{f_i,g_j\}=\sum_{i,j>0}\frac{1}{\lambda}\{\phi_\lambda^*f_i,\phi_\lambda^*g_j\}=\sum_{i,j>0}\lambda^{i+j-1}\{f_i,g_j\}.$$

Because $\lambda^k h = h$ for $k > 0$ and generic $\lambda \in \mathbb{C}^*$ only for the zero function, thus we can conclude $\{f, g\} = 0$. Thus $h_Z$ is a completely integrable system.

The two examples given in this paper and in [4] lead us to speculate whether it is possible to associate with every variety $Z$ satisfying the three assumptions above another variety $\widetilde{Z}$ such that:

1. $\widetilde{Z}$ is a quasi projective nonsingular variety endowed with a holomorphic symplectic structure.
2. The affine reduction map $h_{\widetilde{Z}} : \widetilde{Z} \longrightarrow \operatorname{Spec} \Gamma(\widetilde{Z}, \mathcal{O}_{\widetilde{Z}})$ is birational (hence semismall in view of [22], Lemma 2.11).
3. There is a natural isomorphism $\phi : H^*(Z, \mathbb{Q}) \simeq H^*(\widetilde{Z}, \mathbb{Q})$.
4. The cohomology groups $H^*(\widetilde{Z}, \mathbb{Q})$ have a Hodge structure of split Hodge-Tate type.
5. Under the isomorphism $\phi$, the perverse filtration on $Z$ associated with the map $h$ corresponds to the halved weight filtration on $H^*(\widetilde{Z}, \mathbb{Q})$: a class of perversity $p$ on $Z$ would correspond to a class of type $(p, p)$ on $\widetilde{Z}$.

Let us remark that, if the above were true, then the Hodge structure of $\widetilde{Z}$ cannot be pure. In fact, in view of the relative hard Lefschetz theorem, the class $\alpha \in H^2(Z, \mathbb{Q})$ of any $h$-ample class on $Z$ has necessarily perversity 2. It would then follows that $\phi(\alpha) \in H^2(\widetilde{Z}, \mathbb{Q})$ would have type $(2, 2)$. In view of the conditions we have imposed on the affine reduction maps of the two varieties, i.e. the fact that $h_Z$ is a fibration with middle dimensional fibers and $h_{\widetilde{Z}}$ is semismall, we like to think that $Z$ is "as complete as possible," whereas $\widetilde{Z}$ is "as affine as possible."

At present, we do not know how to attack such a question and we still do not know how to formulate a principle that would justify the exchange of filtrations.

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Authors' addresses:

Mark Andrea A. de Cataldo, Department of Mathematics, SUNY at Stony Brook, Stony Brook, NY 11794, USA. e-mail: *mde@math.sunysb.edu*

Tamás Hausel, Section de Mathématiques, École Polytechnique Fédérale de Lausanne, Station 8, CH-1015 Lausanne, SWITZERLAND. e-mail: *tamas.hausel@epfl.ch*

Luca Migliorini, Dipartimento di Matematica, Università di Bologna, Piazza di Porta S. Donato 5, 40126 Bologna, ITALY. e-mail: *migliori@dm.unibo.it*

**Journal *of* Singularities**
**Volume 7 (2013), 39-60**

received 30 December 2011
in revised form 2 March 2013
DOI: 10.5427/jsing.2013.7d

# THE MULTIPLICITY POLAR THEOREM, COLLECTIONS OF 1-FORMS AND CHERN NUMBERS

TERENCE GAFFNEY AND NIVALDO G. GRULHA JR.

Abstract. In this work, we show how the Multiplicity Polar Theorem can be used to calculate Chern numbers for collections of 1-forms.

## 1. Introduction

Given a space with singularities, and a geometric invariant defined for smooth spaces, it is interesting to see whether or not the invariant is well-defined for the singular space, and, if it is, what are the contributions to the invariant from the singularities.

In a series of papers, Ebeling and Gusein-Zade have discussed the meaning of such invariants as the index of a differential form [10], various notions of the index of a vector field [7], and the Chern numbers, and have described the contributions from the singularities in some cases. In [10], they calculated the radial index of a 1-form on a complete intersection singularity. In [17], it is shown that, in the case of a differential 1-form with an isolated singularity on $X \subset \mathbb{C}^n$, where $X$ is a complex analytic space, that the radial index can be computed using the multiplicity of a pair of modules. The computation of [17] amounts to computing the intersection multiplicity of the graph of the one form $\omega$, which is a subset of the (unprojectivised) conormal bundle of $\mathbb{C}^n$ and the cotangent space of $X$. In contrast to [10], the calculation is valid for any equidimensional space.

It is clear that this springs from earlier work for vector fields and characteristic classes on singular spaces by Schwartz and Brasselet, ([4, 25]), MacPherson ([24]) Seade and others (see for example [2, 27]). The case of a 1-form is analogous to the case of vector fields, and the indices involved concern the Euler characteristic of the singular variety. This can be regarded as a particular Chern number, and the work of Ebeling and Gusein-Zade for collections of 1-forms extends these notions for other Chern numbers.

In [8], Ebeling and Gusein-Zade developed the notion of the Chern number of a singular space using collections of differential 1-forms. Their numbers are well-defined for any equidimensional reduced complex analytic germ, but they only compute the number for ICIS singularities. Their Chern number is again an intersection number. As in the earlier work, the intersection takes place at the level of conormal spaces; they call the points in $X$ which are the projection of the points of intersection, *special points*.

In the case that we have just one 1-form, the Chern number is the Euler obstruction of the differential form ([7], p. 17). This is related to the Euler obstruction of a set and the Euler obstruction of a function as defined by Brasselet, Massey, Parameswaran and Seade in [3]. In [11], the definition of the Euler obstruction of a function was adapted to the case of 1-form, the Euler obstruction of a function was studied by several authors, in this direction we have for example the papers [21],[27],[6],[17]. In [1], the authors determine relations between the local Euler obstruction of an analytic map $f$ defined in [22] and the Chern number of a convenient collection of 1-forms associated to $f$.

In this work, we use the multiplicity polar theorem to calculate Chern numbers for any equidimensional reduced complex analytic germ. This extends the earlier work of [17]. For the Chern number problem, one must work with a set of collections of differential 1-forms, and calculate the order of the point where all of the collections are linearly dependent. Since we want to calculate the number of points at which the forms are linearly dependent after a generic perturbation, this is again a problem involving modules. Because we have a collection of forms we have a collection of modules, so the problem is like a problem in intersection theory, except the spaces are defined by modules not by ideals.

The computation of the Chern numbers is an example of a problem where the underlying vector bundle, which is the tangent bundle in the Chern case, is not defined at every point of $X$. The set $X$ must be modified; so we pass to the Nash modification of $X$ where the tangent bundle of $X$ is defined in order to understand the problem fully. This process of modifying a space to fill in points where a bundle $\xi$ fails to be defined works in general, and our process of calculating intersection numbers also extends. We outline this in the last section. This suggests the easier problem of calculating Chern numbers when the tangent bundle is well-defined at all points, and more generally, intersection numbers of modules. We take this up in section two. The main themes of section four appear in this material.

Also in Section 2, we recall some basic ideas about the theory of integral closure of modules and the statement of the multiplicity polar theorem.

In Section 3, we recall how Ebeling and Gusein-Zade develop the notion of Chern number in their paper.

In Section 4, we introduce the notion of a special point for a collection. Roughly speaking, a point $p \in X$ is called a special point of the collection $\{\omega_j^{(i)}\}$ of 1-forms on the variety $X$ if there exists a point in the fiber of the Nash modification over $p$ such that the restriction of the 1-forms $\omega_1^{(i)}, \cdots, \omega_{d-k_i+1}^{(i)}$ to the point are linearly dependent for each $i = 1, \cdots, s$. We next see how special points can be viewed as intersections, and, hence, have an associated intersection number, if isolated. We then begin to solve the "module intersection theoretic problem" for the computation of the Chern numbers described above. We prove a "Gysin" type theorem, (Proposition 4.7) that is, under suitable genericity hypotheses, we can do our calculations on a single space which represents the intersection of all but the last spaces defined by our collection and use the last module associated with the collection restricted to this space for our computations. We also prove a genericity result (Proposition 4.10) which shows that by deforming just the last collection of differential forms, we can ensure the set of collections is generic in an appropriate sense.

We begin Section 5 by recalling a result of Ebeling and Gusein-Zade (Proposition 5.1) relating Chern numbers and special points. In Proposition 5.4 we describe in integral closure terms what it means for $x \in X$ not to be a special point for a collection of forms. After gearing up to apply the multiplicity polar theorem.

In Section 6 we show, in Theorem 6.1, that deforming our last collection allows us to split the contribution of the Chern number from an isolated special point into the multiplicity of a pair of modules and the intersection number of the new collection. Using this as the inductive step, we can write the contribution to the Chern number as a sum of multiplicities of pairs (Corollary 6.2).

We next show that if $X$ is an ICIS, then our formula agrees with that of Ebeling and Gusein-Zade (Corollary 6.3 and the discussion afterwards.)

We close by indicating how our results can be generalized to the case of a bundle $E^k$ defined on a Zariski open, everywhere-dense subset $U$ of an analytic space $X$, $E^k$ a sub-bundle with $k$-dimensional fiber of a bundle, $F^l$, where $F^l$ is defined everywhere.

The authors thank Steven Kleiman for helpful conversations on the connection between their work and the intersection multiplicity of Serre.

## 2. Integral closure of modules

Let $(X, x)$ be a germ of a complex analytic space, $X$ a small representative of the germ, and let $\mathcal{O}_X$ denote the structure sheaf on a complex analytic space $X$. The study of what it means for a collection of 1-forms to have a special point on a singular space depends on the behavior of limiting tangent hyperplanes. The key tool for studying these limits is the theory of integral closure of modules, which we now introduce.

**Definition 2.1.** Suppose $(X, x)$ is the germ of a complex analytic space, $M$ a submodule of $\mathcal{O}^p_{X,x}$. Then $h \in \mathcal{O}^p_{X,x}$ is in the integral closure of $M$, denoted $\overline{M}$, if for all analytic $\phi : (\mathbb{C}, 0) \to (X, x)$, $h \circ \phi \in (\phi^* M)\mathcal{O}_1$. If $M$ is a submodule of $N$ and $\overline{M} = \overline{N}$, we say that $M$ is a reduction of $N$.

To check the definition, it suffices to check along a finite number of curves whose generic point is in the Zariski open subset of $X$ along which $M$ has maximal rank. (Cf. [14])

If a module $M$ has finite colength in $\mathcal{O}^p_{X,x}$, it is possible to attach a number to the module, its Buchsbaum-Rim multiplicity, $e(M, \mathcal{O}^p_{X,x})$. We can also define the multiplicity $e(M, N)$ of a pair of modules $M \subset N$, $M$ of finite colength in $N$, as well, even if $N$ does not have finite colength in $\mathcal{O}^p_X$.

We recall how to construct the multiplicity of a pair of modules using the approach of Kleiman and Thorup [23]. Given a submodule $M$ of a free $\mathcal{O}_{X^d}$ module $F$ of rank $p$, we can associate a subalgebra $\mathcal{R}(M)$ of the symmetric $\mathcal{O}_{X^d}$ algebra on $p$ generators. This is known as the Rees algebra of $M$. If $(m_1, \cdots, m_p)$ is an element of $M$, then $\sum m_i T_i$ is the corresponding element of $\mathcal{R}(M)$. Then $\operatorname{Projan}(\mathcal{R}(M))$, the projective analytic spectrum of $\mathcal{R}(M)$, is the closure of the projectivised row spaces of $M$ at points where the rank of a matrix of generators of $M$ is maximal. Denote the projection to $X^d$ by $c$. If $M$ is a submodule of $N$ or $h$ is a section of $N$, then $h$ and $M$ generate ideals on $\operatorname{Projan}\mathcal{R}(N)$; denote them by $\rho(h)$ and $\rho(\mathcal{M})$. If we can express $h$ in terms of a set of generators $\{n_i\}$ of $N$ as $\sum g_i n_i$, then in the chart in which $T_1 \neq 0$, we can express a generator of $\rho(h)$ by $\sum g_i T_i/T_1$. Having defined the ideal sheaf $\rho(\mathcal{M})$, we blow it up.

On the blow up $B_{\rho(\mathcal{M})}(\operatorname{Projan}\mathcal{R}(N))$, we have two tautological bundles. One is the pullback of the bundle on $\operatorname{Projan}\mathcal{R}(N)$. The other comes from $\operatorname{Projan}\mathcal{R}(M)$. Denote the corresponding Chern classes by $c_M$ and $c_N$, and denote the exceptional divisor by $D_{M,N}$. Suppose the generic rank of $N$ (and hence of $M$) is $g$.

Then the multiplicity of a pair of modules $M, N$ is:

$$e(M, N) = \sum_{j=0}^{d+g-2} \int D_{M,N} \cdot c_M^{d+g-2-j} \cdot c_N^j.$$

Kleiman and Thorup show that this multiplicity is well-defined at $x \in X$ as long as $\overline{M} = \overline{N}$ on a deleted neighborhood of $x$. This condition implies that $D_{M,N}$ lies in the fiber over $x$, hence is compact. Notice that when $N = F$ and $M$ has finite colength in $F$ then $e(M, N)$ is the Buchsbaum-Rim multiplicity $e(M, \mathcal{O}^p_{X,x})$. There is a fundamental result due to Kleiman and Thorup, the principle of additivity [23], which states that given a sequence of $\mathcal{O}_{X,x}$-modules $M \subset N \subset P$ such that the multiplicity of the pairs is well-defined, then

$$e(M, P) = e(M, N) + e(N, P).$$

Also if $\overline{M} = \overline{N}$ then $e(M, N) = 0$ and the converse also holds if $X$ is equidimensional. Combining these two results we get that if $\overline{M} = \overline{N}$ then $e(M, N) = e(N, P)$. These results will be used in Section 5.

In studying the geometry of singular spaces, it is natural to study pairs of modules. In dealing with non-isolated singularities, the modules that describe the geometry have non-finite colength, so their multiplicity is not defined. Instead, it is possible to define a decreasing sequence of modules, each with finite colength inside its predecessor, when restricted to a suitable complementary plane. Each pair controls the geometry in a particular codimension.

We also need the notion of the polar varieties of $M$. The *polar variety of codimension* $k$ of $M$ in $X$, denoted $\Gamma_k(M)$, is constructed by intersecting $\operatorname{Projan}\mathcal{R}(M)$ with $X \times H_{g+k-1}$ where $H_{g+k-1}$ is a general plane of codimension $g+k-1$, then projecting to $X$.

Setup: We suppose we have families of modules $M \subset N$, $M$ and $N$ submodules of a free module $F$ of rank $p$ on an equidimensional family of spaces with equidimensional fibers $\mathcal{X}^{d+k}$, $\mathcal{X}$ a family over a smooth base $Y^k$. We assume that the generic rank of $M$, $N$ is $g \le p$. Let $P(M)$ denote $\operatorname{Projan}\mathcal{R}(M)$, $\pi_M$ the projection to $\mathcal{X}$.

We will be interested in computing, as we move from the special point 0 to a generic point, the change in the multiplicity of the pair $(M, N)$, denoted $\Delta(e(M, N))$. We will assume that the integral closures of $M$ and $N$ agree off a set $C$ of dimension $k$ which is finite over $Y$, and assume we are working on a sufficiently small neighborhood of the origin, so that every component of $C$ contains the origin in its closure. Then $e(M, N, y)$ is the sum of the multiplicities of the pair at all points in the fiber of $C$ over $y$, and $\Delta(e(M, N))$ is the change in this number from 0 to a generic value of $y$. If we have a set $S$ which is finite over $Y$, then we can project $S$ to $Y$, and the degree of the branched cover at 0 is $\operatorname{mult}_y S$. (Of course, this is just the number of points in the fiber of $S$ over our generic $y$.)

Let $C(M)$ denote the locus of points where $M$ is not free, *i.e.*, the points where the rank of $M$ is less than $g$, $C(\operatorname{Projan}\mathcal{R}(M))$ its inverse image under $\pi_M$.

We can now state the Multiplicity Polar Theorem. The proof in the ideal case appears in [15]; the general proof appears in [16].

**Theorem 2.2.** *(Multiplicity Polar Theorem) Suppose in the above setup we have that* $\overline{M} = \overline{N}$ *off a set* $C$ *of dimension* $k$ *which is finite over* $Y$. *Suppose further that*

$$C(\operatorname{Projan}\mathcal{R}(M))(0) = C(\operatorname{Projan}\mathcal{R}(M(0))),$$

*except possibly at the points which project to* $0 \in \mathcal{X}(0)$. *Then, for* $y$ *a generic point of* $Y$,

$$\Delta(e(M, N)) = \operatorname{mult}_y \Gamma_d(M) - \operatorname{mult}_y \Gamma_d(N),$$

*where* $\mathcal{X}(0)$ *is the fiber over* 0 *of the family* $\mathcal{X}^{d+k}$, $M(0)$ *is the restriction of the module* $M$ *to* $\mathcal{X}(0)$, *and* $C(\operatorname{Projan}\mathcal{R}(M))(0)$ *is the fiber of* $C(\operatorname{Projan}\mathcal{R}(M))$ *over* 0.

Now, we show how this machinery can be applied to a module intersection problem. Suppose we are given modules $M_1 \subset F_1$ and $M_2 \subset F_2$, $F_i$ free $\mathcal{O}_{X^d,x}$ modules of rank $p_i$, $M_i$ generated by $n_i$ generators. Suppose $C(M_i)$ is equidimensional, the codimension of $C(M_i)$ is $n_i - p_i + 1$, and the sum of the codimensions is $d$, $C(M_i)$ equidimensional. If we deform the generators of $M_i$, how many points do we expect to see where both modules have less than maximal rank?

We can take this number as the intersection number of the two modules.

As further justification, we relate this number to an intersection number at $x$. Let $\mathcal{M}(p, q)$, $p \le q$, be the space of $p \times q$ matrices with complex entries and let $D_{p,q}$ be the subspace of $\mathcal{M}(p, q)$ consisting of matrices of rank less than $p$. The subset $D_{p,q}$ is an irreducible subvariety of $\mathcal{M}(p, q)$ of codimension $q - p + 1$.

Fix a matrix of generators $[M_i]$ of $M_i$.

Then each matrix $[M_i]$ defines a section $\Gamma_{M_i}$ of $\mathbb{C}^n \times \mathcal{M}(p_i, n_i)$ in the obvious way; the pair defines a section $\Gamma_{M_1,M_2}$ of $\mathbb{C}^n \times \mathcal{M}(p_1, n_1) \times \mathcal{M}(p_2, n_2)$. We will assume that

$$X \times D_{p_1,n_1} \times D_{p_2,n_2} \cap \operatorname{Im}(\Gamma_{M_1,M_2})$$

is isolated and lies over $x$. The intersection number of $X \times D_{p_1,n_1} \times D_{p_2,n_2}$ and $\operatorname{Im}(\Gamma_{M_1,M_2})$ at $(x, \Gamma_{M_1,M_2}(x))$ is the number we want to calculate. In this paper we will abbreviate "Zariski open set" by "Z-open set".

**Theorem 2.3.** *Suppose each of the sections* $\Gamma_{M_i}$ *is transverse to* $X^d \times D_{p_i,q_i}$ *on a Z-open set* $U_i$ *such that* $U_i \cap C(M_i)$ *is Z-open and dense in* $C(M_i)$. *Then the intersection number of* $X \times D_{p_1,n_1} \times D_{p_2,n_2}$ *and* $\operatorname{Im}(\Gamma_{M_1,M_2})$ *at* $(x, \Gamma_{M_1,M_2}(x))$ *is* $e(M_1, \mathcal{O}_{C(M_2),x}) = e(M_2, \mathcal{O}_{C(M_1),x})$.

*Proof.* $X$ may be singular, so we assume $X$ is stratified with the canonical Whitney stratification [29]. Then the transversality of $\Gamma_{M_i}$ means transversality to each $S_j \times D_{p_i,q_i}$, $S_j$ a stratum. This ensures that the generic point of each component of $C(M_i)$ is a smooth point of $X$. It also ensures that the codimension of $\mathcal{O}_{C(M_i),x}$ is $n_i - p_i + 1$. Since $X \times D_{p_1,n_1} \times D_{p_2,n_2} \cap \mathrm{Im}(\Gamma_{M_1,M_2})$ is isolated and lies over $x$, the sum of the codimensions of the $C(M_i)$ is $d$.

Let us show that, at $(x, \Gamma_{M_1,M_2}(x))$, the intersection number of $X \times D_{p_1,n_1} \times D_{p_2,n_2}$ and $\mathrm{Im}(\Gamma_{M_1,M_2})$ is $e(M_1, \mathcal{O}_{C(M_2),x})$. The proof of the other half of the inequality is parallel.

Note that the number of generators of $M_1$ as a $\mathcal{O}_{C(M_2),x}$ module is

$$n_1 \;=\; (n_1 - p_1 + 1) + p_1 - 1 \;=\; d - (n_2 - p_2 + 1) + p_1 - 1 \;=\; \dim C(M_2) + p_1 - 1.$$

Then, by Theorem 1.2 of [18], we can find a perturbation of $[M_1]$ by a matrix of generic constants such that the section induced by the new matrix, $[\widetilde{M_1}]$, of $C(M_2) \times \mathcal{M}(p_1, n_1)$ is transverse to $C(M_2) \times D_{p,q}$, and the finite number of points at which $[\widetilde{M_1}]$ has less than maximal rank occur at smooth points of $C(M_2)$ and there are $e(M_1, \mathcal{O}_{C(M_2),x})$ of them. In particular, $x$ is no longer a point where both sections have less than maximal rank. (It is not hard to see from the proof of Theorem 2.2 that in fact these lie in the Z-open dense subset of $C(M_2)$ on which the section $\Gamma_{M_2}$ is transverse to $X \times D_{p_2,n_2}$.) The transversality conditions on $\Gamma_{M_2}$ and $\Gamma_{\widetilde{M_1}}$ imply that the section $\Gamma_{\widetilde{M_1},M_2}$ is transverse to $X \times D_{p_1,n_1} \times D_{p_2,n_2}$ at all points of intersection. The total number of such points counted with multiplicity is the intersection number of $X \times D_{p_1,n_1} \times D_{p_2,n_2}$ and $\mathrm{Im}(\Gamma_{M_1,M_2})$ at $(x, \Gamma_{M_1,M_2}(x))$; the transversality statement implies each point occurs with multiplicity 1.

□

**Corollary 2.4.** Suppose $\mathcal{O}_{X,x}$ is Cohen-Macaulay, then the intersection number of $X \times D_{p_1,n_1} \times D_{p_2,n_2}$ and $\mathrm{Im}(\Gamma_{M_1,M_2})$ at $(x, \Gamma_{M_1,M_2}(x))$ is the colength of the ideal generated by the maximal minors of $[M_i]$, $i = 1, 2$.

*Proof.* Since $\mathcal{O}_{X,x}$ is Cohen-Macaulay and the structure on $\mathcal{O}_{C(M_2),x}$ given by the minors of $[M_2]$ is generically reduced, it is reduced and $\mathcal{O}_{C(M_2),x}$ is Cohen-Macaulay. Then $e(M_1, \mathcal{O}_{C(M_2),x})$ is the colength of the ideal of minors of $[M_1]$ in $\mathcal{O}_{C(M_2),x}$ which gives the result. □

Looking at the proof of the above theorem, in applying the technique of the proof to geometric problems, we see that we need a description of the desired quantity as an intersection number, and a theorem about the transversality of a deformation of $[M_1]$ by a matrix of generic constants.

If $\xi_1$ and $\xi_2$ are vector bundles, we may wish to calculate geometric invariants related to sections of the bundles. If the desired invariant is supported at a point, then locally the sets of sections of our vector bundles are free modules, and we can look at the submodules generated by the given sets of sections. Then the last theorem can be used to calculate the contribution to the invariant at a point where the sections fail to be generic.

In the next couple of sections we will look at a more difficult case, one in which the vector bundle may only be defined on a Z-open subset of $X$. This will involve modifying $X$ to produce a new space on which the bundle is defined, then taking into account the fiber of the modification over $x$.

Before developing these ideas, we mention the connection between the ideas of this section and the intersection multiplicity defined by Serre ([28]). Given modules $M_1 \subset F_1$ and $M_2 \subset F_2$, $F_i$ free $\mathcal{O}_{X^d,x}$ modules of rank $p_i$ as above, Serre's intersection number is the alternating sum of the lengths of the $\mathrm{Tor}^i(F^{p_1}/M_1, F^{p_2}/M_2)$.

**Corollary 2.5.** Under the hypotheses of Theorem 2.3, Serre's intersection number is the same as $e(M_1, \mathcal{O}_{C(M_2),x}) = e(M_2, \mathcal{O}_{C(M_1),x})$.

*Proof.* This holds because under small deformations of the $M_i$ the intersection number does not change; but then, by a small deformation, we can reduce to the ideal case (*i.e.*, the modules have rank one less than maximum at common points where they have less than maximal rank). Then, by Theorem 2.3, the intersection number counts the same points as $e(M_1, \mathcal{O}_{C(M_2),x})$. □

For the case where $\mathcal{O}_{X,x}$ is Cohen-Macaulay more can be said. The following result extends some theorems of Buchsbaum and Rim ([5] 2.4 p.207, 4.3 and 4.5 p.223).

**Corollary 2.6.** Suppose $\mathcal{O}_{X,x}$ is Cohen-Macaulay, then the colength of the ideal generated by the maximal minors of $[M_i]$, $i = 1, 2$, is the length of $F^{p_1}/M_1 \otimes F^{p_2}/M_2$.

*Proof.* We claim that the complex used to compute the $\mathrm{Tor}^i$ is exact, so Serre's intersection number is just the length of $F^{p_1}/M_1 \otimes F^{p_2}/M_2$. To see this, consider the complex for ${O_n}^{p_1}/M_1$. At points where $M_1$ has maximal rank, this complex is exact. Further all the maps have maximal rank. These assertions follow because ${O_n}^{p_1}/M_1 = 0$, and the free resolution of 0 is a trivial complex (lemma 20.1 p491 [13]). Now tensor with ${O_n}^{p_2}/M_2$, and consider the resulting complex. At points where $M_2$ has maximal rank we are tensoring with 0, so the complex is exact. At points, different from the origin, where $M_2$ has less than maximal rank, the complex remains exact, as it is a trivial resolution and the torsion terms are zero as they are independent of the resolution. So the origin is the only point where the complex is not exact; but by the acyclicity lemma, (cf [13] p498) the complex must be exact there as well. The Corollary now follows from the pervious one, because since $X$ is Cohen-Macaulay, and $C(M_2)$ has the right dimension, its ring is Cohen-Macaulay as well, so

$$e(M_1, \mathcal{O}_{C(M_2),x})$$

is just the colength of the ideal generated by the maximal minors of $[M_i]$, $i = 1, 2$. □

## 3. Collections of 1-forms

W. Ebeling and S. M. Gusein-Zade studied indices for collections of 1-forms [7, 8], in this section we will recall some ideas and notation from their papers about these concepts.

If $P$ is a complex analytic manifold of dimension $n$, then its Euler characteristic $\chi(P)$ is the characteristic number

$$\langle c_n(TP), [P]\rangle = (-1)^n \langle c_n(T^*P), [P]\rangle,$$

where $TP$ is the tangent bundle of the manifold $P$, $T^*P$ is the dual bundle, and $c_n$ is the corresponding Chern class and $[P]$ the fundamental class of $P$.

The top Chern class of a vector bundle is the first obstruction to the existence of a non-vanishing section. Other Chern classes are obstructions to the existence of a linearly independent collection of sections. There, instead of 1-forms on a complex variety, we consider collections of 1-forms. Further, to calculate intersections of Chern Classes and hence Chern numbers, we will need collections of collections of 1-forms.

Let $\pi\colon E \to P$ be a complex analytic vector bundle of rank $m$ over a complex analytic manifold $P$ of dimension $n$. It is known that the $(2(n-k)$-dimensional) cycle Poincaré dual to the characteristic classe $c_k(E)$ $(k = 1, \cdots, m)$ is represented by the set of points of the manifold $P$ where $m - k + 1$ generic sections of the vector bundle $E$ are linearly dependent.

We continue to use the notation of section two: Let $\mathcal{M}(p, q)$, $p \le q$, be the space of $p \times q$ matrices with complex entries and let $D_{p,q}$ be the subspace of $\mathcal{M}(p, q)$ consisting of matrices of rank less than $p$. The subset $D_{p,q}$ is an irreducible subvariety of $\mathcal{M}(p, q)$ of codimension $q-p+1$. The complement $W_{p,q} = \mathcal{M}(p, q) \setminus D_{p,q}$ is the Stiefel manifold of $p$-frames in $\mathbb{C}^q$ . It is known that the Stiefel manifold $W_{p,q}$ is $2(q - p)$-connected and $H_{2(q-p)+1}(W_{p,q}) \cong \mathbb{Z}$.

We now develop the notation necessary to handle collections of collections of forms. For the rest of the paper, we will refer to these objects simply as collections.

Let $\mathbf{k} = (k_1, \cdots, k_s)$ be a sequence of positive integers with $\sum_{i=1}^{s} k_i = k$. Consider the space $\mathcal{M}_{m,\mathbf{k}} = \prod_{i=1}^{s} \mathcal{M}(m - k_i + 1, m)$ and the subvariety $D_{m,\mathbf{k}} = \prod_{i=1}^{s} D_{m-k_i+1,m}$ in it. The variety $D_{m,\mathbf{k}}$ consists of sets $\{A_i\}$ of $(m - k_i + 1 \times m)$ matrices such that $\mathrm{rk}\, A_i < m - k_i + 1$ for each $i = 1, \cdots, s$. Since $D_{m,\mathbf{k}}$ is irreducible of codimension $k$, its complement $W_{m,\mathbf{k}} = \mathcal{M}_{m,\mathbf{k}} \setminus D_{m,\mathbf{k}}$ is $(2k - 2)$-connected, $H_{2k-1}(W_{m,\mathbf{k}}) \cong \mathbb{Z}$, and there is a natural choice of a map from an oriented manifold of dimension $2k - 1$ to the manifold $W_{m,\mathbf{k}}$.

Let $(X^d, 0) \subset (\mathbb{C}^n, 0)$ be the germ of a purely $n$-dimensional reduced complex analytic variety at the origin. For $\mathbf{k} = \{k_i\}, i = 1, \cdots, s, j = 1, \cdots, d - k_i + 1$, let $\{\omega_j^{(i)}\}$ be a collection of germs of 1-forms on $(\mathbb{C}^n,0)$. (Note that $\{\omega_j^{(i)}\}$ for a fixed value of $i$, is itself a collection of $d-k_i+1$ 1-forms.) Let $\varepsilon>0$ be small enough so that there is a representative $X$ of the germ $(X,0)$ and representatives $\{\omega_j^{(i)}\}$ of the germs of 1-forms inside the ball $B_\varepsilon(0)\subset\mathbb{C}^n$.

The kind of points whose multiplicity we wish to compute is described in the next section.

## 4. Special Points

**Definition 4.1.** A point $p\in X$ is called a special point of the collection $\{\omega_j^{(i)}\}$ of 1-forms on the variety $X$ if there exists a sequence $p_m$ of points from the non-singular part $X_{\rm reg}$ of the variety $X$ such that the sequence $T_{p_m}X_{\rm reg}$ of the tangent spaces at the points $p_m$ has a limit $L$ (in $G(d,n)$) and the restriction of the 1-forms $\omega_1^{(i)},\cdots,\omega_{d-k_i+1}^{(i)}$ to the subspace $L\subset T_p\mathbb{C}^n$ are linearly dependent for each $i=1,\cdots,s$. The collection $\{\omega_j^{(i)}\}$ of 1-forms has an isolated special point on $(X,0)$ if it has no special point on $X$ in a punctured neighborhood of the origin.

Notice that we require each element in the collection to be linearly dependent when restricted to the same limit plane. Notice also, that if an element of the collection has less than maximal rank at a point, then it is linearly dependent on all planes passing through the point.

The framework of this section is a variation on the setting used in [7]. In developing the properties of special points, it is helpful to work on two levels, one of which is based on the Nash modification. The Nash modification comes into play because the tangent bundle of $X$ is not defined at singular points of $X$. However the Nash bundle is an extension of the tangent bundle on the modified space. We begin to describe this setting.

Let $\{\omega_j^{(i)}\}$ be a collection of germs of 1-forms on $(X,0)$ with an isolated special point at the origin. Let $\nu:\widetilde{X}\to X$ be the Nash transformation of the variety $X$, and $\widetilde{T}$ the Nash bundle. The collection of 1-forms $\{\omega_j^{(i)}\}$ gives rise to a section $\Gamma(\omega)$ of the bundle

$$\widetilde{\mathbb{T}}=\bigoplus_{i=1}^{s}\bigoplus_{j=1}^{n-k_i+1}\widetilde{T}^*_{i,j}$$

where $\widetilde{T}^*_{i,j}$ are copies of the dual Nash bundle $\widetilde{T}^*$ over the Nash transform $\widetilde{X}$ numbered by indices $i$ and $j$.

Let $\mathbb{D}\subset\widetilde{\mathbb{T}}$ be the set of pairs $(x,\{\alpha_j^{(i)}\})$ where $x\in\widetilde{X}$ and the collection $\{\alpha_j^{(i)}\}$ is such that $\alpha_1^{(i)},\cdots,\alpha_{n-k_i+1}^{(i)}$ are linearly dependent for each $i=1,\cdots,s$.

**Definition 4.2.** The local Chern obstruction, $\mathrm{Ch}_{X,0}\{\omega_j^{(i)}\}$, of the collections of germs of 1-forms $\{\omega_j^{(i)}\}$ on $(X,0)$, at the origin, is the obstruction to extend the section $\Gamma(\omega)$ of the fibre bundle $\widetilde{\mathbb{T}}\setminus\mathbb{D}\to\widetilde{X}$ from the preimage of a neighbourhood of the sphere $S_\varepsilon=\partial B_\varepsilon$ to $\widetilde{X}$, more precisely its value, as an element of the cohomology group $H^{2n}(\nu^{-1}(X\cap B_\varepsilon),\nu^{-1}(X\cap S_\varepsilon),\mathbb{Z})$, on the fundamental class of the pair $(\nu^{-1}(X\cap B_\varepsilon),\nu^{-1}(X\cap S_\varepsilon))$.

In the case of a single 1-form, if this is radial, then we are exactly in the setting envisaged by MacPherson to define the local Euler obstruction [24], and otherwise this is essentially the "defect" introduced in [3]. The computation of the local Chern obstruction will be revisited in section 5.

The other setting for the study of special points is closer to $X$, and we describe it next. This setting will allow us to describe the number of special points as an intersection number.

Let $X^d\subset\mathbb{C}^n$, $\mathcal{L}^k$ be the set of collections of 1-forms respecting the partition of $k$ as above $(k=k_1+k_2+\cdots+k_s)$, $\mathbb{D}_X^k\subset\mathbb{C}^n\times\mathcal{L}^k$ be the closure of the set of pairs $(x,\{l_j^i\})$ such that $x\in X_{\rm reg}$ and the restriction of the linear functions $l_1^i,\cdots,l_{n-k_i+1}^i$ to $T_xX_{\rm reg}\subset\mathbb{C}^N$ are linearly dependent for each $i=1,\cdots,s$.

Notice that the fiber of $\mathbb{D}_X^{k_i}$ over a regular point $x$ of $X$ can be identified with the elements of $M(d-k_i+1,n)$ which have less than maximal rank when restricted to $T_xX$. Since $T_xX$ is defined by $n-d$ equations, the fiber of $\mathbb{D}_X^{k_i}$ is itself a fibration over the singular matrices in $M(d-k_i+1,d)$, hence the restriction of $\mathbb{D}_X^{k_i}$ to the regular points of $X$ has a stratification by rank of the collection restricted to $TX_x$, $x \in X_{\mathrm{reg}}$.

The collection of 1-forms $\{\omega_j^{(i)}\}$ defines a section $\Gamma_\omega$ of $\mathbb{C}^n \times \mathcal{L}^k$; we will assume in our results that the image of the projection $\pi_X(\mathbb{D}_X^k \cap \mathrm{Im}(\Gamma_\omega))$ is isolated. Note that this implies that the intersection of $\mathrm{Im}(\Gamma_\omega)$ and $\mathbb{D}_X^k$ is isolated as well, since $\pi_X|_{\mathrm{Im}(\Gamma_\omega)}$ is $1-1$. We will further assume that the sets $\mathrm{Im}(\Gamma_\omega)$ and $\mathbb{D}_X^k$ have complementary dimension viewed as subsets of $\mathbb{C}^n \times \mathcal{L}^k$. Thus, their intersection number is well-defined.

We are interested in computing this intersection number.

As we will see, this amounts to computing the intersection number of a collection of sets defined by modules. The viewpoint of this paper is to compute this intersection number by successively restricting to the intersection of $k-1$ elements of the collection. There is a technical condition which describes the way a "good" collection of these sets meet, given in Definition 4.5, which needs some preparation.

**Definition 4.3.** Given a pair $(x,P)$, $x \in X$, $P$ in $G(d,n)$, the pair is degenerate for the collection $\{\omega_j\}$, $1 \le j \le d-k+1$, at $x$, if $\{\omega_j\}|_P$ is linearly dependent at $x$. Denote the set of degenerate pairs for $\{\omega\}$ by $\mathbb{B}(\omega)$.

**Proposition 4.4.** *Suppose the collection $\{\omega\}$ is linearly independent at the origin. Then $\mathbb{B}(\omega)$ has codimension $k$ in $X \times G(d,n)$.*

*Proof.* We can cover $G(d,n)$ with open sets as follows: pick a coordinate plane $P$ of dimension $d$ and a plane of complementary dimension using the complementary coordinates, which we denote by $\hat{P}$. Clearly, the complementary plane intersects $P$ only at the origin. Consider all planes which are the graphs of a linear map from $P$ to $\hat{P}$. The equations of these graphs give a unique set of equations describing the plane, and thus associate a matrix of size $(n-d \times n)$ to each plane.

These planes are just the planes that intersect $\hat{P}$ at the origin, and thus are a Zariski open subset of $G(d,n)$.

Suppose $U$ is such an open set, then construct the map from $U \times \mathrm{Hom}(n, d-k+1)$ to $\mathrm{Hom}(n, n-k+1)$ by combining the 2 matrices - the element of $\mathrm{Hom}(n, d-k+1)$ and the matrix of equations describing points of $U$.

This matrix has size $(n-k+1 \times n)$ and as a map from

$$U \times \mathrm{Hom}(n, d-k+1) \to \mathrm{Hom}(n, n-k+1)$$

is transverse to the rank stratification. So the codimension of the set of pairs which give matrices of less than maximal rank is $(n)-(n-k+1)+1 = k$.

Working globally, it is clear that the set of degenerate pairs is a fibration over the set of elements of $\mathrm{Hom}(n, d-k+1)$ of maximal rank. So fixing $\omega$ we get that the set of degenerate planes has codimension $k$. (Also fixing a plane $P$, the set of $L \in \mathrm{Hom}(n, d-k+1)$ for which the plane is degenerate also has codimension $k$.) □

If we have a collection of forms $\{\omega_j^{(i)}\}$ with $\sum_{i=1}^s k_i = k$, every element of which is linearly independent at the origin, then $\mathbb{B}(\omega)$ denotes the planes which are degenerate for every element of the collection. It has codimension less than or equal to $k$.

**Definition 4.5.** Given $X^d, 0 \subset \mathbb{C}^n, 0$ with $0 \in S(X)$ and a collection $\{\omega_j^{(i)}\}$with $\sum_{i=1}^s k_i = k$, $k \le d$ such that each element of the collection is linearly independent at 0, we say that the collection is proper for $X^d$ if $\dim(\widetilde{X}(S(X)) \cap \mathbb{B}(\omega)) \le d-k-1$ where $\widetilde{X}(S(X))$ is the restriction of the Nash modification of $X$ to $S(X)$, the singular set of $X$. If this condition holds for a collection of forms linearly independent at 0, with the exception of components of the intersection over the origin, we say the collection is proper on a deleted neighborhood of the origin.

If $X$ is smooth at 0, then we ask $\dim(\widetilde{X}(0) \cap \mathbb{B}(\omega)) \le d-k-1$.

**Remark 4.6.** The dimension $\widetilde{X}(S(X))$ is at most $d-1$; if there is a component of dimension $d-1$, the condition just asks that the component meets $B(\omega)$ properly. Since the dimension of all components of $\widetilde{X} \cap B(\omega)$ is at least $d-k$, the properness condition implies that a point of $\widetilde{X}(S(X)) \cap \mathbb{B}(\omega)$ is in the closure of points of the intersection lying over smooth points of $X$. Note also that if $k=d$, and the collection is proper, then $\widetilde{X}(S(X)) \cap \mathbb{B}(\omega)$ is empty.

For the geometric description we need of special points, it is necessary to lift our constructions to the level of the Nash modification.

On $\widetilde{X} \times \mathcal{L}^k$ we can consider triples $(x, P, L)$ where $P$ is a degenerate plane for $L$. Call the space of triples $\mathbb{D}^k$. It is clearly a fibration over $\widetilde{X}$.

Thinking of $\mathbb{C}^n \times G(d,n) \times \mathcal{L}^k$ as a trivial fibration over $\mathbb{C}^n \times G(d,n)$, we have the section induced by $\omega$ which we denote by $\Gamma_{\omega,G}$. Note that, if we restrict $\Gamma_{\omega,G}$ to $\widetilde{X}$, then

$$\Gamma^{-1}_{\omega,G}(\mathbb{D}^k) = \mathbb{B}(\omega) \cap \widetilde{X}.$$

Now, the image of $\Gamma_{\omega,G}$ has dimension $n+d(n-d)$, while $\mathbb{D}^k$ has codimension $(n-d)+d(n-d)+k$ so the expected dimension of the intersection is $d-k$. Denote the projection of the intersection to $X$ by $S(\omega)$. We can make $k$ a multi-index and make similar constructions; we get the expected dimension of $S(\omega_j^{(i)})$ is $d-(k_1+\cdots+k_s)$.

Suppose $\{\omega_j^{(i)}\}$ is a collection of 1 forms such that the $\sum k_i = d$ and 0 is an isolated special point. Then all of the various $S(\omega_j^{(i)})$ using different subcollections must have the correct expected dimension; for if $S(\omega_j^{(i)})$ is too large for one subcollection, the excess dimension will be passed to the others and 0 will not be isolated.

Denote $\mathbb{D}^k \cap \operatorname{Im}(\Gamma_{\omega,G})$ by $S_N(\omega)$.

We will also be interested in the notion of a restricted special point; given a collection of 1-forms $\omega_1^{(i)}, \cdots, \omega_{d-k_i+1}^{(i)}$, $1 \le i \le s$, we say $p$ is a *restricted special point* of the collection if it is a special point, and the sequence of points $p_m$ are in $S(\omega_j^{(i)})$, $1 \le i \le s-1$. In the next proposition we will prove that if the collection $\omega_j^{(i)}$, $1 \le i \le s-1$, is proper, then every special point is a restricted special point.

### 4.1. **Setup.**

Here we describe our assumption about the collections.

Let $X^d, 0 \subset \mathbb{C}^n, 0$ and $\{\omega_j^{(i)}\}$, a collection of 1-forms with $1 \le i \le s$, $1 \le j \le d-k_i+1$, where $\sum k_i = d$.

Assume the collection is arranged so that the first $r$ collections are 1-forms which are linearly independent at 0. We assume the 1-forms in the remaining collection are all linearly dependent at the origin. We assume the collection has an isolated singularity at the origin, and that the generic point of $S(\omega_j^{(i)})$, $1 \le i \le s-1$, is in $X_{\mathrm{reg}}$. If $r = s$ we also assume that the collection made up of the first $s-1$ elements is proper for $X$.

**Proposition 4.7.** *If, in the above set-up, 0 is a isolated special point of the collection, there exists a curve $C$ on $S(\omega_j^{(i)})$, $1 \le i \le s-1$, generically in $X_{\mathrm{reg}}$, such that $\{\omega\}$ is linearly dependent when restricted to the limiting tangent plane $T$ at the origin, and the origin is the only point on $S(\omega_j^{(i)})$ with this property.*

*Proof.* There are two cases to consider.

Case 1: Assume $r < s$, assume a special point exists. This is also a special point for the collection with the first $s-1$ elements. Thus $S(\omega_j^{(i)})$, $1 \le i \le s-1$, has positive dimension and its generic point is in $X_{\mathrm{reg}}$. For $C$, use any curve on $S(\omega_j^{(i)})$, and let $T$ be the limit tangent plane, $T_t$ the tangent plane to $X$ at point $t$ on curve $C$.

Then $\{\omega_j^{(i)}\}$, $1 \le i \le s-1$, are linearly dependent on $T_t$, since our point is in $S(\omega_j^s)$; hence, they are linearly dependent on $T$.

Since $\{\omega_j^{(s)}\}$ is linearly dependent at zero, they are linearly dependent on $T$ also.

Clearly, any point for which such a curve exists is a special point, so the origin is the only such point.

Case 2: $r = s$. Assume we have a special point then $S(\omega_j^{(i)})$, $i \le s-1$, has positive dimension with generic point in $X_{\text{reg}}$. Denote the limit tangent plane on which all of our collections restrict to be linearly dependent by $T$.

By the properness assumption, no component of $S_N(\omega)$, $i \le s-1$, can lie over $S(X)$; for every component of $S_N(\omega)$ must have dimension $d-k$, $k = \sum k_i$, $1 \le i \le s-1$, while by the properness assumption the points over $S(X)$ must have dimension $d-1-k$ or less.

This implies that there exists a curve $\varphi : \mathbb{C}, 0 \to X, 0$ generically in $X_{\text{reg}} \cap S(\omega_j^i)$ with $i \le s-1$, such that the limiting tangent plane to $X$ along $\phi$ is $T$.

Now all the members of our collection are linearly dependent on $T$ including $\{\omega_j^{(s)}\}$. ☐

The previous proposition explains why we are interested in collections which are proper. The properness condition means that if we have a special point, then it is a restricted special point as we can realize the limiting plane on which the collection is dependent as a limit of tangents to $X$ along a curve in some $S(\omega_j^{(i)})$. This is the key to our ability to study the intersection number of $\text{Im}(\Gamma_\omega)$ and $\mathbb{D}_X^k$ by restricting to $S(\omega_j^{(i)})$.

There is a converse to the proposition which requires a stronger genericity condition.

**Proposition 4.8.** *Suppose in the setup of this section, the collection made up of the first $s-1$ elements is proper for $X$, if the elements of the collection are linearly independent at the origin. If they are not linearly independent, assume they are proper for $X$ on a deleted neighborhood of the origin. Suppose the origin is the only point where there exists a curve $C$ on $S(\omega_j^{(i)})$, $1 \le i \le s-1$, generically in $X_{\text{reg}}$, such that $\{\omega\}$ is linearly dependent when restricted to the limiting tangent plane $T$ at the origin. Then the origin is an isolated special point of the collection.*

*Proof.* Clearly the origin is a special point. If it were non-isolated, then we could apply the previous proposition to find curves detecting the nearby special points as well. ☐

These two propositions show that when studying the behavior of special points, with the right genericity requirements, we can restrict from $X$ to $S(\omega_j^{(i)})$, $1 \le i \le s-1$, and having done so consider only the last element of the collection.

The next proposition serves as a "moving lemma".

This proposition and its corollary, together with the multiplicity polar theorem, will show that the invariant of the next section computes the intersection number of $\text{Im}(\Gamma_\omega)$ and $\mathbb{D}_X^k$. The argument we give is adapted from that appearing in [18], Theorem 1.2, p. 187.

To prove our proposition we want to consider the map:

$$\Theta : \mathbb{D}_X^{k_s}|_{S(\omega^{(i)})_{i \le s-1}} \times M(d-k_s+1, n) \to M(d-k_s+1, n)$$

given by

$$\Theta((x, L), M) = L - (\omega(x) + M).$$

If we resolve the singularities of the set $\mathbb{D}_X^{k_s}|_{S(\omega)}$, then the composition $\Theta \circ \pi_{\mathbb{D}_X^{k_s}}$ is a submersion because of the contribution from the $M$ term.

In resolving these singularities there may be multiple components. For example, for $X^2$ a surface in $\mathbb{C}^3$ with an isolated singularity at the origin, then if $s = 2$ and $\omega$ consists of two forms then $\mathbb{D}_X^{k_2}|_{S(\omega)}$ has $(0, M(2,3))$ as a component. This follows because the polar curve of $X^2$ is non-empty, which implies that the generic element of $M(2,3)$ has less than maximal rank on a curve on $X^2$, hence lies in $\mathbb{D}_X^{k_2}$ along that curve. However, there will be a unique component for each component of $S(\omega)$ which surjects onto that component. Denote the components of $\mathbb{D}_X^{k_s}|_{S(\omega)}$ which surject onto $S(\omega)$ by $\mathbb{D}_{S(\omega)}^{k_s}$. The fiber of these components over the origin are those collections of forms which are the limits of forms degenerate along a curve in $S(\omega)$.

Let $C$ denote $\Theta^{-1}(0) \cap (\mathbb{D}^{k_s}_{S(\omega)} \times M(d-k_s+1,n))$ and consider the projection $p$ from $C$ to $M(d-k_s+1,n)$.

Now

$$\dim C \;=\; \dim S(\omega^{(i)})_{i \le s-1} + \dim(\text{generic fiber of } D^{k_s}) \;=\; k_s + [(d-k_s+1)(n) - k_s].$$

By hypothesis, we have an isolated singularity at 0, so the dimension of $S(\omega^{(i)})_{i \le s}$ must be 0. This implies the dimension of $S(\omega^{(i)})_{i \le s-1}$ is $k_s$, the minimum possible, because otherwise, adding another form to the collection will not lower the dimension of $S(\omega^{(i)})_{i \le s}$ to 0.

The expression in [ ] above holds because the codimension of $D^{k_s}$ in $M(,)$ is just

$$-[(d-k_s+1)+(n-d)]+n+1 = k_s,$$

so the map $C \to M(d-k_s+1,n)$ is a map between equidimensional spaces. Assuming that $0 \in X$ is an isolated special point of the collection, the fiber over 0 of $p$ is a single point $\omega(0)$.

Earlier in this section, we began to look at the intersection number of $\mathrm{Im}(\Gamma_\omega)$ and $\mathbb{D}^k_X$. Restricting to $S(\omega^{(i)})_{i \le s-1}$, we can also look at the intersection number of $\mathrm{Im}(\Gamma_{\omega^{(s)}})$ and $\mathbb{D}^{k_s}_{S(\omega)}$. Our moving lemma will be used to calculate this piece of the intersection number of $\mathrm{Im}(\Gamma_\omega)$ and $\mathbb{D}^k_X$.

**Definition 4.9.** A special point of a collection $\{\omega^{(i)}_j\}$ of germs of 1-forms on $X$ is non-degenerate, if the section $\Gamma_{\omega^{(i)}_j}$, $1 \le i \le s$, meets $\mathbb{D}^k_X$ transversally at the point.

We can now state our proposition.

**Proposition 4.10.** *Given a collection as in the set up of this section, assume that the section $\Gamma_{\omega^{(i)}_j}$, $1 \le i \le s-1$, meets $\mathbb{D}^k_X$ transversely on a Z-open subset of $S(\omega^{(i)})_{i \le s-1}$. Then for generic $M$, the collection $\{\omega^{(s)} + tM\}$ meets $\mathbb{D}^{k_s}_{S(\omega)}$ transversely at all points close to the origin for $t$ sufficiently small, $t \ne 0$. The number of such points is just the degree of the projection from $C$ to $M(d-k_s+1,n)$ over the origin in $M(d-k_s+1,n)$. Further, each such point is a non-degenerate point of the collection $\{\omega^{(i)}_j\}$, $1 \le i \le s$.*

*Proof.* Pick $M$ in the complement of the $\Delta(p)$, the discriminant of the projection from $C$ to $M(d-k_s+1,n)$, such that the line between 0 and $M$ does not intersect these sets close to 0.

Over the points of this line close to 0, the number of points is the degree of $p$ and $p$ is a submersion at each point. This implies that the map obtained by fixing the $M$ term in $\Theta$ is a submersion also. Note that the dimension of the source and target of this map are the same, hence the map is in fact a diffeomorphism.

We are interested in exploring the consequences of this fact.

Let us first consider the case where at the points on the fiber of $p$ over $tM$, $t$ small, $x$ is in the regular part of $S(\omega^{(i)})_{i \le s-1}$, and the element in $\mathbb{D}^{k_s}$ has rank one less than maximal when restricted to $TX_x$. Then the resolution of $D^{k_s}|_{S(\omega)}$ is an equivalence at such points because $D^{k_s}|_{S(\omega)}$ is smooth there, so we can work on the tangent space of $D^{k_s}|_{S(\omega)}$. At each point this splits into a direct sum–the part along $S(\omega^{(i)})_{i \le s-1}$, and the part along the fiber. There is a similar decomposition of the tangent space of the target–the part which can be identified with the fiber in the source, and the normal space to this. The differential is the identity on the tangent space to the fiber, so since the differential is surjective, the restriction of the differential to the tangent space to $S(\omega^{(i)})_{i \le s-1}$ must surject onto the normal space to the fiber. In turn this implies that the section induced from $\omega + tM$ meets $D^k$ transversely. In fact, since for transversality we just need the tangent vectors to $S(\omega^{(i)})_{i \le s-1}$, and the other elements of the collection intersect $\mathbb{D}^k_X$ transversely, the collection $\{(\omega^{(i)})_{i \le s-1}, \omega^s + tM\}$ meets $\mathbb{D}^k_X$ transversely.

In the general case, note that the assumptions we made above coincide with the resolution being an equivalence. If the resolution is not an equivalence, then some tangent vectors on the resolution will be in the kernel of the differential of the projection, hence the differential will not be surjective, contradicting our choice of $M$. So we only need to consider the above special case.

□

**Remark 4.11.** If $\{\omega^{(i)}\}$, $1 \leq i \leq r$, is the maximal subset of our collections which meet properly at the origin, then we can choose $M$ so that $\{\omega^{(i)}, \omega^s + tM\}$, $1 \leq i \leq r$, also meet properly at the origin for $t \neq 0$. This will be implicit in our application of our moving lemma.

Denote the collection obtained by moving our last element by $\widetilde{\omega}$. From the last propostion we have:

**Corollary 4.12.** *In the set-up of last proposition we have*

$$\mathrm{Im}(\Gamma_\omega) \cdot \mathbb{D}^k_X = \Gamma(\omega_s) \cdot \mathbb{D}^{k_s}|_{S(\omega^{(i)})_{i \leq s-1}} + \mathrm{Im}(\Gamma_{\widetilde{\omega}}) \cdot \mathbb{D}^k_X.$$

*Proof.* The effect of moving $\{\omega^{(s)}\}$ is to split off points from the intersection $\mathrm{Im}(\Gamma_\omega) \cdot \mathbb{D}^k_X$. The first intersection number on the right is the degree of the projection from $C$ to $M(d - k_s + 1, n)$, and this is the number of points split off from the intersection number on the left hand side of the equation. Moving $\{\omega^{(s)}\}$ ensures that the intersection $\Gamma(\omega_s + tM) \cdot \mathbb{D}^{k_s}|_{S(\omega^{(i)})_{i \leq s-1}}$ is void at the origin, *i.e.*, the intersection point at the origin has split into non-degenerate points. The second term on the right is the remaining points at the origin. □

**Corollary 4.13.** *In the set-up of last proposition, suppose in addition that the collection* $\{\omega^{(i)}\}$, $1 \leq i \leq s-1$, *is proper. Then*

$$\mathrm{Im}(\Gamma_\omega) \cdot \mathbb{D}^k_X = \Gamma(\omega_s) \cdot \mathbb{D}^{k_s}|_{S(\omega^{(i)})_{i \leq s-1}}.$$

*Proof.* Since the collection $\{\omega^{(i)}, \widetilde{\omega^{(s)}}\}$ , $1 \leq i \leq s-1$, is proper, the intersection of $\Gamma_{\{\omega^{(i)}, \widetilde{\omega^{(s)}}\}}$ and $\mathbb{D}^k_X$ is empty.

□

**Remark 4.14.** If $X^{n-1}$ is a hypersurface and $\omega_i$ a collection of forms with $(n-1) - k + 1$ elements, which are linearly independent at the origin, then it is easy to check if $\widetilde{X}(0) \cap B(\omega)$ has dimension $(n-1) - k - 1$.

Suppose $\dim \widetilde{X}(0) \cap B(\omega) \geq (n-1) - k$. To each point in $B(\omega)$ there corresponds a unique point in $\mathrm{Proj}(\omega)$, the projectivized row space of $\omega_i$.

Note that points of $\mathrm{Proj}(\omega)$ corresponding to points of $\widetilde{X}(0) \cap B(\omega)$ are limiting tangent hyperplanes to $X$ at the origin, so the set of points of $\mathrm{Proj}(\omega)$ which are limiting tangent hyperplanes has dimension $\geq (n-1) - k = dim \mathrm{Proj}(\omega)$ so every point is a limiting tangent hyperplane.

This is true if and only if $\mathrm{JM}(f, \sum \alpha_i \omega_i)$ fails to be a reduction of $\mathrm{JM}(f) \oplus \mathcal{O}_X$ for all $\alpha_i$. This can be checked using curves.

**Remark 4.15.** We continue with the hypersurface isolated singularity case.

Suppose $j+1$ collections $\{\omega^i\}$ $1 \leq i \leq j+1$ are in general position *i.e.*, all are linearly independent at 0 and $\dim \cap \mathrm{Proj}(\{\omega^i\})$ is $(n-1) - \sum_i k_i$. Suppose the properness condition holds for the first $j$ elements but fails for the collection. A dimension count shows that a whole component of $\cap Proj\{\omega^i\}$ must lie in the fiber of the Nash modification over the origin. Again this is easy to check.

## 5. Computing Chern Numbers

In this section,we will begin to connect the machinery of section 2 to the computation of Chern numbers of a collection of forms, preparing for the next section which contains our main results.

Ebeling and Gusein-Zade proved this next proposition.

**Proposition 5.1.** [7] *Let $X$ be a representative of the germ of a complex analytic space, then the local Chern obstruction* $\mathrm{Ch}_{X,0}\{\omega^{(i)}_j\}$ *of a collection* $\{\omega^{(i)}_j\}$ *of germs of holomorphic 1-forms is equal to the number of special points on $X$ of a generic deformation of the collection.*

If $X$ is defined by $F : \mathbb{C}^n \to \mathbb{C}^p$, then the Jacobian module of $X$ denoted $\mathrm{JM}(X)$, is the submodule of $\mathcal{O}_X^p$ generated by the partial derivatives of $F$. Given a collection of 1-forms with $r$ elements defined on $X$, form the $p+r$ by $n$ matrix $D(F,\omega)$ by augmenting the Jacobian matrix $DF$ at the bottom with the 1-forms from the collection. Call the submodule of the free module $\mathcal{O}_X^{p+r}$, generated by the columns of $\begin{pmatrix} D(F) \\ \omega \end{pmatrix}$, the augmented Jacobian module and denote it by $\mathrm{JM}(X,\omega)$.

Note that this construction works in general. Given a submodule $M$ of a free module $F$, one can select a matrix of generators, and augment the matrix using linear forms. The points at which the new matrix has less than maximal rank is independent of the choice of generators of $M$ as the row space does not change.

In the next lemma, we begin to relate the theory of integral closure and the infinitesimal limiting geometry of our sets of forms.

**Lemma 5.2.** *Let $X$ be a representative of the germ of a complex analytic space, and let*

$$\mathcal{L} = \{l_1, l_2, \cdots, l_s\}$$

*be a collection of linear forms. Consider the hyperplanes defined by the forms $\sum a_i l_i$. None of these hyperplanes is a limiting tangent hyperplane to $X, 0$ at the origin if and only if*

$$\overline{\mathrm{JM}(X)_p} = \overline{\mathrm{JM}(X)},$$

*where $p$ is a submersion whose kernel is the intersection of the kernels of $l_1, \cdots, l_s$.*
*(Here $\mathrm{JM}(X)_p$ is the submodule of $\mathrm{JM}(X)$ generated by $\frac{\partial}{\partial v_i} f$ where the $v_i$ span the kernel of $p$.)*

*Proof.* Let us prove this result in the special case when $p$ is a linear projection on the last $s$ variables.

If $\mathrm{JM}(X)_p$ is a reduction of $\mathrm{JM}(X)$, then so is $\mathrm{JM}(X)_h$, because $\ker(h) \supset \ker(p)$, where $h = \sum a_i l_i$. Hence, the hyperplane defined by $h$ is not a limiting tangent hyperplane.

Let us prove now that, if $\overline{\mathrm{JM}(X)_h} = \overline{\mathrm{JM}(X)}$ for all $h$, then $\overline{\mathrm{JM}(X)_p} = \overline{\mathrm{JM}(X)}$.

Let $K = \ker(p)$, we will show $\mathrm{JM}(X)_p \subset \mathcal{O}_X^k$ is a reduction of $\mathrm{JM}(X) \subset \mathcal{O}_X^k$ if every hyperplane that contains $K$ is not a limiting tangent hyperplane.

Suppose $\mathrm{JM}(X)_p$ is not a reduction. This implies that there exist a map $\phi : (\mathbb{C}, 0) \to (X, 0)$ and a non-zero $l : \mathbb{C}^n \to \mathbb{C}$, such that if $N$ is the matrix of generators of $\mathrm{JM}(X)_p$ and $M$ is the matrix of generators of $\mathrm{JM}(X)$, then the ideal $< \eta_1(t), \cdots, \eta_n(t) >$ generated by the components of $(l \cdot N) \circ \varphi(t)$ has larger order than the ideal $< m_1(t), \cdots, m_n(t) >$ generated by the components of $l \cdot M \circ \varphi$. Denote the order of $l \cdot M \circ \varphi$ by $k$. Then

$$\lim 1/t^k < m_1(t), \cdots, m_n(t) >$$

defines a limiting tangent hyperplane. Since $m_1 = \eta_1, \cdots, m_p = \eta_p$, and the order of these terms is greater than $k$, it follows that $T$ is a limiting tangent hyperplane which contains the kernel of $p$.

□

Given a collection of linear forms $\mathcal{L} = \{l_1, l_2, \cdots, l_s\}$, we let $\mathrm{JM}(X, \mathcal{L})$ denote the module whose matrix of generators is gotten by adding as rows $\{l_1, l_2, \cdots, l_s\}$ to the jacobian matrix of a set of generators of $I(X)$. In a similar way, let $(M, \mathcal{L})$ denote the module whose matrix of generators is gotten by adding as rows the $\{l_1, l_2, \cdots, l_s\}$ to a matrix of generators of $M$.

**Proposition 5.3.** *Let $\mathcal{L} = \{l_1, l_2, \cdots, l_s\}$ be a collection of linear 1-forms linearly independent at the origin. Consider the hyperplanes defined by the forms $\sum a_i l_i$. None of these hyperplanes is a limiting tangent hyperplane to $X, 0$ at the origin if and only if $\overline{\mathrm{JM}(X, \mathcal{L})} = \overline{\mathrm{JM}(X) \oplus \mathcal{O}_X^s}$.*

*Proof.* It suffices to show that $\mathrm{JM}(X, \mathcal{L})$ is a reduction of $\mathrm{JM}(X) \oplus \mathcal{O}_X^s$ if and only if $\mathrm{JM}(X)_p$ is a reduction of $\mathrm{JM}(X)$. Suppose $\mathrm{JM}(X, \mathcal{L})$ is a reduction of $\mathrm{JM}(X) \oplus \mathcal{O}_X^s$, then $\overline{\mathrm{JM}(X, \mathcal{L})}$ contains $\mathrm{JM}(X) \oplus 0$. Restricting to curves, this implies $\overline{\mathrm{JM}(X)_p}$ contains $\mathrm{JM}(X)$.

Suppose $\mathrm{JM}(X)_p$ is a reduction of $\mathrm{JM}(X)$. Then $\overline{\mathrm{JM}(X,\mathcal{L})}$ contains $\mathrm{JM}(X)\oplus 0$. Let $\{v_i\}$ be a collection of vectors such that $l_i(v_j)=\delta_{i,j}$, then $\overline{\mathrm{JM}(X,\mathcal{L})}$ contains $\begin{pmatrix} D(F)(v_i) \\ \mathcal{L}(v_i) \end{pmatrix}$ and $\mathrm{JM}(X)\oplus 0$, so it contains $\mathrm{JM}(X)\oplus \mathcal{O}_X^s$ □

The previous two propositions can be easily generalized using the same proof. Given $M$ a submodule of a free module $F$, $\operatorname{Projan}\mathcal{R}(M)$ has a canonical projection to $X$ which is a fibration over the Z-open subset $U_M$ of $X$ on which $M$ has maximal rank. The fiber of this map consists of hyperplanes. Call the planes in the fibers over $U_M$, $M$-planes. The planes in the fibers over $C(M)$ then, are limiting $M$-planes. Then the analogues of the previous two results are:

**Proposition 5.4.** *Let $X$ be a representative of the germ of a complex analytic space, and let $\mathcal{L}=\{l_1,l_2,\cdots,l_s\}$ be a collection of linear forms linearly independent at the origin. Consider the hyperplanes defined by the forms $\sum a_i l_i$. Then the following statements are equivalent:*

*1) None of these hyperplanes is a limiting $M$-hyperplane to $X,0$ at the origin.*

*2) If $p$ is a submersion whose kernel is the intersection of the kernels of $l_1,\cdots,l_s$ then*
$$\overline{M_p}=\overline{M}.$$

*3) There is an equality of modules:*
$$\overline{(M,\mathcal{L})}=\overline{M}\oplus\mathcal{O}_X^s.$$

Let $\{\omega_j^{(i)}\}$ be a collection of 1-forms on the variety $X$, for simplicity we will denote $S(\omega_j^{(i)})$ with $1\le i\le s-1$ by $\mathcal{C}$. (Recall $\mathcal{C}$ is the set of points where all of the elements of the collection $\{\omega_j^{(i)}\}$ with $1\le i\le s-1$ are singular.) In the next proposition we are interested in characterizing those collections for which the origin is not a special point or restricted special point.

**Proposition 5.5.** *Let $(X,0)$ be the germ of an equidimensional reduced analytic variety, $X$ a representative of the germ and $\{\omega_j^{(i)}\}$ a collection of 1-forms; assume the generic point of each component of $\mathcal{C}$ lies in $X_{\mathrm{reg}}$. Assume also the last collection $\{\omega_j^{(s)}\}$ is linearly independent at $0$. The origin is not a restricted special point of the collection $\{\omega_j^{(i)}\}$ if and only if $\mathrm{JM}(X,\omega^{(s)})|_{\mathcal{C}}$ is a reduction of $\mathrm{JM}(X)|_{\mathcal{C}}\oplus\mathcal{O}_{\mathcal{C}}^{d-k_s+1}$.*

*If all of the collections are linearly independent at the origin, and we assume the first $s-1$ elements are proper, then the origin is not a special point of the collection $\{\omega_j^{(i)}\}$ if and only if $\mathrm{JM}(X,\omega^{(s)})|_{\mathcal{C}}$ is a reduction of $\mathrm{JM}(X)|_{\mathcal{C}}\oplus\mathcal{O}_{\mathcal{C}}^{d-k_s+1}$.*

*Proof.* Let $\mathcal{L}=\{l_1,l_2,\cdots,l_{d-k_s+1}\}$ be a collection of linear 1-forms such that $\omega_i^s(0)=l_i$. As in Lemma 3.3 of [17], using the integral form of Nakayama's lemma we have that $\mathrm{JM}(X,\omega^{(s)})|_{\mathcal{C}}$ is a reduction of $\mathrm{JM}(X)|_{\mathcal{C}}\oplus\mathcal{O}_{\mathcal{C}}^{d-k_s+1}$ if and only if $\mathrm{JM}(X,\mathcal{L})|_{\mathcal{C}}$ is a reduction of $\mathrm{JM}(X)|_{\mathcal{C}}\oplus\mathcal{O}_{\mathcal{C}}^{d-k_s+1}$. So we can work with $\mathcal{L}$.

Now we apply the previous proposition, where $M=\mathrm{JM}(X)$ restricted to $\mathcal{C}$. Then the limiting $M$-hyperplanes are just the tangent hyperplanes to $X$ as the generic point of each component of $\mathcal{C}$ is in $X_{\mathrm{reg}}$. If some combination of the $\omega_i^s(0)=l_i$ is a limiting tangent hyperplane to $X$, then that combination is zero when restricted to the limiting tangent plane, and the collection is linearly dependent.

If we assume properness, then since every special point is a restricted special point, the result follows.

□

We will need a refinement of this result for later. The key point in the above argument, is that $\mathrm{JM}(X,\omega^{(s)})|_{\mathcal{C}}$ is a reduction of $\mathrm{JM}(X)|_{\mathcal{C}}\oplus\mathcal{O}_{\mathcal{C}}^{d-k_s+1}$ if and only if none of the hyperplanes defined by $\{\omega^{(s)}(0)\}$ is a limiting tangent hyperplane to $X,0$ at the origin along curves on $\mathcal{C}$. Given the collection $\{\omega^{(s)}\}$ , we can deform it to $\{\omega^{(s)}(0)\}$ by using the linear deformation. This fixes the one jet of the collection. Denote this family of collections by $\{\omega_L^{(s)}\}$. Denote the family of sections defined by fixing the first $s-1$ collections and deforming the last one using the linear deformation by $\Gamma_{\{\omega_{s-1,L}\}}$.

**Proposition 5.6.** *Assume* $\mathrm{JM}(X,\omega^{(s)})|_{\mathcal{C}}$ *is a reduction of* $\mathrm{JM}(X)|_{\mathcal{C}} \oplus \mathcal{O}_{\mathcal{C}}^{d-k_s+1}$. *Then, the intersection* $\mathrm{Im}(\Gamma_{\{\omega_{s-1,L}\}}) \cdot \mathbb{D}_X^k$ *is constant in the linear deformation.*

*Proof.* Since $\mathrm{JM}(X,\omega^{(s)})|_{\mathcal{C}}$ is a reduction of $\mathrm{JM}(X)|_{\mathcal{C}} \oplus \mathcal{O}_{\mathcal{C}}^{d-k_s+1}$, the same is true for any member of the family $\{\omega_L^{(s)}\}$. Suppose for some parameter value $t_0$ that the intersection number changes, *i.e.*, a point splits off. This gives a curve of points in $\mathcal{C}$, where at each point $p$, a member of $\{\omega_L^{(s)}\}$ is degenerate when restricted to some plane which is a point over $p$ in the Nash modification. This implies that $\{\omega_L^{(s)}\}(t_0)$ is degenerate when restricted to some plane which is a point over 0 in the Nash modification. As this plane can be reached through a curve on $\mathcal{C}$, it contradicts that $\mathrm{JM}(X,\omega_L^{(s)}(t_0))|_{\mathcal{C}}$ is a reduction of $\mathrm{JM}(X)|_{\mathcal{C}} \oplus \mathcal{O}_{\mathcal{C}}^{d-k_s+1}$. □

**Definition 5.7.** $H_{d-1}(X)$, by definition, consists of all elements of $\mathcal{O}_X^p$ which are in the integral closure of $\mathrm{JM}(X^d)$ except at the origin. A related ideal is $H_{c-1}(X,C^c)$ where $C$ is a subset of $X$ of dimension $c$. It consists of all elements of $\mathcal{O}_C^p$ which are in the integral closure of $\mathrm{JM}(X)$ restricted to $C$ except at the origin.

In general, $H_i(X)$ consists of all elements of $\mathcal{O}_X^p$ which are in the integral closure of $\mathrm{JM}(X)$ off a set of codimension $i+1$. Sometimes we write $H_i(\mathrm{JM}(X))$. The meaning of $H_i(X,C^c)$ is similar.

**Proposition 5.8.** *Let* $(X,0)$ *be the germ of an equidimensional reduced analytic variety,* $X$ *a representative of the germ and* $\{\omega_j^{(i)}\}$ *a collection of 1-forms; assume the generic point of each component of* $\mathcal{C}$ *lies in* $X_{\mathrm{reg}}$. *Assume also the last collection* $\{\omega_j^{(s)}\}$ *is linearly independent at* 0. *The origin is at most an isolated restricted special point of the collection if and only if* $\mathrm{JM}(X,\mathcal{L})|_{\mathcal{C}}$ *is a reduction of* $H_{c-1}(X,\mathcal{C})|_{\mathcal{C}} \oplus \mathcal{O}_{\mathcal{C}}^{d-k_s+1}$ *except possibly at* $x$.

*If in addition, the first* $s-1$ *collections are proper on a deleted neighborhood of the origin, then the origin is at most an isolated special point.*

*Proof.* Suppose the origin is an isolated restricted special point. Let $U$ be a neighborhood of 0 in $X$ such that $x$ is the only restricted special point. Then by proposition 4.7, $\Gamma_{\omega^{(s)}}$ misses $T^*(X)|_{\mathcal{C}}$ on $U \setminus \{0\}$.

Then $\mathrm{JM}(X,\mathcal{L})|_{\mathcal{C}}$ is a reduction of $\mathrm{JM}(X)|_{\mathcal{C}} \oplus \mathcal{O}_{\mathcal{C}}^{d-k_s+1}$ at all $x \in U$, $x \neq 0$ by the previous proposition.

Hence by definition it is a reduction of $H_{c-1}(X,\mathcal{C})|_{\mathcal{C}} \oplus \mathcal{O}_{\mathcal{C}}^{d-k_s+1}$ except possibly at $x$.

On the other hand assume the reduction criterion holds at each point of $U-0$. This implies $\mathrm{JM}(X,\mathcal{L})|_{\mathcal{C}}$ is a reduction of $\mathrm{JM}(X)|_{\mathcal{C}} \oplus \mathcal{O}_{\mathcal{C}}^{d-k_s+1}$ as this last module is a submodule of $H_{c-1}(X,\mathcal{C})|_{\mathcal{C}} \oplus \mathcal{O}_{\mathcal{C}}^{d-k_s+1}$. This implies that there are no restricted special points on $U$ except possibly the origin.

If in addition, the first $s-1$ collections are proper on a deleted neighborhood of the origin, then the lack of restricted special points on $U-0$ is equivalent to a lack of special points.

□

The last proposition leaves open the question as to whether the origin is a restricted special point if the reduction criterion holds. The next proposition settles this point.

**Proposition 5.9.** *Suppose the origin is at most an isolated restricted special point. Then*

$$\begin{aligned} &e(\mathrm{JM}(X,\omega^{(s)})|_{\mathcal{C}}, \mathrm{JM}(X)|_{\mathcal{C}} \oplus \mathcal{O}_{\mathcal{C}}^{d-k_s+1}, 0) \\ &= e(\mathrm{JM}(X,\omega^{(s)})|_{\mathcal{C}}, H_{c-1}(X,\mathcal{C}) \oplus \mathcal{O}_{\mathcal{C}}^{d-k_s+1}, 0) \\ &\quad - e(\mathrm{JM}(X,\mathcal{L})|_{\mathcal{C}}, H_{c-1}(X,\mathcal{C}) \oplus \mathcal{O}_{\mathcal{C}}^{d-k_s+1}, 0) \end{aligned}$$

*where* $\mathcal{L}$ *is a collection of linear 1-forms such that* 0 *is not a restricted special point for it.*

*The origin is not a restricted special point if and only if*

$$e(\mathrm{JM}(X,\omega^{(s)})|_{\mathcal{C}}, H_{c-1}(X,\mathcal{C}) \oplus \mathcal{O}_{\mathcal{C}}^{d-k_s+1}, 0)$$
$$= e(\mathrm{JM}(X,\mathcal{L})|_{\mathcal{C}}, H_{c-1}(X,\mathcal{C}) \oplus \mathcal{O}_{\mathcal{C}}^{d-k_s+1}, 0)$$

*where $\mathcal{L}$ is a collection of linear 1-forms such that 0 is not a restricted special point for it.*

*Proof.* Since the origin is at most an isolated restricted special point all three multiplicities are well-defined. Then, the proof is based on a fundamental result due to Kleiman and Thorup, the principle of additivity [23]. Given a sequence of $\mathcal{O}_X$ modules $A \subset B \subset C$ such that the multiplicity of the pairs is well-defined, then

$$e(A,C) = e(A,B) + e(B,C).$$

The result follows by setting $A = \mathrm{JM}(X,\omega^{(s)})|_C$, $B = \mathrm{JM}(X)|_C \oplus \mathcal{O}_C^{d-k_s+1}$, and

$$C = H_{d-1}(X,\mathcal{C}) \oplus \mathcal{O}_C^{d-k_s+1}.$$

Using the fact that 0 is not a restricted special point for $\mathcal{L}$ we have that the multiplicity of $(\mathrm{JM}(X,\mathcal{L})|_C, H_{d-1}(X)|_C \oplus \mathcal{O}_C^{d-k_s+1})$ and $(\mathrm{JM}(X)|_C \oplus \mathcal{O}_C^{d-k_s+1}, H_{d-1}(X)|_C \oplus \mathcal{O}_C^{d-k_s+1})$ are the same.

The origin is not a restricted special point by 5.5 if and only if $\mathrm{JM}(X,\omega^{(s)})|_C$ is a reduction of $\mathrm{JM}(X)|_C \oplus \mathcal{O}_C^{d-k_s+1}$. The reduction statement holds at 0 if and only if

$$e(\mathrm{JM}(X,\omega^{(s)})|_{\mathcal{C}}, H_{c-1}(X,\mathcal{C}) \oplus \mathcal{O}_{\mathcal{C}}^{d-k_s+1}, 0) = 0,$$

which is true if and only if

$$e(\mathrm{JM}(X,\omega^{(s)})|_{\mathcal{C}}, H_{c-1}(X,\mathcal{C}) \oplus \mathcal{O}_{\mathcal{C}}^{d-k_s+1}, 0) \ = \ e(\mathrm{JM}(X,\mathcal{L})|_{\mathcal{C}}, H_{c-1}(X,\mathcal{C}) \oplus \mathcal{O}_{\mathcal{C}}^{d-k_s+1}, 0),$$

where $\mathcal{L}$ is a collection of linear 1-forms such that 0 is not a restricted special point for it.

□

Our next step to apply the Multiplicity Polar Theorem is to show that the polar curve of $\mathrm{JM}(X,\omega^{(s)} + tM)|_{\mathcal{C}}$ is empty.

**Proposition 5.10.** *Let $X^d, 0 \subset \mathbb{C}^n, 0$ and $\{\omega\}$ a collection of 1-forms $\{\omega_j^{(i)}\}$, $1 \le i \le s$, $1 \le j \le d - k_i + 1$, $\sum k_i = d$. Assume further the collection has an isolated singularity at the origin, and that the generic point of $S(\omega_j^{(i)})$, $1 \le i \le s-1$ is in $X_{\mathrm{reg}}$. Then, the polar curve of the module $\mathrm{JM}(X,\omega^{(s)} + tM)|_{\mathcal{C}}$ is empty for $\mathfrak{C} = \mathcal{C} \times \mathbb{C}$, where $M$ is a collection of generic linear forms.*

*Proof.* The polar variety of codimension $k$ of $M$ in $X$ denoted $\Gamma_k(M)$ is constructed by intersecting $\operatorname{Projan} \mathcal{R}(M)$ with $X \times H_{g+k-1}$ where $H_{g+k-1}$ consists of the set of hyperplanes which contain a general plane of dimension $g+k-1$, and $g$ is the generic rank of $\mathrm{JM}(X,\omega^{(s)} + tM)$, then projecting to $X$. Note that if $M$ has $n$ generators, so that $\operatorname{Projan} \mathcal{R}(M)$ is contained in $X \times \mathbb{P}^{n-1}$, and the dimension of $\operatorname{Projan} \mathcal{R}(M)$ is greater than or equal to $n+1$ then the polar varieties of $M$ of codimension $n$ or more are empty, because the codimension of a point in $\mathbb{P}^{n-1}$ is $n-1$.

With this observation in mind, the next step is to compute the dimension of

$$\operatorname{Projan} \mathcal{R}(\mathrm{JM}(X,\omega^{(s)} + tM)|_{\mathcal{C}}).$$

This dimension is the dimension of the base plus the generic rank of $\begin{pmatrix} D(F) \\ \omega \end{pmatrix}$ minus 1. Now the generic rank of the jacobian matrix is $n-d$, while the generic rank of the jacobian matrix augmented by the $\{\omega_j^s\}$ is $(n-d) + (d-k_s+1) = n - k_s + 1$. This follows because the generic point of $\mathcal{C}$ is a smooth point of $X$ hence the jacobian matrix has maximal rank there. Because 0 is an isolated singularity, it follows that the augmented matrix generically has maximal rank. Thus we have, since $g = n - k_s + 1$,

$$\dim \operatorname{Projan} \mathcal{R}(\mathrm{JM}(X,\omega^{(s)} + tM)|_{\mathcal{C}}) = k_s + 1 + (n - k_s + 1)) - 1 = n + 1.$$

Since the dimension of Projan $\mathcal{R}(\mathrm{JM}(X,\omega^{(s)}+tM)|_{\mathcal{C}})$ is greater than or equal to the number of generators, there is no polar curve for $\mathrm{JM}(X,\omega^{(s)}+tM)$.

□

**Proposition 5.11.** *Suppose $X$ is smooth and $\omega$ is a 1-form such that $\omega$ has a Morse point at 0, then $e(\mathrm{JM}(X,\omega),\mathrm{JM}(X)\oplus\mathcal{O}_X,0)=1$.*

*Proof.* Since $X$ is a smooth manifold, the number of equations of $X$ is $n-d$, so the matrix of generators of $\mathrm{JM}(X,\omega)$ has $n-d+1$ rows, $n$ columns, and a matrix of generators of $\mathrm{JM}(X)\oplus\mathcal{O}_X$ also has $n-d+1$ rows with the same $n-d$ first rows. We may assume the equations for $X$ are $z_1=\cdots=z_{n-d}=0$.

Then the Rees algebra of $\mathrm{JM}(X)\oplus\mathcal{O}_X$ is $\mathcal{O}_X[S_1,\ldots,S_{n-d},S_{n-d+1}]$, while the ideal corresponding to the inclusion of the Rees algebra of $\mathrm{JM}(X,\omega)$ in that of $\mathrm{JM}(X)\oplus\mathcal{O}_X$ is $(S_i,z_jS_{n-d+1})$ where $1\le i\le n-d$, $n-d<j\le n$. Now in this example, we know that

$$1=e(\mathrm{JM}(X,f),\mathcal{O}_X^{n-d+1})=e(\mathrm{JM}(X,f),\mathrm{JM}(X)\oplus\mathcal{O}_X)+e(\mathrm{JM}(X)\oplus\mathcal{O}_X,\mathcal{O}_X^{n-d+1}),$$

while $e(\mathrm{JM}(X)\oplus\mathcal{O}_X,\mathcal{O}_X^{n-d+1})=0$ since the two modules are the same. This uses the additivity of the multiplicity, the fact that $\omega$ is Morse on $X$, and the fact that the multiplicity of $e(\mathrm{JM}(X,\omega),\mathcal{O}_X^{n-d+1})$ is the colength of its ideal of maximal minors.

Now we want to show that we get the same result even if the number of equations is larger than $n-d$. (This happens for example, if we are working at a smooth point of a space which is singular at the origin.) Suppose our choice of generators for $I(X)$ has $p$ generators, $p\ge n-d$. By a change of coordinates we can assume the equations have the form $x_1=\cdots=x_{n-d}=g_{n-d+1}=\cdots=g_p=0$, where the matrix of generators of $\mathrm{JM}(X)$ must have the last $d$ columns 0. Then the Rees algebra of $\mathrm{JM}(X)\oplus\mathcal{O}_X$ is the same as before, as is the ideal induced by $\mathrm{JM}(X,\omega)$, so the multiplicity of the pair is the same. □

**Proposition 5.12.** *Let $\{\omega_j^{(i)}\}$ be a collection of 1-forms, with $1\le i\le s$, $1\le j\le d-k_i+1$, $\sum k_i=d$ such that, restricted to $X^d$, $\{\omega^{(s)}\}$ has a non-degenerate special point at $x$, $x$ a smooth point of $\mathcal{C}$ and $X$. Then*

$$e(\mathrm{JM}(X,\omega^{(s)})|_{\mathcal{C}},\mathrm{JM}(X)|_{\mathcal{C}}\oplus\mathcal{O}_{\mathcal{C}^{n-k_s+1}},x)=1.$$

*Proof.* Let us suppose, that $X$ is a smooth manifold and the number of equations of $X$ is $n-d$ so that the matrix of generators of $\mathrm{JM}(X,\omega^{(s)})$ has $n-k_s+1$ rows, $n$ columns, and a matrix of generators of $\mathrm{JM}(X)\oplus\mathcal{O}_{\mathcal{C}}^{d-k_s+1}$ also has $n-k_s+1$ rows with the same $n-d$ first rows. We may assume the equations for $X$ are $x_1=\cdots=x_{n-d}=0$. Since we assume $\mathcal{C}$ is smooth at $x$, and it has dimension $k_s$, assume that the last $k_s$ coordinates on $X$ define $\mathcal{C}$. We may assume that the collection $\omega^{(s)}$ has form $\{dx_{n-d+i},dh\}$ where $1\le i\le d-k_s$ and $h=\sum_{j=1}^{k_s}x^2_{n-k_s+j}$.

As in the last Proposition, in this example, we know that

$$\begin{aligned}1&=e(\mathrm{JM}(X,\omega^{(s)})|_C,\mathcal{O}_C^{n-k_s+1})\\&=e(\mathrm{JM}(X,\omega^{(s)}),\mathrm{JM}(X)|_C\oplus\mathcal{O}_C)+e(\mathrm{JM}(X)|_C\oplus\mathcal{O}_C^{d-k_s+1}),\mathcal{O}_C^{n-k_s+1}),\end{aligned}$$

while $e(\mathrm{JM}(X)|_C\oplus\mathcal{O}_C^{d-k_s+1},\mathcal{O}_C^{n-k_s+1})=0$ since the two modules are the same. This uses the additivity of the multiplicity, the fact that $\omega$ is non-degenerate on $X$, and the fact that the multiplicity of $e(\mathrm{JM}(X,\omega^{(s)}),\mathcal{O}_X^{n-k_s+1})$ is the colength of its ideal of maximal minors, and as in the last Proposition, the general result follows.

□

## 6. Main Result

Before giving our main result, it is useful to consider the difference between the case of a vector bundle well-defined at all points, and a bundle like the tangent bundle to a singular space which is not well-defined at $S(X)$. In the second case, we get a special point if $\widetilde{X}(0)\cap B(\omega)$ is non-empty. If we alter the last collection of forms, then we can make the last collection generic on $\mathcal{C}$, but the singular locus of the modified forms may still be non-empty. In this case the intersection number $\mathrm{Im}(\Gamma_{\widetilde{\omega}})$ and $\mathbb{D}^k_X$ may still be non-zero at the origin.

In the first case, the analogue of $\widetilde{X}(0)$ consists of a single point, so by altering the last collection of forms we can ensure that the intersection number $\mathrm{Im}(\Gamma_{\widetilde{\omega}})$ and $\mathbb{D}^k_X$ is zero at the origin.

This phenomena is the reason that the formula for the Chern numbers for the Nash bundle has many terms, while that of a vector bundle on $X$ has only one.

The next theorem is the key step in the proof of our main result. It allows us to fix each of the collections in turn, until we are left with collections which are linearly independent at the origin and which are proper. Of course, this last collection has no special points.

**Theorem 6.1.** *Let $(X^d,0)\subset(\mathbb{C}^n,0)$ be the germ of an equidimensional reduced analytic variety, with representative $X$, $\{\omega^{(i)}_j\}$, a collection of 1-forms with $1\le i\le s$, $1\le j\le d-k_i+1$, $\sum k_i=d$. Assume further the collection has an isolated singularity at the origin, and that the generic point of $S(\omega^{(i)}_j)$, $1\le i\le s-1$, is in $X_{\mathrm{reg}}$. We have that,*

$$\mathrm{Ch}_{X,0}\{\omega^{(i)}_j\}=e(\mathrm{JM}(X,\omega^{(s)})|_{\mathcal{C}},\mathrm{JM}(X)|_{\mathcal{C}}\oplus\mathcal{O}^{d-k_s+1}_{\mathcal{C}},0)+\mathrm{Im}(\Gamma_{\widetilde{\omega}})\cdot\mathbb{D}^k_X.$$

*Proof.* Let us consider the family of sets $\mathcal{C}\times\mathbb{C}$. Let $\pi_{\mathcal{C}}$ denote the projection from $\mathcal{C}\times\mathbb{C}$ to $\mathcal{C}$, and $\pi_t$ the projection to $\mathbb{C}$. By conservation of number and taking $M$ as in the Proposition 4.10, $\Gamma(\omega^{(s)})\cdot T^*(X)|_{\mathcal{C}}$ is just

$$\Gamma(\omega^{(s)}+tM)\cdot T^*(X)|_{\mathcal{C}}$$

for $t$ close to 0, and this is just the number of non degenerate special points of

$$\{(\omega^{(s)})_{1\le i\le s-1},\omega^{(s)}+tM\}$$

for $t\neq 0$, and the intersection number $\mathrm{Im}(\Gamma_{\widetilde{\omega}})\cdot\mathbb{D}^k_X$. (Recall that the collection $\widetilde{\omega}$ was defined before Cor 4.12.) To show that the Multiplicity Polar theorem applies, we must also show that

$$C(\mathrm{Projan}(\mathcal{R}(\mathrm{JM}(X\times\mathbb{C},\omega_t)_{\pi_t}|_{\mathcal{C}\times\mathbb{C}})))(0)=C(\mathrm{Projan}(\mathcal{R}(\mathrm{JM}(X,\omega)|_{\mathcal{C}}))),$$

except possibly over $(0,0)\in\mathcal{C}\times 0$. Since $N=\mathrm{JM}(X)|_{\mathcal{C}\times C}\mathcal{O}_{\mathcal{C}\times C}\oplus\mathcal{O}^{d-k_s+1}_{\mathcal{C}\times C}$ as a family of modules is independent of $t$, $\mathrm{Projan}\, R(N)$ is a product, hence $C(\mathrm{Projan}\,\mathcal{R}(N))(0)=C(\mathrm{Projan}\,\mathcal{R}(N(0)))$. Now, at any point $p$ of $\mathcal{C}\times 0$ close to the origin, there exists a neighborhood $U$ of $p$ such that on $U$, $\overline{\mathrm{JM}(X\times\mathbb{C},\omega_t)_{\pi_t}|_{\mathcal{C}\times\mathbb{C}}}=N$. This implies that over $U$, $\mathrm{Projan}\,\mathcal{R}(N)$ is finite over $\mathrm{Projan}\,\mathcal{R}(\mathrm{JM}(X\times\mathbb{C},\omega_t)_{\pi_t}|_{\mathcal{C}\times\mathbb{C}})$ and, on $U\cap C\times 0$, $\mathrm{Projan}\,\mathcal{R}(N(0))$ is finite over

$$\mathrm{Projan}\,\mathcal{R}(\mathrm{JM}(X,\omega_0)|_C).$$

Now, since $\mathrm{Projan}\, R(\mathrm{JM}(X,\omega_0)|_C)\subset\mathrm{Projan}\,\mathcal{R}(\mathrm{JM}(X\times\mathbb{C},\omega_t)_{\pi_t}|_{\mathcal{C}\times\mathbb{C}})(0)$, the desired equality follows, for any element of $\mathrm{Projan}\,\mathcal{R}(\mathrm{JM}(X\times\mathbb{C},\omega_t)_{\pi_t}|_{\mathcal{C}\times\mathbb{C}})(0)$ has a preimage in $\mathrm{Projan}\,\mathcal{R}(N)(0)$ which is $\mathrm{Projan}\,\mathcal{R}(N(0))$, and the last set maps to $\mathrm{Projan}\,\mathcal{R}((\mathrm{JM}(X,\omega_0)|_C))$. So, the multiplicity polar theorem applies. Note, that since $\mathrm{Projan}\,\mathcal{R}(N)$ is a product, $N$ has no polar curve, and by 5.10 we know that $\mathrm{JM}(X\times\mathbb{C},\omega_t)_{\pi_t}|_{\mathcal{C}\times\mathbb{C}})$ has no polar curve either. Now, by Proposition 4.10 we have, $\mathrm{Ch}_{X,0}(\omega^{(i)}_j)=\Gamma(\omega^s+tM)\cdot T^*(X)|_C$.

Then, using the Multiplicity Polar Theorem we have,

$$\mathrm{Ch}_{X,0}(\omega^{(i)}_j)=e(\mathrm{JM}(X,\omega^{(s)})|_{\mathcal{C}},\mathrm{JM}(X)|_{\mathcal{C}}\oplus\mathcal{O}^{d-k_s+1}_{\mathcal{C}},0)+\mathrm{Im}(\Gamma_{\widetilde{\omega}})\cdot\mathbb{D}^k_X.$$

□

Suppose the collection is ordered so that the first $r$ collections meet properly and $r$ is the largest integer for which this is true. Let $C_i$ denote $C(\omega^{(1)},\dots,\omega^{(i)},\widetilde{\omega^{(s)}},\dots,\widetilde{\omega^{(i+2)}})$ where $i\le s-1$, and $\widetilde{\omega^{(j)}}$ is a collection of generic linear forms so that the collections $\{\omega^{(i)}\}, i\le r$, $\{\widetilde{\omega^{(j)}}\}$ meet properly.

**Corollary 6.2.** *Suppose the collection is ordered so that the first $r$ collections meet properly and $r$ is the largest integer for which this is true. In the setup of the last Theorem, we have that*

$$\mathrm{Ch}_{X,0}(\omega) = \sum_{i=r}^{i=s-1} e(\mathrm{JM}(X, \omega^{(i+1)})|_{\mathcal{C}_i}, \mathrm{JM}(X)|_{\mathcal{C}_i} \oplus \mathcal{O}_{\mathcal{C}_i}^{d-k_{i+1}+1}, 0)$$

*Proof.* We prove the Corollary by applying the previous theorem multiple times. First, to $\{\omega^{(s)}|_{\mathcal{C}_{s-1}}\}$, then to $\{\omega^{(s-1)}|_{\mathcal{C}_{s-2}}\}$. Finally, when all but one of our collections meet properly, applying the theorem to $\{\omega^{(r+1)}|_{\mathcal{C}_r}\}$ produces only a single term as the intersection number term is 0.

□

Let $C_{i'}$ denote $C(\widetilde{\omega^{(s)}}, \ldots, \widetilde{\omega^{(i+2)}})$. Then, $C_{i'}$ is related to the polar varieties of $X$. For $C(\widetilde{\omega^{(i+2)}})$ is the polar variety of codimension $k(i+2)$, so $C_{i'}$ is the intersection of the corresponding polar varieties. If $X$ is a hypersurface, then in fact this is the polar variety of codimension $\sum_{i+2}^{s} k(j)$. The hypersurface case is special because since $TX_x$, $x \in X_0$ has codimension 1, the kernels of all of the $\widetilde{\omega^{(j)}}$ are contained in $TX_x$ if $x \in C_{i'}$, hence $x$ is in the polar variety defined by the union of the kernels.

**Corollary 6.3.** *In the set up of the last proposition we have*

$$\begin{aligned}\mathrm{Ch}_{X,0}(\omega) &= \sum_{i=r}^{i=s-1} e(\mathrm{JM}(X, \omega^{(i+1)})|_{\mathcal{C}_i}, \mathrm{JM}(X)|_{\mathcal{C}_i} \oplus \mathcal{O}_{\mathcal{C}_i}^{d-k_{i+1}+1}, 0) \\ &= \sum_{i=r}^{i=s-1} e(\mathrm{JM}(X, \omega^{i+1})|_{\mathcal{C}_i}, H_{c_i-1}(X, \mathcal{C}_i) \oplus \mathcal{O}_{\mathcal{C}_i}^{d-k_{i+1}+1}, 0) \\ &\quad - e(\mathrm{JM}(X, \widetilde{\omega^{i+1}})|_{\mathcal{C}_i}, H_{c_i-1}(X, \mathcal{C}_i) \oplus \mathcal{O}_{\mathcal{C}_i}^{d-k_{i+1}+1}, 0).\end{aligned}$$

*Proof.* Apply Proposition 5.9 to expand

$$e(\mathrm{JM}(X, \omega^{(i+1)})|_{\mathcal{C}_i}, \mathrm{JM}(X)|_{\mathcal{C}_i} \oplus \mathcal{O}_{\mathcal{C}}^{d-k_{i+1}+1}, 0).$$

□

We want to specialize our results to the case where $X^d, 0 \subset \mathbb{C}^n$ is an ICIS to compare with those of [7]. Given a collection of holomorphic forms $\omega$ with an isolated singular point at 0, Ebeling and Gusein-Zade define another notion of index in [7]. In the case $X, 0$ is an ICIS, the index amounts to smoothing $X$ as well as making the forms general, then counting the number of singular points of the new collection on the smoothing. This index is an extension of the GSV-index [20, 28].

This index can be calculated as follows: suppose $\omega_j^{(i)}$, $1 \leq i \leq s$, $1 \leq j \leq n_{k_i} + 1$, $\sum k_i = d$, augment the jacobian matrix of $X$ for each $i$ with $\omega_j^{(i)}$, producing $s$ matrices. Form an ideal in $\mathcal{O}_n$, using as generators, the generators of $I(X)$, and the maximal minors of the augmented matrices. Denote the resulting ideal by $I_{X,\omega_j^{(i)}}$. Then the index, denoted $\mathrm{ind}_{X,0}(\{\omega\})$ is just the colength of $I_{X,\omega_j^{(i)}}$ in $\mathcal{O}_n$. ([7], Theorem 20.) Using this index they show that

$$\mathrm{Ch}_{X,0}(\omega_j^{(i)}) = \mathrm{ind}_{X,0}(\{\omega\}) - \mathrm{ind}_{X,0}(\{l\})$$

where $l = \{l_j^{(i)}\}$ is a generic collection of forms. ([7] Cor. 4.)

We will see that this formula can be recovered from the last corollary. If $X$ is an ICIS, and the $\mathcal{C}_i$ have the minimal dimension then the $\mathcal{C}_i$ are Cohen-Macaulay, with ideal the ideal of $X$ and the maximal minors of the augmented matrices. Further, the matrix of generators of $\mathrm{JM}(X)$ has maximal rank except at the origin when restricted to $\mathcal{C}_i$. This implies that $H_{c_i-1}(X, \mathcal{C}_i)|_{\mathcal{C}_i}$ is free, so $e(\mathrm{JM}(X, \omega^{(i+1)})|_{\mathcal{C}_i}, H_{c_i-1}(X, \mathcal{C}_i) \oplus \mathcal{O}_{\mathcal{C}_i}^{d-k_{i+1}+1}, 0) = e(\mathrm{JM}(X, \omega^{(i+1)})|_{\mathcal{C}_i})$. Since $\mathcal{O}_{\mathcal{C}i}$ is Cohen-Macaulay, the last multiplicity is just the colength of the ideal formed by the maximal minors of the augmented matrices formed from $\mathrm{JM}(X)$, the collection of forms used to define $\mathcal{C}_i$ and $\omega^{(i+1)}$. It follows that

$$e(\mathrm{JM}(X,\omega^{(i+1)})|_{\mathcal{C}_i}) = \mathrm{ind}(\{\omega^{(1)},\ldots,\omega^{(i)},\omega^{i+1},\widetilde{\omega^{(s)}},\ldots,\widetilde{\omega^{(i+2)}}\}).$$

Now the sum on the right hand side of the last corollary telescopes to

$$\mathrm{Ch}_{X,0}(\omega_j^{(i)}) = \mathrm{ind}_{X,0}(\{\omega\}) - \mathrm{ind}(\{\omega^{(1)},\ldots,\omega^{(r)},\widetilde{\omega^{(s)}},\ldots,\widetilde{\omega^{(r+1)}}\}).$$

Since the last collection on the right hand side is proper, an argument similar to that of Prop. 5.5 shows that the last term is $\mathrm{ind}_{X,0}(\{l\})$.

In the case of surfaces it is not hard to compute with our formula, and we give some examples. As preparation we give two versions of our formula for the case of surfaces which are not ICIS.

The general case of our theorem becomes:

**Corollary 6.4.** *Let $(X,0) \subset (\mathbb{C}^n,0)$ be the germ of a purely 2 dimensional reduced analytic variety, with representative $X$, $\{\omega\}$, a collection of sets of 1-forms $\{\omega^{(i)}\}$ $1 \le i \le 2$, each with two elements. Assume further the collection has an isolated singularity at the origin, and that the generic point of $S(\omega_j^{(1)})$ is in $X_{\mathrm{reg}}$. We have that,*

$$\mathrm{Ch}_{X,0}\{\omega_j^{(i)}\} = e(\mathrm{JM}(X,\omega^{(2)})|_{\mathcal{C}}, \mathrm{JM}(X)|_{\mathcal{C}} \oplus \mathcal{O}_{\mathcal{C}}^{d-k_s+1}, 0) + \mathrm{Im}(\Gamma_{\widetilde{\omega}}) \cdot \mathbb{D}_X^k.$$

*Further,*

$$\mathrm{Ch}_{X,0}\{\omega_j^{(i)}\} = e(\mathrm{JM}(X,\omega^{(2)})|_{\mathcal{C}_1}, H_0(X,\mathcal{C}_1) \oplus \mathcal{O}_{\mathcal{C}_1}^{n-d}, 0)$$
$$-e(\mathrm{JM}(X,\widetilde{\omega^{(2)}})|_{\mathcal{C}_1}, H_0(X,\mathcal{C}_1) \oplus \mathcal{O}_{\mathcal{C}_1}^{n-d}, 0) + e(\mathrm{JM}(X,\omega^{(1)})|_{\mathcal{C}_0}, H_0(X,\mathcal{C}_0) \oplus \mathcal{O}_{\mathcal{C}_0}^{n-d}, 0)$$
$$-e(\mathrm{JM}(X,\widetilde{\omega^{(1)}})|_{\mathcal{C}_0}, H_0(X,\mathcal{C}_0) \oplus \mathcal{O}_{\mathcal{C}_0}^{n-d}, 0).$$

*Proof.* This is Theorem 5.13 and Corollary 5.15 for the surface case. □

Further simplification is possible, if $X$ is a complete intersection.

**Corollary 6.5.** *Suppose in addition $X$ is a complete intersection. Then*

$$\mathrm{Ch}_{X,0}\{\omega_j^{(i)}\} = e(\mathrm{JM}(X,\omega^{(2)})|_{\mathcal{C}_1}, 0) - e(\mathrm{JM}(X,\widetilde{\omega^{(2)}})|_{\mathcal{C}_1}, 0)$$
$$+e(\mathrm{JM}(X,\omega^{(1)})|_{\mathcal{C}_0}, 0) - e(\mathrm{JM}(X,\widetilde{\omega^{(1)}})|_{\mathcal{C}_0}, 0)$$
$$= \Gamma^1(\omega^{(1)}) \cdot \Gamma^1(\omega^{(2)}) - \Gamma^1(\widetilde{\omega}^{(1)}) \cdot \Gamma^1(\widetilde{\omega}^{(2)})$$

*Proof.* The first equality holds because $X$ is a complete intersection, and the generic point of $\mathcal{C}_i$ lies in the regular part of $X$, the Jacobian module of $X$ has maximal rank off the origin, so $H_0(X,\mathcal{C}_1) \oplus \mathcal{O}_{\mathcal{C}_1}^{n-d}$ is just $\mathcal{O}_{\mathcal{C}_1}^n$ so the multiplicity of this pair is just the ordinary Buchsbaum-Rim multiplicity. Since our curves are reduced their rings are Cohen Macaulay, so the multiplicity of $\mathrm{JM}(X,\omega^{(2)})|_{\mathcal{C}_1}$ at 0 is just the colength of the ideal generated by the determinant of the matrix of generators of $\mathrm{JM}(X,\omega^{(2)})|_{\mathcal{C}_1}$. This determinant on $X$ defines the union of $S(X)$ and $\Gamma^1(\omega^{(2)})$ since it does so generically. Thus, this colength is just the intersection of $\Gamma^1(\omega^{(1)})$ with $\Gamma^1(\omega^{(2)})$ and $S(X)$. Applying this insight to each of the terms of the first equality and canceling terms results in the next equality. □

We give an example using this result.

**Example 6.6.** *Let $(X,0) \subset \mathbb{C}^3$ be the germ of a singular surface defined by the function $f : \mathbb{C}^3, 0 \to \mathbb{C}, 0$, where $f(x,y,z) = y^2 - x^3$. Take the collection of 1-forms $\omega = \{\omega^1, \omega^2\}$, where $\omega^1 = \{(0,x^3,z^2),(z^3,0,x^2)\}$, and $\omega^2 = \{(y^2,z^3,0),(0,y^3,z^2)\}$. Then the local Chern obstruction of this collection is* 47.

We will show this using the second equality in the last Corollary. The matrix of generators of $\mathrm{JM}(X,\omega^1)$ and $\mathrm{JM}(X,\omega^2)$ are respectively,

$$\begin{pmatrix} -3x^2 & 2y & 0 \\ 0 & x^3 & z^2 \\ z^3 & 0 & x^2 \end{pmatrix}, \begin{pmatrix} -3x^2 & 2y & 0 \\ y^2 & z^3 & 0 \\ 0 & y^3 & z^2 \end{pmatrix}$$

Calculating the determinants of the matrix of generators of $\mathrm{JM}(X,\omega^2)$ and $\mathrm{JM}(X,\omega^1)$, we get $z^2(2y^3+3x^2z^3)$ and $-3x^7+2z^5y$. Since we are only interested in the polar curves of $\omega^i$, we use the defining equation to get the equivalent forms $z^2x^2(2xy+3z^3)$ and $y(-3xy^3+2z^5)$. So the equations of the polar curves of our collection are $z^2(2xy+3z^3)=0$ and $-3xy^3+2z^5=0$. To calculate the intersection multiplicity, pull back to the normalization using the map

$$n(t,z)=(t^2,t^3,z).$$

So we want the intersection multiplicity of $z^2(2t^5+3z^3)=0$ and $-3t^{11}+2z^5=0$, which is (2)(11)+25=47. Since our underlying space is Whitney equisingular, the polar curves of $X$ are empty, so the term we have computed is the only term in the corollary, so the local Chern obstruction of this collection is 47. (Notice that in this example, one component of the polar of $\mathrm{JM}(X,\omega^2)$ is not reduced. Nonetheless, a careful reading of the proof of our main result shows that in this simple case the main result continues to hold.)

We describe briefly how the work of this section can be generalized. Start with an analytic space $X$, and a bundle $E^k$ defined on a Zariski open, everywhere-dense subset $U$ of $X$, $E^k$ a sub-bundle with $k$-dimensional fiber of a bundle, $F^l$, where $F^l$ is defined everywhere. Form the relative Nash transformation $N(X,E,F)$ of $X$ as follows: form the bundle over $X$ of $k$ planes in the fiber of $F$, consider the image of the section of this bundle formed from the fibers of $E^k$, and take its closure. The relative Nash transformation has a canonical bundle $\xi$ on it which is a sub-bundle of the pullback of $F^l$ to $N(X,E,F)$, $\xi$ and the pullback of $E$ to $N(X,E,F)$ agree restricted to $U$. By construction and restriction, sections of $F^*$ give sections of $E^*|U$, and $\xi^*$. If a collection of sections of $E^*|U$ arise in this way from a collection of sections of $F^*$, and the collection has an isolated special point at $x\in X$, then we can compute the contribution to the Chern number of the dual from our set of sections (hence to $\xi$) as we did in this section to the dual of the Nash bundle. As in the Nash bundle case, the contribution will be a sum depending on the polar varieties of $E$ relative to $F$ and their intersections. These polar varieties provide some measure of the geometry of $E$ at its singular points on $X$.

## Acknowledgments

The authors acknowledge the referees for their suggestions and remarks. The second author acknowledges the financial support given by FAPESP grant 2009/08774-0, the program USP-COFECUB grant 07.1.12081.1.7. and CNPq grants 305560/2010-7 and 200430/2011-4.

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**Journal *of* Singularities**
**Volume 7 (2013), 61-87**

received 10 March 2013
DOI: 10.5427/jsing.2013.7e

# TOPOLOGICAL INVARIANTS AND MODULI OF GORENSTEIN SINGULARITIES

SERGEY NATANZON AND ANNA PRATOUSSEVITCH

Abstract. We describe all connected components of the space of hyperbolic Gorenstein quasi-homogeneous surface singularities. We prove that any connected component is homeomorphic to a quotient of $\mathbb{R}^d$ by a discrete group.

## 1. Introduction

In this paper, we study moduli spaces of hyperbolic Gorenstein quasi-homogeneous surface singularities (GQHSS). Quasi-homogeneous surface singularities and their graded affine coordinate rings were studied in great detail in the 1970's by Dolgachev, Milnor, Neumann, Pinkham and others. They described a correspondence between hyperbolic quasi-homogeneous surface singularities and (singular) complex line bundles on Riemann orbifolds, where a Riemann orbifold is a quotient $\mathbb{H}/\Gamma$ of the hyperbolic plane $\mathbb{H}$ by a Fuchsian group $\Gamma$. According to the work of Dolgachev [Dol83b], hyperbolic GQHSS of level $m$ are in 1-to-1 correspondence with $m$-th roots of tangent bundles of Riemann orbifolds, i.e., with (singular) complex line bundles on Riemann orbifolds such that their $m$-th tensor power coincides with the tangent bundle. For more details of this correspondence, see section 2.1. In this paper, we complete the classification of hyperbolic Gorenstein quasi-homogeneous surface singularities by studying their parameter space.

We determine the number and describe the topology of connected components of the space of all hyperbolic Gorenstein quasi-homogeneous surface singularities. We show that the space is connected if $g=0$ or if $g>1$ and $m$ is odd and that the space has two connected components if $g>1$ and $m$ is even. We also determine the number of connected components in the case $g=1$. Moreover, we prove that any component is homeomorphic to a quotient of $\mathbb{R}^d$ by a discrete group action.

On the other hand, we obtain a description of the topology of the moduli space of higher spin structures on Riemann orbifolds. Because of their role in the modern models of 2D-gravitation, moduli spaces of this kind have enjoyed a great deal of interest recently, in particular, in relation to the Witten conjecture; see [Jar00, JKV01, FSZ10] and [Wit93]. Classification of classical spin structures on non-compact Riemann surfaces and the corresponding moduli space were studied in [Nat91, Nat04].

2010 *Mathematics Subject Classification.* Primary 32S25, 14H60, 30F35; Secondary 30F60.

*Key words and phrases.* Gorenstein singularities, $\mathbb{Q}$-Gorenstein singularities, quasi-homogeneous surface singularities, higher spin structures, moduli spaces, Arf functions, lifts of Fuchsian groups.

The first author was partially supported by the grants RFBR 10-01-00678, NSh 8450.2012.1, Russian Federation Government Grant No. 2010-220-01-077. This study was carried out within "The National Research University Higher School of Economics" Academic Fund Programme in 2013–14, research grant No. 12-01-0122. The second author was partially supported by SFB 611 (Bonn), RCMM (Liverpool).

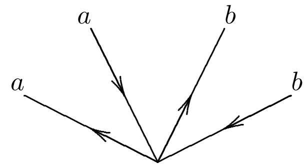


Figure 1: $\hat{\sigma}(ab) = \hat{\sigma}(a) + \hat{\sigma}(b) - 1$

The main technical tool is the following: We assign (Theorem 5.9) to a hyperbolic GQHSS of level $m$ with corresponding orbifold $P$ a function on the space of homotopy classes of simple contours on $P$ with values in $\mathbb{Z}/m\mathbb{Z}$, the associated higher $m$-Arf function.

The higher $m$-Arf functions are described by simple geometric properties:

**Definition:** Let $P$ be a Riemann orbifold and $p \in P$. Let $\pi(P, p)$ be the orbifold fundamental group of $P$ (see Definition 4.3). We denote by $\pi^0(P, p)$ the set of all non-trivial elements of $\pi(P, p)$ that can be represented by simple contours. A *(higher) $m$-Arf function* is a function

$$\sigma : \pi^0(P, p) \to \mathbb{Z}/m\mathbb{Z}$$

satisfying the following conditions

1. $\sigma(bab^{-1}) = \sigma(a)$ for any elements $a, b \in \pi^0(P, p)$,
2. $\sigma(a^{-1}) = -\sigma(a)$ for any element $a \in \pi^0(P, p)$ that is not of order 2,
3. $\sigma(ab) = \sigma(a) + \sigma(b)$ for any elements $a$ and $b$ which can be represented by a pair of simple contours in $P$ intersecting in exactly one point $p$ with $\langle a, b \rangle \neq 0$,
4. $\sigma(ab) = \sigma(a) + \sigma(b) - 1$ for any elements $a, b \in \pi^0(P, p)$ such that the element $ab$ is in $\pi^0(P, p)$ and the elements $a$ and $b$ can be represented by a pair of simple contours in $P$ intersecting in exactly one point $p$ with $\langle a, b \rangle = 0$ and a neighbourhood of the point $p$ with the contours is homeomorphic to the one shown in Figure 1.
5. For any elliptic element $c$ of order $p$ we have $p \cdot \sigma(c) + 1 \equiv 0 \bmod m$.

In order to be able to state our main results we need to give some definitions.

**Definition:** (For a detailed discussion of Fuchsian groups, signatures and standard sets of generators, see Section 4.2.) Let $\Gamma$ be a Fuchsian group such that the corresponding orbifold $P = \mathbb{H}/\Gamma$ is of signature $(g : p_1, \ldots, p_r)$. Let $\sigma : \pi^0(P, p) \to \mathbb{Z}/m\mathbb{Z}$ be a higher $m$-Arf function. We define the *Arf invariant* $\delta = \delta(P, \sigma)$ of $\sigma$ as follows: If $g > 1$ and $m$ is even then we set $\delta = 0$ if there is a standard set of generators $\{a_1, b_1, \ldots, a_g, b_g, c_{g+1}, \ldots, c_{g+r}\}$ of the orbifold fundamental group $\pi(P, p)$ such that

$$\sum_{i=1}^{g} (1 - \sigma(a_i))(1 - \sigma(b_i)) \equiv 0 \bmod 2$$

and we set $\delta = 1$ otherwise. If $g > 1$ and $m$ is odd then we set $\delta = 0$. If $g = 0$ then we set $\delta = 0$. If $g = 1$ then we set

$$\delta = \gcd(m, p_1 - 1, \ldots, p_r - 1, \sigma(a_1), \sigma(b_1)),$$

where $\{a_1, b_1, c_2, \ldots, c_{r+1}\}$ is a standard set of generators of the orbifold fundamental group $\pi(P, p)$. The *type of the higher $m$-Arf function* $(P, \sigma)$ is the tuple $(g, p_1, \ldots, p_r, \delta)$, where $\delta$ is the Arf invariant of $\sigma$ defined above.

**Definition:** We denote by $S^m(t) = S^m(g, p_1, \ldots, p_r, \delta)$ the set of all GQHSS of level $m$ such that the associated higher Arf function is of type $t = (g, p_1, \ldots, p_r, \delta)$.

The following theorem summarizes the main results:

**Theorem:**

1) Two hyperbolic GQHSS are in the same connected component of the space of all hyperbolic GQHSS if and only if they are of the same type. In other words, the connected components of the space of all hyperbolic GQHSS are those sets $S^m(t)$ that are not empty.
2) The set $S^m(t)$ is not empty if and only if $t = (g, p_1, \dots, p_r, \delta)$ has the following properties:
   (a) The orders $p_1, \dots, p_r$ are prime with $m$ and satisfy the condition
   $$(p_1 \cdots p_r) \cdot \left( \sum_{i=1}^{r} \frac{1}{p_i} - (2g-2) - r \right) \equiv 0 \bmod m.$$
   (b) If $g > 1$ and $m$ is odd then $\delta = 0$.
   (c) If $g = 1$ then $\delta$ is a divisor of $\gcd(m, p_1 - 1, \dots, p_r - 1)$.
   (d) If $g = 0$ then $\delta = 0$.
3) Any connected component $S^m(t)$ of the space of all hyperbolic GQHSS of level $m$ and signature $(g : p_1, \dots, p_r)$ is homeomorphic to a quotient of the space $\mathbb{R}^{6g-6+2r}$ by a discrete action of a certain subgroup of the modular group (see Section 6.3 for details).

The paper is organised as follows: In Section 2, we explore the connection between hyperbolic GQHSS, roots of tangent bundles of orbifolds and lifts of Fuchsian groups into the coverings $G_m$ of $G = \mathrm{PSL}(2, \mathbb{R})$ (see Definition 2.2). In Section 3, we study algebraic properties of the covering groups $G_m$. We describe level functions induced by a decomposition of the covering $G_m$ into sheets and choosing an enumeration of the sheets and study properties of these functions. In Section 4, we study lifts of Fuchsian groups into $G_m$. In Section 5, we define (higher) $m$-Arf functions. We prove that there is a 1-1-correspondence between the set of $m$-Arf functions and the set of functions associated to the lifts of Fuchsian groups into $G_m$ via the enumeration of the covering sheets. Hence, these two sets are also in 1-1-correspondence with the set of all hyperbolic GQHSS of level $m$. Moreover, we show that the set of all $m$-Arf functions on an orbifold has a structure of an affine space, using an explicit description of the generalised Dehn generators of the group of homotopy classes of surface self-homeomorphisms. In the last section, we find topological invariants of higher Arf functions and prove that they describe the connected components of the moduli space. Furthermore, we show using a version of the theorem of Fricke and Klein [Nat78, Zie81] that any connected component is homeomorphic to a quotient of $\mathbb{R}^d$ by a discrete group action.

Part of this work was done during the stays at Max-Planck-Institute in Bonn and at IHES. We are grateful to both institutions for their hospitality and support. We would like to thank E.B. Vinberg and V. Turaev for useful discussions related to this work. We would like to thank the referees for their valuable remarks and suggestions.

## 2. Gorenstein singularities and lifts of Fuchsian groups

2.1. **Singularities and automorphy factors.** In this section, we recall the results of Dolgachev, Milnor, Neumann and Pinkham [Dol75, Dol77, Mil75, Neu77, Pin77] on the graded affine coordinate rings, which correspond to quasi-homogeneous surface singularities.

**Definition 2.1.** *A negative unramified automorphy factor* $(U, \Gamma, L)$ is a complex line bundle $L$ over a simply-connected Riemann surface $U$ together with a discrete co-compact subgroup $\Gamma \subset \mathrm{Aut}(U)$ acting compatibly on $U$ and on the line bundle $L$, such that the following two conditions are satisfied:

1) The action of $\Gamma$ is free on $L^*$, the complement of the zero-section in $L$.
2) Let $\tilde{\Gamma} \triangleleft \Gamma$ be a normal subgroup of finite index, which acts freely on $U$, and let $E \to P$ be the complex line bundle $E = L/\tilde{\Gamma}$ over the compact Riemann surface $P = U/\tilde{\Gamma}$. Then $E$ is a negative line bundle, i.e., the self-intersection number $P \cdot P$ is negative.

A simply-connected Riemann surface $U$ can be $\mathbb{CP}^1$, $\mathbb{C}$, or $\mathbb{H}$, the real hyperbolic plane. We call the corresponding automorphy factor and the corresponding singularity *spherical*, *Euclidean*, resp. *hyperbolic*.

*Remark.* There always exists a normal freely acting subgroup of $\Gamma$ of finite index [Dol83b]. In the hyperbolic case the existence follows from the theorem of Fox-Bundgaard-Nielsen. If the second assumption in the last definition holds for some normal freely acting subgroup of finite index, then it holds for any such subgroup, see [Dol83b].

*Remark.* In this paper we are going to study the largest class, the class of hyperbolic GQHSS. See a remark at the end of the paper for more information about the other two classes of GQHSS.

The simplest example of such complex line bundle with a group action is the (co)tangent bundle of $U = \mathbb{CP}^1$, resp. $U = \mathbb{H}$, equipped with the canonical action of a subgroup of $\mathrm{Aut}(U)$.

Let $(U, \Gamma, L)$ be a negative unramified automorphy factor and $\tilde{\Gamma}$ a normal subgroup of $\Gamma$ as above. Since the bundle $E = L/\tilde{\Gamma}$ is negative, one can contract the zero section of $E$ to get a complex surface with one isolated singularity corresponding to the zero section. There is a canonical action of the group $\Gamma/\tilde{\Gamma}$ on this surface. The quotient is a complex surface $X(U, \Gamma, L)$ with an isolated singular point $o(U, \Gamma, L)$, which depends only on the automorphy factor $(U, \Gamma, L)$.

The following theorem summarizes the results of Dolgachev, Milnor, Neumann and Pinkham:

**Theorem 2.1.** *The surface $X(U, \Gamma, L)$ associated to a negative unramified automorphy factor $(U, \Gamma, L)$ is a quasi-homogeneous affine algebraic surface with a normal isolated singularity. Its affine coordinate ring is the graded $\mathbb{C}$-algebra of generalised $\Gamma$-invariant automorphic forms*

$$A = \bigoplus_{m \geqslant 0} H^0(U, L^{-m})^{\Gamma}.$$

*All normal isolated quasi-homogeneous surface singularities $(X, x)$ are obtained in this way, and the automorphy factors with $(X(U, \Gamma, L), o(U, \Gamma, L))$ isomorphic to $(X, x)$ are uniquely determined by $(X, x)$ up to isomorphism.*

We now recall the definition of Gorenstein singularities and the characterization of the corresponding automorphy factors.

An isolated singularity of dimension $n$ is Gorenstein if and only if it is Cohen-Macaulay and there exists a nowhere vanishing holomorphic $n$-form on a punctured neighbourhood of the singular point. A normal isolated surface singularity is Gorenstein if and only if there exists a nowhere vanishing holomorphic 2-form on a punctured neighbourhood of the singular point. For example all isolated singularities of complete intersections are Gorenstein.

In [Dol83b], Dolgachev proved the following theorem obtained independently by W. Neumann (see also [Dol83a]).

**Theorem 2.2.** *A quasi-homogeneous surface singularity is Gorenstein if and only if for the corresponding automorphy factor $(U, \Gamma, L)$ there is an integer $m$ (called the level or the exponent of the automorphy factor) such that the $m$-th tensor power $L^m$ is $\Gamma$-equvariantly isomorphic to the tangent bundle $T_U$ of the surface $U$.*

Let $(U, \Gamma, L)$ be an automorphy factor of level $m$, which corresponds to a Gorenstein singularity. The isomorphism $L^m \cong T_U$ induces an isomorphism $E^m \cong T_P$. A simple computation with Chern numbers shows that the possible values of the exponent are $m = -1$ or $m = -2$ for $U = \mathbb{CP}^1$, whereas $m = 0$ for $U = \mathbb{C}$ and $m$ is a positive integer for $U = \mathbb{H}$.

2.2. **Hyperbolic automorphy factors and lifts of Fuchsian groups.** We consider the universal cover $\tilde{G} = \widetilde{\mathrm{PSL}}(2, \mathbb{R})$ of the Lie group

$$G = \mathrm{PSL}(2, \mathbb{R}) = \mathrm{SL}(2, \mathbb{R})/\{\pm 1\},$$

the group of orientation-preserving isometries of the hyperbolic plane. Here our model of the hyperbolic plane is the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ and the action of an element $[\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)] \in \mathrm{PSL}(2, \mathbb{R})$ on $\mathbb{H}$ is by

$$z \mapsto \frac{az + b}{cz + d}.$$

As a topological space $G = \mathrm{PSL}(2, \mathbb{R})$ is homeomorphic to the open solid torus $\mathbb{S}^1 \times \mathbb{C}$, its fundamental group is infinite cyclic. Therefore, for each natural number $m$, there is a unique connected $m$-fold covering

$$G_m = \tilde{G}/(m \cdot Z(\tilde{G}))$$

of $G$, where $\tilde{G}$ is the universal covering of $G$ and $Z(\tilde{G})$ is the centre of $\tilde{G}$. The centre $Z(\tilde{G})$ of $\tilde{G}$ coincides with the pre-image of the identity element of $G$ under the covering map $\tilde{G} \to G$. For $m = 2$, we obtain $G_2 = \mathrm{SL}(2, \mathbb{R})$.

Here is another description of the covering groups $G_m$ of $G$ which fixes a group structure. Let $\mathrm{Hol}(\mathbb{H}, \mathbb{C}^*)$ be the set of all holomorphic functions $\mathbb{H} \to \mathbb{C}^*$.

**Proposition 2.3.** *The $m$-fold covering group $G_m$ of $G$ can be described as*

$$\{(g, \delta) \in G \times \mathrm{Hol}(\mathbb{H}, \mathbb{C}^*) \mid \delta^m(z) = g'(z) \text{ for all } z \in \mathbb{H}\}$$

*with multiplication $(g_2, \delta_2) \cdot (g_1, \delta_1) = (g_2 \cdot g_1, (\delta_2 \circ g_1) \cdot \delta_1)$.*

*Proof.* Let $X$ be the subspace of $G \times \mathrm{Hol}(\mathbb{H}, \mathbb{C}^*)$ in question. One can check that the space $X$ is connected and that the map $X \to G$ given by $(\gamma, \delta) \mapsto \gamma$ is an $m$-fold covering of $G$. Hence, the coverings $X \to G$ and $G_m \to G$ are isomorphic. One can check that the operation described above defines a group structure on $X$ and that the covering map $X \to G$ is a homomorphism with respect to this group structure. $\square$

*Remark.* This description of $G_m$ is inspired by the notion of automorphic differential forms of fractional degree, introduced by J. Milnor in [Mil75]. For a more detailed discussion see [LV80], section 1.8.

**Definition 2.2.** A *lift* of the Fuchsian group $\Gamma$ into $G_m$ is a subgroup $\Gamma^*$ of $G_m$ such that the restriction of the covering map $G_m \to G$ to $\Gamma^*$ is an isomorphism $\Gamma^* \to \Gamma$.

Using the description of the $m$-fold covering group $G_m$ of $G$ in Proposition 2.3, we obtain the following result:

**Proposition 2.4.** *There is a 1-1-correspondence between the lifts of $\Gamma$ into $G_m$ and hyperbolic Gorenstein automorphy factors of level $m$ associated to the Fuchsian group $\Gamma$.*

*Proof.* Theorem 2.2 implies that a hyperbolic Gorenstein automorphy factor of level $m$ (associated to a Fuchsian group $\Gamma$) is an action of the Fuchsian group $\Gamma$ on the trivial complex line bundle $\mathbb{H}\times\mathbb{C}$ over the hyperbolic plane $\mathbb{H}$ given by

$$g\cdot(z,t)=(g(z),\delta_g(z)\cdot t),$$

where $\delta_g:\mathbb{H}\to\mathbb{C}^*$ is a holomorphic function; for any $g\in\Gamma$, we have $\delta_g^m=g'$ and, for any $g_1,g_2\in\Gamma$, we have $\delta_{g_2\cdot g_1}=(\delta_{g_2}\circ g_1)\cdot\delta_{g_1}$. Comparing this description of hyperbolic Gorenstein automorphy factors with Proposition 2.3 yields the result. $\square$

## 3. Level functions on covering groups of $\mathrm{PSL}(2,\mathbb{R})$

3.1. **Classification of elements in $G=\mathrm{PSL}(2,\mathbb{R})$ and its covering groups.** Elements of $G$ can be classified with respect to the fixed point behavior of their action on $\mathbb{H}$. An element is called *hyperbolic* if it has two fixed points, which lie on the boundary $\partial\mathbb{H}=\mathbb{R}\cup\{\infty\}$ of $\mathbb{H}$. One of the fixed points of a hyperbolic element is attracting, the other fixed point is repelling. The *axis* $\ell(g)$ of a hyperbolic element $g$ is the geodesic between the fixed points of $g$, oriented from the repelling fixed point to the attracting fixed point. The element $g$ preserves the geodesic $\ell(g)$. We call a hyperbolic element with attracting fixed point $\alpha$ and repelling fixed point $\beta$ *positive* if $\alpha<\beta$. The *shift parameter* of a hyperbolic element $g$ is the minimal displacement $\inf_{x\in\mathbb{H}} d(x,g(x))$. An element is called *parabolic* if it has one fixed point, which is on the boundary $\partial\mathbb{H}$. We call a parabolic element $g$ with fixed point $\alpha$ *positive* if $g(x)>x$ for all $x\in\mathbb{R}\backslash\{\alpha\}$. An element that is neither hyperbolic nor parabolic is called *elliptic*. It has one fixed point that is in $\mathbb{H}$. Given a base-point $x\in\mathbb{H}$ and a real number $\varphi$, let $\rho_x(\varphi)\in G$ denote the rotation through angle $\varphi$ counter-clockwise about the point $x$. Any elliptic element is of the form $\rho_x(\varphi)$, where $x$ is the fixed point. Thus, we obtain a $2\pi$-periodic homomorphism $\rho_x:\mathbb{R}\to G$ (with respect to the additive structure on $\mathbb{R}$). Elements of $\tilde{G}$, resp. $G_m$, can be classified with respect to the fixed point behavior of action on $\mathbb{H}$ of their image in $G$. We say that an element of $\tilde{G}$, resp. $G_m$, is *hyperbolic*, *parabolic*, resp. *elliptic*, if its image in $G$ has this property.

3.2. **Central elements in covering groups of $G=\mathrm{PSL}(2,\mathbb{R})$.** The homomorphism $\rho_x:\mathbb{R}\to G$ lifts to a unique homomorphism $r_x:\mathbb{R}\to\tilde{G}$ into the universal covering group. Since $\rho_x(2\pi\ell)=\mathrm{id}$ for $\ell\in\mathbb{Z}$, it follows that the lifted element $r_x(2\pi\ell)$ belongs to the centre $Z(\tilde{G})$ of $\tilde{G}$. Note that the element $r_x(2\pi\ell)$ depends continuously on $x$. But the centre of $\tilde{G}$ is discrete, so this element must remain constant, thus $r_x(2\pi\ell)$ does not depend on $x$. We obtain that $Z(\tilde{G})=\{r_x(2\pi\ell)\bigm|\ell\in\mathbb{Z}\}$. Let $u=r_x(2\pi)$ for some (and hence for any) $x$ in $\mathbb{H}$. The element $u$ is one of the two generators of the centre of $\tilde{G}$ since any other element of the centre is of the form $r_x(2\pi\ell)=(r_x(2\pi))^\ell=u^\ell$.

3.3. **Definition of a level function.** Let $\Delta$ be the set of all elliptic elements of order 2 in $G$. Let $\Xi$ be the complement in $G$ of the set $\Delta$. The space $G$ is homeomorphic to the open solid torus $\mathbb{S}^1\times\mathbb{C}$. Jankins and Neumann gave an explicit homeomorphism (see [JN85], Appendix). The image of the set $\Delta$ under this homeomorphism is $\{*\}\times\mathbb{C}$. From this description, it follows in particular that the subset $\Xi$ is simply-connected. The pre-image $\tilde{\Xi}$ of the subset $\Xi$ in $\tilde{G}$ consists of infinitely many connected components, each of them is homeomorphic to $\Xi$. Each connected component of the subset $\tilde{\Xi}$ contains one and only one pre-image of the identity element of $G$, i.e., one and only one element of the centre of $\tilde{G}$.

**Definition 3.1.** If an element of $\tilde{G}$ is contained in the same connected component of the set $\tilde{\Xi}$ as the central element $u^k$, $k\in\mathbb{Z}$, we say that the element is *of level $k$* and set the *level function $s$*

on this element to be equal to $k$. For pre-images of elliptic elements of order 2 we set $s(r_x(\xi)) = k$ for $\xi = \pi + 2\pi k$.

*Remark.* For elliptic elements we have $s(r_x(\xi)) = k \iff \xi \in (-\pi + 2\pi k, \pi + 2\pi k]$.

**Definition 3.2.** We define the *level function* $s_m$ on elements of $G_m$ with values in $\mathbb{Z}/m\mathbb{Z}$ by

$$s_m(C \mod (m \cdot Z(\tilde{G}))) = s(C) \mod m \text{ for } C \in \tilde{G}.$$

**Definition 3.3.** The canonical lift of an element $\bar{C}$ in $G$ into $\tilde{G}$ is an element $\tilde{C}$ in $\tilde{G}$ such that $\pi(\tilde{C}) = \bar{C}$ and $s(\tilde{C}) = 0$. The canonical lift of an element $\bar{C}$ in $G$ into $G_m$ is an element $\tilde{C}$ in $G_m$ such that $\pi(\tilde{C}) = \bar{C}$ and $s_m(\tilde{C}) = 0$.

3.4. **Properties of the level function.** In this subsection, we study the behavior of the level function $s_m$ under inversion (Lemma 3.1), conjugation (Lemma 3.2) and multiplication in some special cases (Lemma 3.3). We shall obtain further statements about the behavior of the level function under multiplication in Corollary 4.7.

In this section, we shall repeatedly use the following fact: Connected components of the set $\tilde{\Xi}$ are separated from each other by connected components of the set $\tilde{\Delta}$ of all pre-images of (elliptic) elements of order 2. If a path $\gamma$ in $\tilde{G}$ avoids all pre-images of elements of order 2, i.e., avoids $\tilde{\Delta}$, then it means that the path $\gamma$ remains in the same component of the set $\tilde{\Xi}$ and therefore the level function $s$ is constant along $\gamma$.

**Lemma 3.1.** *The equation $s(A^{-1}) = -s(A)$ is satisfied for any element $A$ in $\tilde{G}$ with the exception of the pre-images of elliptic elements of order 2. The equation $s_m(A^{-1}) = -s_m(A)$ is satisfied for any element $A$ in $G_m$ with the exception of the pre-images of elliptic elements of order 2.*

*Proof.* We shall prove the statement about the level function $s$ on $\tilde{G}$, the statement about the level function $s_m$ on $G_m$ follows immediately. Let $A \in \tilde{G}$ and let $k = s(A)$, then $A$ is in the same connected component of $\tilde{\Xi}$ as $u^k$. Let $\gamma$ be the path in $\tilde{\Xi}$ that connects $A$ with $u^k$. Let the path $\delta$ be given by $\delta(t) = (\gamma(t))^{-1}$. The path $\delta$ connects $A^{-1}$ with $u^{-k}$. Since the path $\gamma$ remains in the same component of $\tilde{\Xi}$, it avoids $\tilde{\Delta}$. Consequently, the path $\delta$ also avoids $\tilde{\Delta}$, i.e., it remains in the same component of $\tilde{\Xi}$. Thus the element $A^{-1}$ is in the same connected component of $\tilde{\Xi}$ as $u^{-k}$, i.e., $s(A^{-1}) = -k = -s(A)$. ∎

**Lemma 3.2.** *For any elements $A$ and $B$ in $\tilde{G}$, we have $s(BAB^{-1}) = s(A)$. For any elements $A$ and $B$ in $G_m$, we have $s_m(BAB^{-1}) = s_m(A)$.*

*Proof.* We shall prove the statement about the level function $s$ on $\tilde{G}$, the statement about the level function $s_m$ on $G_m$ follows immediately. An element $B \in \tilde{G}$ can be connected to the unit element in $\tilde{G}$ via a path $\beta$. The path $\gamma$ given by $\gamma(t) = \beta(t) \cdot A \cdot (\beta(t))^{-1}$ connects the elements $A$ and $B \cdot A \cdot B^{-1}$. If $A$ is not in $\tilde{\Delta}$ then any conjugate $\gamma(t)$ of $A$ is not in $\tilde{\Delta}$, hence the path $\gamma$ remains in the same component of the set $\tilde{\Xi}$. If $A$ is in $\tilde{\Delta}$ then any conjugate $\gamma(t)$ of $A$ is also in $\tilde{\Delta}$, hence the path $\gamma$ remains in the same component of the set $\tilde{\Delta}$. In both cases $s$ is constant along $\gamma$, in particular $s(B \cdot A \cdot B^{-1}) = s(A)$. ∎

**Lemma 3.3.** *If the axes of two hyperbolic elements $A$ and $B$ in $\tilde{G}$ intersect, then*

$$s(AB) = s(A) + s(B).$$

*If the axes of two hyperbolic elements $A$ and $B$ in $G_m$ intersect, then*

$$s_m(AB) = s_m(A) + s_m(B).$$

*Proof.* Let $\ell_A$, resp. $\ell_B$, be the axes of $A$, resp. $B$. Let $x$ be the intersection point of $\ell_A$ and $\ell_B$. Any hyperbolic transformation with the axis $\ell_A$ is a product of a rotation by $\pi$ at some point $y \neq x$ on $\ell_A$ and a rotation by $\pi$ at the point $x$. Similarly any hyperbolic transformation with the axis $\ell_B$ is a product of a rotation by $\pi$ at the point $x$ and a rotation by $\pi$ at some point $z \neq x$ on $\ell_B$. Hence the product of any hyperbolic transformation with the axis $\ell_A$ and any hyperbolic transformation with the axis $\ell_B$ is a product of a rotation by $\pi$ at a point $y \neq x$ on $\ell_A$ and a rotation by $\pi$ at a point $z \neq x$ on $\ell_B$, i.e., it is a hyperbolic transformation with an axis going through the points $y$ and $z$. Thus the product of two hyperbolic elements with distinct but intersecting axes is always a hyperbolic element.

We shall prove the statement about the level function $s$ on $\tilde{G}$, the statement about the level function $s_m$ on $G_m$ follows immediately. Assume without loss of generality that the elements $A, B \in \tilde{G}$ satisfy the conditions $s(A) = s(B) = 0$. We want to show that $s(AB) = 0$. Let us deform the elements $A$ and $B$. If we are decreasing the shift parameters while keeping the same axes, then the product tends to the identity element. On the other hand, we have just explained that the product remains hyperbolic, i.e., stays in $\tilde{\Xi}$. Therefore, the value of $s$ on the product remains constant, i.e., $s(AB) = s(\mathrm{id}) = 0$. □

## 4. Level functions on lifts of Fuchsian groups

4.1. **Lifting elliptic cyclic subgroups.** A lift of the Fuchsian group $\Gamma$ into $G_m$ is a subgroup $\Gamma^*$ of $G_m$ such that the restriction of the covering map $G_m \to G$ to $\Gamma^*$ is an isomorphism $\Gamma^* \to \Gamma$. In this subsection, we shall discuss the lifts of cyclic Fuchsian groups generated by an elliptic element.

**Lemma 4.1.** *Let $\Gamma$ be an elliptic cyclic Fuchsian group of order $p$. The group $\Gamma$ is generated by an element $\gamma = \rho_x(2\pi/p)$ for some $x \in \mathbb{H}$.*

1) *Let us assume that $p$ and $m$ are relatively prime. Then the lift $\Gamma^*$ of $\Gamma$ into $G_m$ exists and is unique. There is a unique element $n \in \mathbb{Z}/m\mathbb{Z}$ such that $p \cdot n + 1 \equiv 0$ modulo $m$. The lift $\Gamma^*$ is generated by the pre-image $\tilde{\gamma}$ of $\gamma = \rho_x(2\pi/p)$ in $G_m$ such that $s_m(\tilde{\gamma}) = n$.*
2) *If $p$ and $m$ are not relatively prime, then the group $\Gamma$ cannot be lifted into $G_m$.*

*Proof.* To lift $\Gamma$ into $G_m$, we have to find an element $\tilde{\gamma}$ in the pre-image of $\gamma$ in $G_m$ such that $\tilde{\gamma}^p = 1$. The pre-image of $\gamma$ in $G_m$ can be described as the coset $\{u^n \cdot r_x(2\pi/p) \mid n \in \mathbb{Z}/m\mathbb{Z}\}$. For the element $r_x(2\pi/p)$, we obtain $(r_x(2\pi/p))^p = r_x(2\pi) = u$. Hence, for an element $u^n \cdot r_x(2\pi/p)$ we obtain

$$(u^n \cdot r_x(2\pi/p))^p = u^{np}(r_x(2\pi/p))^p = u^{np+1}.$$

Therefore, $(u^n \cdot r_x(2\pi/p))^p = 1$ if and only if $n \cdot p + 1 \equiv 0 \bmod m$. There exists an $n \in \mathbb{Z}/m\mathbb{Z}$ with $n \cdot p + 1 \equiv 0 \bmod m$ if and only if the numbers $p$ and $m$ are relatively prime. Hence for not relatively prime $p$ and $m$ it is impossible to lift $\Gamma$ into $G_m$. For relatively prime $p$ and $m$, there is a unique lift of $\Gamma$ into $G_m$ generated by $u^n \cdot r_x(2\pi/p)$ with $n \cdot p + 1 \equiv 0 \bmod m$. □

4.2. **Finitely generated Fuchsian groups.** In this section, we are going to describe finitely generated (co-compact) Fuchsian groups using standard sets of generators. The following definitions follow [Zie81]:

**Definition 4.1.** A *Riemann factor surface* or *Riemann orbifold* $(P, Q)$ of signature

$$(g; l_h, l_p, l_e : p_1, \dots, p_{l_e})$$

is a topological surface $P$ of genus $g$ with $l_h$ holes and $l_p$ punctures and a set

$$Q = \{(x_1, p_1), \dots, (x_{l_e}, p_{l_e})\}$$

of points $x_i$ in $P$ equipped with orders $p_i$ such that $p_i \in \mathbb{Z}$, $p_i \geqslant 2$ and $x_i \neq x_j$ for $i \neq j$. The set $Q$ is called the *marking* of the Riemann orbifold $(P, Q)$.

**Definition 4.2.** Let $(P, Q = \{(x_1, p_1), \ldots, (x_{l_e}, p_{l_e})\})$ be a Riemann orbifold. Two curves $\gamma_0$ and $\gamma_1$, which do not pass through exceptional points $x_i \in Q$, are called *Q-homotopic* if $\gamma_0$ can be deformed into $\gamma_1$ by a finite sequence of the following processes:

(a) Homotopic deformations with fixed starting point such that during the deformation no exceptional point is encountered.

(b) Omitting a subcurve of $\gamma_i$ which does not contain the starting point of $\gamma_i$ and is of the form $\delta^{\pm p_i}$, where $\delta$ is a curve on $P$ which bounds a disk that contains exactly one exceptional point $x_i$ in the interior.

(c) Inserting into $\gamma_i$ a subcurve which does not contain the starting point of $\gamma_i$ and is of the form $\delta^{\pm p_i}$, where $\delta$ is a curve on $P$ which bounds a disk that contains exactly one exceptional point $x_i$ in the interior.

Two curves $\gamma_0$ and $\gamma_1$ which do not pass through exceptional points $x_i \in Q$ are called *freely Q-homotopic* if the base point may be moved during the deformations.

**Definition 4.3.** Let $(P, Q = \{(x_1, p_1), \ldots, (x_{l_e}, p_{l_e})\})$ be a Riemann orbifold and let $p$ be in $P \backslash \{x_1, \ldots, x_{l_e}\}$. Then, the set of $Q$-homotopy classes of curves starting and ending in $p$ forms a group. This group is called the *Q-fundamental group* or the *orbifold fundamental group* and denoted by $\pi^Q(P, p)$ or $\pi^{\mathrm{orb}}(P, p)$ or simply $\pi(P, p)$.

**Definition 4.4.** Let $\Gamma$ be a Fuchsian group. The quotient $P = \mathbb{H}/\Gamma$ is a surface and the projection $\Psi : \mathbb{H} \to P$ is a branched cover. Let $Q$ consist of the branching points and the corresponding orders. Then, $(P, Q)$ is a Riemann orbifold. We say that the Riemann orbifold $(P, Q)$ is defined by $\Gamma$.

**Proposition 4.2.** *Let $\Gamma$ be a Fuchsian group, $(P, Q)$ the corresponding Riemann orbifold and $p \in P$ not an exceptional point. Then, $\pi(P, p) \cong \Gamma$.*

**Definition 4.5.** A *canonical system of curves* on a Riemann orbifold

$$(P, Q = \{(x_1, p_1), \ldots, (x_{l_e}, p_{l_e})\})$$

of signature $(g; l_h, l_p, l_e : p_1, \ldots, p_{l_e})$ is a set of simply closed curves

$$\{\tilde{a}_1, \tilde{b}_1, \ldots, \tilde{a}_g, \tilde{b}_g, \tilde{c}_{g+1}, \ldots, \tilde{c}_n\}$$

based at a point $p \in P$, where $n = g + l_h + l_p + l_e$, with the following properties:

1) The contour $\tilde{c}_{g+i}$ encloses a hole in $P$ for $i = 1, \ldots, l_h$, a puncture for $i = l_h + 1, \ldots, l_h + l_p$ and the marking point $x_i$ for $i = l_h + l_p + 1, \ldots, n - g$.
2) Any two curves only intersect at the point $p$.
3) A neighbourhood of the point $p$ with the curves is homeomorphic to the one shown in Figure 2.
4) The system of curves cuts the surface $P$ into $l_h + l_p + l_e + 1$ connected components of which $l_p + l_e$ are homeomorphic to a disc with a hole, $l_h + 1$ are discs. The last disc has boundary $\tilde{a}_1 \tilde{b}_1 \tilde{a}_1^{-1} \tilde{b}_1^{-1} \ldots \tilde{a}_g \tilde{b}_g \tilde{a}_g^{-1} \tilde{b}_g^{-1} \tilde{c}_g \ldots \tilde{c}_n$.

If $\{\tilde{a}_1, \tilde{b}_1, \ldots, \tilde{a}_g, \tilde{b}_g, \tilde{c}_{g+1}, \ldots, \tilde{c}_n\}$ is a canonical system of curves, then we call the corresponding set $\{a_1, b_1, \ldots, a_g, b_g, c_{g+1}, \ldots, c_n\}$ of elements in the orbifold fundamental group $\pi(P, p)$ a *standard set of generators* or a *standard basis* of $\pi(P, p)$.

**Definition 4.6.** A *sequential set of signature* $(0; l_h, l_p, l_e : p_1, \ldots, p_{l_e})$ with $l_h + l_p + l_e = 3$ is a triple of elements $(C_1, C_2, C_3)$ in $G$ such that the element $C_i$ is hyperbolic for $i = 1, \ldots, l_h$, parabolic for $i = l_h + 1, \ldots, l_h + l_p$ and elliptic of order $p_{i - l_h - l_p}$ for $i = l_h + l_p + 1, \ldots, l_h + l_p + l_e = 3$,

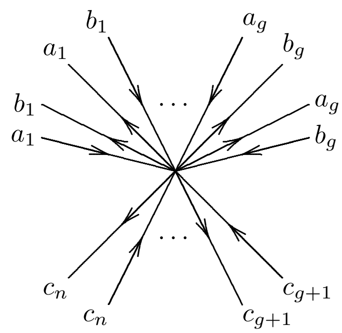


Figure 2: Canonical system of curves

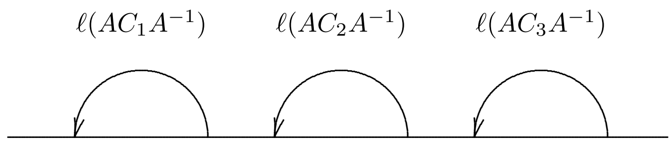


Figure 3: Axes of a sequential set of signature $(0;3,0,0)$

their product is $C_1 \cdot C_2 \cdot C_3 = 1$, and for some element $A \in G$ the elements $\{C_i' = AC_iA^{-1}\}_{i=1,2,3}$ are positive, have finite fixed points and satisfy $C_1' < C_2' < C_3'$. (Figure 3 illustrates the position of the axes of the elements $C_i'$ for a sequential set of signature $(0;3,0,0)$, i.e., when all elements are hyperbolic.)

**Definition 4.7.** A *sequential set of signature* $(0; l_h, l_p, l_e : p_1, \ldots, p_{l_e})$ is a tuple of elements $(C_1, \ldots, C_{l_h+l_p+l_e})$ in $G$ such that the element $C_i$ is hyperbolic for $i = 1, \ldots, l_h$, parabolic for $i = l_h + 1, \ldots, l_h + l_p$ and elliptic of order $p_{i-l_h-l_p}$ for $i = l_h + l_p + 1, \ldots, l_h + l_p + l_e$, and for any $i \in \{2, \ldots, l_h + l_p + l_e - 1\}$ the triple $(C_1 \cdots C_{i-1}, C_i, C_{i+1} \cdots C_{l_h+l_p+l_e})$ is a sequential set.

**Definition 4.8.** A *sequential set of signature* $(g; l_h, l_p, l_e : p_1, \ldots, p_{l_e})$ is a tuple of elements

$$(A_1, \ldots, A_g, B_1, \ldots, B_g, C_{g+1}, \ldots, C_{g+l_h+l_p+l_e})$$

in $G$ such that

(1) the elements $A_1, \ldots, A_g, B_1, \ldots, B_g$ are hyperbolic,
(2) the element $C_{g+i}$ is hyperbolic for $i = 1, \ldots, l_h$, parabolic for $i = l_h + 1, \ldots, l_h + l_p$, and elliptic of order $p_{i-l_h-l_p}$ for $i = l_h + l_p + 1, \ldots, l_h + l_p + l_e$,
(3) and the tuple

$$(A_1, B_1A_1^{-1}B_1^{-1}, \ldots, A_g, B_gA_g^{-1}B_g^{-1}, C_{g+1}, \ldots, C_{g+l_h+l_p+l_e})$$

is a sequential set of signature $(0; 2g + l_h, l_p, l_e : p_1, \ldots, p_{l_e})$.

The relation between sequential sets, standard bases, canonical systems of curves and Fuchsian groups was studied in [Nat72]. Details for the case of Fuchsian groups with elliptic elements can be found in [Zie81]. We recall here the results:

**Theorem 4.3.** *Let $V$ be a sequential set of signature $(g; l_h, l_p, l_e : p_1, \ldots, p_{l_e})$. For $i = 1, \ldots, l_e$ let $y_i \in \mathbb{H}$ be the fixed point of the corresponding elliptic element of order $p_i$ in $V$. Let* $P =$

*$\mathbb{H}/\Gamma$ and let $\Psi : \mathbb{H} \to P$ be the natural projection. Let $Q = \{(\Psi(y_1), p_1), \ldots, (\Psi(y_{l_e}), p_{l_e})\}$. Then the sequential set $V$ generates a Fuchsian group $\Gamma$ such that the Riemann factor surface $(P = \mathbb{H}/\Gamma, Q)$ is of signature $(g; l_h, l_p, l_e : p_1, \ldots, p_{l_e})$. The natural projection $\Psi : \mathbb{H} \to P$ maps the sequential set $V$ to a canonical system of curves on the factor surface $(P, Q)$.*

**Theorem 4.4.** *Let $\Gamma$ be a Fuchsian group such that the factor surface $P = \mathbb{H}/\Gamma$ is of signature $(g; l_h, l_p, l_e : p_1, \ldots, p_{l_e})$ with $l_h + l_p + l_e = n$. Let $p$ be a point in $P$ which does not belong to the marking. Let $\Psi : \mathbb{H} \to P$ be the natural projection. Choose $q \in \Psi^{-1}(p)$ and let $\Phi : \Gamma \to \pi(P, p)$ be the induced isomorphism. Let*

$$v = \{\tilde{a}_1, \tilde{b}_1, \ldots, \tilde{a}_g, \tilde{b}_g, \tilde{c}_{g+1}, \ldots, \tilde{c}_n\} =: \{\tilde{d}_1, \ldots, \tilde{d}_{g+n}\}$$

*be a canonical system of curves on $P$, then*

$$V = \Phi^{-1}(v) = \{\Phi^{-1}([\tilde{d}_1]), \ldots, \Phi^{-1}([\tilde{d}_{g+n}]))\}$$

*is a sequential set of signature $(g; l_h, l_p, l_e : p_1, \ldots, p_{l_e})$.*

4.3. **Lifting Fuchsian groups of genus 0.**

**Lemma 4.5.** *Let $(0; l_h, l_p, l_e : p_1, \ldots, p_{l_e})$ with $l_h + l_p + l_e = n$ be the signature of the sequential set $(\bar{C}_1, \ldots, \bar{C}_n)$. For $i = 1, \ldots, n$, let $\tilde{C}_i$ be the canonical lift of $\bar{C}_i$ into $\tilde{G}$, resp. $G_m$. Let $u$ be the generator of the centre $Z(\tilde{G})$, resp. $Z(G_m)$, defined in section 3.2. The element $u$ is given by the element $r_x(\pi)$, resp. its projection into $G_m$. Then, the elements $\tilde{C}_1, \ldots, \tilde{C}_n$ satisfy the following relations: $\tilde{C}^{p_i}_{l_h+l_p+i} = u$ for $i = 1, \ldots, l_e$ and $\tilde{C}_1 \cdot \cdots \cdot \tilde{C}_n = u^{n-2}$.*

*Proof.* A canonical lift $\tilde{C}_{l_h+l_p+i}$ of an elliptic element $\bar{C}_{l_h+l_p+i}$ is of the form $r_x(2\pi/p_i)$ for some $x$. Hence, $\tilde{C}^{p_i}_{l_h+l_p+i} = (r_x(2\pi/p_i))^{p_i} = r_x(2\pi) = u$. We will now use a geometric approach due to Milnor [Mil75]. Let $\Pi$ be the canonical fundamental polygon for the group generated by the elements $\bar{C}_1, \ldots, \bar{C}_n$ such that the generators $\bar{C}_i$ can be described as products $\bar{C}_i = \sigma_i\sigma_{i+1}$ of reflexions $\sigma_1, \ldots, \sigma_n$ in the edges of the polygon $\Pi$ (suitably numbered). Then, $\sigma_i^2 = \mathrm{id}$ and, therefore,

$$\bar{C}_1 \cdots \bar{C}_n = (\sigma_1\sigma_2)(\sigma_2\sigma_3) \cdots (\sigma_{n-1}\sigma_n)(\sigma_n\sigma_1) = \mathrm{id}\,.$$

Lifting the elements $\bar{C}_i$ to their canonical lifts $\tilde{C}_i$ in $\tilde{G}$, it follows that the product $\tilde{C}_1 \cdots \tilde{C}_n$ belongs to the centre $Z(\tilde{G})$. As we vary the polygon $\Pi$ continuously, this central element must also vary continuously. But $Z(\tilde{G})$ is a discrete group, so $\tilde{C}_1 \cdots \tilde{C}_n$ must remain constant. In particular, we can shrink the polygon $\Pi$ down towards a point $x$. In the course of this continuous deformation of the fundamental polygon $\Pi$, the hyperbolic and parabolic elements of the sequential set will become elliptic. As we continue shrinking the polygon $\Pi$ towards the point $x$, the angles of the polygon tend to the angles $\beta_1, \ldots, \beta_n$ of some Euclidean $n$-sided polygon. Thus, the element $\tilde{C}_i \in \tilde{G}$ tends towards the limit $r_x(2\beta_i)$, while the product $\tilde{C}_1 \cdots \tilde{C}_n$ tends towards the product $r_x(2\beta_1) \cdots r_x(2\beta_n) = r_x(2\beta_1 + \cdots + 2\beta_n)$. Therefore, using the formula $\beta_1 + \cdots + \beta_n = (n-2)\pi$ for the sum of the angles of a Euclidean $n$-sided polygon, we see that the constant product $\tilde{C}_1 \cdots \tilde{C}_n$ must be equal to $r_x(2(n-2)\pi) = u^{n-2}$. Thus $\tilde{C}^{p_i}_{l_h+l_p+i} = u$ and $\tilde{C}_1 \cdots \tilde{C}_n = u^{n-2}$. Projecting into $G_m$ we get the corresponding statement in $G_m$. □

**Lemma 4.6.** *Let $(C_1, \ldots, C_n)$ be an $n$-tuple of elements in $G_m$ such that their images $(\bar{C}_1, \ldots, \bar{C}_n)$ in $G$ form a sequential set of signature $(0; l_h, l_p, l_e : p_1, \ldots, p_{l_e})$ with $l_h + l_p + l_e = n$. Then, $C_1 \cdots C_n = e$ if and only if $\sum_{i=1}^{n} s_m(C_i) \equiv -(n-2)$ modulo $m$.*

*Proof.* For $i = 1, \ldots, n$, let $\tilde{C}_i$ be the canonical lift of $\bar{C}_i$ into $G_m$. The elements $C_i$ can be written in the form $C_i = \tilde{C}_i \cdot u^{s_m(C_i)}$; therefore, the product $C_1 \cdots C_n$ is equal to

$$(\tilde{C}_1 \cdots \tilde{C}_n) \cdot u^{s_m(C_1)+\cdots+s_m(C_n)}.$$

Using Lemma 4.5, we obtain that this product is equal to $u^{n-2+s_m(C_1)+\cdots+s_m(C_n)}$. Hence, the product $C_1\cdots C_n$ is equal to $e$ if and only if the exponent of $u$ in the last equation is divisible by $m$, i.e., if $\sum_{i=1}^n s_m(C_i)\equiv -(n-2)$ modulo $m$. □

**Corollary 4.7.** *Let $(C_1,C_2,C_3)$ be a triple of elements in $G_m$ with $C_1\cdot C_2\cdot C_3=e$. Let $\bar{C}_i$ be the image of the element $C_i$ in $G$. Let $(\bar{C}_1,\bar{C}_2,\bar{C}_3)$ be a sequential set of signature*

$$(0;l_h,l_p,l_e:p_1,\dots,p_{l_e}),$$

*with $l_h+l_p+l_e=3$. Then,*

$$\begin{aligned}&s_m(C_1\cdot C_2)=s_m(C_1)+s_m(C_2)+1, \text{ if the element } C_3 \text{ is not of order } 2, \text{and}\\&s_m(C_1\cdot C_2)=-s_m(C_1)-s_m(C_2)-1, \text{ if the element } C_3 \text{ is of order } 2.\end{aligned}$$

*Proof.* According to Lemma 4.6 the elements $C_i$ satisfy

$$s_m(C_1)+s_m(C_2)+s_m(C_3)\equiv -1 \bmod m.$$

On the other hand, $C_1C_2C_3=e$ implies $C_1C_2=C_3^{-1}$; hence,

$$s_m(C_1C_2)=s_m(C_3^{-1})=-s_m(C_3)=s_m(C_1)+s_m(C_2)+1,$$

if the element $C_3$ is not of order 2, and

$$s_m(C_1C_2)=s_m(C_3^{-1})=s_m(C_3)=-s_m(C_1)-s_m(C_2)-1,$$

if the element $C_3$ is of order 2. □

### 4.4. **Lifting sets of generators of Fuchsian groups.**

**Lemma 4.8.** *Let $\Gamma$ be a Fuchsian group of signature $(g;l_h,l_p,l_e:p_1,\dots,p_{l_e})$ generated by the sequential set $\bar{V}=\{\bar{A}_1,\bar{B}_1,\dots,\bar{A}_g,\bar{B}_g,\bar{C}_{g+1},\dots,\bar{C}_n\}$, where $n=g+l_h+l_p+l_e$. Let $V=\{A_1,B_1,\dots,A_g,B_g,C_{g+1},\dots,C_n\}$ be a set of lifts of the elements of the sequential set $\bar{V}$ into $G_m$, i.e., the image of $A_i$, $B_i$, resp. $C_j$, in $G$ is $\bar{A}_i$, $\bar{B}_i$, resp. $\bar{C}_j$. Then, the subgroup $\Gamma^*$ of $G_m$ generated by $V$ is a lift of $\Gamma$ into $G_m$ if and only if*

$$[A_1,B_1]\cdots[A_g,B_g]\cdot C_{g+1}\cdots C_n=e,\quad C^{p_i}_{g+l_h+l_p+i}=e\quad \text{for } i=1,\dots,l_e.$$

*Proof.* For any choice of the set of lifts $V$, the restriction of the covering map $G_m\to G$ to the group $\Gamma^*$ generated by $V$ is a homomorphism with image $\Gamma$. If the conditions of the lemma hold true, then the group $\Gamma^*$ satisfies the same relations as the group $\Gamma$; hence, this homomorphism is injective. □

**Lemma 4.9.** *Let $\{A_1,B_1,\dots,A_g,B_g,C_{g+1},\dots,C_n\}$ be a tuple of elements in $G_m$ such that the images $\{\bar{A}_1,\bar{B}_1,\dots,\bar{A}_g,\bar{B}_g,\bar{C}_{g+1},\dots,\bar{C}_n\}$ in $G$ form a sequential set of signature*

$$(g;l_h,l_p,l_e:p_1,\dots,p_{l_e}),$$

*with $g+l_h+l_p+l_e=n$. Then,*

$$\prod_{i=1}^g[A_i,B_i]\cdot\prod_{j=g+1}^n C_j=e\iff \sum_{j=g+1}^n s_m(C_j)\equiv(2-2g)-(n-g)\bmod m.$$

*(in the case $n=g$, this means $2-2g\equiv 0\bmod m$) and, for any $i=1,\dots,l_e$,*

$$C^{p_i}_{g+l_h+l_p+i}=e\iff p_i\cdot s_m(C_{g+l_h+l_p+i})+1\equiv 0\bmod m.$$

*Proof.* The case $g = 0$ was discussed in Lemma 4.6. We shall now reduce the general case to the case $g = 0$. By definition of sequential sets, the set

$$(\bar{A}_1, \bar{B}_1\bar{A}_1^{-1}\bar{B}_1^{-1}, \dots, \bar{A}_g, \bar{B}_g\bar{A}_g^{-1}\bar{B}_g^{-1}, \bar{C}_{g+1}, \dots, \bar{C}_n)$$

is a sequential set of signature $(0; 2g + l_h, l_p, l_e)$; hence,

$$\prod_{i=1}^{g} [A_i, B_i] \cdot \prod_{i=g+1}^{n} C_i = \prod_{i=1}^{g} (A_i \cdot B_i A_i^{-1} B_i^{-1}) \cdot \prod_{i=g+1}^{n} C_i = e$$

if and only if

$$\begin{aligned}\sum_{i=1}^{g} (s_m(A_i) + s_m(B_i A_i^{-1} B_i^{-1})) + \sum_{i=g+1}^{n} s_m(C_i) &\equiv -(2g + (n-g) - 2) \\ &\equiv (2 - 2g) - (n - g) \bmod m.\end{aligned}$$

Invariance of the level function $s_m$ under conjugation (Lemma 3.2) implies that

$$s_m(B_i A_i^{-1} B_i^{-1}) = s_m(A_i^{-1}).$$

Since $A_i$ is not an element of order 2, $s_m(A_i^{-1}) = -s_m(A_i)$ and, hence,

$$s_m(A_i) + s_m(B_i A_i^{-1} B_i^{-1}) = s_m(A_i) - s_m(A_i) = 0.$$

The last statement of the lemma follows from Lemma 4.1. □

**Proposition 4.10.** *Let $\Gamma$ be a Fuchsian group of signature $(g : p_1, \dots, p_r)$. Let*

$$\bar{V} = \{\bar{A}_1, \bar{B}_1, \dots, \bar{A}_g, \bar{B}_g, \bar{C}_{g+1}, \dots, \bar{C}_n\}$$

*be a sequential set that generates $\Gamma$. Then, there exist lifts of $\Gamma$ into $G_m$ if and only if the signature $(g : p_1, \dots, p_r)$ satisfies the following liftability conditions:* $\gcd(p_i, m) = 1$ *for $i = 1, \dots, r$ and*

$$(p_1 \cdots p_r) \cdot \left( \sum_{i=1}^{r} \frac{1}{p_i} - (2g - 2) - r \right) \equiv 0 \bmod m.$$

*Moreover, if the liftability conditions are satisfied, then any set of lifts $\{A_i, B_i\}$ of $\{\bar{A}_i, \bar{B}_i\}$ into $G_m$ can be extended in a unique way to a set $\{A_i, B_i, C_j\}$ of lifts of $\{\bar{A}_i, \bar{B}_i, \bar{C}_j\}$ that generates a lift of $\Gamma$ into $G_m$; hence, there are $m^{2g}$ different lifts of $\Gamma$ into $G_m$.*

*Proof.* In this proof all congruences will be modulo $m$. Let us first assume that there exists a lift of $\Gamma$ into $G_m$. Let $\{A_i, B_i, C_j\}$ be a set of lifts of $\bar{V}$ as in Lemmas 4.8 and 4.9. Let $n_i = s_m(C_{g+i})$. Then according to Lemma 4.9 we have $p_i \cdot n_i + 1 \equiv 0$ for $i = 1, \dots, r$ and

$$(2g - 2) + r + \sum_{i=1}^{r} n_i \equiv 0.$$

The congruence $p_i \cdot n_i + 1 \equiv 0$ implies that $p_i$ is prime with $m$ for $i = 1, \dots, r$. Furthermore, since

$$(p_1 \cdots p_r) \cdot n_i \equiv \frac{p_1 \cdots p_r}{p_i} \cdot (p_i \cdot n_i) \equiv \frac{p_1 \cdots p_r}{p_i} \cdot (-1) \equiv -(p_1 \cdots p_r) \cdot \frac{1}{p_i},$$

we obtain that

$$(p_1 \cdots p_r) \cdot \left( \sum_{i=1}^{r} \frac{1}{p_i} - (2g - 2) - r \right) \equiv -(p_1 \cdots p_r) \cdot \left( \sum_{i=1}^{r} n_i + (2g - 2) + r \right) \equiv 0.$$

Now, let us assume that the liftability conditions are satisfied. We want to construct a lift of $\Gamma$ into $G_m$. Since $p_i$ is prime with $m$, we can choose $n_i \in \mathbb{Z}/m\mathbb{Z}$ such that $p_i \cdot n_i + 1 \equiv 0$ for $i = 1, \dots, r$. Then,

$$(p_1 \cdots p_r) \cdot n_i \equiv -(p_1 \cdots p_r) \cdot \frac{1}{p_i}$$

and, hence,

$$(p_1 \cdots p_r) \cdot \left( (2g-2) + r + \sum_{i=1}^{r} n_i \right) \equiv (p_1 \cdots p_r) \cdot \left( (2g-2) + r - \sum_{i=1}^{r} \frac{1}{p_i} \right) \equiv 0.$$

Since $p_1 \cdots p_r$ is prime with $m$, we conclude that $(2g-2)+r+\sum\limits_{i=1}^{r} n_i \equiv 0$, i.e., $\sum\limits_{i=1}^{r} n_i \equiv (2-2g)-r$. Let $V = \{A_i, B_i, C_j\}$ be any set of lifts of $\bar{V}$ such that $s_m(C_{g+i}) = n_i$ for $i = 1, \dots, r$. We have $p_i \cdot n_i + 1 \equiv 0$ for $i = 1, \dots, r$ and $\sum\limits_{i=1}^{r} n_i \equiv (2-2g) - r$. Hence, according to Lemma 4.9, the set $V$ generates a lift of $\Gamma$ into $G_m$. Since Lemma 4.9 does not impose any conditions on the values $s_m(A_i)$ and $s_m(B_i)$ for $i = 1, \dots, g$, any of $m^{2g}$ choices of these $2g$ values leads to a different lift of $\Gamma$ into $G_m$. $\square$

## 5. Higher Arf functions

In [NP05, NP09], we introduced the notion of a higher Arf function and used it to study moduli spaces of higher spin bundles on Riemann surfaces. In this section, we will introduce higher Arf functions on orbifolds, and study their connection with Gorenstein automorphy factors.

### 5.1. Definition of higher Arf functions on orbifolds.

In this subsection, we will define higher Arf functions on orbifolds (compare with subsection 4.1 in [NP09]).

Let $\Gamma$ be a Fuchsian group of signature $(g; l_h, l_p, l_e : p_1, \dots, p_{l_e})$ and $P = \mathbb{H}/\Gamma$ the corresponding orbifold. Let $p \in P$. Let $\Psi : \mathbb{H} \to P$ be the natural projection. Let $\pi(P, p)$ be the orbifold fundamental group of $P$ (see definition 4.3). Choose $q \in \Psi^{-1}(p)$ and let $\Phi : \Gamma \to \pi(P, p)$ be the induced isomorphism. Let $\Gamma^*$ be a lift of $\Gamma$ in $G_m$.

**Definition 5.1.** Let us consider a function $\hat{\sigma}_{\Gamma^*} : \pi(P, p) \to \mathbb{Z}/m\mathbb{Z}$ such that the following diagram commutes

$$\begin{array}{ccc} \Gamma & \xrightarrow{\cong} & \Gamma^* \\ {\scriptstyle\Phi}\big\downarrow & & \big\downarrow{\scriptstyle s_m|_{\Gamma^*}} \\ \pi(P,p) & \xrightarrow{\hat{\sigma}_{\Gamma^*}} & \mathbb{Z}/m\mathbb{Z} \end{array}$$

**Lemma 5.1.** *Let $\alpha$, $\beta$ and $\gamma$ be simple contours in $P$ intersecting pairwise in exactly one point $p$. Let $a$, $b$ and $c$ be the corresponding elements of $\pi(P, p)$. We assume that $a$, $b$ and $c$ satisfy the relations $a, b, c \neq 1$ and $abc = 1$. Let $\langle \cdot, \cdot \rangle$ be the intersection form on $\pi(P, p)$. Then for $\hat{\sigma} = \hat{\sigma}_{\Gamma^*}$*

1. *If the elements $a$ and $b$ can be represented by a pair of simple contours in $P$ intersecting in exactly one point $p$ with $\langle a, b \rangle \neq 0$, then $\hat{\sigma}(ab) = \hat{\sigma}(a) + \hat{\sigma}(b)$.*
2. *If $ab$ is in $\pi^0(P, p)$ and the elements $a$ and $b$ can be represented by a pair of simple contours in $P$ intersecting in exactly one point $p$ with $\langle a, b \rangle = 0$ and a neighbourhood of the point $p$ with the contours is homeomorphic to the one shown in Figure 4, then $\hat{\sigma}(ab) = \hat{\sigma}(a) + \hat{\sigma}(b) + 1$ if the element $ab$ is not of order 2, and $\hat{\sigma}(ab) = -\hat{\sigma}(a) - \hat{\sigma}(b) - 1$ if the element $ab$ is of order 2.*

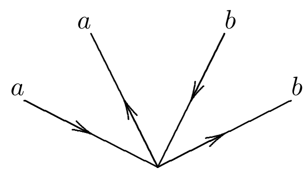


*Figure 4:* $\hat{\sigma}(ab) = \hat{\sigma}(a) + \hat{\sigma}(b) + 1$

3. *If ab is in* $\pi^0(P,p)$ *and the elements a and b can be represented by a pair of simple contours in P intersecting in exactly one point p with* $\langle a, b\rangle = 0$ *and a neighbourhood of the point p with the contours is homeomorphic to the one shown in Figure 1, then* $\hat{\sigma}(ab) = \hat{\sigma}(a) + \hat{\sigma}(b) - 1$.
4. *For any standard set of generators* $v = \{a_1, b_1, \ldots, a_g, b_g, c_{g+1}, \ldots, c_n\}$ *of* $\pi(P,p)$*, we have*

$$\sum_{i=g+1}^{n} \hat{\sigma}(c_i) \equiv (2-2g) - (n-g) \bmod m.$$

5. *For any elliptic element* $c_{g+l_h+l_p+i}$*,* $i = 1, \ldots, l_e$*, we have* $p_i \cdot \hat{\sigma}(c_{g+l_h+l_p+i}) + 1 \equiv 0 \bmod m$.

*Proof.* According to Theorem 4.4, either the set $V = \{\Phi^{-1}(a), \Phi^{-1}(b), \Phi^{-1}(c)\}$ or the set $V^{-1} = \{\Phi^{-1}(a^{-1}), \Phi^{-1}(b^{-1}), \Phi^{-1}(c^{-1})\}$ is sequential. This sequential set can be of signature $(0 : *, *, *)$ or $(1 : *)$.

- If $V$ is a sequential set of signature $(1 : *)$, then according to Lemma 3.3 we obtain $\hat{\sigma}(ab) = \hat{\sigma}(a) + \hat{\sigma}(b)$.
- If $V$ is a sequential set of signature $(0 : *, *, *)$, then according to Corollary 4.7 we obtain

$$\hat{\sigma}(ab) = \hat{\sigma}(a) + \hat{\sigma}(b) + 1 \text{ if } ab \text{ is not of order } 2,$$
$$\hat{\sigma}(ab) = -\hat{\sigma}(a) - \hat{\sigma}(b) - 1 \text{ if } ab \text{ is of order } 2.$$

- If $V^{-1}$ is a sequential set of signature $(0 : *, *, *)$, then according to Corollary 4.7 we obtain,

$$\hat{\sigma}(b^{-1}a^{-1}) = \hat{\sigma}(a^{-1}) + \hat{\sigma}(b^{-1}) + 1 \text{ if } ab \text{ is not of order } 2,$$
$$\hat{\sigma}(b^{-1}a^{-1}) = -\hat{\sigma}(a^{-1}) - \hat{\sigma}(b^{-1}) - 1 \text{ if } ab \text{ is of order } 2.$$

Therefore, for the element $ab$ not of order 2, we obtain

$$\hat{\sigma}(ab) = -\hat{\sigma}((ab)^{-1}) = -\hat{\sigma}(b^{-1}a^{-1}) = -(\hat{\sigma}(a^{-1}) + \hat{\sigma}(b^{-1}) + 1)$$
$$= -\hat{\sigma}(a^{-1}) - \hat{\sigma}(b^{-1}) - 1 = \hat{\sigma}(a) + \hat{\sigma}(b) - 1$$

and, for the element $ab$ of order 2, we obtain

$$\hat{\sigma}(ab) = \hat{\sigma}((ab)^{-1}) = \hat{\sigma}(b^{-1}a^{-1}) = -\hat{\sigma}(a^{-1}) - \hat{\sigma}(b^{-1}) - 1 = \hat{\sigma}(a) + \hat{\sigma}(b) - 1.$$

To prove properties 4 and 5 of $\hat{\sigma}$, we apply Lemma 4.9. $\square$

We now formalize the properties of the function $\hat{\sigma}$ in the following definition:

**Definition 5.2.** We denote by $\pi^0(P,p)$ the set of all non-trivial elements of $\pi(P,p)$ that can be represented by simple contours. An *m-Arf function* is a function

$$\sigma : \pi^0(P,p) \to \mathbb{Z}/m\mathbb{Z}$$

satisfying the following conditions

1. $\sigma(bab^{-1}) = \sigma(a)$ for any elements $a, b \in \pi^0(P,p)$,
2. $\sigma(a^{-1}) = -\sigma(a)$ for any element $a \in \pi^0(P,p)$ that is not of order 2,

3. $\sigma(ab)=\sigma(a)+\sigma(b)$ for any elements $a$ and $b$ which can be represented by a pair of simple contours in $P$ intersecting in exactly one point $p$ with $\langle a,b\rangle\neq 0$,
4. $\sigma(ab)=\sigma(a)+\sigma(b)-1$ for any elements $a,b\in\pi^0(P,p)$ such that the element $ab$ is in $\pi^0(P,p)$ and the elements $a$ and $b$ can be represented by a pair of simple contours in $P$ intersecting in exactly one point $p$ with $\langle a,b\rangle=0$ and a neighbourhood of the point $p$ with the contours is homeomorphic to the one shown in Figure 1.
5. For any elliptic element $c$ of order $p$, we have $p\cdot\sigma(c)+1\equiv 0 \bmod m$.

The following property of $m$-Arf functions follows immediately from Properties 4 and 2 in Definition 5.2:

**Proposition 5.2.** *Let $a$ and $b$ be elements of $\pi^0(P,p)$ such that the element $ab$ is in $\pi^0(P,p)$ and the elements $a$ and $b$ can be represented by a pair of simple contours in $P$ intersecting in exactly one point $p$ with $\langle a,b\rangle=0$ and a neighbourhood of the point $p$ with the contours is homeomorphic to the one shown in Figure 4. Then,*

$$\sigma(c_1c_2)=\sigma(c_1)+\sigma(c_2)+1, \textit{ if } c_1c_2=c_3^{-1} \textit{ is not of order } 2,$$
$$\sigma(c_1c_2)=-\sigma(c_1)-\sigma(c_2)-1, \textit{ if } c_1c_2=c_3^{-1} \textit{ is of order } 2.$$

**Lemma 5.3.** *Let $\Gamma$ be a hyperbolic polygon group of signature $(0:p_1,\dots,p_r)$, $r>3$. Let $\{c_1,\dots,c_r\}$ be a standard set of generators of $\Gamma$. Then, the element $c_1c_2$ is not elliptic.*

*Proof.* Let $\Pi$ be the canonical fundamental polygon for the group generated by the elements $c_1,\dots,c_r$ such that the generators $c_i$ can be described as products $c_i=\sigma_i\sigma_{i+1}$ of reflexions $\sigma_1,\dots,\sigma_r$ in the edges of the polygon $\Pi$ (suitably numbered). Then,

$$c_1c_2=(\sigma_1\sigma_2)(\sigma_2\sigma_3)=\sigma_1\sigma_3.$$

The product of two reflexions $\sigma_1\sigma_3$ is an elliptic element if and only if the axes of the reflexions intersect in $\mathbb{H}$. Since $r>3$, the sides of the polygon $\Pi$ that correspond to the reflexions $\sigma_1$ and $\sigma_3$ are not next to each other. Let us assume that the axes intersect and let $Q$ be the hyperbolic polygon enclosed between by the axes and the polygon $\Pi$. All angles of the polygon $\Pi$ are acute. One angle of the polygon $Q$ is the angle between the intersecting axes, two angles are larger than $\pi/2$, all other angles of $Q$ are larger than $\pi$. Hence, the sum of the angles of $Q$ is larger that it should be for a hyperbolic polygon. $\square$

**Proposition 5.4.** *For any standard set of generators*

$$v=\{a_1,b_1,\dots,a_g,b_g,c_{g+1},\dots,c_n\}$$

*of $\pi(P,p)$ with $n=g+l_h+l_p+l_e$, we have*

$$\sum_{j=g+1}^{n}\sigma(c_j)\equiv(2-2g)-(n-g) \bmod m.$$

*Proof.* We discuss the case $g=0$ first and, then, we reduce the general case to the case $g=0$.

- Let $g=0$. We prove that the statement is true for lifts of sequential sets of signature $(0:p_1,\dots,p_r)$ by induction on $r$. In the case $r=3$, Proposition 5.2 implies

$$\sigma(c_1c_2)=\sigma(c_1)+\sigma(c_2)+1 \text{ if } c_1c_2=c_3^{-1} \text{ is not of order } 2,$$
$$\sigma(c_1c_2)=-\sigma(c_1)-\sigma(c_2)-1 \text{ if } c_1c_2=c_3^{-1} \text{ is of order } 2.$$

  If the element $c_3$ is not of order 2, then Property 2 of $m$-Arf functions implies $\sigma(c_1c_2)=\sigma(c_3^{-1})=-\sigma(c_3)$. Combining $\sigma(c_1c_2)=\sigma(c_1)+\sigma(c_2)+1$ and $\sigma(c_1c_2)=-\sigma(c_3)$, we obtain $\sigma(c_1)+\sigma(c_2)+\sigma(c_3)=-1$. If the element $c_3$ is of order 2, then $\sigma(c_1c_2)=\sigma(c_3^{-1})=$

$\sigma(c_3)$. Combining $\sigma(c_1c_2) = -\sigma(c_1) - \sigma(c_2) - 1$ and $\sigma(c_1c_2) = \sigma(c_3)$, we obtain that $\sigma(c_1) + \sigma(c_2) + \sigma(c_3) = -1$. Let us now assume that the statement is true for $r \leqslant k-1$ and consider the case $r = k$. By our assumption,

$$\sigma(c_1 \cdot c_2) + \sigma(c_3) + \cdots + \sigma(c_k) = 2 - (k-1) = (2-k) + 1.$$

Moreover, according to Lemma 5.3, the element $c_1c_2$ cannot be of order 2. Hence, by Proposition 5.2, we have $\sigma(c_1c_2) = \sigma(c_1) + \sigma(c_2) + 1$. The last two equations imply that

$$\sigma(c_1) + \cdots + \sigma(c_k) = 2 - k.$$

- We now consider the general case. The set

$$(a_1, b_1a_1^{-1}b_1^{-1}, \ldots, a_g, b_ga_g^{-1}b_g^{-1}, c_{g+1}, \ldots, c_{g+l_h+l_p+l_e})$$

is a standard set of generators of an orbifold of signature $(0 : 2g + l_h, l_p, l_e : p_1, \ldots, p_{l_e})$; hence,

$$\begin{aligned}&\sum_{i=1}^{g}(\sigma(a_i) + \sigma(b_ia_i^{-1}b_i^{-1})) + \sum_{i=g+1}^{g+l_h+l_p+l_e}\sigma(c_i)\\ &= 2 - (2g + l_h + l_p + l_e) = (2-2g) - (l_h + l_p + l_e).\end{aligned}$$

Properties 1 and 2 of $m$-Arf functions imply that $\sigma(b_ia_i^{-1}b_i^{-1}) = \sigma(a_i^{-1}) = -\sigma(a_i)$ and, hence, $\sigma(a_i) + \sigma(b_ia_i^{-1}b_i^{-1}) = 0$. □

**Definition 5.3.** Let $\hat{\sigma}_{\Gamma^*} : \pi(P,p) \to \mathbb{Z}/m\mathbb{Z}$ be the function associated to a lift $\Gamma^*$ as in Definition 5.1, then the function $\sigma_{\Gamma^*} = \hat{\sigma}_{\Gamma^*}|_{\pi^0(P,p)}$ is an $m$-Arf function according to Lemmata 5.1, 3.1 and 3.2. We call the function $\sigma_{\Gamma^*}$ the *$m$-Arf function associated to the lift* $\Gamma^*$.

5.2. **Higher Arf functions and self-homeomorphisms of orbifolds.** Let $\Gamma$ be a Fuchsian group of signature $(g : p_1, \ldots, p_r)$ and $P = \mathbb{H}/\Gamma$ the corresponding orbifold. Let $p \in P$. Let $\Psi : \mathbb{H} \to P$ be the natural projection. Choose $q \in \Psi^{-1}(p)$ and let $\Phi : \Gamma \to \pi^0(P,p)$ be the induced isomorphism. Let $\Gamma^*$ be a lift of $\Gamma$ in $G_m$. Consider the following transformations of a standard set of generators $v = \{a_1, b_1, \ldots, a_g, b_g, c_{g+1}, \ldots, c_{g+r}\}$ of $\pi^0(P,p)$ to another standard set of generators $v' = \{a'_1, b'_1, \ldots, a'_g, b'_g, c'_{g+1} \ldots, c'_{g+r}\}$:

1. $a'_1 = a_1b_1$.
2. $a'_1 = (a_1a_2)a_1(a_1a_2)^{-1}$,
   $b'_1 = (a_1a_2)a_1^{-1}a_2^{-1}b_1(a_1a_2)^{-1}$,
   $a'_2 = a_1a_2a_1^{-1}$,
   $b'_2 = b_2a_2^{-1}a_1^{-1}$.
3. $a'_g = (b_g^{-1}c_{g+1})b_g^{-1}(b_g^{-1}c_{g+1})^{-1}$,
   $b'_g = (b_g^{-1}c_{g+1}b_g)c_{g+1}^{-1}b_ga_gb_g^{-1}(b_g^{-1}c_{g+1}b_g)^{-1}$,
   $c'_{g+1} = b_g^{-1}c_{g+1}b_g$.
4. $a'_k = a_{k+1}, \quad b'_k = b_{k+1}$,
   $a'_{k+1} = (c_{k+1}^{-1}c_k)a_k(c_{k+1}^{-1}c_k)^{-1}$,
   $b'_{k+1} = (c_{k+1}^{-1}c_k)b_k(c_{k+1}^{-1}c_k)^{-1}$.
5. $c'_k = c_{k+1}, \quad c'_{k+1} = c_{k+1}^{-1}c_kc_{k+1}$.

Here, $c_i = [a_i, b_i]$ for $i = 1, \dots, g$; in (4), we consider $k \in \{1, \dots, g\}$; in (5), we consider
$$k \in \{g+1, \dots, g+r\}$$
such that $\operatorname{ord}(c_k) = \operatorname{ord}(c_{k+1})$. If $a'_i$, $b'_i$, resp. $c'_i$, is not described explicitly, this means $a'_i = a_i$, $b'_i = b_i$, resp. $c'_i = c_i$.

We will call these transformations generalised Dehn twists. Each generalised Dehn twist induces a homotopy class of self-homeomorphisms of the orbifold $P$, which maps elliptic fixed points to elliptic fixed points of the same order. The group of all homotopy classes of self-homeomorphisms of the orbifold $P$ is generated by the homotopy classes of generalised Dehn twists as described above (compare [Zie73]).

Now, we will compute the values of an $m$-Arf function $\sigma$ on the standard set of generators $v'$ from the values of $\sigma$ on the standard set of generators $v$ for each of the generalised Dehn twists described above.

**Lemma 5.5.** *Let $\sigma : \pi^0(P,p) \to \mathbb{Z}/m\mathbb{Z}$ be an $m$-Arf function. Let $D$ be a generalised Dehn twist of the type described above. Suppose that $D$ maps the standard set of generators*
$$v = \{a_1, b_1, \dots, a_g, b_g, c_{g+1}, \dots, c_{g+r}\}$$
*into the standard set of generators*
$$v' = D(v) = \{a'_1, b'_1, \dots, a'_g, b'_g, c'_{g+1}, \dots, c'_{g+r}\}.$$
*Let $\alpha_i, \beta_i, \gamma_i$, resp. $\alpha'_i, \beta'_i, \gamma'_i$, be the values of $\sigma$ on the elements of $v$, resp. $v'$. Then, for the Dehn twists of types 1–5, we obtain*

1. $\alpha'_1 = \alpha_1 + \beta_1$.
2. $\beta'_1 = \beta_1 - \alpha_1 - \alpha_2 - 1, \quad \beta'_2 = \beta_2 - \alpha_2 - \alpha_1 - 1.$
3. $\alpha'_g = -\beta_g, \quad \beta'_g = \alpha_g - \gamma_{g+1} - 1.$
4. $\alpha'_k = \alpha_{k+1}, \quad \beta'_k = \beta_{k+1}, \quad \alpha'_{k+1} = \alpha_k, \quad \beta'_{k+1} = \beta_k.$
5. $\gamma'_k = \gamma_{k+1}, \quad \gamma'_{k+1} = \gamma_k.$

*Proof.* We assume that the Dehn twist $D$ belongs to one of the types described in the definition above. In the following computations, we illustrate the position of the contours on the surface with figures showing the position of the axes of the corresponding elements in $\Gamma$. Let $\{A_1, B_1, \dots, A_g, B_g, C_{g+1}, \dots, C_{g+r}\}$ be the sequential set corresponding to the standard set of generators $v$. In the first case, according to Property 3 of $m$-Arf functions, we obtain $\sigma(a'_1) = \sigma(a_1 b_1) = \sigma(a_1) + \sigma(b_1)$.

In the second case, according to Property 1, we obtain
$$\begin{aligned}\sigma(a'_1) &= \sigma((a_1a_2)a_1(a_1a_2)^{-1}) = \sigma(a_1),\\ \sigma(b'_1) &= \sigma((a_1a_2)a_1^{-1}a_2^{-1}b_1(a_1a_2)^{-1}) = \sigma(a_1^{-1}a_2^{-1}b_1)\\ &= \sigma(a_1(a_1^{-1}a_2^{-1}b_1)a_1^{-1}) = \sigma(a_2^{-1}b_1a_1^{-1}).\end{aligned}$$
The mutual position of the axes of the elements $A_2^{-1}$ and $B_1A_1^{-1}$ is as in Figure 5; hence, Property 4 implies $\sigma(b'_1) = \sigma(a_2^{-1} \cdot (b_1a_1^{-1})) = \sigma(a_2^{-1}) + \sigma(b_1a_1^{-1}) - 1$. According to Property 3, we have $\sigma(b_1a_1^{-1}) = \sigma(b_1) + \sigma(a_1^{-1})$. Thus, using Property 2, we obtain
$$\sigma(b'_1) = \sigma(a_2^{-1}) + \sigma(b_1) + \sigma(a_1^{-1}) - 1 = \sigma(b_1) - \sigma(a_1) - \sigma(a_2) - 1.$$
Similarly, we show that $\sigma(a'_2) = \sigma(a_2)$ and $\sigma(b'_2) = \sigma(b_2) - \sigma(a_2) - \sigma(a_1) - 1$.

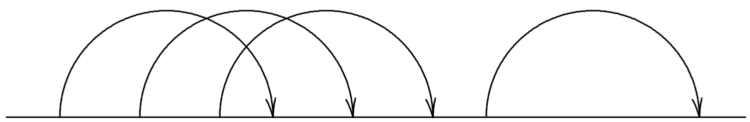


Figure 5: Axes of $B_1A_1^{-1}$ and $A_2^{-1}$

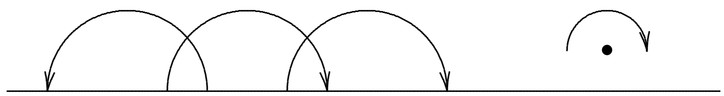


Figure 6: Axis of $B_gA_gB_g^{-1}$ and fixed point of $C_{g+1}^{-1}$

In the third case, we obtain according to Properties 2 and 1

$$\begin{aligned}\sigma(a'_g) &= \sigma((b_g^{-1}c_{g+1})b_g^{-1}(b_g^{-1}c_{g+1})^{-1}) = \sigma(b_g^{-1}) = -\sigma(b_g),\\ \sigma(b'_g) &= \sigma((b_g^{-1}c_{g+1}b_g)c_{g+1}^{-1}b_ga_gb_g^{-1}(b_g^{-1}c_{g+1}b_g)^{-1}) = \sigma(c_{g+1}^{-1}b_ga_gb_g^{-1}),\\ \sigma(c'_{g+1}) &= \sigma(b_g^{-1}c_{g+1}b_g) = \sigma(c_{g+1}).\end{aligned}$$

The actions of the elements $C_{g+1}^{-1}$ and $B_gA_gB_g^{-1}$ on $\mathbb{H}$ are illustrated in Figure 6. According to Properties 4 and 1, we obtain

$$\sigma(b'_g) = \sigma(c_{g+1}^{-1}\cdot(b_ga_gb_g^{-1})) = \sigma(c_{g+1}^{-1}) + \sigma(b_ga_gb_g^{-1}) - 1 = \sigma(c_{g+1}^{-1}) + \sigma(a_g) - 1.$$

In the fourth and fifth cases, computations are easy; we use only Property 1 of $m$-Arf functions. □

5.3. **Correspondence between higher Arf functions and hyperbolic Gorenstein automorphy factors.** Let $\Gamma$ be a Fuchsian group of signature $(g : p_1, \dots, p_r)$ and $P = \mathbb{H}/\Gamma$ the corresponding orbifold. Let $p \in P$. Let $\Psi : \mathbb{H} \to P$ be the natural projection. Choose $q \in \Psi^{-1}(p)$ and let $\Phi : \Gamma \to \pi^0(P, p)$ be the induced isomorphism.

**Lemma 5.6.** *The difference $\sigma_1 - \sigma_2 : \pi^0(P, p) \to \mathbb{Z}/m\mathbb{Z}$ of two Arf functions $\sigma_1$ and $\sigma_2$ induces a linear function $\ell : H_1(P; \mathbb{Z}/m\mathbb{Z}) \to \mathbb{Z}/m\mathbb{Z}$.*

*Proof.* The proof is analogous to the proof of the corresponding statement for higher Arf functions on Fuchsian groups without torsion (see Lemma 4.5 in [NP09]). The main observation is the fact that according to Lemma 5.5 the action of the generalised Dehn twists on the tuples of values of a higher Arf function on elements of a standard set of generators are by affine-linear maps; therefore, the action on the tuples of differences of values of two higher Arf functions is by linear maps. □

**Corollary 5.7.** *The set* $\mathrm{Arf}^{P,m}$ *of all* $m$*-Arf functions on* $\pi^0(P,p)$ *has the structure of an affine space, i.e., the set* $\{\sigma-\sigma_0 \mid \sigma\in\mathrm{Arf}^{P,m}\}$ *is a free module over* $\mathbb{Z}/m\mathbb{Z}$ *for any* $\sigma_0\in\mathrm{Arf}^{P,m}$.

**Corollary 5.8.** *An* $m$*-Arf function is uniquely determined by its values on the elements of some standard set of generators of* $\pi^0(P,p)$.

**Theorem 5.9.** *Let* $\Gamma$ *be a Fuchsian group of signature* $(g:p_1,\dots,p_r)$ *and* $P=\mathbb{H}/\Gamma$ *the corresponding orbifold. Let* $p\in P$. *There is a 1-1-correspondence between*

*1) hyperbolic Gorenstein automorphy factors of level* $m$ *associated to the Fuchsian group* $\Gamma$.
*2) lifts of* $\Gamma$ *into* $G_m$.
*3)* $m$*-Arf functions* $\sigma:\pi^0(P,p)\to\mathbb{Z}/m\mathbb{Z}$.

*Proof.* According to Proposition 2.4, there is a 1-1-correspondence between hyperbolic Gorenstein automorphy factors of level $m$ associated to the Fuchsian group $\Gamma$ and the lifts of $\Gamma$ into $G_m$. In Definition 5.1, we attached to any lift $\Gamma^*$ of $\Gamma$ into $G_m$ an $m$-Arf function $\sigma_{\Gamma^*}$ on $P$. On the other hand, we can attach to any $m$-Arf function $\sigma$ a subset of $G_m$

$$\Gamma^*_\sigma=\{g\in G_m \mid \pi(g)\in\Gamma,\ s_m(g)=\sigma(\Phi(\pi(g)))\},$$

where $\pi:G_m\to G$ is the covering map. It remains to prove that this subset of $G_m$ is actually a lift of $\Gamma$. Let

$$v=\{a_1,b_1,\dots,a_g,b_g,c_{g+1},\dots,c_{g+r}\}=\{d_1,\dots,d_{2g+r}\}$$

be a standard set of generators of $\pi(P,p)$ and let

$$\bar V=\{\Phi^{-1}(d_1),\dots,\Phi^{-1}(d_{2g+r})\}$$

be the corresponding sequential set. Let $\{D_j\}_{j=1,\dots,2g+r}$ be a lift of the sequential set $\bar V$, i.e., $\pi(D_j)=\Phi^{-1}(d_j)$, such that $s_m(D_j)=\sigma(d_j)$. Then, we obtain according to Proposition 5.4 that

$$\sum_{i=g+1}^{g+r}s_m(C_i)=\sum_{i=g+1}^{g+r}\sigma(c_i)\equiv(2-2g)-r \bmod m.$$

Hence, by Lemma 4.9, we obtain

$$\prod_{i=1}^{g}[A_i,B_i]\cdot\prod_{i=g+1}^{g+r}C_i=e.$$

This, and the fact that, for any $i=1,\dots,r$,

$$p_i\cdot s_m(C_{g+i})+1=p_i\cdot\sigma(c_{g+i})+1\equiv 0 \bmod m,$$

imply, according to Lemma 4.8, that the subgroup $\Gamma^*$ of $G_m$ generated by $V$ is a lift of $\Gamma$ into $G_m$. Let us compare the corresponding Arf function $\sigma_{\Gamma^*}$ with the Arf function $\sigma$. We have $\sigma_{\Gamma^*}(d_j)=s_m(D_j)=\sigma(d_j)$ for all $j$, i.e., the Arf functions $\sigma_{\Gamma^*}$ and $\sigma$ coincide on the standard set of generators $v$. Thus, by Lemma 5.6 the Arf functions $\sigma_{\Gamma^*}$ and $\sigma$ coincide on the whole $\pi^0(P,p)$. From the definition of $\sigma_{\Gamma^*}$ and $\Gamma^*_\sigma$, we see that this implies that $\Gamma^*=\Gamma^*_\sigma$; hence, $\Gamma^*_\sigma$ is indeed a lift of $\Gamma$ into $G_m$. It is clear from the definitions that the mappings $\Gamma^*\mapsto\sigma_{\Gamma^*}$ and $\sigma\mapsto\Gamma^*_\sigma$ are inverse to each other. □

**Corollary 5.10.** *Let* $P$ *be a Riemann orbifold of signature* $(g:p_1,\dots,p_r)$. *Let*

$$v=\{a_1,b_1,\dots,a_g,b_g,c_{g+1},\dots,c_{g+r}\}$$

*be a standard set of generators of* $\pi(P,p)$. *An* $m$*-Arf function on* $\pi^0(P,p)$ *exists if and only if the signature* $(g:p_1,\dots,p_r)$ *satisfies the liftability conditions described in Proposition 4.10. Moreover, if the liftability conditions are satisfied then any possible tuple of* $2g$ *values in* $\mathbb{Z}/m\mathbb{Z}$

*can be realised in a unique way as a set of values on $a_i, b_i$ of an $m$-Arf function on $\pi^0(P,p)$, hence there are $m^{2g}$ different $m$-Arf functions on $\pi^0(P,p)$.*

*Proof.* The statement follows immediately from Theorem 5.9 and Proposition 4.10. □

## 6. Moduli spaces of Gorenstein singularities

We study the moduli space of Gorenstein quasi-homogeneous surface singularities (GQHSS). Using Proposition 2.4, we define the moduli space of GQHSS of level $m$ as the space of conjugacy classes of subgroups $\Gamma^*$ in $G_m$ such that the restriction of the covering map $G_m \to G = \mathrm{PSL}(2,\mathbb{R})$ to $\Gamma^*$ is an isomorphism between $\Gamma^*$ and a Fuchsian group $\Gamma$. The projection $\Gamma^* \mapsto \Gamma$ from the moduli space of GQHSS of level $m$ to the moduli space of Riemann orbifolds is a finite ramified covering.

### 6.1. **Topological classification of higher Arf functions.**

There is a 1-1-correspondence (see Theorem 5.9) between automorphy factors of level $m$ and $m$-Arf functions on $\pi^0(P,p)$. This correspondence allows us to reduce the problem of finding the number of connected components of the moduli space of GQHSS of level $m$ to the problem of finding the number of orbits of the action of the group of self-homeomorphisms on the set of $m$-Arf functions. We describe the orbit of an $m$-Arf function under the action of the group of homotopy classes of surface self-homeomorphisms.

Let $P$ be a Riemann orbifold of signature $(g : p_1, \dots, p_r)$. Let $p \in P$.

**Definition 6.1.** Let $\sigma : \pi^0(P,p) \to \mathbb{Z}/m\mathbb{Z}$ be an $m$-Arf function. We define the *Arf invariant* $\delta = \delta(P,\sigma)$ of $\sigma$ as follows: If $g > 1$ and $m$ is even, then we set $\delta = 0$ if there is a standard set of generators $\{a_1, b_1, \dots, a_g, b_g, c_{g+1}, \dots, c_{g+r}\}$ of the orbifold fundamental group $\pi(P,p)$ such that

$$\sum_{i=1}^{g}(1-\sigma(a_i))(1-\sigma(b_i)) \equiv 0 \bmod 2$$

and we set $\delta = 1$ otherwise. If $g > 1$ and $m$ is odd, then we set $\delta = 0$. If $g = 0$, then we set $\delta = 0$. If $g = 1$, then we set $\delta = \gcd(m, p_1 - 1, \dots, p_r - 1, \sigma(a_1), \sigma(b_1))$, where $\{a_1, b_1, c_2, \dots, c_{r+1}\}$ is a standard set of generators of the fundamental group $\pi(P,p)$.

*Remark.* It is not hard to see that $\delta$ does not change under the transformations described in Lemma 5.5, i.e., it is indeed an invariant of the Arf function.

*Proof.* Let $D$, $v$, $v'$, $\alpha_i$, $\beta_i$, $\gamma_i$, $\alpha'_i$, $\beta'_i$, $\gamma'_i$ be as in Lemma 5.5. Let us first consider the case $g > 1$: For a Dehn twist of type 1, we have

$$\begin{aligned}(1-\alpha'_1)(1-\beta'_1) &= (1-(\alpha_1+\beta_1))(1-\beta_1)\\ &= (1-\alpha_1)(1-\beta_1) - \beta_1(1-\beta_1) \equiv (1-\alpha_1)(1-\beta_1) \bmod 2.\end{aligned}$$

For a Dehn twist of type 2, we have

$$\begin{aligned}&(1-\alpha'_1)(1-\beta'_1) + (1-\alpha'_2)(1-\beta'_2)\\ &= (1-\alpha_1)(1-\beta_1+\alpha_1+\alpha_2+1) + (1-\alpha_2)(1-\beta_2+\alpha_1+\alpha_2+1)\\ &= (1-\alpha_1)(1-\beta_1) + (1-\alpha_2)(1-\beta_2) + (2-(\alpha_1+\alpha_2))((\alpha_1+\alpha_2)+1)\\ &\equiv (1-\alpha_1)(1-\beta_1) + (1-\alpha_2)(1-\beta_2) \bmod 2.\end{aligned}$$

For a Dehn twist of type 3, since $m$ is even and $p_1 \cdot \gamma_{g+1} + 1 \equiv 0 \bmod m$, we have that $\gamma_{g+1}$ is odd. Then, $\gamma_{g+1} + 1 \equiv 0 \bmod 2$ and $1 + \beta_g \equiv 1 - \beta_g \bmod 2$ imply

$$\begin{aligned}(1-\alpha_g')(1-\beta_g') &= (1+\beta_g)(1-\alpha_g+(\gamma_{g+1}+1)) \\ &\equiv (1+\beta_g)(1-\alpha_g) \equiv (1-\beta_g)(1-\alpha_g) \bmod 2.\end{aligned}$$

Dehn twists of type 4 do not change $\delta$ since they only permute $(\alpha_k, \beta_k)$ with $(\alpha_{k+1}, \beta_{k+1})$. Dehn twists of type 5 do not change $\delta$ since they only permute $\gamma_i$.

Let us now consider the case $g = 1$: Dehn twists of types 2 and 4 involve pairs $a_i, b_i$ and $a_j, b_j$, i.e., they are not applicable in the case $g = 1$. Dehn twists of type 5 do not change $\delta$ since they do not change $\alpha_i, \beta_i$. For a Dehn twist of type 1, we obtain $\alpha_1' = \alpha_1 + \beta_1$ and $\beta_1' = \beta_1$. Thus,

$$\gcd(\alpha_1', \beta_1') = \gcd(\alpha_1 + \beta_1, \beta_1) = \gcd(\alpha_1, \beta_1)$$

and, therefore, $\gcd(m, p_1 - 1, \dots, p_r - 1, \alpha_1', \beta_1') = \gcd(m, p_1 - 1, \dots, p_r - 1, \alpha_1, \beta_1)$. For a Dehn twist of type 3, we obtain $\alpha_1' = -\beta_1$ and $\beta_1' = \alpha_1 - \gamma_2 - 1$. Let $d$ be a common divisor of $m, p_1 - 1, \dots, p_r - 1, \alpha_1, \beta_1$, i.e

$$m \equiv \alpha_1 \equiv \beta_1 \equiv 0 \bmod d, \quad p_1 \equiv \dots \equiv p_r \equiv 1 \bmod d.$$

We know that $p_1 \cdot \gamma_2 + 1 \equiv 0 \bmod m$, but $m \equiv 0 \bmod d$; hence, $p_1 \cdot \gamma_2 + 1 \equiv 0 \bmod d$. Since $p_1 \equiv 1 \bmod d$, we obtain that $\gamma_2 + 1 \equiv 0 \bmod d$. Hence, $d$ is a common divisor of

$$m, p_1 - 1, \dots, p_r - 1, \alpha_1' = -\beta_1, \beta_1' = \alpha_1 - (\gamma_2 + 1).$$

Similarly, every common divisor of

$$m, p_1 - 1, \dots, p_r - 1, \alpha_1', \beta_1'$$

is a common divisor of $m, p_1 - 1, \dots, p_r - 1, \alpha_1, \beta_1$. Thus, the greatest common divisors coincide. □

**Definition 6.2.** By the *type of the $m$-Arf function* $(P, \sigma)$, we mean the tuple $(g, p_1, \dots, p_r, \delta)$, where $\delta$ is the Arf invariant of $\sigma$ defined above.

**Lemma 6.1.** *Let $\sigma : \pi^0(P, p) \to \mathbb{Z}/m\mathbb{Z}$ be an $m$-Arf function.*

*(a) If $g > 1$, then there is a standard set of generators*

$$v = \{a_1, b_1, \dots, a_g, b_g, c_{g+1}, \dots, c_{g+r}\}$$

*of $\pi(P, p)$ such that $(\sigma(a_1), \sigma(b_1), \dots, \sigma(a_g), \sigma(b_g)) = (0, \xi, 1, \dots, 1)$ with $\xi \in \{0, 1\}$. If $m$ is odd then the set of generators can be chosen in such a way that $\xi = 1$, i.e., so that $(\sigma(a_1), \sigma(b_1), \dots, \sigma(a_g), \sigma(b_g)) = (0, 1, 1, \dots, 1)$.*

*(b) If $g = 1$, then there is a standard set of generators $v = \{a_1, b_1, c_2, \dots, c_{g+1}\}$ of $\pi(P, p)$ such that $(\sigma(a_1), \sigma(b_1)) = (\delta, 0)$, where $\delta$ is the Arf invariant of $\sigma$.*

*Proof.* The proof is along the lines of the proofs of Lemma 5.1 and Lemma 5.2 in [NP09]. Using generalised Dehn twists of types 1, 2 and 4, we can show that a set of generators can be chosen in the desired way. The last step in the proof of Lemma 5.1 in [NP09] was to show that, if $m$ and $\sigma(c_{g+i})$ were even, then we could transform a set of generators with

$$(\sigma(a_1), \sigma(b_1), \dots, \sigma(a_g), \sigma(b_g)) = (0, 0, 1, \dots, 1)$$

into a set of generators with $(\sigma(a_1), \sigma(b_1), \dots, \sigma(a_g), \sigma(b_g)) = (0, 1, 1, \dots, 1)$. However, in the situation we are considering now, we know that $\sigma(c_{g+i})$ satisfies the equation $p_i \cdot \sigma(c_{g+i}) + 1 \equiv 0$ modulo $m$. Therefore, if $m$ is even then $\sigma(c_{g+i})$ must be odd. Hence, this last reduction step is not possible in the case considered here. □

**Theorem 6.2.** *A tuple $t = (g, p_1, \ldots, p_r, \delta)$ is the type of a hyperbolic $m$-Arf function on a Riemann orbifold of signature $(g : p_1, \ldots, p_r)$ if and only if it satisfies the following conditions:*
*(a) The liftability conditions: The orders $p_1, \ldots, p_r$ are prime with $m$ and satisfy the condition*

$$(p_1 \cdots p_r) \cdot \left( \sum_{i=1}^{r} \frac{1}{p_i} - (2g-2) - r \right) \equiv 0 \bmod m.$$

*(b) If $g > 1$ then $\delta \in \{0, 1\}$.*
*(c) If $g > 1$ and $m$ is odd then $\delta = 0$.*
*(d) If $g = 1$ then $\delta$ is a divisor of $\gcd(m, p_1 - 1, \ldots, p_r - 1)$.*
*(e) If $g = 0$ then $\delta = 0$.*

*Proof.* Let us first assume that the tuple $t$ is a type of a hyperbolic $m$-Arf function on an orbifold of signature $(g : p_1, \ldots, p_r)$. Then, according to Corollary 5.10, the signature $(g : p_1, \ldots, p_r)$ satisfies the liftability conditions. If $g > 1$ and $m$ is odd, then, according to Lemma 6.1, there is a standard set of generators

$$\{a_1, b_1, \ldots, a_g, b_g, c_{g+1}, \ldots, c_{g+r}\}$$

of $\pi^0(P, p)$ such that

$$(\sigma(a_1), \sigma(b_1), \ldots, \sigma(a_g), \sigma(b_g)) = (0, 1, 1, \ldots, 1).$$

Hence, $\delta(P, \sigma) = 0$ by definition. If $g = 1$, then $\delta$ is a divisor of $m, p_1 - 1, \ldots, p_r - 1$ by definition. If $g = 0$, then $\delta = 0$ by definition.

Now, let us assume that $t = (g, p_1, \ldots, p_r, \delta)$ satisfies the conditions (a)-(e). Let $P$ be a Riemann orbifold of signature $(g : p_1, \ldots, p_r)$ and let

$$\{a_1, b_1, \ldots, a_g, b_g, c_{g+1}, \ldots, c_{g+r}\}$$

be a standard set of generators of $\pi^0(P, p)$. According to Corollary 5.10, any tuple of $2g$ values in $\mathbb{Z}/m\mathbb{Z}$ can be realised as a set of values on $a_i, b_i$ of an $m$-Arf function on $\pi^0(P, p)$. In particular, if $g > 1$, then, for any $\delta \in \{0, 1\}$, there exists an $m$-Arf function $\sigma^\delta$ such that $(\sigma^\delta(a_1), \sigma^\delta(b_1), \ldots, \sigma^\delta(a_g), \sigma^\delta(b_g)) = (0, 1 - \delta, 1, \ldots, 1)$ and, if $g = 1$, then, for any divisor $\delta$ of $m, p_1 - 1, \ldots, p_r - 1$, there exists an $m$-Arf function $\sigma^\delta$ such that $(\sigma^\delta(a_1), \sigma^\delta(b_1)) = (\delta, 0)$. Let $g > 1$. If $\delta = 0$, then the equation $\delta(\sigma^0) = 0$ is satisfied by definition. If $\delta = 1$ and $m$ is even, it remains to prove that $\delta(\sigma^1) = 1$. To this end, we recall that $\sum_{i=1}^{g} (1 - \sigma(a_i))(1 - \sigma(b_i)) \bmod 2$ is preserved under the Dehn twists and, hence, is equal to 1 modulo 2 for any standard set of generators. Now, let $g = 1$. Then, $\delta(\sigma^\delta) = \gcd(m, p_1 - 1, \ldots, p_r - 1, \delta, 0) = \delta$, since $\delta$ is a divisor of $\gcd(m, p_1 - 1, \ldots, p_r - 1)$. □

6.2. **Teichmüller spaces of Fuchsian groups.** We recall the results on the moduli spaces of Fuchsian groups from [Zie81]. Let $\Gamma_{g;p_1,\ldots,p_r}$ be the group generated by the elements

$$v = \{a_1, b_1, \ldots, a_g, b_g, c_{g+1}, \ldots, c_{g+r}\}$$

with defining relations

$$\prod_{i=1}^{g} [a_i, b_i] \prod_{i=g+1}^{g+r} c_i = 1, \quad c_{g+1}^{p_1} = \cdots = c_{g+r}^{p_r} = 1.$$

We denote by $\tilde{T}_{g;p_1,\ldots,p_r}$ the set of monomorphisms $\psi : \Gamma_{g;p_1,\ldots,p_r} \to \mathrm{Aut}(\mathbb{H})$ such that

$$\psi(v) = \{a_1^\psi, b_1^\psi, \ldots, a_g^\psi, b_g^\psi, c_{g+1}^\psi, \ldots, c_{g+r}^\psi\}$$

is a sequential set of signature $(g; p_1, \dots, p_r)$. For $(g; p_1, \dots, p_r)$ to be a signature of a group of hyperbolic isometries, we have to assume that $\sum\limits_{i=1}^{r} \frac{1}{p_i} < r + (2g-2)$. The group $\operatorname{Aut}(\mathbb{H})$ acts on $\tilde{T}_{g;p_1,\dots,p_r}$ by conjugation. We set $T_{g;p_1,\dots,p_r} = \tilde{T}_{g;p_1,\dots,p_r} / \operatorname{Aut}(\mathbb{H})$. We parametrise the space $\tilde{T}_{g;p_1,\dots,p_r}$ by the fixed points and shift parameters of the elements of the sequential sets $\psi(v)$. We use here the following analogue of a version [Nat78, Nat04] of the Theorem of Fricke and Klein [FK65]:

**Theorem 6.3.** *The space $T_{g;p_1,\dots,p_r}$ is diffeomorphic to an open domain in the space $\mathbb{R}^{6g-6+2r}$ which is homeomorphic to $\mathbb{R}^{6g-6+2r}$.*

For an element $\psi : \Gamma_{g;p_1,\dots,p_r} \to \operatorname{Aut}(\mathbb{H})$ of $\tilde{T}_{g;p_1,\dots,p_r}$, we write

$$\widetilde{\operatorname{Mod}}^{\psi} = \widetilde{\operatorname{Mod}}^{\psi}_{g;p_1,\dots,p_r} = \{\alpha \in \operatorname{Aut}(\Gamma_{g;p_1,\dots,p_r}) \mid \psi \circ \alpha \in \tilde{T}_{g;p_1,\dots,p_r}\}.$$

One can show that $\widetilde{\operatorname{Mod}}^{\psi}$ does not depend on $\psi$; hence, we write $\widetilde{\operatorname{Mod}}$ instead of $\widetilde{\operatorname{Mod}}^{\psi}$. Let $I\widetilde{\operatorname{Mod}}$ be the subgroup of all inner automorphisms of $\Gamma_{g;p_1,\dots,p_r}$ and let

$$\operatorname{Mod}_{g;p_1,\dots,p_r} = \operatorname{Mod} = \widetilde{\operatorname{Mod}} / I\widetilde{\operatorname{Mod}}.$$

We now recall the description of the moduli space of Riemann orbifolds

**Theorem 6.4.** *The group $\operatorname{Mod} = \operatorname{Mod}_{g;p_1,\dots,p_r}$ and the group of homotopy classes of orientation preserving self-homeomorphisms of the orbifold of signature $(g : p_1, \dots, p_r)$ are naturally isomorphic. The group $\operatorname{Mod}_{g;p_1,\dots,p_r}$ acts naturally on $T_{g;p_1,\dots,p_r}$ by diffeomorphisms. This action is discrete. The quotient set of $T_{g;p_1,\dots,p_r}$ by the action of $\operatorname{Mod}_{g;p_1,\dots,p_r}$ can be identified naturally with the moduli space $M_{g;p_1,\dots,p_r}$ of Riemann orbifolds of signature $(g : p_1, \dots, p_r)$.*

### 6.3. **Connected components of the moduli space.**

**Definition 6.3.** We denote by $S^m(t) = S^m(g, p_1, \dots, p_r, \delta)$ the set of all GQHSS of level $m$ and signature $(g : p_1, \dots, p_r)$ such that the associated $m$-Arf function is of type $t = (g, p_1, \dots, p_r, \delta)$.

**Theorem 6.5.** *Let $t = (g, p_1, \dots, p_r, \delta)$ be a tuple that satisfies the conditions of Theorem 6.2, i.e., the space $S^m(t)$ is not empty. Then, the space $S^m(t)$ is homeomorphic to*

$$T_{g;p_1,\dots,p_r} / \operatorname{Mod}^m_{g;p_1,\dots,p_r}(t),$$

*where $T_{g;p_1,\dots,p_r}$ is homeomorphic to $\mathbb{R}^{6g-6+2r}$ and $\operatorname{Mod}^m_{g;p_1,\dots,p_r}(t)$ acts on $T_{g;p_1,\dots,p_r}$ as a subgroup of finite index in the group $\operatorname{Mod}_{g;p_1,\dots,p_r}$.*

*Proof.* Let us consider an element $\psi$ of the space $T_{g;p_1,\dots,p_r}$, i.e., consider a homomorphism $\psi : \Gamma_{g;p_1,\dots,p_r} \to \operatorname{Aut}(\mathbb{H})$. To the homomorphism $\psi$, we attach an orbifold $P_\psi = \mathbb{H}/\psi(\Gamma_{g;p_1,\dots,p_r})$, a standard set of generators

$$v_\psi = \psi(v) = \{a_1^\psi, b_1^\psi, \dots, a_g^\psi, b_g^\psi, c_{g+1}^\psi, \dots, c_{g+r}^\psi\}$$

of $\pi(P_\psi, p)$ and an $m$-Arf function $\sigma_\psi$ on this surface given by

$$\begin{aligned}
&(\sigma_\psi(a_1^\psi), \sigma_\psi(b_1^\psi)) = (\delta, 0) \quad \text{if} \quad g = 1, \\
&(\sigma_\psi(a_1^\psi), \sigma_\psi(b_1^\psi), \sigma_\psi(a_2^\psi), \sigma_\psi(b_2^\psi), \dots, \sigma_\psi(a_g^\psi), \sigma_\psi(b_g^\psi)) \\
&\ = (0, 1-\delta, 1, \dots, 1) \quad \text{if} \quad g > 1.
\end{aligned}$$

By Theorem 5.9, the $m$-Arf function $\sigma_\psi$ on the orbifold $P_\psi$ corresponds to a lift of $\psi(\Gamma_{g;p_1,\dots,p_r})$ into $G_m$. This correspondence defines a map $T_{g;p_1,\dots,p_r} \to S^m(t)$. According to Theorem 6.2, this map is surjective. Let $\operatorname{Mod}^m_{g;p_1,\dots,p_r}(t)$ be the subgroup of $\operatorname{Aut}(P_\psi) = \operatorname{Mod}_{g;p_1,\dots,p_r}$ that preserves the $m$-Arf function $\sigma_\psi$. For any point in $S^m(t)$, its pre-image in $T_{g;p_1,\dots,p_r}$ consists of an orbit of the subgroup $\operatorname{Mod}^m_{g;p_1,\dots,p_r}(t)$. Thus, $S^m(t) = T_{g;p_1,\dots,p_r}/\operatorname{Mod}^m_{g;p_1,\dots,p_r}(t)$. ☐

Summarizing the results of Theorems 6.2 and 6.5, we obtain the following:

**Theorem 6.6.**

1) *Two hyperbolic GQHSS are in the same connected component of the space of all hyperbolic GQHSS if and only if they are of the same type. In other words, the connected components of the space of all hyperbolic GQHSS are those sets $S^m(t)$ that are not empty.*
2) *The set $S^m(t)$ is not empty if and only if $t = (g, p_1, \dots, p_r, \delta)$ has the following properties:*
   (a) *The orders $p_1, \dots, p_r$ are prime with $m$ and satisfy the condition*
   $$(p_1 \cdots p_r) \cdot \left( \sum_{i=1}^{r} \frac{1}{p_i} - (2g-2) - r \right) \equiv 0 \bmod m.$$
   (b) *If $g > 1$ and $m$ is odd, then $\delta = 0$.*
   (c) *If $g = 1$, then $\delta$ is a divisor of $\gcd(m, p_1 - 1, \dots, p_r - 1)$.*
   (d) *If $g = 0$, then $\delta = 0$.*
3) *Any connected component $S^m(t)$ of the space of all hyperbolic GQHSS of level $m$ and signature $(g : p_1, \dots, p_r)$ is homeomorphic to*
   $$\mathbb{R}^{6g-6+2r}/\operatorname{Mod}^m_{g;p_1,\dots,p_r}(t),$$
   *where $\operatorname{Mod}^m_{g;p_1,\dots,p_r}(t)$ is a subgroup of finite index in the group $\operatorname{Mod}_{g;p_1,\dots,p_r}$ and acts discretely on $\mathbb{R}^{6g-6+2r}$.*

## 7. Concluding Remarks

1) **Higher Spin Structures on General Riemann Orbifolds:** Combining the results in this paper on moduli spaces of higher spin structures on compact Riemann orbifolds with the results on moduli spaces of higher spin structures on Riemann surfaces with holes and punctures in [NP09], we obtain a description of moduli spaces of higher spin structures on Riemann orbifolds with holes and punctures.
2) **$\mathbb{Q}$-Gorenstein singularities:** A normal isolated singularity of dimension at least 2 is $\mathbb{Q}$-*Gorenstein* if there is a natural number $r$ such that the divisor $r \cdot \mathcal{K}_X$ is defined on a punctured neighbourhood of the singular point by a function. Here, $\mathcal{K}_X$ is the canonical divisor of $X$. According to [Pra07], hyperbolic $\mathbb{Q}$-Gorenstein quasi-homogeneous surface singularities are in 1-to-1 correspondence with groups of the form $C^* \times \Gamma^*$, where $C^*$ is a lift of a finite cyclic group of order $r$ into $G_m$ and $\Gamma^*$ is a lift of a Fuchsian group $\Gamma$ into $G_m$. The lift of a finite cyclic group is unique; hence, hyperbolic $\mathbb{Q}$-Gorenstein quasi-homogeneous surface singularities are in 1-to-1 correspondence with lifts of a Fuchsian group into $G_m$. Thus, the moduli space of hyperbolic $\mathbb{Q}$-Gorenstein quasi-homogeneous surface singularities coincides with the moduli space of hyperbolic Gorenstein quasi-homogeneous surface singularities as described in Theorem 6.6.
3) **Spherical and Euclidean Automorphy Factors:** For a spherical Gorenstein automorphy factor $(\mathbb{CP}^1, \Gamma, L)$, the group of automorphisms is $\operatorname{Aut}(U) = \operatorname{Aut}(\mathbb{CP}^1) = \operatorname{PSU}(2)$. The finite subgroups of $\operatorname{SU}(2)$ are the cyclic groups, the dihedral groups and the symmetry groups of the regular polyhedra, i.e., the tetrahedral, octahedral and icosahedral groups. The corresponding singularities are $A_k$, $D_k$, $E_6$, $E_7$, $E_8$. For a Euclidean Gorenstein automorphy factor $(\mathbb{C}, \Gamma, L)$,

the group $\Gamma$ is contained in the translation subgroup of $\mathrm{Aut}(\mathbb{C})$ and can be identified with a sublattice $\mathbb{Z}\cdot 1+\mathbb{Z}\cdot\tau$ of the additive group $\mathbb{C}$, where $\tau\in\mathbb{C}$ and $\mathrm{Im}(\tau)>0$, see [Dol83b]. The corresponding singularities are $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$. All GQHSS other than $A_k$, $D_k$, $E_6$, $E_7$, $E_8$, $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$ belong to the class of hyperbolic GQHSS, which is studied in this paper.

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National Research University Higher School of Economics, Vavilova Str. 7, 117312 Moscow, Russia

Belozersky Institute of Phys.-Chem. Biology, Moscow State University, Korp. A, Leninske Gory, 11899 Moscow, Russia

Institute of Theoretical and Experimental Physics (ITEP), Moscow, Russia
*E-mail address*: natanzon@mccme.ru

Department of Mathematical Sciences, University of Liverpool, Liverpool L69 7ZL, UK
*E-mail address*: annap@liv.ac.uk

**Journal *of* Singularities**
**Volume 7 (2013), 88-107**

received 28 September 2012
in revised form 21 March 2013
DOI: 10.5427/jsing.2013.7f

# BRIANÇON-SPEDER EXAMPLES AND THE FAILURE OF WEAK WHITNEY REGULARITY

KARIM BEKKA AND DAVID TROTMAN

## 1. Introduction

In [3, 5] we introduced a weakened form of Whitney's condition $(b)$, motivated by the work of M. Ferrarotti on metric properties of Whitney stratified sets [11, 12]. The resulting weakly Whitney stratified sets retain many properties of Whitney stratified sets, in particular they are locally topologically trivial along strata [26, 16], because they are Bekka $(c)$-regular (see section 5) and so they have the structure of abstract stratified sets [3, 4], and thus are triangulable [14]. Weakly Whitney stratified sets also have many metric properties known to hold for Whitney stratified sets [7]. Orro and Trotman [20], Parusiński [23], Pflaum [24], and Schürmann [25] have described and developed further useful properties of weakly Whitney stratified sets.

There are real algebraic varieties with weakly Whitney regular stratifications which are not Whitney regular, and we give such an example in section 3 below. No examples are known among complex analytic varieties however, so that the natural question arises : do Whitney regularity and weak Whitney regularity coincide in the complex case? As a test, in this paper we study the weak Whitney regularity of the well-known Briançon-Speder examples, consisting of Milnor number constant families of complex surface singularities in $\mathbb{C}^3$ which are not Whitney regular [9], although they are $(c)$-regular because they are weighted homogeneous with constant weights.

We investigate systematically all of these (infinitely many) Briançon-Speder examples, and establish in particular that none of the examples are weakly Whitney regular. We determine **all** the complex curves along which Whitney $(b)$-regularity fails and **all** the complex curves along which weak Whitney regularity fails. It turns out that for each example there are a finite number of curves $\gamma_i$ with the property that weak Whitney regularity fails along every curve tangent to one of the $\gamma_i$ at the origin, while weak Whitney regularity holds along all other curves. For example, the classical Briançon-Speder example

$$f_t(x, y, z) = x^5 + txy^6 + y^7 z + z^{15}$$

for which $\mu(f_t) = 364$, has 16 such curves $\gamma_1, \ldots, \gamma_{16}$, where each $\gamma_i(s)$ is of the form

$$(s^8, as^5, 4a^{-7}s^5, -5a^{-6}s^2) \in \mathbb{C}^4,$$

with $a^{16} = -8$ (hence the 16 distinct complex solutions).

It should be of interest to interpret these curves in the light of other studies of the metric geometry of singular complex surfaces, for example the recent work of Birbrair, Neumann and Pichon characterising their inner bilipschitz geometry [8], and the work of Neumann and Pichon characterising outer bilipschitz triviality [19], or the work of Garcia Barroso and Teissier on the local concentrations of curvature [13].

Further evidence that weak Whitney regularity and Whitney regularity might be equivalent for complex analytic stratifications, at least for complex analytic hypersurfaces, comes from a recent result of the second author with Duco van Straten [29] that equimultiplicity of a family of complex analytic hypersurfaces follows from weak Whitney regularity.

The second author acknowledges the support of the University of Rennes 1 during several visits to Rennes, when much of the work in this paper was done.

## 2. Definitions.

We start by recalling the Whitney conditions.

Let $X, Y$ be two submanifolds of a riemannian manifold $M$ and take $y \in X \cap Y$.

**Condition** $(a)$: The triple $(X, Y, y)$ satisfies Whitney's condition $(a)$ if for each sequence of points $\{x_i\}$ of $X$ converging to $y \in Y$ such that $T_{x_i}X$ converges to $\tau$ (in the corresponding grassmannian in $TM$), then $T_yY \subset \tau$.

**Condition** $(b)$: The triple $(X, Y, y)$ satisfies Whitney's condition $(b)$ if for each local diffeomorphism $h : \mathbb{R}^n \to M$ onto a neighbourhood $U$ of $y$ in $M$ and for each sequence of points $\{(x_i, y_i)\}$ of $h^{-1}(X) \times h^{-1}(Y)$ converging to $(h^{-1}(y), h^{-1}(y))$, such that the sequence $\{T_{x_i}h^{-1}(X)\}$ converges to $\tau$ in the corresponding grassmannian and the sequence $\{\overline{x_iy_i}\}$ converges to $\ell$ in $\mathbb{P}^{n-1}(\mathbb{R})$, then $\ell \subset \tau$.

**Condition** $(b^\pi)$: The triple $(X, Y, y)$ satisfies Whitney's condition $(b^\pi)$ if for each local diffeomorphism $h : \mathbb{R}^n \to M$ onto a neighbourhood $U$ of $y$ in $M$ and for each sequence of points $\{x_i\}$ of $h^{-1}(X)$ converging to $h^{-1}(y)$, such that the sequence $\{T_{x_i}h^{-1}(X)\}$ converges to $\tau$ in the corresponding grassmannian and the sequence $\{\overline{x_i\pi(x_i)}\}$ converges to $\ell$ in $\mathbb{P}^{n-1}(\mathbb{R})$, then $\ell \subset \tau$.

One says that $(X, Y)$ satisfies condition $(a)$ (resp.$(b)$, $(b^\pi)$) if $(X, Y, y)$ satisfies $(a)$ (resp. $(b)$, $(b^\pi)$) at each $y \in X \cap Y$.

*Remark* 2.1. It is an easy exercise to check that condition $(b)$ implies condition $(a)$ [16]. Also $(b)$ is equivalent to both $(a)$ and $(b^\pi)$ holding [18].

We now introduce a regularity condition $(\delta)$, obtained by weakening condition $(b)$.

Given a euclidean vector space $V$, and two vectors $v_1, v_2 \in V^* = V - \{0\}$, define the sine of the angle $\theta(v_1, v_2)$ between them by :

$$\sin\theta(v_1, v_2) = \frac{||v_1 \wedge v_2||}{||v_1||.||v_2||}$$

where $v_1 \wedge v_2$ is the usual vector product and $||.||$ is the norm on $V$ induced by the euclidean structure. Given two vector subspaces $S$ and $T$ of $V$ we define the sine of the angle between $S$ and $T$ by :

$$\sin\theta(S, T) = \sup\{\sin\theta(s, T) : s \in S^*\}$$

where

$$\sin\theta(s, T) = \inf\{\sin\theta(s, t) : t \in T^*\}.$$

If $\pi_T : V \longrightarrow T^\perp$ is the orthogonal projection onto the orthogonal complement of $T$, then

$$\sin\theta(s, T) = \frac{||\pi_T(s)||}{||s||}.$$

The definition for lines is similar to that for vectors - take unit vectors on the lines.

One verifies easily that :

$$\sin\theta(v_1, v_3) \le \sin\theta(v_1, v_2) + \sin\theta(v_2, v_3)$$

for all $v_1, v_2, v_3 \in V^*$, and

$$\sin\theta(S_1 + S_2, T) \le \sin\theta(S_1, T) + \sin\theta(S_2, T),$$

for subspaces $S_1, S_2$, $T$ of $V$ such that $S_1$ is orthogonal to $S_2$.

**Condition** $(\delta)$: We say that the triple $(X, Y, y)$ satisfies condition $(\delta)$ if there exists a local diffeomorphism $h : \mathbb{R}^n \longrightarrow M$ to a neighbourhood $U$ of $y$ in $M$, and there exists a real number $\delta_y$, $0 \leq \delta_y < 1$, such that for every sequence $\{x_i, y_i\}$ of $h^{-1}(X) \times h^{-1}(Y)$ which converges to $(h^{-1}(y), h^{-1}(y))$ such that the sequence $\overline{x_i y_i}$ converges to $l$ in $\mathbb{P}^{n-1}(\mathbb{R})$, and the sequence $T_{x_i} h^{-1}(X)$ converges to $\tau$, then $\sin\theta(l, \tau) \leq \delta_y$.

*Remark* 2.2. Clearly condition $(b)$ implies $(\delta)$ : just take $\delta_y = 0$.

**Definition 2.3.** *A weakly Whitney stratification of a subspace $A$ of a manifold $M$ is a locally finite partition of $A$ into connected submanifolds, called the strata, such that :*

*(1). Frontier Condition : If $X$ and $Y$ are distinct strata such that $\overline{X} \cap Y \neq \emptyset$, that is $X$ and $Y$ are adjacent, then $Y \subset \overline{X}$.*

*(2). Each pair of adjacent strata satisfies condition $(a)$.*

*(3). Each pair of adjacent strata satisfies condition $(\delta)$.*

**Examples.** (1). Every Whitney stratification is weakly Whitney regular.

(2). Let $X$ be the open logarithmic spiral with polar equation,

$$\{(r, \theta) \in \mathbb{R}^2 \,|\, r = e^{\frac{t}{\tan(\beta)}}, \theta = t (\mathrm{mod}\, 2\pi)\} \quad \text{where } 0 < \beta < \frac{\pi}{2}\}$$

and let $Y = \{0\} \subset \mathbb{R}^2$. Condition $(a)$ is trivially satisfied for $(X, Y, \{0\})$, and condition $(\delta)$ is also satisfied, but condition $(b)$ fails because the angle $\theta(\overline{x0}, T_x X) = \beta$ is constant and nonzero for all $x$ in $X$. So this is a weakly Whitney regular stratification which is not Whitney regular.

(3). If $X$ is the open spiral with polar equation

$$\{(r, t) \in \mathbb{R}^2 \,|\, r = e^{-\sqrt{t}}, t \geq 0\}$$

and $Y = \{0\} \subset \mathbb{R}^2$, then the stratified space $X \cup Y$ is not weakly Whitney.

*Remark* 2.4. In the definition of weakly Whitney stratification, we could further weaken condition $(\delta)$ as follows : If $\pi$ is a local $C^1$ retraction associated to a $C^1$ tubular neighbourhood of $Y$ near $y$, a condition $(\delta^\pi)$ is obtained from the definition of $(\delta)$ by replacing the sequence $\{y_i\}$ by the sequence $\{\pi(x_i)\}$. Clearly $(b^\pi$ implies $(\delta^\pi)$. Recall that $(b) \Longleftrightarrow (b^\pi) + (a)$ [18], as noted above in Remark 2.1.

**Lemma 2.5.** $(\delta) + (a) \Longleftrightarrow (\delta^\pi) + (a)$.

*Proof.* Clearly $(\delta) \Longrightarrow (\delta^\pi)$, so it suffices to show that $(\delta^\pi) + (a) \Longrightarrow (\delta)$. In the definition of $(\delta)$ decompose the limiting vector $l$ as the sum of a vector $l_1$ tangent to $Y$ at $y$, and a vector $l_2$ tangent to $\pi^{-1}(y)$ at $y$. Then

$$\sin\theta(l, \tau) = \sin\theta(l_1 + l_2, \tau) \leq \sin\theta(l_1, \tau) + \sin\theta(l_2, \tau).$$

By condition $(a)$, $\sin\theta(l_1, \tau) = 0$, hence $\sin\theta(l, \tau) \leq \sin\theta(l_2, \tau)$, which is less than or equal to $\delta_y$ by hypothesis, implying $(\delta)$. □

Using Lemma 2.5 will make checking weak Whitney regularity easier.

## 3. Real algebraic examples.

Because many of the important applications of Whitney stratifications arise in real algebraic geometry and real singularity theory, it is necessary to know how weak Whitney regularity compares with Whitney regularity for semi-algebraic or real algebraic stratifications, as well as for complex algebraic/analytic stratifications. The following simple example illustrates that weak Whitney regularity is strictly weaker than Whitney regularity for real algebraic stratifications. No such example is currently known in the case of complex algebraic stratifications, and this will be the motivation for the calculations in sections 7, 8 and 9 of this paper.

**Example 3.1.** *Let $V = \{(x,y,t) \in \mathbb{R}^3 \,|\, y^6 = t^6x^2 + x^6\}$, let $Y$ denote the $t$-axis, and let $X = V \setminus Y$. One can check that the triple $(X, Y, (0,0,0))$ satisfies conditions $(a)$ and $(\delta)$, but not condition $(b)$. See [6] for details.*

The following example illustrates the independence of the conditions $(a)$ and $(\delta)$ in the case of real algebraic stratifications.

**Example 3.2.** *Let $V = \{(x,y,t) \in \mathbb{R}^3 \,|\, y^{20} = t^4x^6 + x^{10}\}$, let $Y$ denote the $t$-axis and let $X = V \setminus Y$. Then the triple $(X, Y, (0,0,0))$ satisfies condition $(\delta)$, but not condition $(a)$. For details see [6].*

## 4. Some properties of weakly Whitney stratified spaces.

Like Whitney stratified spaces, weakly Whitney stratified spaces are filtered by dimension.

**Proposition 4.1.** *Suppose that a triple $(X, Y, y)$, $y \in Y \cap \overline{X}$, satisfies conditions $(a)$ and $(\delta)$. Then $\dim Y < \dim X$.*

**Definition 4.2.** *If $(A, \Sigma)$, $(B, \Sigma')$ are weakly Whitney stratified spaces in $M$, then $(A, \Sigma)$ and $(B, \Sigma')$ are said to be in general position if for each pair of strata $X \in \Sigma$ and $X' \in \Sigma'$, $X$ and $X'$ are in general position in $M$, i.e. the natural map :*

$$T_xM \longrightarrow T_xM/T_xX \oplus T_xM/T_xX'$$

*is surjective for all $x \in X \cap X'$.*

**Proposition 4.3.** *Let $V$ be a submanifold of $M$ in general position with respect to $(A, \Sigma)$. Then $(A \cap V, \Sigma \cap V)$ is weakly Whitney regular, if $(A, \Sigma)$ is weakly Whitney regular.*

A proof is given in [6]. A stronger statement, in the case of two stratified sets transverse to each other, is given in [22].

If $A$ is locally closed and $(A, \Sigma)$ is weakly Whitney (without assuming the frontier condition) then the stratified space $(A, \Sigma_c)$, whose strata are the connected components of the strata of $\Sigma$, automatically satisfies the frontier condition. See [3, 4] for the $(c)$-regular case, which includes the case of weakly Whitney stratifications, as remarked below.

**Proposition 4.4.** *Let $f : M \to M'$ be a $C^1$ map, and let $(A, \Sigma)$ be a weakly Whitney stratified space in $M'$. If $f$ is transverse to each stratum $X \in \Sigma$, then the pull-back $(f^{-1}(A), f^{-1}(\Sigma))$ is weakly Whitney stratified.*

See [6] for proofs.

## 5. $(c)$-regularity of weakly Whitney stratifications.

In this section we recall the fact that weakly Whitney stratified spaces are $(c)$-regular. It follows [3, 4] that they can be given the structure of abstract stratified sets in the sense of Thom-Mather [16], implying in particular local topological triviality along strata and triangulability [14].

Let $(U, \phi)$ be a $C^1$ chart at $y$ for a submanifold $Y \subseteq M$ where $\dim Y = d$,

$$\phi : (U, U \cap Y, y) \longrightarrow (\mathbb{R}^n, \mathbb{R}^d \times \{0\}^{n-d}, 0).$$

Then $\phi$ defines a tubular neighbourhood $T_\phi$ of $U \cap Y$ in $U$, induced by the standard tubular neighbourhood of $\mathbb{R}^d \times \{0\}^{n-d}$ in $\mathbb{R}^n$ :

- with retraction $\pi_\phi = \phi^{-1} \circ \pi_d \circ \phi$ where $\pi_d : \mathbb{R}^n \to \mathbb{R}^d$ is the canonical projection,
- and distance function $\rho_\phi = \rho \circ \phi : U \to \mathbb{R}^+$ where $\rho : \mathbb{R}^n \to \mathbb{R}^+$ is the function defined by $\rho(x_1, \cdots, x_n) = \Sigma_{i=d+1}^n x_i^2$.

It is well-known (see [16, 27, 28]) that if a pair $(X, Y)$ of submanifolds of $M$ satisfies Whitney's condition $(b)$ then for *any* sufficiently small tubular neighbourhood $T_Y$ of $Y$ in $M$, the map

$$(\pi_Y, \rho_Y)|_{X\cap T_Y} : X \cap T_Y \longrightarrow Y \times \mathbb{R}$$

is a submersion. In fact this property characterises $(b)$-regularity [27]. For comparison, when the pair $(X, Y)$ is weakly Whitney, there exists *some* tubular neighbourhood $T_Y$ such that the map

$$(\pi_Y, \rho_Y)|_{X\cap T_Y} : X \cap T_Y \longrightarrow Y \times \mathbb{R}$$

is a submersion.

**Proposition 5.1.** *Let $X, Y$ be two submanifolds of $M$, such that $Y \subset \overline{X}$ and let $y \in Y$. If the triple $(X, Y, y)$ satisfies the weak Whitney conditions, then there exists a $C^1$ chart $(U, \phi)$ at $y$ for $Y$ in $M$ and a neighbourhood $U'$ of $y$, $U' \subset U$, such that $(\pi_\phi, \rho_\phi)|_{U'\cap X}$ is a submersion.*

**Corollary 5.2.** *Let $X, Y$ be two submanifolds of $M$ such that $Y \subset \overline{X}$ and the pair $(X, Y)$ satisfies the conditions $(a)$ and $(\delta)$. Then there exists a tubular neighbourhood $T_Y$ of $Y$ in $M$ such that $(\pi_Y, \rho_Y)|_X : X \cap T_Y \longrightarrow Y \times \mathbb{R}$ is a submersion.*

**Proposition 5.3.** *Every weakly Whitney stratified space is $(c)$-regular, and hence is locally topologically trivial along strata.*

For the proofs see [6]. We note that, when weak Whitney regularity holds, the control function in the definition of $(c)$-regularity can be chosen to be a standard distance function arising from a tubular neighbourhood. This means that weak Whitney regularity is a much stronger condition than mere $(c)$-regularity, for which the control function may be weighted homogeneous or even infinitely flat along $Y$.

## 6. Complex stratifications.

In Example 3.1 we saw an example of a weakly Whitney regular real algebraic stratification in $\mathbb{R}^3$ which is not Whitney $(b)$-regular. We are now interested in comparing weak Whitney regularity and Whitney regularity of *complex* analytic or *complex* algebraic stratifications, the main question being whether the extra 'rigidity' of complex analytic varieties prevents the existence of weakly Whitney complex analytic stratifications which are not Whitney regular.

Let $F$ be an analytic function germ from $\mathbb{C}^n \times \mathbb{C}$ to $\mathbb{C}$, defined in a neighbourhood of 0,

$$\begin{array}{rccc} F: & \mathbb{C}^n \times \mathbb{C}, 0 & \longrightarrow & \mathbb{C}, 0 \\ & (x,t) & \longmapsto & F(x,t) \end{array}$$

where $F(0, t) = 0$. We denote by $\pi$ the projection on the second factor, and let $V = F^{-1}(0)$, $Y = \{0\}^n \times \mathbb{C}$ and $V_t = \{x \in \mathbb{C}^n \mid F(x, t) = 0\}$. We assume that each $V_t$ has an isolated singularity at $(0, t)$, the critical set of the restriction of $\pi$ to $V$ is $Y$, and $X = V \setminus Y$ is an analytic complex manifold of dimension $n$.

For each point $(x, t) \in X$ we have

$$T_{(x,t)}X = \left\{ (u, v) \in \mathbb{C}^n \times \mathbb{C} \;\middle|\; \sum_{i=1}^n u_i \frac{\partial F}{\partial x_i}(x,t) + v \frac{\partial F}{\partial t}(x,t) = 0 \right\} = \left(\mathbb{C}\overline{\mathrm{grad}} F\right)^\perp.$$

Let $\mathrm{grad}F = (\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}, \frac{\partial F}{\partial t})$, $\mathrm{grad}_x F = (\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n})$ and

$$\|\mathrm{grad}_x F\|^2 = \sum_{i=1}^n \|\frac{\partial F}{\partial x_i}\|^2.$$

The following characterisations of conditions $(a)$, $(b^\pi)$ and $(\delta^\pi)$ are straightforward.

Whitney's condition $(a)$

The pair $(X, Y)$ satisfies Whitney's condition $(a)$ at 0 if and only if

$$\lim_{\substack{(x,t)\to 0\\ (x,t)\in X}} \left( \frac{\frac{\partial F}{\partial t}(x,t)}{\|\mathrm{grad}_x F(x,t)\|} \right) = 0.$$

Whitney's condition $(b^\pi)$

The couple $(X, Y)$ satisfies Whitney's condition $(b^\pi)$ at 0 if and only if

$$\lim_{\substack{(x,t)\to 0\\ (x,t)\in X}} \left( \frac{\sum_{i=1}^n x_i \frac{\partial F}{\partial x_i}(x,t)}{\|x\|\|\mathrm{grad}_x F(x,t)\|} \right) = 0.$$

Condition $(\delta^\pi)$

The pair $(X, Y)$ satisfies the $(\delta^\pi)$ condition at 0 if and only if there exists a real number $0 \le \delta < 1$ such that

$$\lim_{\substack{(x,t)\to 0\\ (x,t)\in X}} \left( \frac{\sum_{i=1}^n x_i \frac{\partial F}{\partial x_i}(x,t)}{\|x\|\|\mathrm{grad}_x F(x,t)\|} \right) \le \delta.$$

Recall that Whitney's condition $(b)$ implies $(a) + (\delta^\pi)$.

**Question.** Is the converse true in the complex hypersurface case, i.e. does $(a) + (\delta^\pi)$ imply $(b)$ or, equivalently, does $(a) + (\delta^\pi)$ imply $(b^\pi)$ ?

*Remark* 6.1. Because weak Whitney regularity implies local topological triviality along strata, if we wish to decide whether weak Whitney regularity and Whitney regularity are equivalent for complex hypersurfaces, we can restrict to studying families of isolated singularities of complex hypersurfaces with constant Milnor number (Milnor number is a topological invariants). But we know by the fundamental result of Lê Dung Tràng and K. Saito [15] that a family of complex hypersurfaces with isolated singularities has constant Milnor number if and only if

$$\lim_{(x,t)\to 0} \left( \frac{\frac{\partial F}{\partial t}(x,t)}{\|\mathrm{grad}_x F(x,t)\|} \right) = 0,$$

which implies condition $(a)$.

The following lemma due to Briançon and Speder [10] gives an equivalent condition to $(b^\pi)$ when $(a)$ is satisfied.

Let $\gamma : ([0,1], 0) \to (\mathbb{C}^n \times \mathbb{C}, 0)$, be a germ of an analytic arc and $\nu$ the valuation along $\gamma$ in the local ring $\mathcal{O}_{n+1,0}$.

**Notation.** Let $\nu(x) := \inf\{\nu(x_i) | 1 \le i \le n\}$ and $\nu(J_x(F)) := \inf\{\nu(\frac{\partial F}{\partial x_i} | 1 \le i \le n\}$.

**Lemma 6.2.** *The following statements are equivalent:*

*(i) the pair $(X, Y)$ satisfies $(b^\pi)$ at 0,*

*(ii)*

$$\lim_{\substack{(x,t)\to 0\\ (x,t)\in X}} \left( \frac{t\frac{\partial F}{\partial t}(x,t)}{\|x\|\|\mathrm{grad}_x F(x,t)\|} \right) = 0.$$

*In other words, the following statements are equivalent:*

*(i)* $\nu(\sum_{i=1}^n x_i \frac{\partial F}{\partial x_i}) > \nu(x) + \nu(J_x(F))$

*(ii)* $\nu(t) + \nu(\frac{\partial F}{\partial t}) > \nu(x) + \nu(J_x(F))$

*where $\nu$ is the valuation along germs of analytic arcs $\gamma : [0,1] \to X$.*

*Proof.* For $s \in [0,1]$, $\gamma(s) = (x_1(s), \dots, x_n(s), t(s))$.

Since $F \circ \gamma \equiv 0$, we have

$$\sum_{i=1}^{n} x_i'(s) \frac{\partial F \circ \gamma}{\partial x_i}(s) = -t'(s) \frac{\partial F \circ \gamma}{\partial t}(s). \qquad (*)$$

If $a = \nu(x)$ and $b = \nu(J_x(F))$, there exist two non zero vectors of $\mathbb{C}^n$, $A$ and $B$, such that

$$(x_1(s), \dots, x_n(s)) = As^a + \dots$$

and

$$\left(\frac{\partial F \circ \gamma}{\partial x_1}(s), \dots, \frac{\partial F \circ \gamma}{\partial x_n}(s)\right) = Bs^b + \dots.$$

We suppose $(i)$ holds. Then since

$$\sum_{i=1}^{n} x_i(s) \frac{\partial F \circ \gamma}{\partial x_i}(s) = \langle A, B \rangle s^{a+b} + \dots,$$

we must have $\langle A, B \rangle = 0$.

From (*) we have

$$t'(s) \frac{\partial F \circ \gamma}{\partial t}(s) = -\sum_{i=1}^{n} x_i'(s) \frac{\partial F \circ \gamma}{\partial x_i}(s) = -a\langle A, B \rangle s^{a+b-1} + \dots.$$

Then

$$\nu(t) + \nu(\frac{\partial F}{\partial t}) = \nu(t'(s) \frac{\partial F \circ \gamma}{\partial t}) + 1 > (a+b-1) + 1 = a+b.$$

We suppose now that (ii) holds. Then since

$$t'(s) \frac{\partial F \circ \gamma}{\partial t} = -a\langle A, B \rangle s^{a+b-1} + \dots,$$

we must have again $\langle A, B \rangle = 0$, which is exactly condition (i). □

## 7. The Briançon and Speder example with $\mu = 364$.

In this section we study the original example, due to Briançon and Speder [9], of a topologically trivial family of isolated complex hypersurface singularities which is not Whitney regular. The examples of Briançon and Speder given in [9] were the *only* such examples known, until very recently.

Initially we shall carry out explicit calculations for the most well-known example of Briançon and Speder, analysed in their celebrated note of January 1975 :

$$F(x, y, z, t) = F_t(x, y, z) = x^5 + txy^6 + y^7 z + z^{15}$$

for which $\mu(F_t) = 364$ for all $t$ near 0.

**Theorem 7.1.** *The Briançon and Speder example $F(x, y, z, t) = x^5 + txy^6 + y^7 z + z^{15}$ is not weakly Whitney regular.*

*Proof.* Let $F(x, y, z, t) = x^5 + txy^6 + y^7 z + z^{15}$. Then $F$ is a quasihomogenous $\mu$-constant family of type $(3, 2, 1; 15)$. Thus the stratification $(F^{-1}(0) \setminus (0t), (0t))$ is $(a)$-regular by Remark 6.1

We shall construct an explicit analytic path $\gamma(s) = (x(s), y(s), z(s), t(s))$ contained in $F^{-1}(0)$ such that

$$\Delta(x, y, z, t) = \left( \frac{\sum_{i=1}^{n} x_i \frac{\partial F}{\partial x_i}(x, y, z, t)}{\|x\| \|\mathrm{grad}_x F(x, y, z, t)\|} \right)$$

tends to 1 when $(x, y, z, t)$ tends to 0 along $\gamma(s)$. This means that condition $(\delta^\pi)$ is not satisfied at 0, by the characterisation given in section 6. By Lemma 2.4 it then follows using $(a)$-regularity that $(\delta)$ is not satisfied at 0, so that weak Whitney regularity fails.

Following [9] and [27] we take

$$\begin{aligned} x(s) &= s^8 \\ y(s) &= as^5 \\ z(s) &= \frac{4}{a^7}\lambda s^5 \\ t(s) &= -\frac{5}{a^6}s^2 \end{aligned}$$

with $a \neq 0$.

For $\gamma(s)$ to lie on $F^{-1}(0)$ we must have that

$$F(\gamma(s)) = (1 - \frac{5}{a^6}a^6 + 4\lambda + (\frac{4}{a^7})^{15}\lambda^{15}s^{35})s^{40} \equiv 0,$$

so that

$$G(\lambda, s) = -4 + 4\lambda + (\frac{4}{a^7})^{15}\lambda^{15}s^{35} \equiv 0.$$

ince $\frac{\partial G}{\partial \lambda}(\lambda, 0) = 4 \neq 0$, it follows by the implicit function theorem that $\lambda$ is a function of $s$ for $s$ near 0.

Note that $\lambda(0) = 1$.

Then we have along $\gamma(s)$ near $s = 0$,

$$\begin{aligned} \frac{\partial F}{\partial x} &= 5x^4 + ty^6 = 5s^{32} - \frac{5}{a^6}a^6 s^{32} = 0, \\ \frac{\partial F}{\partial y} &= 6txy^5 + 7y^6 z = \left(\frac{-30}{a} + \frac{28}{a}\lambda\right)s^{35}, \\ \frac{\partial F}{\partial z} &= y^7 + 15z^{14} = a^7 s^{35} + 15(\frac{4}{a^7})^{14}\lambda^{14}s^{70} \sim a^7 s^{35}. \end{aligned}$$

Because $\lambda(0) = 1$, the limit of the orthogonal secant vectors

$$\frac{(x, y, z)}{\|(x, y, z)\|}$$

is

$$(0 : a : \frac{4}{a^7}) = (0 : a^8 : 4),$$

and the limit of the normal vectors

$$\frac{\mathrm{grad}_x F(x, y, z, t)}{\|\mathrm{grad}_x F(x, y, z, t)\|}$$

is

$$(0 : \frac{-2}{a} : a^7) = (0 : -2 : a^8).$$

Then $\Delta(\gamma(s))$ tends to 1 if and only if $(0 : a^8 : 4) = (0 : -2 : a^8)$, i.e.

$$\frac{a^8}{4} = \frac{-2}{a^8} \Longleftrightarrow a^{16} = -8.$$

It follows that $(\delta^\pi)$ is not satisfied along $\gamma$ if and only if $a^{16} = -8$. Choosing $a$ to be one of these 16 complex numbers, we have the desired conclusion, i.e. that $(\delta^\pi)$ fails. It follows as above that weak Whitney regularity fails, proving the theorem. □

Note that in the proof above we cannot exclude the possibility that there are other curves on which $(\delta^\pi)$ fails. To clarify the situation, in the next section we make a systematic study of all curves $\gamma(s)$ on $F^{-1}(0)$ and passing through the origin.

A similar calculation as in the theorem above for the simpler $\mu$-constant family

$$F(x,y,z,t) = x^3 + txy^3 + y^4z + z^9,$$

for which $\mu = 56$ (also due to Briançon and Speder [9]), shows that $(\delta^\pi)$ fails for this example too. Our systematic study to determine all curves on which $(\delta^\pi)$ fails for this simpler example will be extended to a more general study, given in section 9 below, of an infinite family of examples, of which $x^3 + txy^3 + y^4z + z^9$ is the first, again defined by Briançon and Speder in their celebrated 1975 note [9].

In what follows we determine the initial terms of *all* curves along which condition $(\delta)$ fails, or equivalently along which $(\delta^\pi)$ fails, by Lemma 2.5.

## 8. Failure of weak Whitney regularity: a complete analysis.

Take again

$$F(x,y,z,t) = x^5 + txy^6 + y^7z + z^{15}.$$

Let $\gamma : ([0,1],0) \to (F^{-1}(0),0) \subset (\mathbb{C}^n \times \mathbb{C}, 0)$ be a germ of an analytic arc and let $\nu$ be the valuation along $\gamma$.

Let $X = (x,y,z)$ and

$$J_X F = (\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}).$$

We will use the notations

$$\nu(X) := \inf\{\nu(x), \nu(y), \nu(z)\},$$

and

$$\nu(J_X(F)) := \inf\{\nu(\frac{\partial F}{\partial x}), \nu(\frac{\partial F}{\partial y}), \nu(\frac{\partial F}{\partial z})\}.$$

We begin by determining the curves along which condition $(b^\pi)$ holds (because $(a)$-regularity holds, by Remark 6.1, we know that $(b^\pi)$ is equivalent to $(b)$, by Remark 2.1).

By Lemma 6.2, the $\mu$-constant property and the Lê-Saito theorem (see Remark 6.1), if $\nu(t) \geq \nu(X)$ then

$$\begin{aligned} \nu(t) + \nu(\frac{\partial F}{\partial t}) &\geq \nu(X) + \nu(\frac{\partial F}{\partial t}) \\ &> \nu(X) + \nu(J_X(F))), \end{aligned}$$

so that $(b^\pi)$ holds by Lemma 6.2, and hence Whitney's condition $(b)$ holds also.

We can therefore suppose from now on that $\nu(t) < \nu(X)$.

If

$$\nu(\frac{\partial F}{\partial x}) = \inf\{4\nu(x), \nu(t) + 6\nu(y)\} \tag{1}$$

we have:

(1). when $4\nu(x) \geq \nu(t) + 6\nu(y)$, so that $\nu(x) > \nu(y)$, then

$$\begin{aligned}\nu(t) + \nu(\frac{\partial F}{\partial t}) &= \nu(t) + \nu(x) + 6\nu(y)\\ &> \nu(y) + \nu(t) + 6\nu(y)\\ &= \nu(y) + \nu(\frac{\partial F}{\partial x}) && \text{(by (1))}\\ &\geq \nu(X) + \nu(J_X(F)),\end{aligned}$$

When however

$$\nu(\frac{\partial F}{\partial x}) > \inf\{4\nu(x), \nu(t) + 6\nu(y)\} \tag{2}$$

then we must have

$$4\nu(x) = \nu(t) + 6\nu(y). \tag{3}$$

Because $F \circ \gamma \equiv 0$, we have that

$$x^5 + txy^6 = -y^7 z - z^{15}. \tag{4}$$

On the other hand

$$x^5 + ty^6 x = -4x^5 + x\frac{\partial F}{\partial x}$$

and (2) imply that

$$\nu(x^5 + ty^6 x) = 5\nu(x).$$

Hence, by (4),

$$5\nu(x) \geq \inf\{7\nu(y) + \nu(z), 15\nu(z)\}$$

and, unless $\nu(y) = 2\nu(z)$, it follows that

$$5\nu(x) = \inf\{7\nu(y) + \nu(z), 15\nu(z)\}. \tag{5}$$

(i) If $\nu(y) > 2\nu(z)$, it follows that $\nu(x) = 3\nu(z)$. Then, by (1) and using that

$$\frac{\partial F}{\partial z} = y^7 + 15z^{14} \tag{6}$$

it follows that

$$\begin{aligned}\nu(t) + \nu(\frac{\partial F}{\partial t}) &= \nu(t) + \nu(x) + 6\nu(y)\\ &> \nu(t) + 15\nu(z)\\ &> 15\nu(z)\\ &= \nu(z) + \nu(\frac{\partial F}{\partial z})\\ &\geq \nu(X) + \nu(J_X(F)),\end{aligned}$$

and hence $(b^\pi)$ holds.

(ii) If $\nu(y) = 2\nu(z)$,

(a) and $\nu(x) = 3\nu(z)$, then from (3) we obtain

$$12\nu(z) = \nu(t) + 12\nu(z),$$

i.e. $\nu(t) = 0$, which is impossible;

(b) and $\nu(x) > 3\nu(z)$, then

$$\nu(y^7 + z^{14}) > 14\nu(z)$$

and from (6) it follows that

$$\nu(\frac{\partial F}{\partial z}) = 14\nu(z)$$

so that

$$\begin{aligned}\nu(t) + \nu(\frac{\partial F}{\partial t}) &= \nu(t) + \nu(x) + 6\nu(y)\\ &> \nu(t) + 15\nu(z)\\ &> 15\nu(z)\\ &= \nu(z) + \nu(\frac{\partial F}{\partial z})\\ &\geq \nu(X) + \nu(J_X(F)),\end{aligned}$$

so that $(b^\pi)$ holds, using Lemma 6.2 again.

(iii) If $\nu(y) < 2\nu(z)$, we have from Equation (5) that

$$5\nu(x) = 7\nu(y) + \nu(z). \tag{7}$$

Subtracting (2) from (7) gives

$$\nu(x) + \nu(t) = \nu(y) + \nu(z). \tag{8}$$

We can suppose now that

$$\nu(x) + \nu(t) = \nu(y) + \nu(z)$$

and

$$\nu(y) < 2\nu(z).$$

We carry on with the last cases:

(I) If $\nu(z) > \nu(y)$ we have

$$\nu(z^{14}) > \nu(y^7),$$

so that

$$\nu(\frac{\partial F}{\partial z}) = 7\nu(y). \tag{9}$$

Then (1), (8) and (9) give

$$\begin{aligned}\nu(t) + \nu(\frac{\partial F}{\partial t}) &= \nu(t) + \nu(x) + 6\nu(y)\\ &= \nu(z) + 7\nu(y)\\ &> \nu(y) + 7\nu(y)\\ &= \nu(y) + \nu(\frac{\partial F}{\partial z})\\ &\geq \nu(X) + \nu(J_X(F)),\end{aligned}$$

and we have that $(b^\pi)$ holds, by Lemma 6.2.

(II) If $2\nu(z) > \nu(y) > \nu(z)$, then

$$\nu(\frac{\partial F}{\partial z}) = 7\nu(y) > 6\nu(y) + \nu(z),$$

so that

$$\frac{\partial F}{\partial y} = 6txy^6 + 7y^7z = 6txy^6 - 7(x^5 + txy^6 + z^{15})$$

or

$$\frac{\partial F}{\partial y} = -7x^5 - txy^6 - 7z^{15}. \tag{10}$$

Now because $2\nu(z) > \nu(y)$ we have that

$$\begin{aligned} 15\nu(z) &> \nu(z) + 7\nu(y), \\ &= \nu(x) + \nu(t) + 6\nu(y) && \text{(by (8))} \\ &= \nu(txy^6). && (11) \end{aligned}$$

Also by (2)

$$\nu(\frac{\partial F}{\partial x}) > 4\nu(x).$$

This means that

$$\nu(5x^4 + ty^6) > 4\nu(x)$$

which implies in turn that

$$\nu(-7x^5 - txy^6) = \nu(txy^6). \tag{12}$$

It follows from (10), (11) and (12) that

$$\nu(y\frac{\partial F}{\partial y}) = \nu(txy^6),$$

i.e.

$$\nu(\frac{\partial F}{\partial y}) = \nu(txy^5). \tag{13}$$

Then

$$\begin{aligned} \nu(t) + \nu(\frac{\partial F}{\partial t}) &= \nu(t) + \nu(x) + 6\nu(y) \\ &= \nu(txy^5) + \nu(y) \\ &> \nu(txy^5) + \nu(z) \\ &= \nu(\frac{\partial F}{\partial y}) + \nu(z) && \text{(by (13))} \\ &\geq \nu(X) + \nu(J_X(F)), \end{aligned}$$

and again $(b^\pi)$ holds, by Lemma 6.2

**Résumé: a germ of arc $(x(s), y(s), z(s), t(s))$ along which Whitney condition $(b)$ is not satisfied must fulfil the following conditions:**

- $\nu(x) > \nu(y) = \nu(z) > \nu(t)$
- $\nu(x) + \nu(t) = \nu(z) + \nu(y)$
- $4\nu(x) = \nu(t) + 6\nu(y)$.

Resolving these equations we find that

$$5\nu(x) = 8\nu(y) \text{ and } 5\nu(t) = 2\nu(y),$$

so that the set of germs of analytic arcs along which Whitney condition $(b)$ is not satisfied is contained in the set

$$\begin{aligned}\mathcal{A} := \{\ \gamma(s) = (x(s), y(s), z(s), t(s)) : [0,1] \to \mathbb{C}^n \times \mathbb{C} \mid \\ x(s) = a_1 s^{\alpha_1} + \cdots \\ y(s) = a_2 s^{\alpha_2} + \cdots \\ z(s) = a_3 s^{\alpha_3} + \cdots \\ t(s) = a_4 s^{\alpha_4} + \cdots\end{aligned}$$

with $5\alpha_1 = 8\alpha, \alpha_2 = \alpha_3 = \alpha, 5\alpha_4 = 2\alpha, \alpha \equiv 0[5], \text{and } a_i \in \mathbb{C}^*$satisfying some conditions $\}$

It remains to characterize the subset of arcs along which the $(\delta)$ condition is not satisfied, or equivalently along which the $(\delta^\pi)$ condition is not satisfied, using $(a)$-regularity and Lemma 2.5. Let $\gamma \in \mathcal{A}$. We may suppose $a_1 = 1$, and write $a_2 = a, a_3 = b$ and $a_4 = c$.Then

$$F \circ \gamma(s) = (s^{8\alpha} + ...) + (c.a^6 s^{8\alpha} + \ldots) + (a^7 b s^{8\alpha} + \ldots) + (b^{15} s^{15\alpha} + \ldots) \equiv 0$$

so that

$$s^{8\alpha}(1 + c.a^6 + a^7.b + s(\ldots + b^{15} s^{7\alpha - 1} + \ldots)) \equiv 0,$$

and we must have

$$1 + c.a^6 + a^7.b = 0.$$

Thus along $\gamma(s)$ near $s = 0$ we have,

$$\begin{aligned}\frac{\partial F}{\partial x} &= 5x^4 + ty^6 = s^{\frac{32}{5}\alpha}(5 + ca^6) + \ldots \\ \frac{\partial F}{\partial y} &= 6txy^5 + 7y^6 z = a^5(6c + 7ab)s^{7\alpha} + \ldots \\ \frac{\partial F}{\partial z} &= y^7 + 15z^{14} = a^7 s^{7\alpha} + \ldots + 14b^{14} s^{14\alpha} + \ldots.\end{aligned}$$

But now, using (2), $\nu(\frac{\partial F}{\partial x}) > \frac{32}{5}\alpha$ imposes the condition $5 + ca^6 = 0$. It follows that

$$c = -\frac{5}{a^6}, b = \frac{4}{a^7}$$

and

$$a^5(6c + 7ab) = -\frac{2}{a}.$$

The limit of orthogonal secant vectors

$$\frac{(x, y, z)}{\|(x, y, z)\|}$$

is

$$(0 : a : b) = (0 : a^8 : 4),$$

and the limit of normal vectors

$$\frac{\mathrm{grad}_x F(x, y, z, t)}{\|\mathrm{grad}_x F(x, y, z, t)\|}$$

is

$$(0 : a^5(6c + 7ab) : a^7) = (0 : -\frac{2}{a} : a^7) = (0 : -2 : a^8).$$

As at the end of section 7 we deduce that $(\delta^\pi)$ is not satisfied along $\gamma$ if and only if

$$(0 : a^8 : 4) = (0 : -2 : a^8),$$

or equivalently when $a^{16} = -8$. Choosing $a$ to be one of these 16 complex numbers, we have the desired conclusion, namely that $(\delta^\pi)$ fails precisely on those curves

$$\gamma(s) = (x(s), y(s), z(s), t(s))$$

whose initial terms are

$$(s^8, as^5, 4a^{-7}s^5, -5a^{-6}s^2).$$

By Lemma 2.5 and $(a)$-regularity, these are precisely the curves on which $(\delta)$ fails, that is to say we have identified all of the curves on which weak Whitney regularity fails to hold.

## 9. Other Briançon and Speder examples

We perform similar calculations for the infinite family of examples, also due to Briançon and Speder [9] :

$$F(x, y, z, t) = x^3 + txy^\alpha + y^\beta z + z^{3\alpha},$$

where $\alpha \geq 3$ and $3\alpha = 2\beta + 1$.
The functions $f_t(x, y, z) = F_t(x, y, z)$ are quasihomogenous of type $(\alpha, 2, 1; 3\alpha)$ with isolated singularity at the origin, for each $t$, and so each

$$\mu_t = (3\alpha - 1)(3\alpha - 2) = 2\beta(2\beta - 1),$$

by the Milnor-Orlik formula [17]. Thus $f_t$ is a $\mu$-constant family.

We are again hunting for analytic arc germs where condition $(\delta)$ fails.

Clearly

$$\frac{\partial F}{\partial x} = 3x^2 + ty^\alpha$$
$$\frac{\partial F}{\partial y} = \alpha txy^{\alpha-1} + \beta y^{\beta-1}z$$
$$\frac{\partial F}{\partial z} = y^\beta + 3\alpha z^{3\alpha-1} = y^\beta + 3\alpha z^{2\beta}.$$

Let $\gamma : ([0, 1], 0) \to (F^{-1}(0), 0) \subset (\mathbb{C}^n \times \mathbb{C}, 0)$, be a germ of an analytic arc and $\nu$ the valuation along $\gamma$.

As above we let $X = (x, y, z)$ and $J_X F = (\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z})$, then write

$$\nu(X) := \inf\{\nu(x), \nu(y), \nu(z)\}$$

and

$$\nu(J_X(F)) := \inf\{\nu(\frac{\partial F}{\partial x}), \nu(\frac{\partial F}{\partial y}), \nu(\frac{\partial F}{\partial z})\}.$$

We begin by determining along which curves condition $(b^\pi)$ holds. Note that again Remark 2.1 implies that for the examples studied here $(b^\pi)$ is equivalent to Whitney's condition $(b)$, because Whitney's condition $(a)$ holds by the Lê -Saito theorem (Remark 6.1).

Suppose that $\nu(t) \geq \nu(X)$. Again by the $\mu$-constant condition and Remark 6.1,

$$\nu(t) + \nu(\frac{\partial F}{\partial t}) > \nu(t) + \nu(\frac{\partial F}{\partial x})$$

so that, since $\nu(t) \geq \nu(X)$,

$$\nu(t) + \nu(\frac{\partial F}{\partial t}) > \nu(X) + \nu(\frac{\partial F}{\partial x}),$$

so that

$$\nu(t) + \nu(\frac{\partial F}{\partial t}) > \nu(X) + \nu(J_X(F))$$

and by Lemma 6.2 $(b^\pi)$ holds, and Whitney's condition $(b)$ holds using $(a)$ and Remark 2.1.

**We shall assume from now on that $\nu(t) < \nu(X)$.**

$$\nu(\frac{\partial F}{\partial x}) = \inf\{2\nu(x), \nu(t) + \alpha\nu(y)\} \tag{14}$$

we have:

(1) either $2\nu(x) \geq \nu(t) + \alpha\nu(y)$, and then we must have $\nu(x) > \nu(y)$ and

$$\begin{aligned} \nu(t) + \nu(\frac{\partial F}{\partial t}) &= \nu(t) + \nu(x) + \alpha\nu(y) \\ &> \nu(y) + \nu(t) + \alpha\nu(y) \\ &\geq \nu(X) + \nu(\frac{\partial F}{\partial x}) && \text{(by (14))} \\ &\geq \nu(X) + \nu(J_X(F)), \end{aligned}$$

so that as in section 8 we obtain that $(b^\pi)$ holds, using Lemma 6.2;

(2) or we have $2\nu(x) < \nu(t) + \alpha\nu(y)$, and then

$$\begin{aligned} \nu(t) + \nu(\frac{\partial F}{\partial t}) &= \nu(t) + \nu(x) + \alpha\nu(y) \\ &> \nu(x) + 2\nu(x) \\ &\geq \nu(x) + \nu(\frac{\partial F}{\partial x}) && \text{(by (14))} \\ &\geq \nu(X) + \nu(J_X(F)), \end{aligned}$$

and again $(b^\pi)$ holds by Lemma 6.2.

**It follows that from now on we are reduced to studying the case when**

$$\nu(\frac{\partial F}{\partial x}) > \inf\{2\nu(x), \nu(t) + \alpha\nu(y)\}, \tag{15}$$

and hence that

$$2\nu(x) = \nu(t) + \alpha\nu(y). \tag{16}$$

Now we are assuming that $F \circ \gamma \equiv 0$, i.e.

$$x^3 + txy^\alpha = -y^\beta z - z^{3\alpha}. \tag{17}$$

Write

$$x^3 + txy^\alpha = -2x^3 + x\frac{\partial F}{\partial x}.$$

Then (15) implies that

$$\nu(x^3 + txy^\alpha) = 3\nu(x).$$

Using (17) we see that

$$3\nu(x) \geq \inf\{\beta\nu(y) + \nu(z), 3\alpha\nu(z)\}$$

and that

$$3\nu(x) = \inf\{\beta\nu(y) + \nu(z), 3\alpha\nu(z)\} \quad \text{if} \quad \nu(y) \neq 2\nu(z), \tag{18}$$

using that $3\alpha - 1 = 2\beta$.

(i) If $\nu(y) > 2\nu(z)$ then, by (18), $\nu(x) = \alpha\nu(z)$. Also

$$\nu(\frac{\partial F}{\partial z}) = (3\alpha - 1)\nu(z). \tag{19}$$

Then

$$\begin{aligned}
\nu(t) + \nu(\frac{\partial F}{\partial t}) &= \nu(t) + \nu(x) + \alpha\nu(y) \\
&> \nu(t) + 3\alpha\nu(z) \\
&> \nu(z) + \nu(\frac{\partial F}{\partial z}) && \text{(by (19))} \\
&\geq \nu(X) + \nu(J_X(F))
\end{aligned}$$

so that $(b^\pi)$ holds by Lemma 6.2.

(ii) If $\nu(y) = 2\nu(z)$, then from (16) it follows immediately that $\nu(x) > \alpha\nu(z)$.
Now

$$\begin{aligned}
\nu(\frac{\partial F}{\partial y}) &= \inf\{\nu(t) + \nu(x) + (\alpha - 1)\nu(y), (\beta - 1)\nu(y) + \nu(z)\} \\
&= \inf\{3\nu(x) - \nu(y), (2\beta - 1)\nu(z)\} && \text{(by (16))} \\
&= \inf\{3\nu(x) - 2\nu(z), (3\alpha - 2)\nu(z)\} && \text{(since } \nu(y) = 2\nu(z)) \\
&= (3\alpha - 2)\nu(z) && \text{(since } \nu(x) > \alpha\nu(z).)
\end{aligned}$$

Then

$$\begin{aligned}
\nu(t) + \nu(\frac{\partial F}{\partial t}) &= \nu(t) + \nu(x) + \alpha\nu(y) \\
&= 3\nu(x) && \text{(by (16))} \\
&> 3\alpha\nu(z) \\
&= \nu(z) + (3\alpha - 1)\nu(z) \\
&> \nu(z) + \nu(\frac{\partial F}{\partial y}) \\
&\geq \nu(X) + \nu(J_X(F)),
\end{aligned}$$

and $(b^\pi)$ holds by Lemma 6.2.

(iii) If $\nu(y) < 2\nu(z)$, we have

$$3\nu(x) = \beta\nu(y) + \nu(z),$$

and (16) gives

$$\nu(x) + \nu(t) = (\beta - \alpha)\nu(y) + \nu(z).$$

**Thus we can suppose from now on that**

$$\nu(x) + \nu(t) = (\beta - \alpha)\nu(y) + \nu(z) \tag{20}$$

**and**

$$\nu(y) < 2\nu(z).$$

We carry on with the last cases:

(I) If $\nu(z) > \nu(y)$ we have
$$\nu(z^{3\alpha-1}) = \nu(z^{2\beta}) > \nu(y^{\beta})$$
so that
$$\nu(\frac{\partial F}{\partial z}) = \beta\nu(y).$$
Then
$$\begin{aligned}\nu(t) + \nu(\frac{\partial F}{\partial t}) &= \nu(t) + \nu(x) + \alpha\nu(y)\\ &= \nu(z) + \beta\nu(y)\\ &> \nu(y) + \nu(\frac{\partial F}{\partial z})\\ &\geq \nu(X) + \nu(J_X(F)),\end{aligned}$$
and $(b^{\pi})$ holds by Lemma 6.2

(II) If $2\nu(z) > \nu(y) > \nu(z)$, then
$$\nu(z^{3\alpha-1}) = \nu(z^{2\beta}) > \nu(y^{\beta})$$
so that
$$\nu(\frac{\partial F}{\partial z}) = \beta\nu(y) > (\beta-1)\nu(y) + \nu(z).$$
Now
$$\begin{aligned}y\frac{\partial F}{\partial y} &= \alpha txy^{\alpha} + \beta y^{\beta}z\\ &= \alpha txy^{\alpha} - \beta(x^3 + txy^{\alpha} + z^{3\alpha}) && \text{(on } F^{-1}(0))\\ &= -\beta x^3 - (\beta-\alpha)txy^{\alpha} - \beta z^{3\alpha},\\ &= x(ty^{\alpha} - \frac{\beta}{3}(\frac{\partial F}{\partial x})) - \beta z^{3\alpha} && (21)\end{aligned}$$
since $3\alpha = 2\beta + 1$.
Also
$$\begin{aligned}\nu(z^{3\alpha}) &= 3\alpha\nu(z)\\ &= \nu(z) + 2\beta\nu(z)\\ &> \nu(z) + \beta(y)\\ &= \nu(txy^{\alpha}) && \text{(by (20))}\end{aligned}$$
so we have that
$$\nu(z^{3\alpha}) > \nu(txy^{\alpha}). \tag{22}$$
From (15) and (16),
$$\nu(\frac{\partial F}{\partial x}) > \nu(t) + \alpha\nu(y). \tag{23}$$
Using (21), (22) and (23) we find that
$$\nu(y\frac{\partial F}{\partial y}) = \nu(txy^{\alpha}),$$
and thus
$$\nu(\frac{\partial F}{\partial y}) = \nu(txy^{\alpha-1}). \tag{24}$$

Then

$$\begin{aligned}\nu(t)+\nu(\frac{\partial F}{\partial t}) &= \nu(t)+\nu(x)+\alpha\nu(y)\\ &= \nu(txy^{\alpha-1})+\nu(y)\\ &> \nu(txy^{\alpha-1})+\nu(z)\\ &= \nu(\frac{\partial F}{\partial y})+\nu(z) && \text{(by (24))}\\ &\geq \nu(X)+\nu(J_X(F)),\end{aligned}$$

and again $(b^{\pi})$ holds by Lemma 6.2.

**Résumé: a germ of arc along which Whitney condition $(b)$ is not satisfied must fulfil the following conditions:**

- $\nu(x) > \nu(y) = \nu(z) > \nu(t)$
- $\nu(x)+\nu(t) = (\beta-\alpha)\nu(y)+\nu(z)$
- $2\nu(x) = \nu(t)+\alpha\nu(y)$

Finally the set of germs of analytic arcs along which Whitney condition $(b)$ is not satisfied is contained in the set

$\mathcal{A} := \{\ \gamma(s) = (x(s), y(s), z(s), t(s)) : [0,1] \to \mathbb{C}^n \times \mathbb{C} \mid$

$$\begin{aligned} x(s) &= a_1 s^{\alpha_1} + \cdots\\ y(s) &= a_2 s^{\alpha_2} + \cdots\\ z(s) &= a_3 s^{\alpha_3} + \cdots\\ t(s) &= a_4 s^{\alpha_4} + \cdots \end{aligned}$$

$3\alpha_1 = (\beta+1)m, \alpha_2 = \alpha_3 = m, 3\alpha_4 = m, \alpha \equiv 0[3], a_i \in \mathbb{C}^*$ satisfying some conditions $\}$ .

It remains to characterize the subset of arcs along which the $(\delta)$ condition is not satisfied.

Let $\gamma \in \mathcal{A}$. We may suppose $a_1 = 1$, and write $a_2 = a, a_3 = b$ and $a_4 = c$.

Now

$$\begin{aligned} F\circ\gamma(s) &= (s^{(\beta+1)m}+...)+(c.a^{\alpha}s^{(\beta+1)m}+\ldots)+(a^{\beta}bs^{(\beta+1)m}+\ldots)+(b^{3\alpha}s^{3\alpha m}+\ldots)\\ &\equiv 0,\end{aligned}$$

so then

$$s^{(\beta+1)m}(1+a^{\alpha}.c+a^{\beta}.b+s(\ldots+b^{3\alpha}s^{\beta m}+\ldots))\equiv 0,$$

and we must have

$$1+c.a^{\alpha}+b.a^{\beta}=0.$$

Hence along $\gamma(s)$ near $s=0$ we have,

$$\begin{aligned} \frac{\partial F}{\partial x} &= 3x^2+ty^{\alpha} = s^{\frac{2(\beta+1)m}{3}\alpha}(3+ca^{\alpha})+\ldots\\ \frac{\partial F}{\partial y} &= \alpha txy^{\alpha-1}+\beta y^{\beta-1}z = (\alpha ca^{\alpha-1}+\beta b.a^{\beta-1})s^{\beta m}+\ldots\\ \frac{\partial F}{\partial z} &= y^{\beta}+3\alpha z^{3\alpha-1} = y^{\beta}+3\alpha z^{2\beta} = a^{\beta}s^{\beta m}+\ldots+(2\beta)b^{2\beta)}s^{(2\beta)m}+\ldots.\end{aligned}$$

However, the condition

$$\nu(\frac{\partial F}{\partial x}) > \beta m$$

implies that

$$3+ca^{\alpha}=0$$

and it follows that

$$c = -\frac{3}{a^{\alpha}},\ b = \frac{2}{a^{\beta}}$$

and so

$$\alpha c.a^{\alpha-1} + \beta b.a^{\beta-1} = -\frac{1}{a}.$$

The limit of orthogonal secant vectors

$$\frac{(x, y, z)}{\|(x, y, z)\|}$$

is thus

$$(0 : a : b) = (0 : a : \frac{2}{a^{\beta}}),$$

and the limit of normal vectors

$$\frac{\mathrm{grad}_x F(x, y, z, t)}{\|\mathrm{grad}_x F(x, y, z, t)\|}$$

is

$$(0 : \alpha c.a^{\alpha-1} + \beta b.a^{\beta-1} : a^{\beta}) = (0 : -\frac{1}{a} : a^{\beta}).$$

It follows that $(\delta)$ is not satisfied along $\gamma$ if and only if

$$a^{2\beta+2} = -2.$$

Choosing $\alpha$ to be one of these $2\beta+2 = 3\alpha+1$ complex numbers, we have the desired conclusion, i.e. that $(\delta)$ fails.

## 10. Other examples.

A Milnor number constant family,

$$F_t(x, y, z) = z^{12} + zy^3x + ty^2x^3 + x^6 + y^5,$$

with $\mu = 166$, which is also not Whitney regular over the $t$-axis, was studied by E. Artal Bartolo, J. Fernandez de Bobadilla, I. Luengo and A. Melle-Hernandez in a recent paper [2]. Also a series of Milnor number constant but non Whitney regular families, depending on a parameter $\ell$, was given by Abderrahmane [1] as follows:

$$F_t^{\ell}(x, y, z) = x^{13} + y^{20} + zx^6y^5 + tx^6y^8 + t^2x^{10}y^3 + z^{\ell},$$

for integers $\ell \geq 7$. Here $\mu = 153\ell + 32$, while $\mu^2(F_0) = 260$ and $\mu^2(F_t) = 189$, according to Abderrahmane. We do not yet know whether weak Whitney regularity holds or fails for these examples.

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**Journal *of* Singularities**
**Volume 7 (2013), 108-133**

received 28 May 2012
DOI: 10.5427/jsing.2013.7g

# ON THE CLASSIFICATION OF RATIONAL SURFACE SINGULARITIES

JAN STEVENS

Abstract. A general strategy is given for the classification of graphs of rational surface singularities. For each maximal rational double point configuration we investigate the possible multiplicities in the fundamental cycle. We classify completely certain types of graphs. This allows to extend the classification of rational singularities to multiplicity 8. We also discuss the complexity of rational resolution graphs.

## Introduction

The topological classification of complex surface singularities amounts to classifying resolution graphs. Such a graph represents a complex curve on a surface, and the simplest case is when this curve is rational; then the singularity is called rational and the graph in fact determines the analytical type of the singularity up to equisingular deformations.

Classification of singularities tends to lead to long lists, but making them is not a purpose on its own. Sometimes one wants a list to prove statements by case-by-case checking. If the lists become too unwieldy, as in the case on hand, their main use will be to provide an ample supply of examples to test conjectures on. With this objective the most useful description of rational resolution graphs is as a list of parts, together with assembly instructions, guaranteeing that the result is a rational graph. For a special class of rational singularities, those with almost reduced fundamental cycle, such a classification exists [13, 4].

As prototype of our classification and to fix notations we first treat the special case. The fundamental cycle ([1], see also Definition 1.5) can be seen as divisor on the exceptional set of the resolution, with positive coefficients (and it is this divisor which should be rational as non-reduced curve). It is characterised numerically as the minimal positive cycle intersecting each exceptional curve non-positively, and can therefore be computed using the intersection form encoded in the graph. The fundamental cycle is called almost reduced if it is reduced at the non-$(-2)$'s. So higher multiplicities can only occur on the maximal rational double point (RDP) configurations. The classification splits in two parts: one has to determine the multiplicities on the RDP-configurations and how they can be attached to the rest of the graph. The explicit list of graphs can be found in the paper by Gustavsen [4]. Blowing down the RDP-configurations to rational double point singularities gives the canonical model or RDP-resolution. Its exceptional set can again by described by a graph. Our classification strategy in general is to first find the graphs for the RDP-resolution, and then determine which rational double point (RDP) configurations can occur.

The first new results in this paper are on graphs, where each RDP-configuration is attached to at most one non-reduced non-$(-2)$. The possible graphs for the RDP-resolution are easy to describe, but here a new phenomenon occurs, that not every candidate graph can be realised by a rational singularity. In particular, if the graph contains only one non-$(-2)$, this vertex has multiplicity at most 6 in the fundamental cycle. These considerations apply to all multiplicities, but only for a restricted class of singularities; they cover all singularities of low multiplicity.

Our result extends the classification of rational singularities of multiplicity 4 [14], and allows to recover the classification by Tosun et al. for multiplicity 5 [16].

Multiplicity 6 necessitates the study of RDP-configurations, connecting two non-reduced non-$(-2)$'s. We first determine the conditions under which the multiplicities in the fundamental cycle become as high as possible. We do this for each RDP-configuration separately. The existence depends on the rest of the graph. Then we use the same computations to treat the case that the non-$(-2)$'s have multiplicity exactly two. This allows us to complete the classification of rational singularities of multiplicity 6. The same methods work for multiplicity 7 and 8, but we do not treat these cases explicitly, except for one new case, of three non-reduced non-$(-2)$'s, with which we conclude our classification.

We do not claim that it is feasible to treat all multiplicities with our methods. Our last result, on multiplicity 8, gives a glimpse of what is needed in general. To use induction over the number of non-$(-2)$'s, one needs detailed knowledge on the graphs for lower multiplicity, and it does not suffice to compute with RDP-configurations separately. We include (at the end of the first section) a non-trivial example of a rational graph, of multiplicity 37; the graph of the canonical model is rather simple. This example comes from a paper by Karras [6], which maybe contains the deepest study of the structure of resolution graphs in the literature. He proves that every rational singularity deforms into a cone over a rational normal curve of the same multiplicity. My main motivation for taking up the classification again lies in the same direction. The ultimate goal is to study the Artin component of the semi-universal deformation. Over this component a simultaneous resolution exists (or, without base change, a simultaneous canonical model). This is one motivation of our classification strategy of first finding the graph for the RDP-resolutions. The analytical type of the total space over the Artin component (up to smooth factors) is an interesting invariant of the singularity. In his thesis [13] Ancus Röhr turned the problem of formats around and defined the format as just this invariant. He showed that the format determines the exceptional set of the canonical model of the singularity. Examples in this paper cast doubt on our earlier conjecture that the converse holds.

RDP-configurations can be of type $A$, $D$ and $E$. Our computations show that one cannot reach high multiplicities in the fundamental cycle using configurations of type $D$ and $E$. With this goal it suffices to look at configurations of type $A$. Indeed, the picture which arises from our classifications, is that for most purposes it suffices to look at rather simple configurations of type $A$.

One answer to the question how complex a graph can be is that of Lê and Tosun [10], who take the number of rupture points (vertices with valency at least 3) as measure. We give a simplified proof of their estimate, that this number is bounded by $m-2$, where $m$ is the multiplicity of the singularity. Our argument shows that the highest complexity is attained by graphs with reduced fundamental cycle.

The structure of this paper is as follows. In the first section we review some properties of resolution graphs. The next section gives the classification of singularities with almost reduced fundamental cycle. Section 3 is about complexity in the sense of [10]. Then we discuss the format of a rational singularity, following [13]. Our computations use a special way to compute the fundamental cycle, which we explain in Section 5. The case, where each RDP-configuration is attached to at most one non-reduced non-$(-2)$, is treated in Section 6, while the following section describes RDP-configurations on general graphs. In the final section we complete the classification for multiplicity 6 and treat the case of three non-reduced non-$(-2)$'s in multiplicity 8.

## 1. Rational graphs

In this section we review some properties of resolution graphs. References are Artin [1], Wagreich [17] and Wall [18], and for rational singularities in addition Laufer [8].

The topological type of a normal complex surface singularity is determined by and determines the resolution graph of the minimal good resolution [12]. A resolution graph can be defined for any resolution.

**Definition 1.1.** Let $\pi\colon (M,E)\to(X,p)$ be a resolution of a surface singularity with exceptional divisor $E=\bigcup_{i=1}^r E_i$. The *resolution graph* $\Gamma$ is a weighted graph with vertices corresponding to the irreducible components $E_i$. Each vertex has two weights, the self-intersection $-b_i=E_i^2$, and the arithmetic genus $p_a(E_i)$, the second traditionally written in square brackets and omitted if zero. There is an edge between vertices if the corresponding components $E_i$ and $E_j$ intersect, weighted with the intersection number $E_i\cdot E_j$ (only written out if larger than one).

Other definitions, which record more information, are possible: one variant is to have an edge for each intersection point $P\in E_i\cap E_j$, with weight the local intersection number $(E_i\cdot E_j)_P$. These subtleties need not concern us here, as the exceptional divisor of a rational singularity is a simple normal crossings divisor.

We call the vertices of the graph $\Gamma$ also for $E_i$. This should cause no confusion. From the context it will be clear whether we consider $E_i$ as vertex or as curve. In fact, we use $E_i$ also in a third sense. The classes of the curves $E_i$ form a preferred basis of $H:=H_2(M,\mathbb{Z})$. Following algebro-geometric tradition the elements of $H$ are called *cycles*. They are written as linear combinations of the $E_i$.

The resolution graph (as defined above) is also the graph of the quadratic lattice $H:=H_2(M,\mathbb{Z})$, in the sense of [11]. The intersection form on $M$ gives a negative definite quadratic form on $H$. Let $K\in H^2(M,\mathbb{Z})=H^{\#}$ be the canonical class. It can be written as rational cycle in $H_{\mathbb{Q}}=H\otimes\mathbb{Q}$ by solving the adjunction equations $E_i\cdot(E_i+K)=2p_a(E_i)-2$. The function $-\chi(A)=\frac12 A\cdot(A+K)$, $A\in H$, makes $H$ into a *quadratic lattice* [11, 1.4]. We prefer to work with the genus $p_a(A)=1-\chi(A)$. Note that the genus function determines the intersection form, as

$$p_a(A+B)=p_a(A)+p_a(B)+A\cdot B-1\ .$$

The data $(H,p_a)$ is equivalent to $(H,\{E_i\cdot E_j\},\{p_a(E_i)\})$, encoded in the resolution graph $\Gamma$. Sometimes we identify $H$ with the free abelian group on the vertex set of $\Gamma$, and talk about cycles on $\Gamma$.

**Definition 1.2.** A cycle $A=\sum a_iE_i$ (in $H$ or $H_{\mathbb{Q}}$) is *effective* or non-negative, $A\ge 0$, if all $a_i\ge 0$. There is a natural inclusion $j\colon H\to H^{\#}$, given by $j(A)(B)=-A\cdot B$ (note the minus sign, because of negative definiteness). A cycle $A$ is *anti-nef*, if $j(A)\ge 0$ in $H^{\#}$, i.e., $A\cdot E_i\le 0$ for all $i$. The anti-nef elements in $H$ form a semigroup $\mathcal{E}$ and one writes $\mathcal{E}^+$ for $\mathcal{E}\setminus\{0\}$.

If $A$ is anti-nef, then $A\ge 0$. Indeed, write $A=A_+-A_-$ with $A_+$, $A_-$ non-negative cycles with no components in common. Then $0\le -A\cdot A_-=A_-^2-A_+\cdot A_-\le A_-^2$, so by negative definiteness $A_-=0$. Furthermore, if $A\in\mathcal{E}^+$, then $A\ge E$, where $E=\sum E_i$ is the reduced exceptional cycle. Indeed, if the support of $A$ is not the whole of $E$, then there exists an $E_i$ intersecting $A$ strict positively, as $A>0$, and $E$ is connected.

**Definition 1.3.** Given two cycles $A=\sum a_iE_i$, $B=\sum b_iE_i$, their *infimum* is the cycle $\inf(A,B)=\sum c_iE_i$ with $c_i=\min(a_i,b_i)$ for all $i$. This definition extends to subsets of $\mathcal{E}$.

**Lemma 1.4.** *Let $\mathcal{W}\subset\mathcal{E}^+$ be a subset. Then* $\inf\,\mathcal{W}\in\mathcal{E}^+$.

*Proof.* Let $W = \inf \mathcal{W}$. Fix an $i$ and choose $A \in \mathcal{W}$ with $a_i$ minimal. Then

$$0 \geq E_i \cdot A = E_i \cdot (A - W) + E_i \cdot W \geq E_i \cdot W,$$

as $A - W \geq 0$ with coefficient 0 at $E_i$. So $W \cdot E_i \leq 0$ for all $i$. As $A \geq E$ for all $A \in \mathcal{W}$, also $W \geq E > 0$. □

**Definition 1.5.** The *fundamental cycle* $Z$ is the cycle $\inf \mathcal{E}^+$.

In other words, the cycle $Z$ is the smallest cycle such that $E_i \cdot Z \leq 0$ for all $i$. It can be computed with a *computation sequence* [8]. Start with any cycle $Z_0$ known to satisfy $Z_0 \leq Z$; one such cycle is $E$. Let $Z_k$ be computed. If $Z_k \neq Z$, then there is an $E_{j(k)}$ with $Z_k \cdot E_{j(k)} > 0$. Define $Z_{k+1} = Z_k + E_{j(k)}$. Then $(Z - Z_k) \cdot E_{j(k)} < 0$, so $E_{j(k)}$ lies in the support of $Z - Z_k$, giving $E_{j(k)} \leq Z - Z_k$. Therefore $Z_{k+1} \leq Z$.

The fundamental cycle depends of course on the chosen resolution, but in an easily controlled way. Therefore it can be used to define invariants of the singularity [17].

Let $\sigma\colon M' \to M$ be the blow-up in a point of $E$, with exceptional divisor $E'_0$. The exceptional divisor of $M' \to X$ is $E' = E'_0 + \sum_{i=1}^r E'_i$, where the $E'_i$, $i \geq 1$ are mapped onto the $E_i$. For a cycle $A = \sum a_i E_i$ on $M$ the pull-back $\sigma^* A$ is defined as

$$\sigma^* A = a_0 E'_0 + A^{\#} \;, \qquad \text{where } A^{\#} = \sum_{i=1}^r a_i E'_i \text{ and } E'_0 \cdot \sigma^* A = 0 \;.$$

In fact, $a_0$ is the multiplicity of $A$ in the point blown up. The main property of the intersection product in this connection is that $\sigma^* A \cdot \sigma^* B = A \cdot B$. This product is then also equal to $\sigma^* A \cdot B^{\#}$.

The canonical cycle on $M'$ satisfies $K' = \sigma^* K + E'_0$. This gives that

$$\sigma^* A \cdot K' = \sigma^* A \cdot (\sigma^* K + E'_0) = \sigma^* A \cdot \sigma^* K = A \cdot K$$

and therefore $p_a(\sigma^* A) = p_a(A)$.

**Lemma 1.6.** *The fundamental cycle $Z'$ on $M'$ is $\sigma^* Z$, the pull back of the fundamental cycle on $M$.*

*Proof.* One has $E'_0 \cdot Z' = 0$, for otherwise $Z' - E'_0$ is anti-nef. Therefore $Z' = \sigma^* Y$ for some cycle $Y$ and $Y \cdot E_i = \sigma^* Y \cdot \sigma^* E_i = Z' \cdot E'_i \leq 0$, so $Z \leq Y$. On the other hand, $\sigma^* Z \in \mathcal{E}'$, so $\sigma^* Y = Z' \leq \sigma^* Z$. □

**Corollary 1.7.** *The genus $p_a(Z)$ and degree $-Z^2$ of the fundamental cycle are invariants of the singularity.*

**Definition 1.8.** The *fundamental genus* of a singularity is the genus $p_a(Z)$ of the fundamental cycle.

A singularity has also an arithmetic genus [17] (the largest value of $p_a(D)$ over all effective cycles $D$), but this is a less interesting invariant. More important is the geometric genus, which is $h^1(\mathcal{O}_M)$, and also the largest value of $h^1(\mathcal{O}_D)$ over all effective cycles $D$.

Rational singularities were introduced by Artin [1] using the geometric genus of singularities. He proved the following characterisation, which we take as definition.

**Definition 1.9.** A normal surface singularity is *rational* if its fundamental genus $p_a(Z)$ is equal to 0.

Artin also proves that the degree $-Z^2$ of the fundamental cycle is equal to the multiplicity $m$ of the singularity. The embedding dimension of $X$ is $m+1$, which is maximal for normal surface singularities of multiplicity $m$.

**Theorem 1.10** (Laufer's rationality criterion)**.** *A resolution graph represents a rational singularity if and only if*

- *each vertex $E_i$ has $p_a(E_i) = 0$,*
- *if a cycle $Z_k$ occurs in a computation sequence and if $Z_k \cdot E_i > 0$, then $Z_k \cdot E_i = 1$.*

For the 'if'-direction it suffices to have the second property for the steps in one computation sequence, starting from a single vertex. The criterion follows from the fact that the genus cannot decrease in a computation sequence, as $p_a(Z_k + E_i) = p_a(Z_k) + p_a(E_i) + Z_k \cdot E_i - 1$.

All irreducible components of the exceptional set have to be smooth rational curves, pairwise intersecting transversally in at most one point. This shows that minimal resolution of a rational singularity is a good resolution.

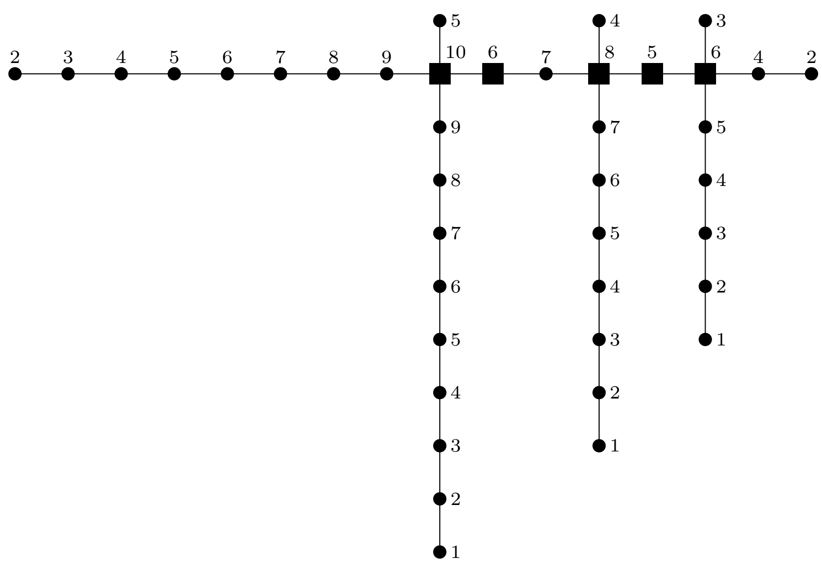
Following Lê–Tosun [10] we call the minimal resolution graph of a rational singularity a *rational graph*. It can be characterised combinatorically as weighted tree (with only vertex weights $-b_i \leq -2$), representing a negative definite quadratic form, such that the genus of the fundamental cycle is 0.

The main invariant of a rational graph is its *degree* $-Z^2$. It is related to the *canonical degree* $Z \cdot K$ by $-Z^2 = Z \cdot K + 2$, as $p_a(Z) = 0$. Let $Z = \sum z_i E_i$, $-b_i = E_i^2$. Then

$$Z \cdot K = \sum z_i(b_i - 2)\ .$$

So the degree is determined by the coefficients $z_i$ of the fundamental cycle at non-$(-2)$-vertices $E_i$.

As example of a rational graph we show the one (of degree 37) occurring in the paper of Karras [6]. Every ■ is a $(-3)$-vertex. The numbers are the coefficients of the fundamental cycle.

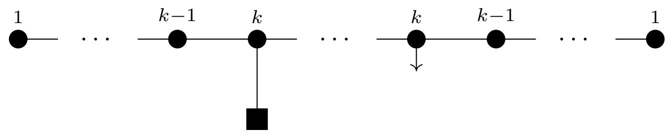



## 2. Almost reduced fundamental cycle

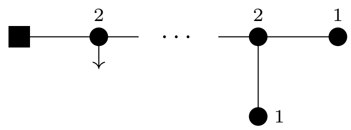
As the lists in the classification become unwieldy, we first treat a simple special case, where only $(-2)$ vertices can have higher multiplicity in the fundamental cycle. Its classification is contained in the thesis of Röhr [13] as part of more general results. The explicit list (Tables 1, 2 and 3) of graphs of RDP-configurations can be found with Gustavsen [4].

Table 1. RDP-configurations, attached to one curve

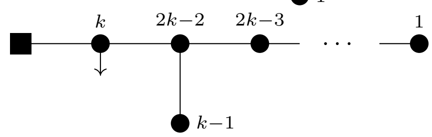
$A_n^k$:



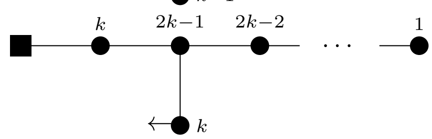

${}^{I}D_k^2$:



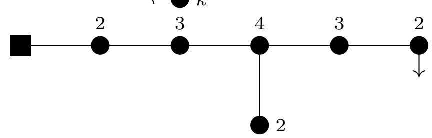

${}^{II}D_{2k}^k$:



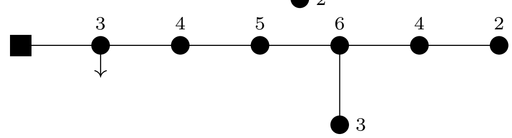

${}^{II}D_{2k+1}^k$:




$E_6^2$:




$E_7^3$:




**Definition 2.1** ([9])**.** A rational singularity has an *almost reduced* fundamental cycle if the fundamental cycle $Z = \sum z_i E_i$ on the minimal resolution is reduced at the non-$(-2)$'s, i.e., $z_i = 1$ if $b_i > 2$.

We also talk about rational graphs with almost reduced fundamental cycle.

One can compute the fundamental cycle starting from the reduced exceptional cycle by only adding curves occurring in rational double point configurations. The computation can be done for each configuration separately. Therefore we start with these configurations.

**Theorem 2.2.** *A maximal rational double point configuration on a rational graph with almost reduced fundamental cycle occurs in Tables 1, 2 or 3.*

*Proof.* By rationality at most one vertex in a rational double point configuration can have valency three in the resolution graph. Furthermore, a non-$(-2)$ can only be attached to a vertex with multiplicity one in the fundamental cycle of the rational double point. One then computes for a graph satisfying these restrictions the fundamental cycle. The lists show that all possibilities occur. □

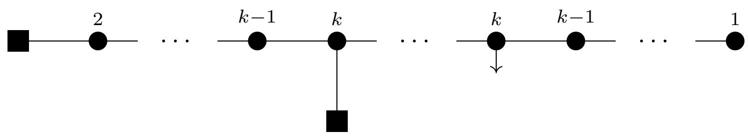
**Remark 2.3.** The list of configurations attached to two curves is obtained from the list of Table 1 by replacing a vertex with multiplicity one by a non-$(-2)$.

The numbers on the graphs in the Tables indicate the coefficients in the fundamental cycle. The squares are not part of the configuration, but stand for the non-$(-2)$'s, to which the configuration is attached. The arrow indicates the curve which intersects the fundamental cycle strict negatively.

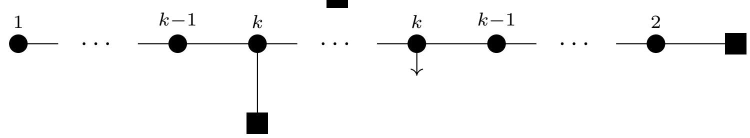
TABLE 2. RDP-configurations, attached to two curves

$A_n^{1,1}$:

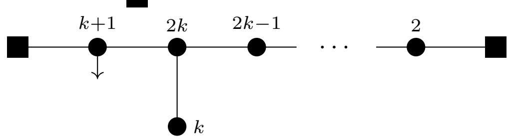

${}^{I}A_n^{2,k}$:




${}^{II}A_n^{k,2}$:




$D_{2k+1}^{k+1,2}$:




$D_{2k}^{k,2}$:

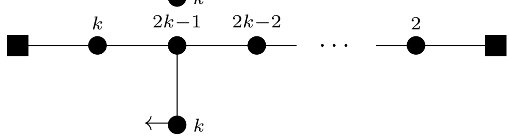
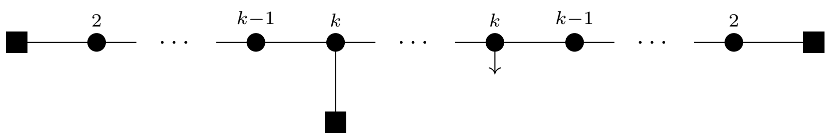



TABLE 3. RDP-configurations, attached to three curves

$A_n^{2,k,2}$:

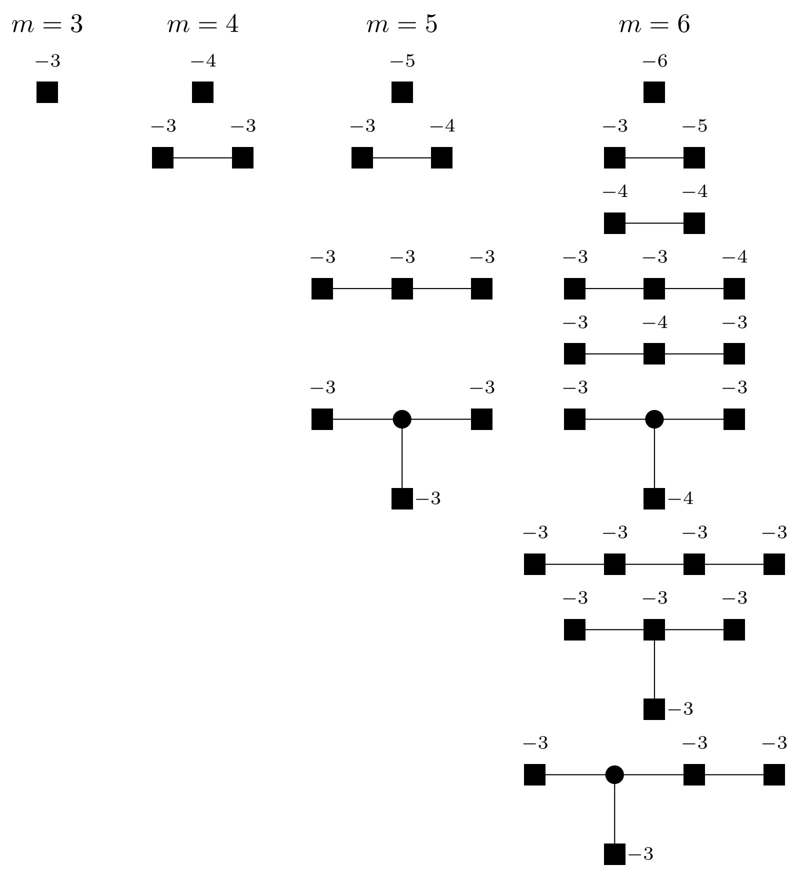



Our notation is a combination of that in [14] and Gustavsen's naming scheme [4], which is based on that of De Jong [5], who gave the list of Table 1, of configurations attached to only one curve. Our ${}^{I}D_k^2$ is called $D_k^{\mathrm{I}}$ there. Our upper indices give the multiplicity at the vertices, which are connected to non-$(-2)$'s. For the $D$-cases we could do without the upper left $I$ or $II$, except that $D_5^2$ can have two meanings.

By blowing down all RDP-configurations on the minimal resolution $M \to X$ one obtains the *canonical model*, or *RDP-resolution*, $\hat{X} \to X$. The only singularities of $\hat{X}$ are rational double points. The reduced exceptional set has two types of singularities, normal crossing of two curves, and three curves intersecting transversally in one point. The last case occurs for an $A_n^{2,k,2}$ -configuration. Again one can form a dual graph $\hat{\Gamma}$, which in this case is a hypertree with edges for the normal crossing points and T-joints for three curves meeting in one point. The canonical model does not determine the multiplicities of the fundamental cycle on the minimal resolution. Therefore we add this multiplicity as second weight (we do not write the weight if it is equal to 1).

We want to draw ordinary graphs. Observe that given a hypertree $\hat{\Gamma}$ for a canonical model, there exists a smallest ordinary tree (i.e., having minimal number of vertices) giving rise to this hypertree: one replaces each $T$-joint by an $A_1^{2,2,2}$-configuration, i.e., by a single $(-2)$-vertex.

TABLE 4. Minimal representatives up to degree 6




In Table 4 we list the graphs of the minimal representatives up to degree $m = 6$. Such a graph has to have an almost reduced fundamental cycle. The necessary and sufficient condition is that for a non-$(-2)$ vertex $E_i$ the sum of its valency $v(i)$ and the number of $(-2)$'s attached to it, is at most $b_i$.

**Classification** (of graphs with almost reduced fundamental cycle)**.** *First classify all hypergraphs of RDP-resolutions with all multiplicities equal to 1, and canonical degree $\sum(b_i-2) = m-2$. Each hypertree with $b_i$ at least the valency of $E_i$ occurs. Let then $\hat\Gamma$ be such a hypergraph. Replace a T-joint by an $A_n^{2,k,2}$ configuration, replace any number of edges by configurations from Table 2 and attach configurations from Table 1 to vertices, in such a way that the total multiplicity in the fundamental cycle of the neighbours of any vertex $v_i$ does not exceed $b_i$. The resulting graph is a rational graph with almost reduced fundamental cycle, and all graphs can be obtained this way.*

## 3. Complexity

Lê and Tosun [10] used the number of rupture points (i.e., vertices with valency at least three, stars in the terminology of [7]) as a measure of the complexity of a rational graph. They showed that it is bounded in terms of the degree $m = -Z^2$ of the graph (that is, the multiplicity of a corresponding rational singularity), more precisely by $m-2$, if the degree $m$ is at least 3. We give here a simplified proof for a sharpened version. It shows that the most complex graphs are already obtained from singularities with reduced fundamental cycle.

**Definition 3.1.** The *complexity* of a rational graph is the weighted number of rupture points, where each rupture point is counted with its valency minus two as multiplicity.

**Theorem 3.2.** *The complexity of a rational graph of degree $m$ at least 3 is at most its canonical degree $m-2$.*

The proof uses the following observation [10, Thm. 8].

**Lemma 3.3.** *The graph, obtained from a rational graph, by making some vertex weights more negative, is again rational and the fundamental cycle of the new graph is reduced at the changed vertices.*

*Proof.* We can obtain the new graph as subgraph of the graph of the resolution of the original singularity, blown up in smooth points of the relevant exceptional curves. Its fundamental cycle can be computed by first computing the fundamental cycle of the subgraph. By Laufer's rationality criterion the remaining curves intersect this cycle with multiplicity one. □

*Proof of Theorem 3.2.* Step 1: reduction to the case of almost reduced fundamental cycle. Consider the cycle $Y$, which has multiplicity 1 at the non-$(-2)$'s and multiplicities on the RDP-configurations as in Tables 1, 2 and 3. A vertex $E_i$ with $E_i \cdot Y > 0$ is a non-$(-2)$ and has coefficient $z_i > 1$ in the fundamental cycle. For those $E_i$ we increase $b_i$ by one. By the previous lemma we get the same underlying graph with the same complexity, but with almost reduced fundamental cycle, namely $Y$. The contribution of $E_i$ to the canonical degree $Z \cdot K$ changes from $z_i(b_i-2)$ to $b_i - 1$ and $(b_i-1) - z_i(b_i-2) = 1 - (z_i-1)(b_i-2) \leq 0$ with equality if and only if $z_i = 2$ and $b_i = 3$. So the degree does not increase.

Step 2: reduction to the case of reduced fundamental cycle. Consider a RDP-configuration, where $Z$ is not reduced. Make the self-intersection of the unique rupture point in the configuration into $-3$. This increases the canonical degree by 1. For all non-$(-2)$'s $E_j$ to which the configuration is connected we increase the self-intersection by 1 (decrease $b_j$ by 1). This decreases the canonical degree by at least 1 (here we use that $m > 2$). If $b_j$ was equal to 3, then $E_j$ might be connected to at most one other RDP-configuration, but without rupture point. The result is a longer chain of $(-2)$'s. Proceeding in this way we obtain without increasing the degree the same underlying graph, but with reduced fundamental cycle.

Step 3. For a graph with reduced fundamental cycle the valency of a vertex is at most $b_i$. So the complexity is bounded by $\sum(b_i - 2) = Z \cdot K$. □

## 4. The format of a rational singularity

If a singularity is not a hypersurface, its equations can be written in many ways, some of which have a special meaning. The standard example is the cone over the rational normal curve of degree four, whose equations are the minors of

$$\begin{pmatrix} z_0 & z_1 & z_2 & z_3 \\ z_1 & z_2 & z_3 & z_4 \end{pmatrix} ,$$

but also the $2 \times 2$ minors of the symmetric matrix

$$\begin{pmatrix} z_0 & z_1 & z_2 \\ z_1 & z_2 & z_3 \\ z_2 & z_3 & z_4 \end{pmatrix} .$$

In fact, perturbing these matrices gives two different ways of deforming the singularity, leading to the two components of the versal deformation. We say that we can write the total spaces in a determinantal format. In a naive interpretation a format is a way of writing or coding (efficiently) the equations of a singularity. Another point of view is that we have a high-dimensional variety (like the generic determinantal), from which the singularity is derived by specialising the equations. This will lead us to the definition of a format, given by Ancus Röhr [13]. We start with:

**Definition 4.1** ([2])**.** Let $Y \subset \mathbb{C}^N$ be a singularity. A germ $X \subset \mathbb{C}^M$ is a *singularity of type* $Y$, if there exists a map $\phi\colon \mathbb{C}^M \to \mathbb{C}^N$, such that $\phi^*(Y) = X$, which induces a complete intersection morphism $\phi\colon X \to Y$.

For a singularity $X$ of minimal multiplicity (in particular, for a rational surface singularity) of multiplicity at least 3 the existence of a complete intersection morphism $X \to Y$ already implies that $X$ is of type $Y$ [13, 2.4.2]. The singularity $Y$ has the same minimal multiplicity. Indeed, $X$ is cut out by equations with independent linear part, for otherwise the multiplicity increases.

*Deformations of type* $Y$ of $X$ are obtained by unfolding the map $\phi$: for every map

$$\Phi\colon \mathbb{C}^M \times (S,0) \to \mathbb{C}^N,$$

extending $\phi$, the map $\pi\colon \Phi^*Y \to (S,0)$ is flat [2, 4.3.4]. In general such deformations will not fill out a component of the deformation space, but one can turn the problem around and start from the total space of the deformation over a smooth component. This total space is then rigid [15, p. 101].

A rational singularity has always a smoothing component with smooth base space. This is the *Artin component*, over which simultaneous resolution exists after base change. This simultaneous resolution is a versal deformation of the resolution $M$ of $X$. A base change is not needed, if one considers instead deformations of the canonical model $\hat{X} \to X$.

We therefore concentrate on the Artin component. As it is smooth, the singularity $X$ itself is cut out by a regular sequence from the total space $Y$ of the deformation over the Artin component. Therefore the singularity is of type $Y$. By a result of Ephraim [3] one can write every reduced singularity $Y$ in a unique way (up to isomorphism) as product of a singularity $F$ and a smooth germ of maximal dimension.

**Definition 4.2** ([13])**.** The *format* $F(X)$ of a rational surface singularity $X$ is the unique germ $F$ in a decomposition $Y = F \times \mathbb{C}^k$, with $k$ maximal, of the total space $Y$ over the Artin component of $X$.

Let $\hat{\pi}\colon (\hat{X}, \hat{Z}) \to (X,p)$ be the RDP-resolution of a rational singularity $X$ of multiplicity $m$; it can be obtained by blowing up a canonical ideal. It gives an embedding of $\hat{X} \hookrightarrow \mathbb{P}^{m-2}_X$ over $X$ and with it an embedding of the exceptional set $\hat{Z} = \hat{\pi}^{-1}(p)$ in $\mathbb{P}^{m-2}$, as arithmetically Cohen-Macaulay scheme of genus 0 and degree $m-2$ [13, 2.6.3]. Röhr calls the cone over $\hat{Z}$ the *canonical cone* of $X$. One can also obtain $\hat{Z}$ by blowing up a canonical ideal of $F$. This implies that the canonical cone of a rational surface singularity is determined up to isomorphy by its format. We conjectured that the converse also holds. This would imply that the singularities in Remark 6.8 have the same format.

Röhr proves that quasi-determinantal singularities can be recognised from the resolution graph [13, Satz 4.2.1]. The condition is that the graph contains the graph of a cyclic quotient singularity of the same multiplicity. Equivalently one can say that the graph $\hat{\Gamma}$ of the canonical model is a chain, with everywhere multiplicity 1. The proof is based on a criterion for RDP-configurations to be deformed on the resolution without changing the format [13, Satz 3.3.1]. This criterion also applies to rational singularities with almost reduced fundamental cycle: all RDP-configurations can be deformed away, except $A_1^{2,2,2}$. The graph of the resulting singularity is the minimal tree for the given hypertree $\hat{\Gamma}$. Note that the canonical cone can have moduli, so also the formats. The graph can therefore at most determine an equisingularity class of formats.

## 5. Computation of the fundamental cycle

In this section we describe, following Röhr [13, 1.3], a special way to compute the fundamental cycle, for a given rational graph. We single out a vertex $E_0$, which we call *central vertex*. The computation is done in steps, where each time the multiplicity at $E_0$ is increased by one.

We decompose the complement of a vertex $E_0$ in a rational graph $\Gamma$ in irreducible components: $\Gamma \setminus \{E_0\} = \cup_{i=1}^{k} \Gamma_i$. We suppose that $k > 1$; the case $k = 1$ can be reduced to it by blowing up a point of the curve $E_0$.

We construct the fundamental cycle inductively. To start with, let $E_0 + Y_i^{(1)}$ be the fundamental cycle on $\{E_0\} \cup \Gamma_i$; as $k > 1$, the support of $Y_i^{(1)}$ is $\Gamma_i$: one can compute $Z$ starting from $E_0 + Y_i^{(1)}$, so the coefficient at $E_0$ is one. Define $Z^{(1)} = E_0 + \sum Y_i^{(1)}$. Then $Z^{(1)} \cdot E_j \le 0$ for all $j \ne 0$.

Let $Z^{(s)}$ be constructed with $Z^{(s)} \cdot E_j \le 0$ for all $j \ne 0$, with coefficient $s$ at $E_0$ and satisfying $Z^{(s)} \le Z$. If $Z^{(s)} \cdot E_0 \le 0$, then $Z^{(s)}$ is the fundamental cycle $Z$. Otherwise, consider the set of vertices $E_{i,j} \in \Gamma_i$ such that $Z^{(s)} \cdot E_{i,j} = 0$ and let $\Gamma_i^{(s+1)}$ be the connected component of this set adjacent to $E_0$. Let $E_0 + Y_i^{(s+1)}$ be the fundamental cycle on $\{E_0\} \cup \Gamma_i^{(s+1)}$. As $Y_i^{(s+1)} \le Y_i^{(1)}$, the support of $Y_i^{(s+1)}$ does not contain $E_0$. Now define

$$Z^{(s+1)} = Z^{(s)} + E_0 + \sum Y_i^{(s)} \ .$$

Then $Z^{(s+1)} \cdot E_j \le 0$ for all $j \ne 0$, the coefficient at $E_0$ is $s+1$ and $Z^{(s+1)} \le Z$; indeed $Z^{(s+1)}$ can be constructed from $Z^{(s)}$ by first adding $E_0$ and then continuing in the manner of a computation sequence without ever adding $E_0$ again.

This construction ends with the fundamental cycle.

If $k = 1$, we blow up a point of the curve $E_0$, introducing a $\Gamma_2$. But this can be avoided, as in fact the same description as above holds, with the only difference that for $k = 1$ the cycle $E_0 + Y_1^{(s)}$ is not the fundamental cycle on $\{E_0\} \cup \Gamma_1^{(s)}$ (in particular, $E_0 + Y_1^{(1)}$ is not the fundamental cycle on $\Gamma$), but $Y_1^{(s)}$ is the cycle constructed from $Z^{(s-1)} + E_0$ in the manner of a computation sequence without ever adding $E_0$.

Let $m_i^{(s)} \le m_i^{(1)}$ be the coefficient of $Y_i^{(s)}$ at the vertex in $\Gamma_i$ adjacent to $E_0$. As $E_0 \cdot Z^{(s)} = 1$ for $s < l$, where $Z^{(l)} = Z$ is the last step of the computation, we have $\sum_i m_i^{(1)} = b_0 + 1$, $\sum_i m_i^{(s)} = b_0$ for $1 < s < l$ and $\sum_i m_i^{(l)} < b_0$.

**Example 5.1.** Consider an $E_6$-configuration, connected to a non-$(-2)$ vertex $E_0$. We compute the $Z^{(s)}$. We only write the multiplicities of $E_0$ (in boldface) and of the irreducible components of the configuration.

$$\begin{array}{cccccc} \mathbf{1} & 2 & 3 & 4 & 3 & 2 \\ & & & 2 & & \end{array} \qquad \begin{array}{cccccc} \mathbf{2} & 4 & 5 & 6 & 4 & 2 \\ & & & 3 & & \end{array} \qquad \begin{array}{cccccc} \mathbf{3} & 4 & 5 & 6 & 4 & 2 \\ & & & 3 & & \end{array}$$

The sequence $(m_1^{(s)})$ is $(2, 2, 0)$ and therefore an $E_6$-configuration can only be connected to a curve with multiplicity at most 3. We observe that the same sequence can be obtained from $2A_2^1$, two chains of length two.

## 6. One non-reduced curve

The goal of this section is to give the elements for the classification of rational graphs, where each RDP-configuration is attached to at most one non-reduced non-$(-2)$-vertex. We first classify the possible multiplicities at RDP-configurations. These depend only on the multiplicity of the non-$(-2)$, and the computation can again be done for each configuration separately. The candidates for graphs of RDP-resolutions can be found from the graphs with almost reduced fundamental cycle, but not every candidate arises from a rational graph.

Let $E_0$ be a non-reduced non-$(-2)$, with multiplicity $z_0$ in the fundamental cycle. According to the previous section, we can compute the fundamental cycle in $z_0$ steps, each time increasing the multiplicity of $E_0$ by one. We add cycles with support on the subgraphs $\Gamma_i$ and each subgraph gives a multiplicity sequence $(m_i^{(1)}, \dots, m_i^{(z_0)})$. These multiplicities satisfy

$$\sum_i m_i^{(1)} = b_0 + 1, \qquad \sum_i m_i^{(s)} = b_0 \text{ for } 1 < s < z_0, \qquad \sum_i m_i^{(z_0)} < b_0.$$

After the first step we add only cycles with support in RDP-configurations intersecting $E_0$, as all other non-$(-2)$-curves, intersecting such configurations, have multiplicity one. Each $\Gamma_i$ contains at most one RDP-configurations adjacent $E_0$. We include the case that there is no such configuration by calling it $A_0^{1,1}$.

For each RDP-configuration from Tables 1, 2 and 3 we compute the *multiplicity sequence* $(m^{(1)}, \dots, m^{(j)})$. The multiplicities satisfy $m^{(1)} - 1 \le m^{(s)} \le m^{(1)}$ for all $s < j$. We abbreviate a sequence $k, \dots, k$ of $l$ equal multiplicities as $k^l$. An exponent $l = 0$ means that this factor is absent. If the sequence is infinite, and repeating itself, we underline the repeating section. So in Table 6 the entry $(1^{n+1}, \underline{0, 1^n})$ for $LA_n^{1,1}$ should be read as $(1^{n+1}, 0, 1^n, 0, 1^n, 0, \dots)$. The case $n = 0$, of two non-$(-2)$'s intersecting each other, is included. The sequence is then $(1, 0, 0, \dots)$.

For configurations between several vertices only one of the non-$(-2)$'s has higher multiplicity, and we suppose that the other ones have sufficiently negative self-intersection for the graph being rational.

We have to distinguish which of the two or three attached vertices is the non-reduced one. We always draw the graphs as in Tables 2 and 3. In a graph of type ${}^{I}A_n^{2,k}$, ${}^{II}A_n^{k,2}$ or $A_n^{2,k,2}$ the arrowhead (which indicates the curve intersecting the fundamental cycle of the extended configuration negatively) is on the right hand side of the graph. So it makes sense to distinguish between the left, middle or right attached vertex. We denote this by writing an $L$, $M$ or $R$ before the name. For type $D$ we use $L$ and $R$.

It is possible to obtain a multiplicity sequence from different configurations or combinations of configurations. We then speak about equivalent configurations. For each configuration we also determine the simplest equivalent combination of configurations.

**Proposition 6.1.** *The multiplicity sequences and the equivalent configurations for the configurations of Table 1 are as given in Table 5. The different cases arising from the configurations of Table 2 are in Table 6; it gives also the multiplicity at the component attached to the other, reduced non-$(-2)$. If the sequence is infinite, the multiplicity after step $s$ of the computation is given. Table 7 gives the results for $A_n^{2,k,2}$.*

*Proof.* We do here only the case $A_n^k$, for $k > 1$, as the other cases involve similar or easier computations. We write $n = lk + r + (k-1)$ with $l \ge 1$ and $0 \le r \le k-1$. This is possible as the number $n$ satisfies $n \ge 2k - 1$. There is a chain of $lk + r - (k-1) = (l-1)k + r + 1$ $(-2)$-vertices with multiplicity $k$ in $Z^{(1)}$, and the end of this chain not intersecting $E_0$ intersects $Z^{(1)}$ negatively (when $l = 1$ and $r = 0$ there is only one vertex with multiplicity $k$; in this case the multiplicity sequence is $(k, 0)$ and the format is $kA_1^1$, in accordance with the general

TABLE 5.

| name | mult sequence | equivalent to |
|---|---|---|
| $A^1_n$ | $(\underline{1^n, 0})$ | |
| $A^k_{(l+1)k+r-1},\ r < k-1$ | $(k^l, r)$ | $(k-r)A^1_l + rA^1_{l+1}$ |
| $A^2_{2l+2}$ | $(2^l, 1, 1, 2^l, 0)$ | |
| $A^k_{(l+2)k-2},\ k > 2$ | $(k^l, k-1, 1)$ | $A^1_l + (k-1)A^1_{l+1}$ |
| ${}^{I}D^2_k$ | $(2, 0)$ | $2A^1_1$ |
| ${}^{II}D^k_{2k},\ k > 2$ | $(k, 0)$ | $kA^1_1$ |
| ${}^{II}D^2_5$ | $(2, 1, 2, 0)$ | $A^1_1 + A^1_3$ |
| ${}^{II}D^k_{2k+1},\ k > 2$ | $(k, 1)$ | $(k-1)A^1_1 + A^1_2$ |
| $E^2_6$ | $(2, 2, 0)$ | $2A^1_2$ |
| $E^3_7$ | $(3, 0)$ | $3A^1_1$ |

TABLE 6.

| name | mult sequence | other mult | equivalent configuration |
|---|---|---|---|
| $LA^{1,1}_n$ | $(1^{n+1}, \underline{0, 1^n})$ | $\lceil \frac{n+s}{n+1} \rceil$ | |
| $L^{I}A^{2,k}_n$ | $(2, 1^{n-k}, 0)$ | $n-k+2$ | $LA^{1,1}_0 + A^1_{n-k+1}$ |
| $M^{I}A^{2,k}_{(l+1)(k-1)+r}$ $k > 2$ | $(k, (k-1)^{l-1}, r)$ | $\lceil \frac{(l+1)(k-1)+r}{k-1} \rceil$ | $LA^{1,1}_0 + (k-1-r)A^1_l + rA^1_{l+1}$ |
| $M^{II}A^{k,2}_{(l+1)k+r-2}$ $0 \le r < k-1$ | $(k^l, r)$ | $2$ | $LA^{1,1}_{l-1} + rA^1_{l+1} + (k-1-r)A^1_l$ |
| $M^{II}A^{k,2}_{(l+1)k+k-3}$ $k > 2,\ l > 1$ | $(k^l, k-1, 1)$ | $3$ | $LA^{1,1}_{l-1} + (k-1)A^1_{l+1}$ |
| $M^{II}A^{k,2}_{3k-3}$ $k > 2$ | $(k, k-1, 1)$ | $3$ | $LA^{1,1}_1 + A^1_1 + (k-2)A^1_2$ |
| $M^{II}A^{2,2}_{2l+1}$ | $(2^l, \underline{1^2, 2^{l-1}})$ | $\lceil \frac{l+1+s}{l+1} \rceil$ | $LA^{1,1}_l + A^1_l$ |
| $R^{II}A^{k,2}_n$ $k > 2$ | $(2, 1^{k-2}, 0)$ | $k$ | $LA^{1,1}_0 + A^1_{k-1}$ |
| $R^{II}A^{2,2}_n$ | $(2, 0)$ | $2$ | $LA^{1,1}_0 + A^1_1$ |
| $LD^{k+1,2}_{2k+1}$ | $(k+1, 0)$ | $2$ | $LA^{1,1}_0 + kA^1_1$ |
| $RD^{k+1,2}_{2k+1}$ | $(2, 1, 1, \dots)$ | $\lceil \frac{2k+s}{2} \rceil$ | $LA^{1,1}_1 + A^1_1$ |
| $LD^{k,2}_{2k}$ $k > 2$ | $(k, 1)$ | $3$ | $(k-2)A^1_1 + A^1_2 + LA^{1,1}_0$ |
| $RD^{k,2}_{2k}$ | $(2, 1, 1, \dots)$ | $\lfloor \frac{2k+s}{2} \rfloor$ | $LA^{1,1}_1 + A^1_1$ |

formula). The set $\Gamma^{(2)}$ consists of $(l-1)k + r + (k-1)$ vertices. If $l > 1$ this number is at least $2k-1$ and the multiplicities in $Z^{(2)}$ are

$$2, 4, \dots, 2k, 2k, \dots, 2k, 2k-1, 2k-2, \dots, 2, 1\ .$$

Table 7.

| name | mult seq | mult at L | mult at M | mult at R |
|---|---|---|---|---|
| $LA_n^{2,k,2}$ | $(2, 1^{n-k+1}, 0)$ | | $n-k+3$ | 2 |
| $MA_n^{2,k,2}$ | $(k, (k-1)^{l-1}, r)$ | $\lceil \frac{n+1}{k-1} \rceil$ | | 2 |
| $RA_n^{2,k,2}$ | $(2, 1^{k-2}, 0)$ | 2 | $k$ | |

Here $n = (l+1)(k-1) - 1 + r$ with $0 \le r \le k-2$ and $k > 2$.

There are $(l-2)k + r + 1$ vertices with multiplicity $2k$ in $Z^{(2)}$. We continue in this way until there are $r+1$ vertices with multiplicity $lk$ in $Z^{(l)}$; all multiplicities are then

$$l, 2l, \dots, lk, lk, \dots, lk, lk-1, lk-2, \dots, 2, 1 .$$

The set $\Gamma^{(l+1)}$ consists of $r + (k-1)$ vertices (except when $r = 0$; then $\Gamma^{(l+1)}$ is empty). We therefore add the multiplicities

$$1, 2, \dots, r-1, r, \dots, r, r-1, \dots, 2, 1, 0, \dots, 0 .$$

If $r < k-1$ the sequence stops here, the multiplicity sequence is $(k^l, r)$ and the equivalent configuration is $(k-r)A_l^1 + rA_{l+1}^1$. If $r = k-1$ the multiplicities in $Z^{(l+1)}$ are

$$l+1, 2(l+1), \dots, (k-1)(l+1), k(l+1) - 1, k(l+1) - 2, \dots, 2, 1 .$$

We add the multiplicities $0, \dots, 0, 1, \dots, 1$. If $k \ge 3$ the sequence stops here, the multiplicity sequence is $(k^l, k-1, 1)$ and the configuration is equivalent to $A_l^1 + (k-1)A_{l+1}^1$. If $k = 2$, the sequence continues; as $\Gamma^{(l+3)}$ consists of $n-1$ nodes, the multiplicity sequence is $(2^l, 1^2, 2^l, 0)$. There is no easier equivalent configuration for this $A_{2l+2}^2$. □

**Remark 6.2.** The condition $k > 2$ in the tables is included to avoid duplications. For example, as $MA_n^{2,2,2} = LA_n^{2,2,2}$, we can assume that $k > 2$ for $MA_n^{2,k,2}$.

**Remark 6.3.** Note that the tables give the maximal multiplicity sequence for each configuration. If the computation stops earlier (due to other configurations), one gets a simpler equivalent singularity.

**Corollary 6.4.** *Every RDP-configuration, attached to only one vertex, is equivalent to a combination of configurations of type $A_n^1$ and $A_{2l}^2$.*

**Corollary 6.5.** *An RDP-configuration, attached to two or three vertices, of which only one has multiplicity greater than one in the fundamental cycle, is equivalent to a combination of configurations of type $A_n^1$, $A_{2l}^2$, $LA_n^{1,1}$ and $LA_n^{2,2,2}$.*

*Proof.* Table 6 gives the result for configurations between two vertices.

From Table 7 we see that the multiplicities of $LA_n^{2,k,2}$ depend only on $n-k$, so $LA_n^{2,k,2}$ is equivalent to $LA_{n-k+2}^{2,2,2}$. The multiplicities of $RA_n^{2,k,2}$ depend only on $k$, so we can take the smallest $n$, which is $2k-3$. In that case the left and right chain of $(-2)$'s are equally long, so by interchanging $L$ and $R$ we obtain $LA_{2k-3}^{2,k,2}$, which is equivalent to $LA_{k-1}^{2,2,2}$.

For $MA_n^{2,k,2}$ we distinguish between the cases $r = 0$ and $0 < r \le k-2$. In the first case $\lceil \frac{n+1}{k-1} \rceil = l+1$, while $\lceil \frac{n+1}{k-1} \rceil = l+2$ in the second case. For $r = 0$ an equivalent configuration, attached to the vertex $v_M$, is $MA_l^{2,2,2} + (k-2)A_l^1$ and, for $r > 0$, is

$$MA_{l+1}^{2,2,2} + (k-1-r)A_l^1 + (r-1)A_{l+1}^1 .$$

Finally we note that interchanging $M$ and $L$ makes $MA_n^{2,2,2}$ into $LA_n^{2,2,2}$. □

From an arbitrary rational graph we obtain a graph with almost reduced fundamental cycle and the same underlying graph by making some vertex weights $-b_i < -2$ more negative. This process can also be inverted. The possible candidates for graphs (or hypergraphs) of RDP-resolutions with non-reduced fundamental cycle can be obtained from reduced (hyper)-graphs by replacing a $-(b_i+2)$-vertex by a $-(b_i/z_i+2)$-vertex with multiplicity $z_i$, but not all graphs can be realised.

**Proposition 6.6.** *On a rational graph with only one non-$(-2)$ vertex $E_0$ the multiplicity of $E_0$ in the fundamental cycle can at most be* 6*.*

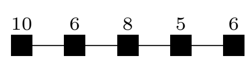
*Proof.* By Corollary 6.4 it suffices to consider only RDP-configurations of type $A_n^1$ and $A_{2l}^2$. If $z_0 > 2$, there is exactly one configuration $\Gamma_i$ with $m_i^{(2)} = m_i^{(1)} - 1$, so it is either $A_1^1$ or $A_4^2$. In the last case $z_0 \le 5$, as $A_4^2$ gives the sequence $(2,1,1,2,0)$. Suppose now that there is exactly one $A_1^1$. If $z_0 > 3$, there is exactly one configuration $\Gamma_i$ with $m_i^{(3)} = m_i^{(2)} - 1 = m_i^{(1)} - 1$, which is either $A_2^1$ or $A_6^2$. In the last case $z_0 = 4$, as we combine the sequences $(2,2,1,1,2,2,0)$ and $(1,0,1,0,\dots)$. The sequence of $A_1^1 + A_2^1$ is $(1+1, 0+1, 1+0, 0+1, 1+1, 0+0) = (2,1,1,1,2,0)$, which shows that $z_0 \le 6$. $\square$

**Remark 6.7.** We realise $z_0 = 6$ for a $(-3)$ with $A_1^1 + A_2^1 + A_4^1 + A_5^1$.

**Remark 6.8.** With $A_4^2$ and $E_0$ a $(-3)$ we can realise $z_0 = 5$ with the configuration $A_4^2 + A_3^1 + A_4^1$. Another way to get $5E_0$ is with $A_1^1 + A_2^1 + 2A_4^1$. It would be interesting to study the formats of the corresponding singularities. We remark that neither is a deformation of the other.

**Classification** (of graphs, where each RDP-configuration is attached to at most one non-reduced non-$(-2)$)**.** *Start by making a list of all possible hypergraphs $\hat{\Gamma}$ of canonical cones, without edges (or hyperedges) between non-reduced vertices. Given $\hat{\Gamma}$, realise this graph (if possible) in all ways, using only configurations $A_n^1$ and $A_{2l}^2$, $A_n^{1,1}$ (including $n = 0$) and $LA_n^{2,2,2}$. Replace (combinations of) RDP-configurations with equivalent ones, as given by the Tables 5, 6 and 7.*

**Remark 6.9.** The computations so far also can help to compute the fundamental cycle for complicated graphs. As example we return to Karras' graph, given at the end of Section 1. The graph for the canonical model is rather simple. Note also that only configurations of type $A_n^1$ occur.




We first simplify the graph. The configuration $A_1^1 + A_2^1$ at $(-3)$ on the right implies that its multiplicity is at most 6. Therefore the $A_5^1$ has no influence on the computation, and we get the same multiplicities, if we remove it and increase the weight $(-3)$ to $(-2)$. We have then a $A_4^2$ attached to the $(-3)$ of multiplicity 5. The $(-3)$ on the left has multiplicity at most 10 because of $A_1^1 + A_9^1$. Again we can remove the $A_9^1$ and increase the weight $(-3)$ to $(-2)$. We have then a $A_{10}^2$ attached to the $(-3)$ of multiplicity 6. By the same argument the $A_7^1$ at the vertex of multiplicity 8 can be removed, so that we end up with two $(-3)$-vertices $E_1$ and $E_2$ with a $A_3^{2,2}$ in between, an $A_{10}^2$ attached to $E_1$ and $A_4^2$ attached to $E_2$.

It remains to compute the fundamental cycle for this configuration. This is best done with the rupture point between $E_1$ and $E_2$ as central vertex. We give the total multiplicities at each step. The multiplicities of the $(-3)$'s are in bold face, while those of the central vertex are underlined.

$$\begin{array}{cccccccccccccccc} & & & & & & & & 1 & & & 1 & & 1 & & \\ 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & \mathbf{1} & 1 & \underline{1} & \mathbf{1} & 2 & 2 & 1 \end{array}$$

$$\begin{array}{cccccccccccccccc}
 & & & & & & & & 2 & & & 1 & & 2 & & \\
1 & 2 & 3 & 4 & 4 & 4 & 4 & 4 & 4 & \mathbf{2} & 2 & \underline{2} & \mathbf{2} & 3 & 2 & 1
\end{array}$$

$$\begin{array}{cccccccccccccccc}
 & & & & & & & & 3 & & & 2 & & 2 & & \\
1 & 2 & 3 & 4 & 5 & 6 & 6 & 6 & 6 & \mathbf{3} & 3 & \underline{3} & \mathbf{2} & 3 & 2 & 1
\end{array}$$

$$\begin{array}{cccccccccccccccc}
 & & & & & & & & 4 & & & 2 & & 2 & & \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 8 & \mathbf{4} & 4 & \underline{4} & \mathbf{3} & 4 & 3 & 2
\end{array}$$

$$\begin{array}{cccccccccccccccc}
 & & & & & & & & 5 & & & 3 & & 2 & & \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \mathbf{5} & 5 & \underline{5} & \mathbf{3} & 4 & 3 & 2
\end{array}$$

$$\begin{array}{cccccccccccccccc}
 & & & & & & & & 5 & & & 3 & & 3 & & \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \mathbf{5} & 6 & \underline{6} & \mathbf{4} & 6 & 4 & 2
\end{array}$$

$$\begin{array}{cccccccccccccccc}
 & & & & & & & & 5 & & & 4 & & 3 & & \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \mathbf{5} & 6 & \underline{7} & \mathbf{5} & 6 & 4 & 2
\end{array}$$

$$\begin{array}{cccccccccccccccc}
 & & & & & & & & 5 & & & 4 & & 3 & & \\
2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \mathbf{6} & 7 & \underline{8} & \mathbf{5} & 6 & 4 & 2
\end{array}$$

Karras computes in a different way, as his goal is to find a smoothable subcycle.

## 7. RDP-configurations on general graphs

In this section we determine the maximal multiplicities that most can occur on an RDP-configuration. We continue to compute for each RDP-configuration separately. For some configurations the multiplicities can become arbitrary high, but what actually happens, depends on the rest of the graph. We do not investigate the exact conditions.

The results apply to the classification of graphs, in which two or three non-reduced non-$(-2)$'s are connected to each other by a single RDP-configuration, but not connected to any other non-reduced non-$(-2)$. In particular, we determine the conditions that the multiplicity of the non-$(-2)$'s does not exceed two. This suffices to give a complete classification of rational graphs of degree 6. We indicate this in the next section.

We first treat configurations attached to exactly two vertices, both of higher multiplicity. Then there are two vertices $E_a$ and $E_b$, of self-intersection $-a$ and $-b$, which are connected by a RDP-configuration $\Delta$. The fundamental cycle $E_a + Z_\Delta + E_b$ on $\{E_a\} \cup \Delta \cup \{E_b\}$ is given in Table 2. Let $n_{\Delta,a}$ be the coefficient of the vertex of $\Delta$, adjacent to $E_a$. Furthermore, let $\Gamma_a$ be the union of the connected components of the complement of the graph, which are connected to $E_a$. Let $E_a + Z_a$ be the fundamental cycle on $\{E_a\} \cup \Gamma_a$, let $n_a$ be the sum of the multiplicities of $Z_a$ at the vertices of $\Gamma_a$, adjacent to $E_a$. Define the corresponding objects for $E_b$.

**Definition 7.1.** In the above situation $E_a$ is a *bad vertex* if $n_{\Delta,a} + n_a = a + 1$.

We borrow the term *bad* from Tosun, see [10, Definition 3.4] and [16, Definition 3.14], where it is used without multiplicities: Tosun calls a vertex bad if its valency is one less then its vertex weight $b$. Karras [6] calls it a *basic center*. If $E_i \cdot Z_\Delta = 0$ for every vertex of $\Delta$, then exactly one of $E_a$ and $E_b$ is a bad vertex (in our sense).

7.1. $A_n^{1,1}$. We call the two vertices $E_L$ and $E_R$, and denote the numbers defined above correspondingly; the vertex weight of $E_L$ is $-b_L$, and that of $E_R$ is $-b_R$. Then $n_{\Delta,L} = n_{\Delta,R} = 1$ and there is exactly one bad vertex, which we suppose to be $E_L$. This means that $n_L = a$, and $\Gamma_L$ is non-empty. We claim that the multiplicity of $\Delta$ in the fundamental cycle can be arbitrarily high. We compute the fundamental cycle with $E_L$ as central vertex. We set $Y_L^{(1)} = Z_L$, $Y_{\Delta,R}^{(1)} = Z_\Delta + E_R + Z_R$. Then $Z^{(1)} = Y_L^{(1)} + E_L + Y_{\Delta,R}^{(1)}$, and $E_L$ is the only vertex with $E_i \cdot Z^{(1)} = 1$. In each next step $Y_L^{(s)} \le Y_L^{(1)}$ and $Y_{\Delta,R}^{(s)} \le Y_{\Delta,R}^{(1)}$. In particular, the multiplicity of the fundamental cycle at $E_R$ does not exceed that at $E_L$. We describe the case that the computation never stops. For the sum $n_L^{(s)}$ of multiplicities in $\Gamma_L$, adjacent to $E_L$, and the multiplicity $n_{R,\Delta}^{(s)}$ we have then either $n_L^{(s)} = a-1$ and $n_{\Delta,R}^{(s)} = 1$, or $n_L^{(s)} = a$ and $n_{\Delta,R}^{(s)} = 0$. As remarked earlier, we do not investigate the conditions which this assumption imposes on $\Gamma_L$ and $\Gamma_R$.

Let $Z_\Delta = E_1 + \cdots + E_n$ with $E_1 \cdot E_L = 1$ and $E_n \cdot E_R = 1$. Suppose the coefficient of $E_R$ in $Y_{\Delta,R}^{(s)}$ is 1, and the coefficient of $E_R$ in $Z^{(s)}$ is $k$. If $E_R \cdot Z^{(s)} = -s_k < 0$, then $Y_{\Delta,R}^{(s+1)} = E_1 + \cdots + E_n$, $Y_{\Delta,R}^{(s+2)} = E_1 + \cdots + E_{n-1}$, ..., $y_{\Delta,R}^{(s+n)} = E_1$ and $Y_{\Delta,R}^{(s+n+1)} = \emptyset$. Then $E_R \cdot Z^{(s+n+1)} = -s_k + 1$. We continue by adding only cycles with support on $\Delta$ until $E_R$ intersects the total computed cycle trivially. In the next step the coefficient of $E_R$ in the added cycle will again be 1. At this stage the coefficients of the total cycle in the neighbourhood of $\Delta$ are as follows.

The coefficient of $E_L$ is $s = k + (n+1)\sum s_i$, the sum of the $n_L^{(j)}$ is

$$(a-1)(n\sum s_i + k - 1) + a(1 + \sum s_i),$$

the coefficient of $E_1$ is $k + n\sum s_i$, that of $E_t$ is $k + (n+1-t)\sum s_i$, that of $E_n$ is $k + \sum s_i$, the coefficient of $E_R$ is $k$, and the sum of the multiplicities of the vertices in $\Gamma_R$, adjacent to $E_R$, is $k(b-1) - \sum s_i$.

We remark that the formulas also work, if $n = 0$. This means that $\Delta = \emptyset$ and $E_L$ is adjacent to $E_R$. Furthermore, if $\sum s_i = 0$, the multiplicities at $E_L$ and $E_R$ are independent of $n$.

7.2. ${}^I A_n^{2,k}$. In this case, and also for ${}^{II} A_n^{k,2}$ and $A_n^{2,k,2}$, it is more convenient to compute the fundamental cycle with the rupture point in the chain of $(-2)$'s as central vertex $E_0$. We therefore use a slightly different notation, consistent with the description of the computation in Section 5. Let $m_L^{(s)}$, $m_M^{(s)}$ and $m_R^{(s)}$ be the multiplicities in step $s$ at the vertices directly to the left, below or to the right of the central vertex. The non-$(-2)$ vertices are $E_L$ with weight $-b_L$, and $E_M$ with weight $-b_M$.

We have $m_L^{(1)} + m_M^{(1)} + m_R^{(1)} = 3$, $m_L^{(s)} + m_M^{(s)} + m_R^{(s)} \le 2$ for $s > 1$, and the computation stops at the first $s$ where this sum is less than 2.

We start by computing the sequence $(m_R^{(s)})$. We apply Proposition 6.1: as we have a $A_{n-k+1}^1$-configuration, the sequence is $(1^{n-k+1}, 0, 1^{n-k+1}, 0, \dots)$. So $m_R^{(s)} = 1$ for $s \ne l(n-k+2)$ and $m_R^{(l(n-k+2))} = 0$ for all $l$.

Next we look at $(m_M^{(s)})$. Let $Z_M$ be the fundamental cycle on the connected component $\Gamma_M$ of $\Gamma \setminus \{E_0\}$, containing $E_M$. For the first step $Z^{(1)}$ of the computation we determine the fundamental cycle $Y_M^{(1)}$ on $\{E_0\} \cup \Gamma_M$: it is $E_0 + Z_M$. The condition that $E_M$ is a bad vertex translates into $E_M \cdot Z_M = 1 - k$, so $E_M \cdot Z^{(1)} = 2 - k$. Therefore we put $E_M \cdot Z^{(1)} = 2 - k - t_1$, where $t_1 \ge 0$ with equality if and only if $E_M$ is a bad vertex. In the next steps $Y_M^{(s)}$ is empty. We find that $E_M \cdot Z^{(k+t_1-1)} = 0$, so $E_M$ is in the support of $Y_M^{(k+t_1)}$. We set $E_M \cdot Z^{(k+t_1)} = 2 - k - t_2$, with $t_2 \ge t_1$. Proceeding this way we find the sequence

$$(1, 0^{k+t_1-2}, 1, 0^{k+t_2-2}, 1, 0^{k+t_3-2}, \dots)\,.$$

On the left side $E_L \cdot Z^{(1)} = -s_1$ with $s_1 = 0$ if and only if $E_L$ is a bad vertex. If $s_1 > 0$, then $E_L$ is not contained in the support of $Y_L^{(2)}$, which is the $A_{k-2}$-configuration between $E_L$ and $E_0$. We continue in the manner of $A^1_{k-2}$, until $E_L \cdot Z^{(s_1(k-1)+1)} = 0$ and $E_L$ is in the support of $Y_L^{(s_1(k-1)+2)}$. Then $E_L \cdot Z^{(s_1(k-1)+2)} = -s_2$ with $s_2 \geq s_1$. The sequence is

$$(1, (1^{k-2}, 0)^{s_1}, 1, (1^{k-2}, 0)^{s_2}, 1, (1^{k-2}, 0)^{s_3}, \dots) .$$

Exactly one of $E_L$ and $E_M$ is a bad vertex. If $k = 2$, both $E_L$ and $E_M$ are connected to $E_0$, so upon relabeling we may assume that the bad vertex is $E_L$. We first treat the other case, that $E_M$ is the bad vertex. Then $t_1 = 0$, and, as just said, we make the assumption that $k > 2$. To obtain a high multiplicity we need that $m_L^{(s)} + m_M^{(s)} = 1$ for $1 < s < n-k+2$, and $m_L^{(n-k+2)} + m_M^{(n-k+2)} = 2$. We achieve $m_L^{(s)} + m_M^{(s)} = 1$ for $s > 1$ by taking $t_1 = \cdots = t_{s_1} = 0$, $t_{s_1+1} = 1$, $t_{s_1+2} = \cdots = t_{s_1+s_2} = 0$, $t_{s_1+s_2+1} = 1$, $t_{s_1+s_2+2} = \cdots = t_{s_1+s_2+s_3} = 0$, $\dots$. The only possibility to get $m_L^{(s)} = m_M^{(s)} = 1$ is by taking $t_{s_1+\dots+s_p+1} = 0$: this gives $s = p + \sum_{i=1}^p s_i(k-1) + k - 1$. We therefore put $n-k+2 = p + \sum_{i=1}^p s_i(k-1) + r$ with $r < 1 + s_{p+1}(k-1)$. If $r \neq k-1$, the computation stops with $s = n-k+2$. If $r = k-1$, we go one step further, as then $m_L^{(n-k+2)} = m_M^{(n-k+2)} = 1$ and $m_R^{(n-k+2)} = 0$, but $m_L^{(n-k+3)} = m_M^{(n-k+3)} = 0$. So the computation always stops.

Suppose now that $E_L$ is the bad vertex; here $k = 2$ is allowed. In this case the computation need not end. We have $s_i = 0, 1$ for all $i$. As $m_M^{(p(k-1)+\sum t_i+1)} = 1$, we obtain $m_L^{(s)} + m_M^{(s)} = 1$ for $s > 1$ by taking $s_1 = \cdots = s_{t_1} = 0$, $s_{t_1+1} = 1$, $s_{t_1+2} = \cdots = s_{t_1+t_2} = 0$, $s_{t_1+t_2+1} = 1$, $s_{t_1+t_2+2} = \cdots = s_{t_1+t_2+t_3} = 0$, $\dots$. We need $m_L^{(s)} + m_M^{(s)} = 2$ for $s = l(n-k+2)$. This is possible if $p(k-1) + \sum_{i=1}^m t_i + 1 = l(n-k+2)$. In case $l = 1$ we then do not set $s_{\sum t_i+1} = 1$, but continue with $s_{\sum t_i+1} = s_{\sum t_i+2} = \cdots = 0$. This gives a shift in the indices of the $s_i$, which we do not compute here.

7.3. ${}^{II}A_n^{k,2}$**.** In this case the non-$(-2)$ vertices are $E_M$ and $E_R$. The sequence $(m_L^{(s)})$ is

$$(1^{k-1}, 0, 1^{k-1}, 0, \dots).$$

So $m_L^{(s)} = 1$ for $s \neq lk$ and $m_L^{(lk)} = 0$ for all $l$. As in the previous case the sequence $(m_M^{(s)})$ is

$$(1, 0^{k+t_1-2}, 1, 0^{k+t_2-2}, 1, 0^{k+t_3-2}, \dots) .$$

We have $E_R \cdot Z^{(1)} = -u_1$ with $u_1 = 0$ if and only if $E_R$ is a bad vertex. If $u_1 > 0$, then $E_R$ is not contained in the support of $Y_R^{(2)}$, which is the $A_{n-k}$ between $E_R$ and $E_0$. The sequence $(m_R^{(s)})$ is

$$(1, (1^{n-k}, 0)^{u_1}, 1, (1^{n-k}, 0)^{u_2}, 1, (1^{n-k}, 0)^{u_3}, \dots) .$$

The computation stops when $m_L^{(s)} + m_M^{(s)} = 0$, or when $m_R^{(s)} = 0$, except when $m_L^{(s)} + m_M^{(s)} = 2$ for that value of $s$. We achieve that $m_L^{(s)} + m_M^{(s)} = 1$ for $s > 1$ by taking $t_1 = 0$ and $t_i = 1$ for $i > 1$. It is possible to have $m_L^{(lk)} = m_M^{(lk)} = 1$ for some $l > 1$, while $m_L^{(pk)} + m_M^{(pk)} = 1$ for $p < l$, by setting $t_l = 0$. If $k > 2$, then $m_L^{(lk+1)} = m_M^{(lk+1)} = 0$, so the computation stops at that point. Therefore the computation stops when $m_R^{(s)}$=0, or if $s = lk$, in the next step. The computation never stops if $u_i = 0$ for all $i$. Note that in that case both $E_M$ and $E_R$ are bad vertices.

If $k = 2$, the situation is a bit different. The sequence $(m_L^{(s)})$ is $(1, 0, 1, 0, \dots)$, $(m_M^{(s)})$ is $(1, 0^{t_1}, 1, 0^{t_2}, 1, 0^{t_3}, \dots)$ and $(m_R^{(s)})$ is

$$(1, (1^{n-2}, 0)^{u_1}, 1, (1^{n-2}, 0)^{u_2}, 1, (1^{n-2}, 0)^{u_3}, \dots) .$$

We always take $t_1 = 0$, and $t_i \le 1$. By taking suitable consecutive $t_i$ equal to zero we can get $m_L^{(2s)} + m_M^{(2s)} = 2$, with this sum always equal to one for odd indices. It is possible that the computation never stops. If $n$ is odd, we need $u_{2l-1} = 0$ for all $l$, while the $u_{2l}$ may be arbitrary. If $n$ is even, then $u_i \le 1$. If $u_i = 0$ then also $u_{i+1} = 0$. If $n = 2$, we see no difference between $E_M$ and $E_R$, and indeed the sequences $(m_M^{(s)})$ and $(m_R^{(s)})$ are of the same shape.

7.4. $D_{2k+1}^{k+1,2}$**.** The configuration is connected to vertices $E_R$ and $E_L$. We claim that *the coefficient $z_L$ of $E_L$ in the fundamental cycle can be at most two.* We compute the fundamental cycle with $E_L$ as central vertex. The relevant information on the cycle $Y_{\Delta,R}^{(1)}$ is given in the entry for $RD_{2k+1}^{k+1,2}$ in Table 6. If the coefficient of $E_R$ is $s$, then the multiplicity of the vertex adjacent to $E_L$ is $m_L^{(1)} = \lfloor \frac{2k+1+s}{2} \rfloor$. We assume that $E_L \cdot Z^{(1)} = 1$. If $s = 2t+1$, then $Y_{\Delta,R}^{(2)} = \emptyset$ and the computation stops with $z_L = 2$ and $z_R = 2t+1$. If $s = 2t+2$, then $\Gamma_{\Delta,R}^{(2)}$ has only an $A_{2k}^{1,1}$-configuration between $E_L$ and $E_R$, so $E_R$ is not a bad vertex for $Y_{\Delta,R}^{(2)}$ and $m_L^{(2)} = 1$. As $\lfloor \frac{2k+1+s}{2} \rfloor = k+1+t \ge 3$, the computation again stops with $z_L = 2$. Depending on whether $E_R \cdot Z^{(1)} = 0$ or less, $z_R = 2t+3$ or $z_R = 2t+2$.

7.5. $D_{2k}^{k,2}$**.** In this case only one of the vertices $E_L$ and $E_R$ is bad. In the symmetric case $k = 2$ we assume that $E_R$ is the bad vertex. We compute as in the previous case with $E_L$ as central vertex. If the coefficient of $E_R$ in $Y_{\Delta,R}^{(1)}$ is $s$ (with $s > 1$ if and only if $E_R$ is bad), then $m_L^{(1)} = \lfloor \frac{2k+s}{2} \rfloor$. If $s = 2t-1$, then $m_L^{(2)} = 1$. If $k \ge 3$, then $\lfloor \frac{2k+s}{2} \rfloor = k+t-1 \ge k \ge 3$. For $k = 2$ we assumed $s > 1$, so $t > 1$ and again $k+t-1 \ge 3$. So the computation stops with $z_L = 2$, and $z_R = 2t-1$ or $z_R = 2t$. If $s = 2t$, then $Y_{\Delta,R}^{(2)} = \emptyset$ and the computation stops with $z_L = 2$ and $z_R = 2t$.

7.6. $A_n^{2,k,2}$**.** As in the cases ${}^{I}A_n^{2,k}$ and ${}^{II}A_n^{k,2}$ we compute with the rupture point in the $A_n^{2,k,2}$-configuration as central vertex $E_0$. The sequence $(m_L^{(s)})$ is

$$(1, (1^{k-2}, 0)^{s_1}, 1, (1^{k-2}, 0)^{s_2}, 1, (1^{k-2}, 0)^{s_3}, \dots)\ ,$$

the sequence $(m_M^{(s)})$ is

$$(1, 0^{k+t_1-2}, 1, 0^{k+t_2-2}, 1, 0^{k+t_3-2}, \dots)\ .$$

and finally $(m_R^{(s)})$ is

$$(1, (1^{n-k+1}, 0)^{u_1}, 1, (1^{n-k+1}, 0)^{u_2}, 1, (1^{n-k+1}, 0)^{u_3}, \dots)\ .$$

First suppose $E_M$ is a bad vertex, i.e., $t_1 = 0$. We may assume that $k > 2$. Then $E_L$ is not a bad vertex, $s_1 > 0$, except possibly if $n$ has the lowest possible value $2k-3$, when there is an arrowhead between $E_M$ and $E_L$ at $E_0$. In that case the chains from $E_0$ to $E_L$ and $E_R$ are equally long. As $n-k+1 = k-2$, not all three of $s_1$, $t_1$ and $u_1$ are zero, so upon relabeling we may assume also here that $s_1 > 0$. As in the case ${}^{I}A_n^{2,k}$ we find that the computation stops with the first 0 in the sequence $(m_R^{(s)})$, or in the step immediately after. It is however possible that there is no 0 in this sequence; this happens if $u_i = 0$ for all $i$.

If $t_1 > 0$, then $s_1 = 0$, and if $n = 2k-3$, also $u_1 = 0$. For most values of $s$ we will have $m_L^{(s)} + m_R^{(s)} = 2$, but we want that $m_L^{(s)} + m_R^{(s)} = 1$ for $s = p(k-1) + \sum_{i=1}^{p} t_i + 1$ for all $p \ge 1$. We determine on which places in the sequence $(m_L^{(s)})$ there are zeroes. Let $\sum_{j=1}^{i-1} s_j < r \le \sum_{j=1}^{i} s_j$. Then the $r$-th zero is on place $r(k-1)+i$. Similarly the $r$-th zero in the sequence $(m_R^{(s)})$ is on place $r(n-k+2)+i$, if $\sum_{j=1}^{i-1} u_j < r \le \sum_{j=1}^{i} u_j$.

If $k = 2$, we may upon relabeling assume that $t_1 > 0$. Then the same description holds. In particular, if $n = 1$, we have the sequences

$$(1, 0^{s_1}, 1, 0^{s_2}, \dots), \quad (1, 0^{t_1}, 1, 0^{t_2}, \dots), \text{ and } \quad (1, 0^{u_1}, 1, 0^{u_2}, \dots).$$

Once again we stress that we do not investigate, which values of $s_i$, $t_i$ and $u_i$ are possible.

7.7. **Multiplicity at most two.** In the previous subsections we have tried to make the multiplicity of the fundamental cycle at non-$(-2)$'s as large as possible. The computations above also tell us when the multiplicity does not exceed two. Now we make the conditions explicit in terms of the multiplicities of the other components of the graph, attached to the two non-$(-2)$'s. Let $E_a$ be one of these vertices. Then as before $\Gamma_a$ is the union of connected components, attached to $E_a$. Let $E_a + Y_a^{(1)}$ be the fundamental cycle on $\{E_a\} \cup \Gamma_a$, and denote by $n_a^{(1)}$ the sum of the multiplicities of $Y_a^{(1)}$ at the vertices of $\Gamma_a$, adjacent to $E_a$. At the stage of the computation of the fundamental cycle, when the multiplicity of $E_a$ has increased to 2, we need the fundamental cycle $E_a + Y_a^{(2)}$ on $\{E_a\} \cup \Gamma_a^{(2)}$, where $\Gamma_a^{(2)}$ is a connected component with vertices satisfying $E_i \cdot (E_a + Y_a^{(1)}) = 0$; then $E_a^{(2)}$ is the sum of the multiplicities of $Y_a^{(2)}$, adjacent to $E_a$.

7.7.1. $A_n^{1,1}$. As before we assume that $E_L$ is the bad vertex. The computation with $E_L$ as central vertex should stop at $s = 2$, so $E_L \cdot Z^{(1)} = 1$ and $E_L \cdot Z^{(2)} \le 0$. As the multiplicity of $E_R$ also should be two, we need $E_R \cdot Z^{(1)} = 0$. This gives us

$$\begin{aligned} n_L^{(1)} &= b_L, & n_R^{(1)} &= b_R - 1, \\ n_L^{(2)} &\le b_L - 2, & n_R^{(2)} &\le b_R - 1. \end{aligned}$$

7.7.2. ${}^{I}A_n^{2,k}$. First consider the case that $E_M$ is the bad vertex, so $t_1 = 0$ and $s_1 > 0$. If $s_1 > 1$, then the computation stops before the multiplicity $z_L$ becomes two, or $z_M$ becomes at least three. Therefore $s_1 = 1$. We have the sequences $(1, 1^{k-2}, 0, 1, 1^{k-2}, 0, \dots)$ and $(1, 0^{k-2}, 1, 0^{k+t_2-2}, 1, \dots)$. As $n - k + 1 \ge k$ we have $n - k + 2 \ge k + 1$. The condition that the multiplicities do not exceed two depend on $n$. If $n - k + 2 \le 2k - 2$, the computation always stops at $s = n - k + 2$. If $n - k + 2 = 2k - 1$, then we need $t_2 \ge 1$ and if $n - k + 2 \ge 2k$, then we $t_2 \ge 2$. Thus

$$\begin{aligned} n_L^{(1)} &= b_L - 2, & n_M^{(1)} &= b_M - k + 1, \\ n_L^{(2)} &\le b_L - 2, & n_M^{(2)} &\le \begin{cases} b_M - k + 1, & \text{if } n \le 3k - 4, \\ b_M - k, & \text{if } n = 3k - 3, \\ b_M - k - 1, & \text{if } n \ge 3k - 2. \end{cases} \end{aligned}$$

If $E_L$ is the bad vertex, we have $s_1 = 0$ and we need $t_1 = 1$. Furthermore $s_2 \ge 1$.

If $n - k + 2 = k + 1$, the computation stops at $s = n - k + 2$. Otherwise we need $s_2 > 1$. This gives

$$\begin{aligned} n_L^{(1)} &= b_L - 1, & n_R^{(1)} &= b_R - k, \\ n_L^{(2)} &\le \begin{cases} b_L - 2, & \text{if } n = 2k - 1, \\ b_L - 3, & \text{if } n \ge 2k, \end{cases} & n_R^{(2)} &\le b_R - k. \end{aligned}$$

7.7.3. ${}^{II}A_n^{k,2}$. If $u_1 > 0$, so $t_1 = 0$, the computation stops too early or the coefficient of $E_M$ becomes too high. We need $u_2 > 0$. The value of $t_2$ depends again on $n$. The results also hold for $k=2$.

$$n_M^{(1)} = b_M - k + 1, \qquad\qquad n_R^{(1)} = b_R - 1,$$

$$n_M^{(2)} \le \begin{cases} b_M - k + 1, & \text{if } n \le 3k-5, \\ b_M - k, & \text{if } n = 3k-4, \\ b_M - k - 1, & \text{if } n \ge 3k-3, \end{cases} \qquad n_R^{(2)} \le b_R - 2.$$

7.7.4. $D_{2k+1}^{k+1,2}$. In the notation of 7.4 we need that $s=2$ and $E_R \cdot Z^{(1)} < 0$. This gives us

$$n_L^{(1)} = b_L - k, \qquad n_R^{(1)} = b_R - 1,$$
$$n_L^{(2)} \le b_L - k, \qquad n_R^{(2)} \le b_R - 3.$$

7.7.5. $D_{2k}^{k,2}$. In this case $s \le 2$ and $z_R = 2$. We first assume $k > 2$. This gives two possibilities. If $s=1$ we obtain

$$n_L^{(1)} = b_L - k + 1, \qquad n_R^{(1)} = b_R - 2,$$
$$n_L^{(2)} \le b_L - k + 1, \qquad n_R^{(2)} \le b_R - 2$$

and for $s=2$

$$n_L^{(1)} = b_L - k, \qquad n_R^{(1)} = b_R - 1,$$
$$n_L^{(2)} \le b_L - k, \qquad n_R^{(2)} \le b_R - 2.$$

The last formula also works for the symmetric case $k=2$, if we assume that $E_R$ is the bad vertex.

7.7.6. $A_n^{2,k,2}$. We have to determine the conditions that at least two multiplicities become 2, whereas none may become 3. We argue as in the cases ${}^{I}A_n^{2,k}$ and ${}^{II}A_n^{k,2}$. If $E_R$ ($t_1 = 0$) is bad we may assume that $k > 2$. If $s_1 > 1$, the multiplicity of $E_L$ remains 1, which is seen by the absence of the entry for $n_L^{(2)}$:

$$n_L^{(1)} \le b_L - 3, \qquad n_M^{(1)} = b_M - k + 1, \qquad\qquad n_R^{(1)} = b_R - 1,$$

$$n_M^{(2)} \le \begin{cases} b_M - k + 1, & \text{if } n \le 3k-6, \\ b_M - k, & \text{if } n \ge 3k-5, \end{cases} \qquad n_R^{(2)} \le b_R - 2.$$

If $s_1 = 1$ and $u_1 < 0$ (so $n_R^{(1)} < b_R - 1$), then $n > 2k-3$; for $n = 2k-3$ one has, if necessary, to interchange $E_L$ and $E_R$. We get

$$n_L^{(1)} = b_L - 2, \qquad n_M^{(1)} = b_M - k + 1, \qquad\qquad n_R^{(1)} \le b_R - 2.$$

$$n_L^{(2)} \le b_L - 2, \qquad m_M^{(2)} \le \begin{cases} b_M - k + 1, & \text{if } n \le 3k-5, \\ b_M - k, & \text{if } n = 3k-4, \\ b_M - k - 1, & \text{if } n \ge 3k-3, \end{cases}$$

It is also possible that all three multiplicities are 2:

$$n_L^{(1)} = b_L - 2, \qquad n_M^{(1)} = b_M - k + 1, \qquad\qquad n_R^{(1)} = b_R - 1.$$

$$n_L^{(2)} \le b_L - 2, \qquad n_M^{(2)} \le \begin{cases} b_M - k + 1, & \text{if } n \le 3k-6, \\ b_M - k, & \text{if } n = 3k-5, \\ b_M - k - 1, & \text{if } n \ge 3k-4, \end{cases} \qquad n_R^{(2)} \le b_R - 2.$$

If $E_M$ is not bad, we allow that $k = 2$.

$$n_L^{(1)} = b_L - 1, \qquad n_M^{(1)} = b_M - k - 1, \qquad n_R^{(1)} = b_R - 1.$$
$$n_L^{(2)} \le b_L - 2 \qquad\qquad\qquad n_R^{(2)} \le b_R - 2.$$

Also now it is possible that all three multiplicities are 2:

$$n_L^{(1)} = b_L - 1, \qquad n_M^{(1)} = b_M - k, \qquad n_R^{(1)} = b_R - 1.$$
$$n_L^{(2)} \le \begin{cases} B_L - 2, & \text{if } n = 2k-3, \\ b_L - 3, & \text{if } n \ge 2k-2, \end{cases} \qquad n_M^{(2)} \le b_M - k, \qquad n_R^{(2)} \le b_R - 2.$$

## 8. Low degree

The classification of rational graphs of degree three was given by Artin [1], degree four by the author [14] and degree five by Tosun et al. [16]. In these cases there is at most one non-reduced non-$(-2)$, so the classification can be written using the results of Sections 2 and 6. For degree six one new case arises, with two non-reduced non-$(-2)$'s; here the results of Subsection 7.7 suffice, as we presently shall make explicit. For degree 7 one can use the same methods; we do not go into detail. Things become more complicated for degree 8, where possibility of three non-reduced non-$(-2)$'s appears. We classify the occurring graphs in this section.

8.1. **Degree six.** We start with the classification of graphs of canonical models. The ones with reduced fundamental cycle are given in Table 4. From it one can also infer the other possibilities: just replace some vertices with weight $-b$ with a vertex of weight $-3$ and multiplicity $b-2$, or the $(-6)$ by a $(-4)$ of multiplicity 2. We do not treat all cases, where there is only one non-$(-2)$ with higher multiplicity, but We give partial results for some cases and as example we list the complete classification in the case of highest multiplicity.

The new case in degree 6 is that there are two $(-3)$-vertices with multiplicity two in the fundamental cycle. The possible configurations are described in Section 7.7. We have to specialise to the case that the vertex weights are 3.

We write $C(m_1, m_2)$ for any combination of RDP-configurations realising the multiplicity sequence $(m_1, m_2)$, and $C(m_1, \le m_2)$ for configurations where the total second multiplicity is at most $m_2$. The notation $C(0,0)$ stands for the empty configuration. These combinations can be found from Table 5; e.g., $(3, \le 1)$ stands for $2A_1^1 + A_n^1$ $(n \ge 1)$, $A_5^3$, $A_6^3$, ${}^{I}D_k^2 + A_1^1$, $A_3^2 + A_1^1$, $A_4^2 + A_1^1$, ${}^{II}D_5^2 + A_1^1$, ${}^{II}D_6^3$, ${}^{II}D_7^3$ and $E_7^3$.

**Proposition 8.1.** *Suppose the graph of the RDP-resolution consist of two $(-3)$-vertices, both with multiplicity 2. Then they are connected by one of the RDP-configurations, listed in Table 8 together with the the other configurations at the left and the right vertex.*

**Proposition 8.2.** *Suppose the graph of the RDP-resolution consist of one $(-3)$-vertex, with multiplicity 4. The following combinations of RDP-configurations are possible.*

$$\begin{array}{lll} A_4^2 + 2A_3^1, & A_4^2 + A_7^2, & A_1^1 + A_6^2 + A_{\ge 3}^1, \\ {}^{II}D_5^2 + A_6^2, & A_1^1 + A_2^1 + A_8^2, & A_1^1 + A_2^1 + A_3^1 + A_{\ge 3}^1, \\ {}^{II}D_5^2 + A_2^1 + A_{\ge 3}^1, & A_1^1 + A_{10}^3, & A_1^1 + A_2^1 + A_7^2. \end{array}$$

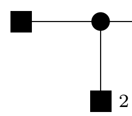
*Proof.* We argue as in the proof of Proposition 6.6. We first consider only RDP-configurations of type $A_n^1$ and $A_{2k}^2$. We need either $A_1^1$ or $A_4^2$. As $A_4^2$ gives the sequence $(2,1,1,2,0)$, we need the sequence $(2,2,2,0)$, so the configuration $2A_3^1$. If there is exactly one $A_1^1$, we further need $A_2^1$ or $A_6^2$. In the first case we can complete with $A_8^2$ or $A_3^1 + A_n^1$ with $n \ge 3$, in the second only with $A_n^1$, $n \ge 3$. Table 5 gives the possible equivalent configurations. □

Table 8.

| name | left | right |
|---|---|---|
| $A_n^{1,1}$ | $C(3,\le 1)$ | $C(2,\le 2)$ |
| ${}^{I}A_3^{2,2}$ | $C(2,\le 1)$ | $C(1,\le 1)$ |
| ${}^{I}A_{\ge 4}^{2,2}$, | $C(2,0)$ | $C(1,\le 1)$ |
| ${}^{I}A_5^{2,3}$ | $C(2,\le 1)$ | $C(0,0)$ |
| ${}^{I}A_{\ge 6}^{2,3}$ | $C(2,0)$ | $C(0,0)$ |
| ${}^{I}A_5^{2,3}$ | $C(1,\le 1)$ | $(1,\le 1)$ |
| ${}^{I}A_6^{2,3}$ | $C(1,\le 1)$ | $C(1,0)$ |
| ${}^{I}A_7^{2,4}$ | $C(1,\le 1)$ | $C(0,0)$ |
| ${}^{I}A_8^{2,4}$ | $C(1,\le 1)$ | $C(0,0)$ |
| ${}^{II}A_2^{2,2}$ | $C(2,\le 1)$ | $C(2,\le 1)$ |
| ${}^{II}A_{\ge 3}^{2,2}$ | $C(2,0)$ | $C(2,\le 1)$ |

| name | left | right |
|---|---|---|
| ${}^{II}A_4^{3,2}$ | $C(1,\le 1)$ | $C(2,\le 1)$ |
| ${}^{II}A_5^{3,2}$ | $C(1,0)$ | $C(2,\le 1)$ |
| ${}^{II}A_6^{4,2}$ | $C(0,0)$ | $C(2,\le 1)$ |
| ${}^{II}A_7^{4,2}$ | $C(0,0)$ | $C(2,\le 1)$ |
| $D_4^{2,2}$ | $C(1,\le 1)$ | $C(2,\le 1)$ |
| $D_5^{3,2}$ | $C(1,\le 1)$ | $C(2,0)$ |
| $D_6^{3,2}$ | $C(0,0)$ | $C(2,\le 1)$ |
| $D_6^{3,2}$ | $C(1,\le 1)$ | $C(1,\le 1)$ |
| $D_7^{4,2}$ | $C(0,0)$ | $C(2,0)$ |
| $D_8^{4,2}$ | $C(0,0)$ | $C(1,\le 1)$ |

Next we consider the case that the hypertree for the RDP-resolution has a $T$-joint. The smallest tree realising it looks as follows.




As drawn, the vertex $E_M$ has multiplicity two. The other cases are also possible, and occur in the classification, but they give basically the same graph.

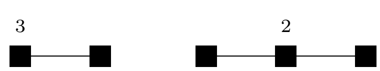
**Proposition 8.3.** *If the hypertree of the RDP-resolution has a $T$-joint and $E_M$ is the vertex of higher multiplicity, the configurations*

$$MA_3^{2,3,2}+C(1,\le 1),\ MA_4^{2,3,2}+C(1,\le 1),\ MA_{\ge 5}^{2,3,2}+C(1,0),\ MA_5^{2,4,2},\ MA_6^{2,4,2},\text{ and } MA_{\ge 7}^{2,4,2}$$

*can be attached to $E_M$; at $E_R$ an $A_n^1$ is possible and also at $E_L$ in the symmetric case of minimal $n=2k-3$. To $E_R$ of higher multiplicity the configurations $RA_n^{2,2,2}+C(2,\le 2)$ and $RA_n^{2,3,2}+C(2,\le 1)$ can be attached; at $E_L$ an $A_n^1$ is possible and also at $E_M$ in the case $k=2$. The last possibility is $LA_{\ge 2}^{2,2,2}+C(2,\le 1)$ with an optional $A_n^1$ at $E_R$.*

*Proof.* We use Table 7. The only thing to note is that we stop the computation earlier, at step two, so in the case $LA_{\ge 3}^{2,2,2}$ the multiplicity at $E_M$ does not reach the value $n+1$, but remains 3. □

For two other cases, with the following graphs for the RDP-resolution,




we only show how they can be realised, using configurations of type $LA_n^{1,1}$ for the connection to other $(-3)$'s, and configurations $C(m_1,\le m_2)$ and $C(m_1,m_2,\le m_3)$; as before this notation stands for any combination of configurations, realising a multiplicity sequence.

**Proposition 8.4.** *Suppose the graph of the RDP-resolution consist of two $(-3)$-vertices, one with multiplicity 3. This type can be realised by attaching to the curve of multiplicity 3 a combination $LA_0^{1,1}+C(3,3,\le 2)$, $LA_1^{1,1}+C(3,2,\le 2)$ or $LA_{\ge 2}^{1,1}+C(3,2,\le 1)$.*

*Two reduced $(-3)$'s with a $(-3)$ of multiplicity 2 in between can be realised by attaching, to the vertex of multiplicity 2,*

$$LA_0^{1,1}+LA_0^{1,1}+C(2,\le 2),\ LA_{\ge 1}^{1,1}+LA_0^{1,1}+C(2,\le 1),\ \text{or } LA_{\ge 1}^{1,1}+LA_{\ge 1}^{1,1}+C(2,0).$$

The three remaining cases are easier.

8.2. **Degree eight.** We consider here only the cases that there are three $(-3)$-vertices, all with multiplicity two in the fundamental cycle. Either all three are connected by a single $A_n^{2,k,2}$ configuration, or they form a chain. The first possibility is a special case of Section 7.7.6.

**Proposition 8.5.** *Suppose the graph of the RDP-resolution consist of three $(-3)$-vertices, all with multiplicity 2, connected by a single $A_n^{2,k,2}$ configuration. Then following values for $n$ and $k$ are possible, with the given other configurations at each vertex.*

| name | left | middle | right |
|---|---|---|---|
| $A_1^{2,2,2}$ | $C(2,\le 1)$ | $C(1,\le 1)$ | $C(2,\le 1)$ |
| $A_n^{2,2,2}$ | $C(2,0)$ | $C(1,\le 1)$ | $C(2,\le 1)$ |
| $A_3^{2,3,2}$ | $C(2,\le 1)$ | $C(0,0)$ | $C(2,\le 1)$ |
| $A_n^{2,3,2}$ | $C(2,0)$ | $C(0,0)$ | $C(2,\le 1)$ |
| $A_3^{2,3,2}$ | $C(1,\le 1)$ | $C(1,\le 1)$ | $C(2,\le 1)$ |
| $A_n^{2,3,2}$ | $C(1,\le 1)$ | $C(1,0)$ | $C(2,\le 1)$ |
| $A_5^{2,4,2}$ | $C(1,\le 1)$ | $C(0,0)$ | $C(2,\le 1)$ |
| $A_6^{2,4,2}$ | $C(1,\le 1)$ | $C(0,0)$ | $C(2,\le 1)$ |

Finally we consider a chain of non-reduced $(-3)$'s. Let the vertices be called $E_L$, $E_M$ and $E_R$. We compute the fundamental cycle as described in Section 5 with $E_M$ as central vertex. The complement $\Gamma\setminus\{E_M\}$ decomposes into the connected components $\Gamma_L$ and $\Gamma_R$, containing respectively $E_L$ and $E_R$, and the union $\Gamma_M$ of the remaining components. We consider the multiplicity sequences $(m_L^{(s)})=(m_L^{(1)},m_L^{(2)})$, $(m_M^{(1)},m_M^{(2)})$ and $(m_R^{(1)},m_R^{(2)})$. We need that $m_L^{(1)}+m_M^{(1)}+m_R^{(1)}=4$ and $m_L^{(2)}+m_M^{(2)}+m_R^{(2)}\le 2$. Upon interchanging $E_L$ and $E_R$ we may assume that $m_L^{(1)}\ge m_R^{(1)}$.

**Proposition 8.6.** *For a chain of three $(-3)$'s with multiplicity 2 in the fundamental cycle the following multiplicity sequences are possible, when computing with the middle vertex as central vertex.*

| $(m_L^{(s)})$ | $(m_M^{(s)})$ | $(m_R^{(s)})$ |
|---|---|---|
| $(3,\le 1)$ | $(0,0)$ | $(1,1)$ |
| $(2,\le 2)$ | $(0,0)$ | $(2,0)$ |
| $(2,1)$ | $(0,0)$ | $(2,1)$ |

| $(m_L^{(s)})$ | $(m_M^{(s)})$ | $(m_R^{(s)})$ |
|---|---|---|
| $(2,0)$ | $(1,\le 1)$ | $(1,1)$ |
| $(2,1)$ | $(1,0)$ | $(1,1)$ |
| $(1,1)$ | $(2,0)$ | $(1,1)$ |

The configurations giving the required values for $(m_M^{(1)},m_M^{(2)})$ can be read off from Table 5. We have $C(1,0)=A_1^1$, $C(1,1)=A_n^1$, $n>1$, and $C(2,0)$ can be $2A_1^1$, $A_3^2$ or ${}^ID_k^2$. For $(m_L^{(1)},m_L^{(2)})$ and $(m_R^{(1)},m_R^{(2)})$ we use Table 6. It suffices to describe the possible configurations for $E_L$. The result is given in Table 9.

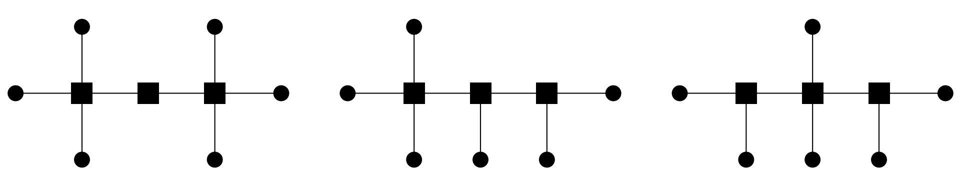
We have to distinguish cases depending on whether $E_L$ is a bad vertex for $\Gamma_L\cup\{E_M\}$ or not. If bad, then the multiplicity of $E_L$ in $Y_L^{(1)}$ is two, and the multiplicity does not increase in the second step. This means that $E_i\cdot Z^{(1)}<0$ for some vertex $E_i$ on the chain between $E_L$ and $E_M$.

This is an extra condition, which excludes a number of cases from Table 6. If $E_L$ is not bad, then its multiplicity in $Y_L^{(1)}$ is one, and $E_i \cdot Z^{(1)} = 0$ for all vertices $E_i$ on the chain between $E_L$ and $E_M$, including $E_L$. In this case $m_L^{(2)}) \geq 1$.

Table 9.

| $(m_L^{(1)}, m_L^{(2)})$ | $E_L$ bad | $E_L$ not bad |
|---|---|---|
| $(1,1)$ | | $LA_n^{1,1} + C(2, \leq 2)$ |
| $(2,0)$ | $LA_n^{1,1} + C(3, \leq 1)$ | |
| $(2,1)$ | $M^{II}A_2^{2,2} + C(2, \leq 1)$<br>$M^{II}A_3^{2,2} + C(2,0)$<br>$M^{II}A_4^{3,2} + C(1, \leq 1)$<br>$M^{II}A_5^{3,2} + C(1,0)$<br>$M^{II}A_6^{4,2} + C(0,0)$<br>$M^{II}A_7^{4,2} + C(0,0)$ | $L^IA_3^{2,2} + C(1, \leq 1)$<br>$LD_4^{2,2} + C(1, \leq 1)$<br>$M^IA_5^{2,3} + C(0,0)$<br>$LD_6^{3,2} + C(0,0)$ |
| $(2,2)$ | $R^{II}A_{\geq 3}^{2,2} + C(2, \leq 2)$<br>$LD_5^{2,2} + C(1, \leq 1)$<br>$LD_7^{3,2} + C(0,0)$ | $L^IA_{\geq 4}^{2,2} + C(1, \leq 1)$<br>$M^IA_{\geq 6}^{2,3} + C(0,0)$ |
| $(3,0)$ | $L^IA_3^{2,2} + C(2,0)$<br>$LD_4^{2,2} + C(2, \leq 1)$<br>$M^IA_5^{2,3} + C(1,0)$<br>$LD_6^{3,2} + C(1, \leq 1)$<br>$M^IA_7^{2,4} + C(0,0)$<br>$LD_8^{4,2} + C(0,0)$ | |
| $(3,1)$ | $L^IA_{\geq 4}^{2,2} + C(2,0)$<br>$R^{II}A_4^{3,2} + C(2,0)$<br>$RD_5^{3,2} + C(2,0)$<br>$M^IA_6^{2,3} + C(1,0)$<br>$M^IA_8^{2,4} + C(0,0)$ | $L^IA_5^{2,3} + C(1, \leq 1)$<br>$RD_6^{3,2} + C(1, \leq 1)$ |

We give the graphs for the simplest ways to realise a chain of three $(-3)$'s with multiplicity 2, depending on $E_L$ or $E_R$ being bad. Again it would be interesting to know whether these singularities have the same format.



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Matematiska vetenskaper, Göteborgs universitet och Chalmers tekniska högskola, 41296 Göteborg, Sweden

*E-mail address*: `stevens@chalmers.se`

**Journal *of* Singularities**
**Volume 7 (2013), 134-156**

received 15 March 2013
in revised form 17 June 2013
DOI: 10.5427/jsing.2013.7h

# Hodge-theoretic splitting mechanisms for projective maps

Mark Andrea A. de Cataldo

**Abstract**

According to the decomposition and relative hard Lefschetz theorems, given a projective map of complex quasi projective algebraic varieties and a relatively ample line bundle, the rational intersection cohomology groups of the domain of the map split into various direct summands. While the summands are canonical, the splitting is certainly not, as the choice of the line bundle yields at least three different splittings by means of three mechanisms in a triangulated category introduced by Deligne. It is known that these three choices yield splittings of mixed Hodge structures. In this paper, we use the relative hard Lefschetz theorem and elementary linear algebra to construct five distinct splittings, two of which seem to be new, and to prove that they are splittings of mixed Hodge structures.

## Contents



# 1 Introduction and main theorem

Let $f : X \to Y$ be a projective map of complex quasi projective varieties, let

$$H := \oplus_{d\geq 0} IH^d(X, \mathbb{Q})$$

be the total intersection cohomology rational vector space of $X$ and let $\eta \in H^2(X, \mathbb{Q})$ be the first Chern class of an $f$-ample line bundle on $X$. We refer to the survey [6] for the background concerning perverse sheaves and the decomposition and relative hard Lefscehtz theorems etc. that we use.

The map $f$ endows $H$ with the perverse Leray filtration $P$. The graded objects

$$H_p := P_pH/P_{p-1}H$$

are non-trivial only in a certain interval $[-r, r]$, with $r = r(f) \in \mathbb{Z}_{\geq 0}$. We thus obtain the two objects $\mathbb{H} := (H, P)$ and $\mathbb{H}_* = \oplus_p(H_p, T[-p])$ ($T[-p]$ = the trivial filtration translated to position $p$) in $\mathscr{V}_{\mathbb{Q}}\mathscr{F}$, the filtered category of finite dimensional rational vector spaces. Obviously, every filtration on a vector space by vector subspaces splits and we have a good (= inducing the identity on the graded pieces) isomorphisms $\varphi : \mathbb{H}_* \cong \mathbb{H}$ in $\mathscr{V}_{\mathbb{Q}}\mathscr{F}$.

The vector space $H$ underlies a natural mixed Hodge structure (MHS). The subspaces $P_pH$ are mixed Hodge substructures (MHSS) so that the graded objects $H_p$ are endowed with a natural MHS. Let $\mathscr{M}HS$ be the Abelian category of rational mixed Hodge structures. It is natural to ask whether there are good-isomorphisms $\varphi : \mathbb{H}_* \cong \mathbb{H}$ in $\mathscr{M}HS\mathscr{F}$, the filtered category of mixed Hodge structures. In English, do we have splittings $\varphi : \oplus_p H_p \cong H$ such that the component $H_p \to H$ is a map of MHS so that, in particular, the image is a MHSS?

In this paper, we list five distinct such mixed-Hodge theoretic good splittings. They are built by using the $f$-ample $\eta$ and they depend on it (see Theorem 1.1.1):

$$\omega_{\mathrm{I}}(\eta),\ \omega_{\mathrm{II}}(\eta),\ \phi_{\mathrm{I}}(\eta),\ \phi_{\mathrm{II}}(\eta),\ \phi_{\mathrm{III}}(\eta)\ :\quad \mathbb{H}_* \xrightarrow{\ \cong\ } \mathbb{H} \qquad \text{in } \mathscr{M}HS\mathscr{F} \tag{1}$$

The key to our approach is the relative hard Lefschetz theorem (RHL). The cup product with $\eta$ induces an arrow $\eta : \mathbb{H} \to \mathbb{H}[2](1)$ in $\mathscr{M}HS\mathscr{F}$. This means that $\eta : H \to H(1)$ (Tate shift (1)) is such that $\eta : P_pH \to P_{p+2}H(1)$ (translation of filtration [2] and Tate shift (1)). RHL yields isomorphisms in $\mathscr{M}HS$:

$$\eta^k\ : H_{-k} \xrightarrow{\ \cong\ } H_k(k), \qquad \forall\, k \geq 0. \tag{2}$$

Our main technical result, which in fact is proved in an elementary way, is as follows: let $\mathscr{A}$ be an Abelian category with shift functors $(n)$ and let $(\mathbb{V}, e)$ be a pair where $e : \mathbb{V} \to \mathbb{V}[2](1)$ is an arrow in the filtered category $\mathscr{A}\mathscr{F}$ inducing isomorphisms $e^k : V_{-k} \cong V_k(k)$, for every $k \geq 0$; then there is a natural isomorphism $\omega_{\mathrm{I}}(e) : \mathbb{V}_* \cong \mathbb{V}$ in $\mathscr{A}\mathscr{F}$.

With this result in hand, we easily verify that we can construct the remaining four splittings within $\mathscr{A}\mathscr{F}$. We then set $\mathscr{A} = \mathscr{M}HS$ and deduce (1). Let us stress again, that we use RHL in an essential way and that the point made in this paper is that once you have this deep result, the splittings (1) stem from elementary linear algebra considerations.

The construction of the three splittings of type $\phi$ is borrowed from [8]. However, it seems that [8] only yields $\phi$-type splittings in $\mathscr{V}_{\mathbb{Q}}\mathscr{F}$, i.e., not necessarily in its refinement $\mathscr{M}HS\mathscr{F}$. By coupling [8] with the theory of mixed Hodge modules, one can indeed prove that the splittings of type $\phi$ take place in $\mathscr{M}HS\mathscr{F}$. By way of contrast, as pointed out above, the constructions of this paper are based on the elementary construction of $\omega_{\mathrm{I}}(e)$ in $\mathscr{A}\mathscr{F}$.

The fact that $\phi_{\mathrm{I}}(\eta)$ is mixed-Hodge theoretic had been proved in [2, 4] (projective and quasi projective case, respectively) by using the properties of the cup product, of Poincaré duality and geometric descriptions of the perverse Leray filtration $P$ on $H$ associated with the map $f$. The proof that $\phi_{\mathrm{I}}(\eta)$ is an isomorphism in $\mathscr{M}HS\mathscr{F}$ that we give here is different since it does not use the aforementioned special features of the geometric situation.

The simple Examples 2.6.4 and 2.6.5 show that the five splittings (1) are, in general, pairwise distinct.

There is a natural condition, the existence of an $e$-good splitting, under which the five splittings coincide; see Definition 2.4.5 and Proposition 2.6.3.

In the paper [7], we proved the following result (auxiliary to the main result of [7]): the Hitchin fibration $f : X \to Y$ for the groups $GL_2(\mathbb{C})$, $SL_2(\mathbb{C})$ and $PGL_2(\mathbb{C})$ associated with any compact Riemann surface of genus $g \geq 2$ and with Higgs bundles of odd degree, presents a natural $f$-ample line bundle $\alpha$ and, in the terminology of the present paper, the splitting $\phi_{\mathrm{I}}(\alpha)$ is $\alpha$-good ([7] shows that (55) holds for $\phi_{\mathrm{I}}(\alpha)$). In particular, in this case, the five splittings coincide.

**Acknowledgments.** I thank Luca Migliorini for useful conversations. This paper was written during my wonderful stay at the Department of Mathematics at the University of Michigan, Ann Arbor, as Frederick W. and Lois B. Gehring Visiting Professor of Mathematics in the Fall of 2011. I thank the Department of Mathematics of the University of Michigan, especially Mircea Mustaţă, for their kind and generous hospitality and gratefully acknowledge partial financial support from the N.S.F., the Frederick W. and Lois B. Gehring Visiting Professorship fund and the David and Lucille Packard foundation.

## 1.1 The main theorem

A splitting $\mathbb{H}_* \cong \mathbb{H}$ in $\mathscr{V}_{\mathbb{Q}}\mathscr{F}$ acquires significance only if we can describe $H_*$. This is the content of the decomposition theorem of Beilinson, Bernstein, Deligne and Gabber (see the survey [6]) which implies the highly non-trivial fact that, up to a simple renumbering of cohomological degrees, we have that $H_p = H(Y, {}^p\mathcal{H}^p(Rf_* IC_X))$, where ${}^p\mathcal{H}^p(Rf_* IC_X)$ is the $p$-th perverse cohomology sheaf of the push-forward $Rf_* IC_X$ of the intersection cohomology complex $IC_X$ of $X$ ($= \mathbb{Q}_X[\dim_{\mathbb{C}} X]$, if $X$ is nonsingular).

Let us briefly discuss how the cohomology groups of these perverse sheaves split according to the decomposition by supports and to the primitive Lefschetz decomposition coming from RHL. Let us start with the one by supports, i.e., the $Z$'s appearing in what follows: each perverse sheaf $\mathcal{H}^p := {}^p\mathcal{H}^p(Rf_* IC_X)$ is a semi-simple perverse sheaf and decomposes canonically by taking supports: $\mathcal{H}^p = \oplus_Z \mathcal{H}^p_Z$, where the sum is finite and the summands are the intersection cohomology complexes with suitable semisimple local coefficients of suitable closed integral subvarieties $Z$ of $Y$.

The RHL induces the primitive Lefschetz decompositions (PLD) of $\mathcal{H}^p$: recall that $p \in [-r, r]$; let $i \geq 0$; set $\mathcal{P}^\eta_{-i,Z} = \mathrm{Ker}\,\{\eta^{i+1} : \mathcal{H}^{-i}_Z \to \mathcal{H}^{i+2}_Z\}$ and, for $0 \leq j \leq i$, set $\mathcal{P}^\eta_{ij,Z} := \mathrm{Im}\,\{\eta^j : \mathcal{P}^\eta_{-i,Z} \to \mathcal{H}^{-i+2j}\}$; then the PLD reads as $\mathcal{H}^p = \oplus_{-i+2j=p} \oplus_Z \mathcal{P}^\eta_{ij,Z}$ (sum subject to $0 \leq j \leq i$). Set $P^\eta_{-i}(-j)_Z := H(Y, \mathcal{P}^\eta_{ij,Z})$; it is a MHSS of $H_{-i+2j}$. Note that $\eta$ induces isomorphisms $\eta^j : (P^\eta_{-i}(0)_Z)(-j) \cong P^\eta_{-i}(-j)_Z$ of MHS. By combining the decomposition by supports with the primitive one, we obtain the splitting $\oplus_{i,j,Z} P^\eta_{-i}(-j)_Z \cong H_*$ which, in view of [1, 2, 4, 5], is a splitting of MHS. We set $\mathbb{P}^\eta_p := (\oplus_{-i+2j=p,Z} P^\eta_{-i}(-j)_Z, T[-p])$ (sum subject to $0 \leq j \leq i$; trivial filtration translated to position $p$) which is an object in $\mathscr{MHSF}$.

By taking into account these refined splittings, we have the corresponding refinements of (1). We note that everything holds just as well, with the same proofs, for intersection cohomology with compact supports.

The main result of this paper is the following

**Theorem 1.1.1** *Let $f: X \to Y$, $\eta$ and $\mathbb{H}$ be as above. There are five distinguished good splittings* (1) *in $\mathscr{MHSF}$:*

$$\bigoplus_{0 \leq j \leq i,\, Z} \mathbb{P}^{\eta}_{-i}(-j)_Z \xrightarrow{\cong} \mathbb{H}_*. \tag{3}$$

*Similarly, for intersection cohomology with compact supports.*

*Proof.* We first work in the abstract setting outlined in §2.1 of the filtered category $\mathscr{AF}$ of an Abelian category $\mathscr{A}$ endowed with a shift functor. In this context, we work with an object $\mathbb{V}$ and an arrow $e: \mathbb{V} \to \mathbb{V}[2](1)$ subject to the HL condition (32).

The splitting $\omega_{\mathrm{II}}(e)$ is obtained by the "dual" procedure as follows. Let $\mathscr{A}^o$ be the category opposite to $\mathscr{A}$; it is Abelian and can be endowed with a shift functor coming from the given one in $\mathscr{A}$. We then have that $e^o: \mathbb{V}^o \to \mathbb{V}^o[2]^o(1)^o$ in $\mathscr{A}^o\mathscr{F}$ satisifes the corresponding HL condition (32). We thus obtain $\omega_{\mathrm{I}}(e^o): \mathbb{V}^o_* \cong \mathbb{V}^o$. We set $\omega_{\mathrm{II}} := ((\omega_{\mathrm{I}}(e^o))^o)^{-1}$.

The splitting $\phi_{\mathrm{I}}(e)$ is constructed in §2.4. The proof is parallel to [8], §2 with the following two changes: (i) instead of using the existence of a splitting arising from [8], §1, which is proved using some basic features of $t$-categories, features that $\mathscr{AF}$ does not present, we use the existence of either of the splittings $\omega(e)$ established above; (ii) we adapt the proof of [8], Lemma 2.1, which again takes place in the context of a $t$-category, to the context of $\mathscr{AF}$.

The splitting $\phi_{\mathrm{II}}(e)$ is obtained in §2.5 by following the procedure "dual" to the one followed to produce $\phi_{\mathrm{I}}(e)$.

Finally, $\phi_{\mathrm{III}}(e)$, which necessitates that we work with a $\mathbb{Q}$-category (= the Hom-sets are rational vector spaces), is constructed in §2.6 by adapting the corresponding construction in [8], §3. This construction is self-dual.

We now specialize to $\mathscr{A} := \mathscr{MHS}$, with shift functor given by the Tate shift and to $(\mathbb{V}, e) := (\mathbb{H}, \eta)$ and conclude, due to the fact that the HL condition (32) is met by the RHL (2). $\square$

# 2 The five splittings

## 2.1 Filtered category associated with an Abelian category

Let $\mathscr{A}$ be an Abelian category whose elements we denote $V, W$, etc. For ease of exposition only, in this paper we make heavy use of the language of sets and elements.

A filtration $F$ on $V$ is a finite increasing filtration, i.e., an increasing sequence of subobjects

$$\ldots \subseteq F_pV \subseteq F_{p+1}V \subseteq \ldots \subseteq$$

of $V$ such that $F_pV = 0$ for $p \ll 0$ and $F_pV = V$ for $p \gg 0$. We set $\mathrm{Gr^F_p V} := F_pV/F_{i-1}V$. We denote by $T$ the trivial filtration on $V$: $T_{-1}V = 0 \subseteq T_0V = V$. Given $n \in \mathbb{Z}$, we denote by $F[n]$ the $n$-th translate of $F$: $F[n]_pV := F_{n+p}V$, so $\mathrm{Gr}^{F[n]}_pV = \mathrm{Gr}^F_{n+p}V$; for example $T[-p]$ is the trivial filtration in position $p$.

Given a pair $\mathbb{V} := (V, F)$ and a subquotient $U$ of $V$, the filtration $F$ on $V$ induces a filtration on $U$, which we still denote by $F$; for example $(\mathrm{Gr}^F_pV, F) = (\mathrm{Gr}^F_pV, T[-p])$. In particular, for every $p \leq q \in \mathbb{Z}$, we have the following pairs associated with $\mathbb{V}$:

$$\mathbb{V}_{\leq p} := \quad (F_p V, F), \qquad \mathbb{V}_{\geq p} := \quad (V/F_{p-1}V, F), \tag{4}$$

$$\mathbb{V}_p := \quad (\mathrm{Gr}^F_p V, F), \qquad \mathbb{V}_{[p,q]} := \quad (F_q V/F_{p-1}V, F).$$

We say that $\mathbb{V}$ has type $[a, b]$, with $a \leq b$, if $\mathrm{Gr}^F_p V = 0$ for every $p \notin [a, b]$.

Given $(V, F)$ and $(W, F)$, an arrow $\mathfrak{l} : V \to W$ in $\mathscr{A}$ is said to be a filtered arrow if it respects the given filtrations, i.e., $\mathfrak{l}$ maps $F_p V$ to $F_p W$, for every $p$.

The filtered category $\mathscr{A}F$ associated with $\mathscr{A}$ is the category with objects the pairs $\mathbb{V} = (V, F)$ and arrows the filtered arrows. In particular, an arrow $\mathfrak{l} : \mathbb{V} \to \mathbb{W}$ induces arrows on the objects listed in (4), e.g., $\mathfrak{l}_i : \mathbb{V}_i \to \mathbb{W}_i$. An arrow $\mathfrak{l}$ in $\mathscr{A}\mathscr{F}$ is an isomorphism if and only if $\mathfrak{l}_i$ is an isomorphism for every $i \in \mathbb{Z}$.

We have functors $[n] : \mathscr{A}\mathscr{F} \to \mathscr{A}\mathscr{F}$, $(V, F) = \mathbb{V} \mapsto \mathbb{V}[n] := (V, F[n])$, etc.

We have the following graded-type objects, in $\mathscr{A}$, $\mathscr{A}\mathscr{F}$ and $\mathscr{A}\mathscr{F}$, respectively:

$$V_* := \oplus_p V_p, \qquad \mathbb{V}_* := \bigoplus_p \mathbb{V}_p, \qquad \widetilde{\mathbb{V}}_* := \bigoplus_p (V_p, T[-p]) \,. \tag{5}$$

We say that $\mathbb{V}$ splits in $\mathscr{A}\mathscr{F}$ it there is an isomorphism in $\mathscr{A}\mathscr{F}$:

$$\varphi \, : \, \mathbb{V}_* \xrightarrow{\cong} \mathbb{V}. \tag{6}$$

We say that a splitting $\varphi$ is good if, for every $p$, the induced map $\varphi_p : V_p \cong V_p$ is the identity.

**Remark 2.1.1** If $\mathbb{V}$ splits, then there is a good splitting: let $\varphi_p : \mathbb{V}_p \cong \mathbb{V}_p$ be the induced isomorphisms and replace $\varphi$ with $\varphi \circ (\sum_p \varphi_p^{-1})$.

The category $\mathscr{A}\mathscr{F}$ is pre-Abelian (additive with kernels and cokernels), hence pseudo-Abelian (every idempotent has a kernel). In particular, given an idempotent $\pi : \mathbb{V} \to \mathbb{V}$, $\pi^2 = \pi$, we have canonical splittings in $\mathscr{A}\mathscr{F}$:

$$\mathbb{V} \; = \; \mathrm{Ker}\,(\mathrm{id} - \pi) \oplus \mathrm{Ker}\,\pi \; = \; \mathrm{Im}\,\pi \oplus \mathrm{Ker}\,\pi. \tag{7}$$

The arrow $\iota : (V, T) \to (V, T)[1]$ induced by the identity is such that the induced arrow $\mathrm{Coim}\,\iota \to \mathrm{Im}\,\iota$ is not an isomorphism so that $\mathscr{A}\mathscr{F}$ is not Abelian.

**Example 2.1.2** The example we have in mind is the one where $\mathscr{A}$ is the Abelian category $\mathscr{M}HS$ of integral (or rational) mixed Hodge structures (MHS) where the arrows are the maps that respect the weight and Hodge filtrations. The Tate shift functor, denoted by (1) is such that if $\mathbb{Z}$ is the pure Hodge structure with weight zero and type $(0, 0)$, then $\mathbb{Z}(1)$ is the pure Hodge structure with weight $-2$ and type $(-1, -1)$. Note that, for example, the cup product with the first Chern class $L$ of a line bundle on a complex algebraic variety $X$ induces, for every $k \geq 0$, a map $L : IH^k(X, \mathbb{Z}) \to IH^{k+2}(X, \mathbb{Z})(1)$. An element $\mathbb{M} = (M, F)$ of $\mathscr{M}HS\mathscr{F}$ is a MHS $M$ (with its weight and Hodge filtrations) equipped with an *additional* filtration $F$ for which $F_p M$ is a mixed Hodge substructure (MHSS) of $M$ for every $p$.

Let $(1) : \mathscr{A} \to \mathscr{A}$, $V \mapsto V(1)$, $\mathfrak{l} \mapsto \mathfrak{l}(1)$, be an additive and exact autoequivalence and, for $m \in \mathbb{Z}$, denote by $(m)$ its $m$-th iterate, called the $m$-shift functor.

By exactness, the shift functors lift to functors (also called shift functors):

$$(m) \,:\, \mathscr{A}F \longrightarrow \mathscr{A}F, \quad \mathbb{V} = (V,F) \longmapsto (V(m),F) =: \mathbb{V}(m), \quad \mathfrak{l} \longmapsto \mathfrak{l}(m), \tag{8}$$

and we have

$$(\mathbb{V}(m))_p = \mathbb{V}_p(m), \qquad \mathrm{Gr}_p^F(\mathfrak{l}(m)) = (\mathrm{Gr}_p^F\,\mathfrak{l})(m). \tag{9}$$

The shift functors commute with the tanslation functors, so that we can write $\mathbb{V}[n](m)$ unambiguously. We have $(\mathbb{V}[n](m))_p = \mathbb{V}_{n+p}(m)$, etc.

For every $p \in \mathbb{Z}$, an arrow $\mathfrak{l} : \mathbb{V} \to \mathbb{W}[n](m)$ induces arrows in $\mathscr{A}$:

$$\mathfrak{l}_p \,:\, V_p \longrightarrow W_{n+p}(m), \tag{10}$$

where it is understood that:

$$W_{n+p}(m) \,=\, \mathrm{Gr}^F_{n+p}(m) \,=\, \mathrm{Gr}_p^{F[n]}W(m). \tag{11}$$

If $\mathbb{V}$ has type $[a,b]$, then we have canonical arrows:

$$\mathbb{V}_a \xrightarrow{i_a} \mathbb{V} \xrightarrow{p_b} \mathbb{V}_b, \tag{12}$$

first inclusion and last quotient, with compositum $\delta_{ab}\mathrm{Id}$.

An arrow $\mathfrak{l}[n](m)$ obtained from an arrow $\mathfrak{l} : \mathbb{V} \to \mathbb{W}$ by shift/translation is simply denoted by $\mathfrak{l} : \mathbb{V}[n](m) \to \mathbb{W}[n](m)$ (e.g., the arrow $\mathfrak{l}_a^{-1}$ in (21) is really $\mathfrak{l}_a^{-1}(-m)$).

Let $\mathbb{V}$ and $\mathbb{W}$ be in $\mathscr{A}\mathscr{F}$, let $m,n \in \mathbb{Z}$, let $\mathfrak{l} : \mathbb{V}_* \to \mathbb{W}_*[n](m)$ be an arrow in $\mathscr{A}\mathscr{F}$ and let $\mathfrak{l}_{pq} : \mathbb{V}_p \to \mathbb{W}_q[n](m)$ be the $(p,q)$-th component of $\mathfrak{l}$. We define the degree $d \in \mathbb{Z}$ homogeneous part $\mathfrak{l}^{\{d\}}$ of $\mathfrak{l}$ by setting:

$$\mathfrak{l}^{\{d\}} := \sum_{q-p=d} \mathfrak{l}_{pq}, \qquad \left(\mathfrak{l} = \sum_d \mathfrak{l}^{\{d\}}\right). \tag{13}$$

Since $\mathbb{V}_p = (V_p, T[-p])$ and $\mathbb{W}_q[n](m) = (W_q(m), T[-q+n])$, we must have

$$\mathfrak{l}^{\{d\}} = 0, \quad \forall d \geq n+1. \tag{14}$$

Let $\mathscr{A}^o$ denote the Abelian category opposite to $\mathscr{A}$.

**Remark 2.1.3** Let $\mathscr{A} = \mathscr{V}_{\mathbb{Q}}$ be the category of finite dimensional rational vector spaces and linear maps. The natural contravariant functor $\mathscr{V}_{\mathbb{Q}} \to \mathscr{V}_{\mathbb{Q}}^o$ can be identified with taking dual vector spaces and transposition of linear maps. Similarly, if we take $\mathscr{A} = \mathscr{M}HS$. This observation may make what follows more down-to-earth and the computations of explicit examples easier.

We have the exact anti-equivalence $(-)^o : \mathscr{A} \to \mathscr{A}^o$, $(V \xrightarrow{f} W) \longmapsto (V^o \xleftarrow{f^o} W^o)$ whose second iterate is the identity functor. We endow $\mathscr{A}^o$ with the additive and exact shift functors $(m)^o$: $V^o \mapsto V^o(m^o) := (V(-m))^o$.

A filtered object $\mathbb{V} = (V, F)$ in $\mathscr{A}\mathscr{F}$ gives rise to a filtered object $\mathbb{V}^o = (V^o, F^o)$ in $\mathscr{A}^o\mathscr{F}$ by setting:

$$F_i^o V^o = (V^o)_{\leq i} := (V_{\geq -i})^o. \tag{15}$$

Contemplation of the following diagram may be useful:

$$\begin{tikzcd}
V_{\leq i-1} \arrow[r, "\text{mono}"] \arrow[dr] & V_{\leq i} \arrow[d] & (V_{\leq i-1})^o & (V_{\leq i})^o \arrow[l, "\text{epi}"'] \\
& V \arrow[d] & & (V)^o \arrow[u] \arrow[ul] \\
& V_{\geq i+1} & V_{\geq i} \arrow[l, "\text{epi}"'] & (V_{\geq i+1})^o \arrow[u] \arrow[r, "\text{mono}"] & (V_{\geq i})^o.
\end{tikzcd} \tag{16}$$

Clearly, $(V^o)_i = (V_{-i})^o$ and we set $F^o[n] := (F[-n])^o$.

We obtain an anti-equivalence $(-)^o : \mathscr{A}\mathscr{F} \to \mathscr{A}^o\mathscr{F}$ whose second iterate is the identity functor. The anti-equivalence $(-)^o$ is anti-compatible with translations, shifts and taking graded pieces, etc., for example:

$$\mathbb{V}^o[n]^o(m)^o \;=\; (\mathbb{V}[-n](-m))^o. \tag{17}$$

An arrow $\mathfrak{l} : \mathbb{V} \to \mathbb{W}[n](m)$ in $\mathscr{A}\mathscr{F}$ yields the arrow $\mathfrak{l}^o : \mathbb{W}^o \to \mathbb{V}^o[n]^o(m)^o$ in $\mathscr{A}^o\mathscr{F}$. This arrow is really $\mathfrak{l}^o[n]^o(m)^o$, but we omit those decorations for arrows.

We record the following fact for use in the next section.

**Lemma 2.1.4** *Let $\mathscr{B}$ be an additive category and let $\rho : B \to B'$ be an arrow in $\mathscr{B}$. Assume that the kernel $\iota_\rho : \operatorname{Ker}\rho \to B$ of $\rho$ exists and that there is a splitting $r : B' \to B$ of $\rho$, i.e., $\rho \circ r = \mathrm{id}_{B'}$. Then the natural arrow*

$$B' \oplus \operatorname{Ker}\rho \xrightarrow{r+\iota_\rho} B \tag{18}$$

*is an isomorphism in $\mathscr{B}$.*

*Proof.* Note that $\rho \circ (1 - r\,\rho) = 0$, so that there is an unique arrow $s : B \to \operatorname{Ker}\rho$ such that $1 - r \circ \rho = \iota_\rho \circ s$. It is easy to verify that the arrow:

$$B \xrightarrow{(\rho, s)} B' \oplus \operatorname{Ker}\rho \tag{19}$$

yields the desired inverse to $r + \iota_\rho$. $\square$

**Remark 2.1.5** Assume, in addition, that $\mathscr{B}$ is pseudo-Abelian, e.g., $\mathscr{B} = \mathscr{A}\mathscr{F}$, and consider the idempotent arrow $\pi := r \circ \rho$. Then we have a canonical isomorphism $B = \operatorname{Im}\pi \oplus \operatorname{Ker}\pi$. This isomorphism can be canonically identified with the one in (19), for $\operatorname{Ker}\pi = \operatorname{Ker}\rho$, and $r$ identifies $B'$ with $\operatorname{Im}\pi$.

## 2.2 A splitting mechanism in $\mathscr{A}\mathscr{F}$

Let $\mathbb{V} = (V, F)$ in $\mathscr{A}\mathscr{F}$ be of type $[a, b]$, let $m \in \mathbb{Z}$ and let:

$$\mathfrak{l} : \mathbb{V} \longrightarrow \mathbb{V}[b-a](m) \tag{20}$$

be an arrow such that the resulting arrow (10) is an isomorphism in $\mathscr{A}$:

$$\mathfrak{l}_a : V_a \xrightarrow{\cong} V_b(m). \tag{21}$$

There is the commutative diagram in $\mathscr{A}\mathscr{F}$ (see (12)):

$$\begin{tikzcd}
\mathbb{V}_a \oplus \mathbb{V}_b \arrow[r, "r", "i_a+\mathfrak{l} i_a \mathfrak{l}_a^{-1}"'] \arrow[drr, "1_{\mathbb{V}_a\oplus\mathbb{V}_b}"'] & \mathbb{V} \arrow[r, "\rho'", "(\mathfrak{l}_a^{-1} p_b \mathfrak{l}, p_b)"'] \arrow[dr, "\rho"] & \mathbb{V}_a \oplus \mathbb{V}_b \arrow[d, "u"', "\begin{pmatrix} 1_{\mathbb{V}_a} & -\mathfrak{l}_a^{-1} p_b \mathfrak{l}^2 i_a \mathfrak{l}_a^{-1} \\ 0 & 1_{\mathbb{V}_b} \end{pmatrix}"] \\
& & \mathbb{V}_a \oplus \mathbb{V}_b
\end{tikzcd} \tag{22}$$

so that:

$$\rho \,:\, \mathbb{V} \xrightarrow{\left(\mathfrak{l}_a^{-1} p_b \mathfrak{l} - \mathfrak{l}_a^{-1} p_b \mathfrak{l}^2 i_a \mathfrak{l}_a^{-1} p_b \;,\; p_b\right)} \mathbb{V}_a \oplus \mathbb{V}_b. \tag{23}$$

The kernel $\mathbb{K}\mathrm{er}\,\rho$ of $\rho$ in $\mathscr{A}\mathscr{F}$ is the kernel $\mathrm{Ker}\,\rho$ of the underlying map in $\mathscr{A}$ with the filtration induced by $(V, F)$. The natural inclusion induces a map in $\mathscr{A}F$:

$$\iota_\rho : \mathbb{K}\mathrm{er}\,\rho \longrightarrow \mathbb{V}. \tag{24}$$

**Remark 2.2.1** Since the arrows $u$ and $\mathfrak{l}_a$ in (22) are isomorphisms, we have that:

$$\mathrm{Ker}\,\rho = \mathrm{Ker}\,\rho' = \mathrm{Ker}\,(p_b \circ \mathfrak{l}) \cap \mathrm{Ker}\,p_b, \tag{25}$$

and similarly, if we take into account the induced filtrations.

**Lemma 2.2.2** *The following arrow is an isomorphism in $\mathscr{A}\mathscr{F}$:*

$$w : \mathbb{V}_a \oplus \mathbb{K}\mathrm{er}\,\rho \oplus \mathbb{V}_b \xrightarrow[\cong]{i_a+\iota_\rho+\mathfrak{l} i_a \mathfrak{l}_a^{-1}} \mathbb{V}. \tag{26}$$

*Proof.* Apply Lemma 2.1.4. $\square$

**Remark 2.2.3** The map $w$ (26) is uniquely determined by, and depends on, $\mathfrak{l}$. However, the component $i_a$, being the inclusion of the first subspace of the filtration, is independent of $\mathfrak{l}$.

The object $\mathbb{K}\mathrm{er}\,\rho$ is of type $[a+1,b-1]$, the inclusion $\iota_\rho$ induces natural isomorphisms:

$$\iota_{\rho_p} : (\mathbb{K}\mathrm{er}\,\rho)_p \xrightarrow{\cong} \mathbb{V}_p\,, \qquad \forall\, p \in [a+1,b-1] \tag{27}$$

and, by taking subquotients, a natural isomorphism:

$$\iota_{\rho_{[a+1,b-1]}} : \mathbb{K}\mathrm{er}\,\rho \xrightarrow{\cong} \mathbb{V}_{[a+1,b-1]}. \tag{28}$$

By combining (26) with (28), we obtain an isomorphism:

$$w_{[a,b]} \;:\; \mathbb{V}_a \oplus \mathbb{V}_{[a+1,b-1]} \oplus \mathbb{V}_b \xrightarrow{\cong} \mathbb{V}, \tag{29}$$

as well as its component:

$$w_{[a+1,b-1]} \;:\; \mathbb{V}_{[a+1,b-1]} \longrightarrow \mathbb{V}. \tag{30}$$

Both isomorphisms induce the identity on the $p$-th graded pieces, for every $p$ in (29), for $p \in [a+1,b-1]$ in (30).

One may picture the content of Lemma 2.2.2 as an unwrapping of the outmost layer $\mathbb{V}_a \oplus \mathbb{V}_b$ of $\mathbb{V}$ via $\mathfrak{l}$. Note that in general, there is no natural non-trivial arrow from a subquotient of an object to the object itself. The arrow (30) is made possible by the HL condition.

### 2.3 The splittings $\omega_{\mathrm{I}}(e)$ and $\omega_{\mathrm{II}}(e)$

Let $\mathbb{V} = (V,F)$ in $\mathscr{A}\mathscr{F}$ be of type $[-r.r]$ for some $r \geq 0$. Up to a translation, this condition can always be met and leads to simplified notation in what follows.

Let

$$e : \mathbb{V} \longrightarrow \mathbb{V}[2](1) \tag{31}$$

be an arrow in $\mathscr{A}\mathscr{F}$. In particular, for every $k \geq 0$, we have the iterations $e^k$ and their graded counterparts: (we drop the shift when denoting a shifted map and, in what follows, we drop subscripts for the maps induced on graded objects):

$$e^k \;:\; \mathbb{V} \longrightarrow \mathbb{V}[2k](k), \qquad e^k = V_j \longrightarrow V_{j+2k}(k). \tag{32}$$

**Assumption 2.3.1** (**Condition HL**) We assume that $(\mathbb{V}, e)$ satisfies the hard Lefschetz-type condition (HL), i.e., that the arrows:

$$e^k : V_{-k} \xrightarrow{\cong} V_k(k), \qquad \forall k \geq 0, \tag{33}$$

are isomorphisms in $\mathscr{A}$.

The following proposition ensures that if the HL condition is met by a given $(\mathbb{V}, e)$, then $\mathbb{V}$ splits, i.e., we have an isomorphism $\mathbb{V} \cong \mathbb{V}$ (6). By keeping with the analogy of Remark 2.2.3, we may say that HL allows to completely unwrap $\mathbb{V}$. Recall the notion of good splitting (on inducing the identity on the graded pieces).

**Proposition 2.3.2** *Let $(\mathbb{V}, e)$ satisfy the HL condition. There is a good splitting:*

$$\omega_{\mathrm{I}}(e) = \omega_{\mathrm{I}} \,:\, \mathbb{V}_* = \bigoplus_{p\in[-r.r]} \mathbb{V}_p \xrightarrow{\cong} \mathbb{V}. \tag{34}$$

*Proof.* By applying Lemma 2.2.2 and to $\mathfrak{l} := e^r$, so that $[a,b]=[-r,r]$ and $m=r$, we obtain

$$w : \mathbb{V}_{-r} \oplus \mathbb{K}\mathrm{er}\,\rho \oplus \mathbb{V}_r \xrightarrow{\cong} \mathbb{V}. \tag{35}$$

The arrow $\mathfrak{l}$ yields the arrow $\tilde{\mathfrak{l}} := \omega_{\mathrm{I}}^{-1} \circ \mathfrak{l} \circ \omega_{\mathrm{I}}$ on the lhs of (35). Keeping in mind that (26) means that the filtration $F$ on $V$ splits, we obtain the $\mathbb{K}\mathrm{er}\,\rho$-component $\mathfrak{l}'$ of $\tilde{\mathfrak{l}}$:

$$\mathfrak{l}' : \mathbb{K}\mathrm{er}\,\rho \longrightarrow \mathbb{K}\mathrm{er}\,\rho[2](1). \tag{36}$$

In view of (27), we have that $\mathfrak{l}'$ satisfies the HL condition.
By using (28) and (29), we replace $\mathbb{K}\mathrm{er}\,\rho$ with $\mathbb{V}_{[-r+1,r-1]}$ and we obtain the desired splitting $\omega_{\mathrm{I}}(e)$ by descending induction on $r$.
By construction (i.e., $\tilde{\mathfrak{l}} = \omega_{\mathrm{I}}^{-1} \circ \mathfrak{l} \circ \omega_{\mathrm{I}}$, (27) and (33)), the isomorphism $\omega_{\mathrm{I}}(e)$ induces the identity on the graded pieces and is thus good. □

The splitting $\omega_{\mathrm{II}}(e)$ is obtained by the dual construction. This is explained in the proof of Theorem 1.1.1.

**Remark 2.3.3** In general, $\omega_{\mathrm{I}}(e) \neq \omega_{\mathrm{II}}(e)$; see Examples 2.6.4 and 2.6.5. In particular, neither of the two constructions $\omega_{\mathrm{I}}(e)$ and $\omega_{\mathrm{II}}(e)$ is self-dual.

The purpose of the next three sections is to show that if $\mathbb{V}$ splits in $\mathscr{A}\mathscr{F}$, then one can use the HL property (33) to construct three additional natural splittings taking place in $\mathscr{A}\mathscr{F}$. Note that these constructions are based solely on the existence of a splitting.

## 2.4 The first Deligne splitting $\phi_{\mathrm{I}}(e)$

Let $(\mathbb{V}, e)$ be as in (31) and assume that it satisfies the HL condition (33). In particular, in view of (34), $\mathbb{V}$ splits in $\mathscr{A}\mathscr{F}$.

Let $i \geq 0$ and define the primitive objects in $\mathscr{A}$:

$$P_{-i} := \mathrm{Ker}\,\left\{e^{i+1} : V_{-i} \longrightarrow V_{i+2}(i+1)\right\}. \tag{37}$$

The subquotient $P_{-i}$ of $V$ inherits the filtration induced by $F$ on $V$, i.e., the trivial filtration translated in position $-i$, and we denote the resulting object in $\mathscr{A}\mathscr{F}$ by:

$$\mathbb{P}_{-i} = (P_{-i}, F) = (P_{-i}, T[i]). \tag{38}$$

We have the natural monomorphisms in $\mathscr{A}$:

$$P_{-i}(-j) \longrightarrow V_{-i}(-j) \xrightarrow{e^j} V_{-i+2j} \tag{39}$$

which, taken together, yield the canonical primitive Lefschetz decompositions (PLD) in $\mathscr{A}$ and $\mathscr{A}\mathscr{F}$, respectively:

$$\bigoplus_{0\leq j\leq i} P_{-i}(-j) \xrightarrow[\cong]{\epsilon} V_*, \tag{40}$$

$$\bigoplus_{0\leq j\leq i} (P_{-i}(-j), T[i-2j]) = \bigoplus_{0\leq j\leq i} \mathbb{P}_{-i}(-j)[-2j] \xrightarrow[\cong]{\epsilon} \mathbb{V}_*.$$

We have the commutative diagram of epimorphisms and monomorphisms in $\mathscr{A}\mathscr{F}$:

$$\begin{tikzcd} & & & \mathbb{P}_{-i} \arrow[r, "\text{mono}"] \arrow[d, "\text{mono}"'] & \mathbb{V}_{-i} \arrow[dl, "\text{mono}"] \\ \mathbb{V} \arrow[r, "\text{epi}"] & \dots \arrow[r, "\text{epi}"] & \mathbb{V}_{\geq -i-1} \arrow[r, "\text{epi}"] & \mathbb{V}_{\geq -i}. & \end{tikzcd} \tag{41}$$

Note that (38) implies that if $\mathfrak{l} : \mathbb{P}_{-i} \to \mathbb{W}$ is an arrow in $\mathscr{A}\mathscr{F}$, then it factors through $\mathbb{W}_{\leq -i}$, i.e., the underlying arrow $\mathfrak{l} : P_{-i} \to W$, factors through $F_{-i}W$.

**Lemma 2.4.1** *Let $(\mathbb{V}, e)$ be as above. There is a unique arrow $f_i : \mathbb{P}_{-i} \to \mathbb{V}$ in $\mathscr{A}\mathscr{F}$ with the following properties:*

1. *it lifts the natural arrow $\mathbb{P}_{-i} \to \mathbb{V}_{\geq -i}$ in (41);*
2. *for every $s > i \geq 0$, the composition of the arrows below is zero:*

$$\mathbb{P}_{-i} \xrightarrow{e^s \circ f_i} \mathbb{V}(s) \longrightarrow \mathbb{V}_{\geq s}(s). \tag{42}$$

*Proof.* The proof is essentially identical to the one of [8], Lemme 2.1 (see also [8], 2.3). We include it, with the necessary changes, for the reader's convenience. Recall that we use the language of sets.

Let $\Phi : \mathscr{A}\mathscr{F} \to \mathscr{B}$ be an additive functor into an Abelian category $\mathscr{B}$. We denote $\Phi(e)$ simply by $e$. Let $i \geq 0$ and $x \in \Phi(V_{\geq -i})$ be such that $0 = e^{i+1}(x) \in \Phi(V_{\geq i+2})$.

*CLAIM 1:* there is a unique lift $y \in \Phi(V_{\geq -i-1})$ of $x$ such that $0 = e^{i+1}(y) \in \Phi(V_{\geq i+1})$.

*Proof.* For every $a \in \mathbb{Z}$, we have the natural maps:

$$\mathbb{V}_a \longrightarrow \mathbb{V}_{\geq a+1} \longrightarrow \mathbb{V}_{\geq a+1}. \tag{43}$$

Since $\Phi$ is additive and, in view of Lemma 2.3.2, $\mathbb{V}$ splits in $\mathscr{A}\mathscr{F}$, we have the short exact sequences in $\mathscr{B}$ stemming from (43):

$$0 \longrightarrow \Phi(\mathbb{V}_a) \longrightarrow \Phi(\mathbb{V}_{\geq a}) \longrightarrow \Phi(\mathbb{V}_{\geq a+1}) \longrightarrow 0.$$

By naturality, we have the following commutative diagram of short exact sequences in $\mathscr{B}$:

$$\begin{tikzcd} 0 \arrow[r] & \Phi(\mathbb{V}_{-i-1}) \arrow[r] \arrow[d, "e^{i+1}", "\cong"'] & \Phi(\mathbb{V}_{\geq -i-1}) \arrow[r] \arrow[d, "e^{i+1}"] & \Phi(\mathbb{V}_{\geq -i}) \arrow[r] \arrow[d, "e^{i+1}"] & 0 \\ 0 \arrow[r] & \Phi(\mathbb{V}_{i+1}(i+1)) \arrow[r] & \Phi(\mathbb{V}_{\geq i+1}(i+1)) \arrow[r] & \Phi(\mathbb{V}_{\geq i}(i+1)) \arrow[r] & 0. \end{tikzcd} \tag{44}$$

The claim follows from a simple diagram-chase starting at $x \in \Phi(\mathbb{V}_{\geq -i})$.

*CLAIM 2:* under the same hypotheses as the ones of CLAIM 1, there is a unique lift $y \in \Phi(\mathbb{V})$ of $x \in \Phi(\mathbb{V}_{\geq i})$ such that $\forall s > i$, we have that $0 = e^s(y) \in \Phi(\mathbb{V}_{\geq s})$.
*Proof.* The element $y$ found in CLAIM 1 satisfies the hypotheses of CLAIM 1 for $i+1$. Since the filtration is finite, we conclude by a repeated use of CLAIM 1.

Let us apply CLAIM 2 to the functor $\Phi(-) := \mathrm{Hom}_{\mathscr{A}\mathscr{F}}(\mathbb{P}_{-i}, -) : \mathscr{A}\mathscr{F} \to \mathrm{Ab}$ (Abelian groups): set $x : \mathbb{P}_{-i} \to \mathbb{V}_{\geq -i}$ to be as in (41).
The statement of the lemma follows by applying CLAIM 2 to $x$: the hypotheses of CLAIM 2 are met in view of the defining property (37) of $P_{-i}$, and the resulting element $y$ is the desired $f_i$. $\square$

Let $\varphi : \mathbb{V}_* \cong \mathbb{V}$ be any splitting. In view of the primitive Lefschetz decomposition (40), we can talk about the components $\varphi_{ij} : \mathbb{P}_{-i}(-j)[-2j] \to \mathbb{V}$. Of course, there are many splittings having components $\varphi_{ii} = f_i$.

We define the *first Deligne isomorphism* $\phi_{\mathrm{I}}(e)$ *associated with* $(\mathbb{V}, e)$ by taking the compositum of the following two isomorphism

$$\phi_{\mathrm{I}}(e) = \phi_{\mathrm{I}} : \mathbb{V}_* \xrightarrow[(40)]{\epsilon^{-1}} \bigoplus_{0 \leq j \leq i} \mathbb{P}_{-i}(-j)[-2j] \xrightarrow[\cong]{\sum e^j \circ f_i} \mathbb{V}. \tag{45}$$

By using the language of elements, if we denote by $p_{ij}$ the typical element in $P_{-i}(-j)$ and we form the typical element $v_* \in V_*$:

$$v_* = \sum_p v_p = \epsilon \left( \sum_p \sum_{-i+2j=p} p_{ij} \right) \in V_*, \tag{46}$$

then

$$\phi_{\mathrm{I}}(e) : v_* \longmapsto \sum_{0 \leq j \leq i} e^j \left( f_i(p_{ij}) \right). \tag{47}$$

In particular, we have that

$$\phi_{\mathrm{I}}(\epsilon(e^l p_{ij})) = e^l \phi_{\mathrm{I}}(\epsilon(p_{ij})), \qquad \forall\, 0 \leq l \leq i-j. \tag{48}$$

**Remark 2.4.2** Let us omit the shifts, translations and filtrations. Let $\varphi : \mathbb{V}_* \cong \mathbb{V}$ be a splitting and $\varphi_i : P_{-i} \to V$ be the resulting components. By (37), we have that

$$e^{i+t} \circ \varphi_i : P_{-i} \to V_{\leq i+2t-1},$$

for every $t > 0$. Lemma 2.4.1 yields $f_i : P_{-i} \to V$ with

$$e^{i+t} \circ f_i : P_{-i} \to V_{\leq i+t-1},$$

for every $t > 0$, i.e., an improvement by $t$ units with respect to an arbitrary splitting, even a good one. The paper [7] exploits this special property of $\phi_{\mathrm{I}}(e)$ in the context of a study of the geometry of the Hitchin fibration.

**Remark 2.4.3** By constuction (see (45) and Remark 2.1.1), the isomorphisms $\phi_{\mathrm{I}}(e)$ and $\omega_{\mathrm{I}}(e)$ are good (6). In general, the two differ from each other; see Examples 2.6.4 and 2.6.5. However, they agree on $\mathbb{V}_{-r} \oplus \mathbb{V}_r$: in fact, both induce the identity on the graded pieces, so that they must agree on $\mathbb{V}_{-r} = \mathbb{P}_{-r}$; by comparing the expression (45) for $\phi_{\mathrm{I}}(e)$ restricted to $\mathbb{V}_r$ with the corresponding one for $\omega_{\mathrm{I}}(e)$, i.e., (35) and (26), we see that they coincide. In particular, if $V_i = 0$ for every $|i| \neq r$, then $\omega_{\mathrm{I}}(e) = \phi_{\mathrm{I}}(e)$.

Let $\varphi : \mathbb{V}_* \cong \mathbb{V}$ be a splitting. The matrix $\tilde{e}(\varphi)$ of $e$ with respect to $\varphi$ is defined by setting:

$$\tilde{e}(\varphi) \,=\, \tilde{e} := \left(\varphi^{-1} \circ e \circ \varphi\right) \,:\, \mathbb{V}_* \longrightarrow \mathbb{V}_*[2](1), \qquad \tilde{e} = \sum_{pq} \tilde{e}_{pq} = \sum \tilde{e}^{\{d\}}. \tag{49}$$

By virtue of (14), we have that

$$\tilde{e}^{\{d\}} = 0, \quad \forall d > 2. \tag{50}$$

Let us assume that $\varphi$ is good. Then

$$\tilde{e}^{\{2\}} = \sum_p e, \quad e : \mathbb{V}_p \xrightarrow{e} \mathbb{V}_{p+2}(1). \tag{51}$$

Note that while $\tilde{e}^{\{2\}}$ is independent of $\varphi$, we have that $\tilde{e}(\varphi)^{\{d\}}$ depends on $\varphi$ for $d \leq 1$.

We have the refinement $\hat{e}(\varphi)$ of the matrix $\tilde{e}(\varphi)$ that takes into account the primitive Lefschetz decomposition (40):

$$\hat{e}(\varphi) \,=\, \hat{e} \,:=\, \left(\epsilon^{-1} \circ \tilde{e} \circ \epsilon\right) \,:\, \bigoplus_{0 \leq j \leq i} \mathbb{P}_{-i}(-j)[-2j] \longrightarrow \bigoplus_{0 \leq j \leq i} \mathbb{P}_{-i}(-j)[-2j][2](1). \tag{52}$$

By taking components, we have arrows

$$\hat{e}(\varphi)^{kl}_{ij} \,:\, \mathbb{P}_{-i}(-j)[-2j] \longrightarrow \mathbb{P}_{-k}(-l)[-2l][2](1). \tag{53}$$

Proposition 2.7 in [8] can be easily adapted to the present context and yield the following characterization of $\phi_{\mathrm{I}}(e)$. For a "visual", see [8], p.119.

**Lemma 2.4.4** *The splitting $\phi_{\mathrm{I}} : \mathbb{V}_* \cong \mathbb{V}$ is characterized among the good ones by the following two conditions:*

1. *for $0 \leq j < i$, we have $\hat{e}(\phi_{\mathrm{I}})^{kl}_{ij} = 0$ except for $\hat{e}(\phi_{\mathrm{I}})^{i,j+1}_{ij} = \mathrm{Id}$;*
2. *for $j = i$, we have the $\hat{e}(\phi_{\mathrm{I}})^{kl}_{ii} = 0$ except, possibly, for $l \leq i$.*

**Definition 2.4.5** We say that a splitting $\varphi : \mathbb{V}_* \cong \mathbb{V}$ is *e-good* if it induces the identity on the graded pieces and $\tilde{e}(\varphi)$ is homogeneous of degree two:

$$\tilde{e}(\varphi) = \tilde{e}(\varphi)^{\{2\}}. \tag{54}$$

Clearly, $\varphi$ is $e$-good if and only we have that

$$\hat{e}(\varphi)^{kl}_{ij} = 0 \qquad \text{except for } \; \hat{e}(\varphi)^{i,j+1}_{ij} = \mathrm{Id}, \; \forall\, 0 \le j \le i-1. \tag{55}$$

We also have that $\varphi$ is $e$-good if and only the composita:

$$\mathbb{P}_{-i} \xrightarrow{\subseteq} \mathbb{V} \xrightarrow{e^{i+1}} \mathbb{V}[2i+2](i+1) \tag{56}$$

are zero for every $i \ge 0$. In this case we say that the $i$-th graded primitive objects $P_{-i}$ are embedded into $V$ via $\varphi$ as bona-fide $i$-th primitive classes: i.e., killed by

$$e^{i+1} : P_{-i} \to V_{\le i+1}(i+1),$$

and not just killed by the subsequent projection to $V_{i+1}(i+1)$.

**Remark 2.4.6** Lemma 2.4.4 implies that if $\varphi$ is $e$-good, then $\varphi = \phi_{\mathrm{I}}(e)$. In particular, if there exists an $e$-good splitting, then it is unique. However, $e$-good splittings do not exist in general: the reader can verify this in Examples 2.6.4 and 2.6.5; in the latter example, one can even take $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$. Proposition 2.6.3 shows that the existence of an $e$-good splitting is rare.

## 2.5 The second Deligne splitting $\phi_{\mathrm{II}}(e)$

Let $(\mathbb{V}, e)$ be as in the beginning of §2.4. In particular, $\mathbb{V}$ admits a splitting in $\mathscr{A}\mathscr{F}$ as in Proposition 2.3.2: in fact, we have three so far $\omega_{\mathrm{I}}(e)$, $\omega_{\mathrm{II}}(e)$ and $\phi_{\mathrm{I}}(e)$.

The first Deligne isomorphism $\phi_{\mathrm{I}}(e^o)$ (45) associated with $(\mathbb{V}^o, e^o)$ in $\mathscr{A}^o\mathscr{F}$ yields, by application of $(-)^o : \mathscr{A}^o\mathscr{F} \to \mathscr{A}\mathscr{F}$, the isomorphism in $\mathscr{A}\mathscr{F}$:

$$(\phi_{\mathrm{I}}(e^o))^o \; : \; \mathbb{V} \xrightarrow{\cong} \mathbb{V}_*. \tag{57}$$

We define the *second Deligne isomorphism associated with* $(\mathbb{V}, e)$ to be

$$\phi_{\mathrm{II}}(e) \; = \; \phi_{\mathrm{II}} \; := \; ((\phi_{\mathrm{I}}(e^o))^o)^{-1} \; : \; \mathbb{V}_* \xrightarrow{\cong} \mathbb{V}. \tag{58}$$

In this context, the analogue of Lemma 2.4.1 reads as follows. Let

$$f'_{ij} \; : \; \mathbb{V} \longrightarrow \mathbb{P}_{-i}(-j)[-2j] \;\; (\overset{\epsilon}{\subseteq} \mathbb{V}_i) \tag{59}$$

be the components of (57) associated with (40) ($\epsilon$ as in (40)).

**Lemma 2.5.1** *For every $i \ge 0$, the arrow $f'_{ii}$ is the unique arrow $\mathbb{V} \to \mathbb{V}_i$ such that:*

1. *by taking the $i$-th graded pieces, $f'_{ii}$ induces the natural projection $\mathbb{V}_i \to \mathbb{P}_{-i}(-i)[-i]$;*
2. *for every $s > i$, the composition below is zero is (see [8], §3.1):*

$$\mathbb{V}_{\le -s} \xrightarrow{\subseteq} \mathbb{V} \xrightarrow{\epsilon^s} \mathbb{V}[2s](s) \xrightarrow{f'_{ii}} \mathbb{P}_{-i}(-i)[-2i][2s](s). \tag{60}$$

By using Lemma 2.5.1 and the explicit formula (47) for $\phi_{\rm I}(e)$, it is easy to deduce the following explicit expression for the arrows $f'_{ii}$:

$$f'_{ii}(\phi_{\rm I}(v_*)) = p_{ii} \qquad (v_* \;=\; \textstyle\sum_{0\leq j'\leq i'} p_{i'j'}). \tag{61}$$

In general, $\phi_{\rm I}(e) \neq \phi_{\rm II}(e)$ and this discrepancy is due to the fact that $f'_{ij}(\phi_{\rm I}(v_*)) \neq p_{ij}$. We now discuss how this discrepancy is measured exactly in terms the matrix $\hat{e}(\phi_{\rm I})$ (53) of $e$.

By combining (45) with (57), we see that there is the commutative diagram (the bottom identification is due to (40)):

$$\begin{tikzcd} \mathbb{V} \arrow[r, "e^{i-j}"] \arrow[d, "f'_{ij}"] & \mathbb{V}[2(i-j)](i-j) \arrow[d, "f'_{ii}"] \\ \mathbb{P}_{-i}(-j)[-2j] \arrow[r, "e^{i-j}=\mathrm{Id}"] & \mathbb{P}_{-i}(-i)[-2i][2i-2j](i-j). \end{tikzcd} \tag{62}$$

We fix $0 \leq j \leq i$. For every $0 \leq s \leq t$, we use (47) and (48) together with (61) and (62) with the goal of determining the value of

$$f'_{ij}\left(e^s f_t(p_{ts})\right). \tag{63}$$

Recalling that (57) induces the identity on the graded pieces, we deduce that:

1. if $t = i$ and $s < j$, then $f'_{ij}\left(e^s f_t(p_{ts})\right) = 0$;

   we can see this, as well as the assertions that follow, on the following diagram (we do not write $\epsilon$):

$$e^s f_t(p_{is}) = \phi_{\rm I}(p_{is}) \longmapsto e^{i-j}\phi_{\rm I}(p_{is}) = \phi_{\rm I}(e^{i-j}p_{is}) \overset{f'_{ii}}{\longmapsto} 0; \tag{64}$$

2. if $t = i$ and $s = j$ then $f'_{ij}\left(e^j f_i(p_{ij})\right) = p_{ij}$;

3. if $t \neq i$ and $s + i - j \leq t$, then $f'_{ij}\left(e^s f_t(p_{ts})\right) = 0$;

4. if $t \neq i$ and $\sigma := s + i - j - t \geq 1$, then

$$f'_{ij}\left(e^s f_t(p_{ts})\right) = f'_{ii}(e^{\sigma} e^t f_t(p_{ts})) \tag{65}$$

   which, recalling the definition (53) of $\hat{e}_{\rm I} = \hat{e}(\phi_{\rm I}(e))$, has the following form:

$$(\hat{e}^{\sigma}_{\rm I})^{ii}_{tt}\,(q_{tt}) \qquad (\text{where } q_{tt} \;:=\; f'_{tt}(e^t f_t(p_{ts}))\,. \tag{66}$$

**Proposition 2.5.2** *The first Deligne isomorphism $\phi_{\rm I}(e)$ is $e$-good (55) if and only if*

$$\phi_{\rm I}(e) = \phi_{\rm II}(e).$$

*Proof.* In view of (57) and 58), we have that the two Deligne isomorphisms coincide if and only if, using the notation in (61), we have that $f'_{ij}(v_*) = p_{ij}$, for every $0 \leq j \leq i$. According to the four points above, the only obstruction to having this latter condition stems from (66) not being zero for some pair $(i,t)$ with $i \neq t$. By reasons of degree, i.e., by (14), since $i \neq t$, we have that $(\hat{e}_{\mathrm{I}})^{ii}_{tt}$ is in degrees $\leq 1$ . If $\phi_{\mathrm{I}}$ is $e$-good, then $\hat{e}_{\mathrm{I}}$ is of pure homogeneous degree 2, so that $e^{\sigma} e^t f_t(p_{ts})) = 0$ and we infer the desired equality.

Conversely, let us assume that the two Deligne isomorphisms coincide. By contradiction let us assume that $\phi_{\mathrm{I}}$ is not $e$-good. According to (55) there are integers $0 \leq t$ and $0 \leq l \leq k$ and a non zero arrow $l \leq t$. Among these non zero arrows $(\hat{e}_I)^{kl}_{tt}$, chose one, $(\hat{e}_I)^{k_o l_o}_{t_o t_o}$, for which the difference $k - l$ attains the minimum value. In the language of elements, what above ensures that there is $0 \neq p_{t_o} \in P_{-t_o}$ such that:

$$e^{t_o+1} f_{t_o}(p_{t_o}) = e^{l_o} f_{k_o}\left(\hat{e}^{k_o l_o}_{t_o t_o}\left(e^{t_o} p_{t_o}\right)\right) + \sum\nolimits^* e^l f_k\left(\hat{e}^{kl}_{t_o t_o}\left(e^{t_o} p_{t_o}\right)\right), \tag{67}$$

where the first term on the r.h.s. is non-zero and $\sum^*$ is the sum over the non-zero terms with $(k,l) \neq (k_o, l_o)$. Since for these latter terms, $k - l \geq k_o - l_o$, we deduce that

$$f'_{k_o k_o}\left(e^{k_o - l_o} \sum\nolimits^*\right) = 0 \in P_{-k_o}(-k_o). \tag{68}$$

On the other hand, since obviously $e^{k_o - l_o} e^{l_o} = e^{k_o}$, we have that:

$$q_{k_o} := f'_{k_o k_o}\left(e^{k_o - l_o} e^{l_o} f_{k_o}\left(\hat{e}^{k_o l_o}_{t_o t_o}\left(e^{t_o} p_{t_o}\right)\right)\right) \neq 0 \in P_{-k_o}(-k_o). \tag{69}$$

In view of (62), we have that

$$f'_{k_o, l_o - 1}\left(e^{t_o} f_{t_o}(p_{t_o})\right) = q_{t_o} \neq 0. \tag{70}$$

Since we are assuming that the two Deligne isomorphisms coincide, by virtue of the first paragraph of this proof, we must have $k_o = t_o$ and $t_o = l_o - 1$. This contradicts $l_o \leq t_o$. $\square$

**Remark 2.5.3** In general, there is no $e$-good splitting; see Examples 2.6.4 and 2.6.5. In particular, neither of the two constructions $\phi_{\mathrm{I}}(e)$ and $\phi_{\mathrm{II}}(e)$ is self-dual.

## 2.6 The third Deligne splitting $\phi_{\mathrm{III}}(e)$

Let $(\mathbb{V}, e)$ be as in §2.4. In particular, $\mathbb{V}$ admits a splitting (34) in $\mathscr{A}\mathscr{F}$ as in Proposition 2.3.2; in fact, we have four, so far. We also assume that the Abelian category $\mathscr{A}$ is $\mathbb{Q}$-linear, i.e., that Hom-groups are rational vector spaces. The reason for this is that, in what follows, one needs to exploit the $\mathrm{sl}_2(\mathbb{Q})$-action arising from the given arrow $e$.

The goal of this section is to construct the third Deligne isomorphism associated with $(\mathbb{V}, e)$. Whereas we omit the detailed presentation of the algebra underlying this construction (see [8], Lemme 3.3 and Proposition 3.5), we review some of the key points, state its characterization and, along the way, indicate the necessary changes.

Let $\mathbb{V}$ and $\mathbb{W}$ be in $\mathscr{A}\mathscr{F}$ and set:

$$L^{[n]}_{(i,j)}(\mathbb{V}, \mathbb{W}) := \mathrm{Hom}_{\mathscr{A}\mathscr{F}}(\mathbb{V}(i), \mathbb{W}(j)[n]). \tag{71}$$

Up to the canonical isomorphism induced by the shift functors, the above depends only on the difference $m := (j-i)$ and we denote the resulting bi-functor by $L^{[n]}_{(m)}$.

Recalling the definition of the graded-type objects (5) and of degree of maps (13) (an arrow $(V_p, T) \to (V_q, T)[n](m)$ in $\mathscr{A}\mathscr{F}$ has degree $d := q-p$), we have the natural decomposition by homogeneous degrees:

$$L^{[n]}_{(m)}(\tilde{\mathbb{V}}_*, \tilde{\mathbb{V}}_*) = \bigoplus_{d=-2r}^{2r} L^{[n],\{d\}}_{(m)}(\tilde{\mathbb{V}}_*, \tilde{\mathbb{V}}_*). \tag{72}$$

The arrow $h := \sum_p p \operatorname{Id}_{(V_p,T)}$ induces the arrow:

$$h : L^{[n]}_{(m)}(\tilde{\mathbb{V}}_*, \tilde{\mathbb{V}}_*) \longrightarrow L^{[n]}_{(m)}(\tilde{\mathbb{V}}_*, \tilde{\mathbb{V}}_*), \quad u \longmapsto h \circ u; \tag{73}$$

this arrow is of homogeneous degree zero, i.e., $\{d\} \mapsto \{d\}$, with respect to (72).
By taking together the graded pieces of the arrow $e : \mathbb{V} \to \mathbb{V}[2](1)$, i.e., set $e' := \sum e_p$, with $e_p : V_p \to V_{p+2}(1)$, we obtain $e' \in L^{[0],\{2\}}_{(1)}(\tilde{\mathbb{V}}_*, \tilde{\mathbb{V}}_*)$ which, in turn, induces the homogeneous degree two arrow:

$$e \; : \; L^{[n]}_{(m)}(\tilde{\mathbb{V}}_*, \tilde{\mathbb{V}}_*) \longrightarrow L^{[n]}_{(m+1)}(\tilde{\mathbb{V}}_*, \tilde{\mathbb{V}}_*), \qquad u \longmapsto e(u) := e' \circ u - u \circ e'. \tag{74}$$

There is a canonical arrow of homogeneous degree $-2$ (this is where we need denominators ([8], p.121):

$$f \; : \; L^{[n]}_{(m)}(\tilde{\mathbb{V}}_*, \tilde{\mathbb{V}}_*) \longrightarrow L^{[n]}_{(m-1)}(\tilde{\mathbb{V}}_*, \tilde{\mathbb{V}}_*). \tag{75}$$

The arrows $(h, e, f)$ in (73), (74) and (75) form an $\mathrm{sl}_2(\mathbb{Q})$-triple turning the rational vector spaces

$$L^{[n]}_j \; : \; \bigoplus_{d \in \mathbb{Z}^{\mathrm{even/odd}}} L^{[n],\{d\}}_{j+d/2}(\tilde{\mathbb{V}}_*, \tilde{\mathbb{V}}_*) \tag{76}$$

into $\mathrm{sl}_2(\mathbb{Q})$-modules; in what above, $j$ is a fixed integer multiple of $1/2$ and the sum is over the integers $d$ with fixed parity, even if $j$ is integral, odd if $j$ is an half-integer. Recalling that the sum is finite, for $|d| \leq 2r$, we have that the corresponding HL statememt reads as follows: ($e^k$ the $k$-th iteration of $e$ (74))

$$e^k \; : \; L^{[n],(-k)}_{(j-k/2)} \xrightarrow{\quad\cong\quad} L^{[n],(k)}_{(j+k/2)} \; . \tag{77}$$

Let $\varphi \; : \mathbb{V}_* \cong \mathbb{V}$ be any good splitting (6) and let $\tilde{e}(\varphi)$ be the associated matrix of $e : \mathbb{V} \to \mathbb{V}[2](1)$ (49). The degree $d$ homogeneous part of $\tilde{e}(\varphi)$ satisfies:

$$\tilde{e}(\varphi)^{\{d\}} := \sum_{q-p=d} \tilde{e}(\varphi)_{pq} \; \in \; L^{[2-d],\{d\}}_{(1)} \tag{78}$$

and is subject to (50) and (51): it is zero for every $d > 2$ and it is the obvious arrow for $d = 2$.

The *the third Deligne isomorphism associated with* $(\mathbb{V}, e)$:

$$\phi_{\mathrm{III}}(e) \; := \phi_{\mathrm{III}} \; : \mathbb{V}_* \xrightarrow{\quad\cong\quad} \mathbb{V}, \tag{79}$$

is the unique good splitting subject to the following conditions: ($e^{1-d}$ is the $(1-d)$-iteration of (74)):

$$e^{1-d}\left(\tilde{e}(\varphi)^{\{d\}}\right) \;=\; 0, \qquad \forall d \leq 1. \tag{80}$$

Let us illustrate how, any good splitting $\varphi : \mathbb{V}_* \cong \mathbb{V}$ can be modified recursively, via HL (77), to obtain a new good splitting subject to (80).

Let $d = 1$. The condition (80) reads $\tilde{e}^{\{1\}} = 0$. Set $\varphi_1 := \varphi(\mathrm{id} + \psi^{\{-1\}})$, where $\psi^{\{-1\}} \in L^{[1],\{-1\}}_{(0)}$ is a variable arrow. We conjugate $e$ and obtain:

$$\left(\mathrm{id} + \psi^{\{-1\}}\right)^{-1} \circ e \circ \left(\mathrm{id} + \psi^{\{-1\}}\right) \equiv \tilde{e}(\varphi)^{\{2\}} + e\left(\psi^{\{-1\}}\right) \quad \text{modulo degree} \leq 0. \tag{81}$$

Note that the last term on the left is in $L^{[1],\{1\}}_{(1)}$. We take the degree 1 part of the r.h.s of (81) and set it equal to zero

$$e\left(\psi^{\{-1\}}\right) = -\tilde{e}(\varphi)^{\{1\}} \qquad \left(\text{equality in } L^{[1],\{1\}}_{(1)}\right). \tag{82}$$

The HL (77) ensures that such a $\psi^{\{-1\}}$ exists and is unique. This determines $\varphi_1$.

Let $d = 0$. The condition (80) reads $e(\tilde{e}^{\{0\}}) = 0$. Set $\varphi_0 := \varphi_1(\mathrm{id} + \psi^{\{-2\}})$, where $\psi^{\{-2\}} \in L^{[1],\{-2\}}_{(0)}$ is a variable arrow. We conjugate $e$ and obtain

$$\left(\mathrm{id} + \psi^{\{-2\}}\right)^{-1} e \left(\mathrm{id} + \psi^{\{-2\}}\right) \equiv \tilde{e}(\varphi_1)^{\{2\}} + e\left(\psi^{\{-2\}}\right) \quad \text{modulo degree} \leq -1. \tag{83}$$

Note that the last term on the left is in $L^{[1],\{0\}}_{(1)}$. We take the degree 0 part of the r.h.s of (83) and set it equal to zero after application of $e$:

$$e^2\left(\psi^{\{-2\}}\right) = -e\left(\tilde{e}(\varphi_1)^{\{0\}}\right) \qquad \left(\text{equality in } L^{[1],\{2\}}_{(2)}\right). \tag{84}$$

The HL (77) ensures that such a $\psi^{\{-2\}}$ exists and is unique. This determines $\varphi_0$.

We repeat this procedure for all decreasing values of $d$ and, recalling that $\mathbb{V}$ has type $[-r, r]$, the procedure ends no later than $d = -2r$.

The unicity of the resulting arrow is verified easily as follows. Let $a, b$ be two good splittings subject to (80). Set

$$c \;:=\; b^{-1}a = \mathrm{Id} + \textstyle\sum_{l\geq 1} c^{\{-l\}} \;:\; \mathbb{V}_* \xrightarrow{\;\;\cong\;\;} \mathbb{V}. \tag{85}$$

We apply the procedure carried out above to $b$, modifying it to $b_1 := b(\mathrm{Id} + c^{\{-1\}})$. We have that $b_1 \equiv a$ modulo degree $\leq -2$, so that, in view of the fact that $a$ also satisfies (80), we must have that $c^{\{-1\}} = 0$. It follows that $b \equiv a$, modulo degree $\leq -1$. We repeat this procedure and kill all the $c^{\{-l\}}$.

In general, $\phi_{\mathrm{I}} \neq \phi_{\mathrm{III}}$, however, by [8], Proposition 3.6 (easily adapted to the present context), we have that: (see Lemma (2.4.1) for the definition of $f_i$)

$$\phi_{\mathrm{III}}(e)_{|\mathbb{P}_{-i}} = \phi_{\mathrm{I}}(e)_{|\mathbb{P}_{-i}} = f_i : \mathbb{P}_i \longrightarrow \mathbb{V}. \tag{86}$$

**Remark 2.6.1** Unlike the four previous splittings $\omega_{\mathrm{I,II}}(e)$ and $\phi_{\mathrm{I,II}}(e)$, the construction leading to $\phi_{\mathrm{III}}(e)$ is self-dual in the sense that we have:

$$\left(\left(\phi_{\mathrm{III}}\left(e^o\right)\right)^o\right)^{-1} = \phi_{\mathrm{III}}(e). \tag{87}$$

This is because an isomorphism $\varphi$ satisfies condition (80) if and only if $\varphi^o$ satisfies the analogous "opposite" conditions.

**Remark 2.6.2** In genereal, the five good splittings

$$\omega_{\mathrm{I}}(e), \quad \omega_{\mathrm{II}}(e), \quad \phi_{\mathrm{I}}(e), \quad \phi_{\mathrm{II}}(e), \quad \phi_{\mathrm{III}}(e) \tag{88}$$

are pairwise distinct; see Examples 2.6.4 and 2.6.5.

**Proposition 2.6.3** *If an $e$-good splitting* (2.4.5) *exists, then it is unique and it coincides with the third Deligne isomorphism. In this case we have:*

$$\omega_I(e) = \omega_{\mathrm{II}}(e) = \phi_{\mathrm{I}}(e) = \phi_{\mathrm{II}}(e) = \phi_{\mathrm{III}}(e). \tag{89}$$

*Proof.* Let $\varphi$ be $e$-good. Then $\tilde{e}(\varphi)^{\{d\}} = 0$ for every $d \leq 1$. It follows that condition (80) is met by $\varphi$, so that $\varphi = \phi_{\mathrm{III}}(e)$ and we have proved uniqueness (see also Remark 2.4.6).
Assume that there is an $e$-good splitting, which, by the above, must coincide with $\phi_{\mathrm{III}}(e)$. By Remark 2.4.6, we have that $\phi_{\mathrm{III}}(e) = \phi_{\mathrm{I}}(e)$. Proposition 2.5.2 implies that $\phi_{\mathrm{I}}(e) = \phi_{\mathrm{II}}(e)$ (this equality can be also seen by using a duality argument similar to the one in (92)).
Let us compare $\phi_{\mathrm{I}}(e)$ with $\omega_{\mathrm{I}}(e)$. By Remark (2.4.3), the two agree on $\mathbb{V}_{-r} \oplus \mathbb{V}_r$.
By comparing the general description (25) of $\operatorname{Ker}\rho$ with the formula (47) for the embedding $\phi_{\mathrm{I}}$, we see, with the aid of (56), that

$$\operatorname{Ker}\rho = \left\{ v \in V \mid v = \sum_{(i,j)\in I_r} e^j f_i(p_{ij}) \right\}, \tag{90}$$

where $I_r$ is the set of the indices subject to $0 \leq j \leq i$ and to $(i,j) \neq (r,0),(r,r)$. It follows that $\operatorname{Ker}\rho$ coincides with $\phi_{\mathrm{I}}(\sum_{I_r} P_{-i}(-j))$. By projecting onto $\sum_{|p|\neq r} V_p$, we deduce that $\phi_{\mathrm{I}}(e)$, restricted to $\sum_{|p|\neq r} V_p$ factors through $\operatorname{Ker}\rho \to V$. Now we repeat for $\operatorname{Ker}\rho$, what we have done above for $V$ and deduce, by descending induction on $r$, that

$$\phi_{\mathrm{I}}(e) = \omega_{\mathrm{I}}(e). \tag{91}$$

We conclude by using a duality argument:

$$\omega_{\mathrm{II}}(e) = ((\omega_{\mathrm{I}}(e^o))^o)^{-1} = ((\phi_{\mathrm{I}}(e^o))^o)^{-1} = \phi_{\mathrm{II}}(e) = \omega_{\mathrm{I}}(e), \tag{92}$$

where: the first equality is by definition; the second equality follows by the fact that there is an $e$-good splitting for $(\mathbb{V}, e)$ if and only there is an $e^o$-good splitting for $(\mathbb{V}^o, e^o)$ and we have proved that $\omega_{\mathrm{I}} = \phi_{\mathrm{I}}$; the third equality is by definition; the final equality is (91). This concludes the proof. □

**Example 2.6.4** $V \cong \mathbb{Q}^3$ with basis $(v_{-2}, v_0, v_2)$, $e : (v_{-2}, v_0, v_2) \longmapsto (v_0, v_2, v_2)$,

$$V_{\leq -2} = V_{\leq -1} = \langle v_{-2} \rangle, \quad V_{\leq 0} = V_{\leq 1} = \langle v_{-2}, v_0 \rangle, \quad V_{\leq 2} = V, \tag{93}$$

$$V_{-2} = \langle [v_{-2}] \rangle, \quad V_0 = \langle [v_0] \rangle, \quad V_2 = \langle [v_2] \rangle. \tag{94}$$

The five splittings associated with $e$ discussed in this paper are:

- $\phi_{\mathrm{I}}(e) : ([v_{-2}], [v_0], [v_2]) \longmapsto (v_{-2}, v_0, v_2)$;
- $\phi_{\mathrm{II}}(e) : ([v_{-2}], [v_0], [v_2]) \longmapsto (v_{-2}, -v_{-2} + v_0, -v_0 + v_2)$;
- $\phi_{\mathrm{III}}(e) : ([v_{-2}], [v_0], [v_2]) \longmapsto (v_{-2}, -\frac{1}{3}v_{-2} + v_0, -\frac{2}{3}v_0 + v_2)$;
- $\omega_{\mathrm{I}}(e) : ([v_{-2}], [v_0], [v_2]) \longmapsto (v_{-2}, -v_{-2} + v_0, v_2)$;
- $\omega_{\mathrm{I}}(e) : ([v_{-2}], [v_0], [v_2]) \longmapsto (v_{-2}, -v_{-2} + v_0, -v_{-2} + v_2)$.

A direct calculation, or Proposition 2.6.3, shows that there is no $e$-good splitting.

**Example 2.6.5** Here is a class of examples from geometry where, unlike the previous example, $e$ is nilpotent.

Let $Y \times Z$ be the product of nonsingular complex projective varieties and let $r := \dim_{\mathbb{C}} Z$. Let $V := H(Y \times Z, \mathbb{Q}) = H(Y, \mathbb{Q}) \otimes H(Z, \mathbb{Q}) = \oplus_{r,s} H^r(Y, \mathbb{Q}) \otimes H^s(Z, \mathbb{Q})$. Let $F'_s V$ be the subspace spanned by the elements of the form $y_r z_\sigma$ with $\sigma \leq s$. Set $F := F'[r]$; this way $(V, F)$ has type $[-r, r]$. Let $\eta \in H^2(Y \times Z, \mathbb{Q})$ be the first Chern class of a line bundle on $Y \times Z$ which is ample when restricted to the fibers of the projection onto $Y$. Denote by $e : (V, F) \to (V, F[2])$ the map $v \mapsto \eta \cup v$.

By using the hard Lefschetz theorem on $Z$, one sees directly that the HL condition (33) holds for $((V, F), e)$. In addition to the five splittings considered in this paper, we also have the Künneth splitting $\kappa$. In general, the six splittings are pairwise distinct. The reader can verify this fact directly by taking $Y = Z$ to be an elliptic curve and the line bundle to be of the form $E \times \zeta + \zeta \times E + \mathfrak{P}$, where $\zeta \in E$ is a point and $\mathfrak{P}$ is a Poincaré bundle (this is to ensure that $\tilde{e}(\kappa)^{\{1\}} \neq 0$, so that, according to (80), we must have $\kappa \neq \phi_{\mathrm{III}}(e)$).

If we take $Y = \mathbb{P}^1 \times \mathbb{P}^2$, we are lead to examples where $\kappa = \phi_{\mathrm{III}}(e)$, but otherwise the isomorphisms of type $\phi$ and $\omega$ are pairwise distinct, and distinct from $\phi_{\mathrm{III}}(e)$. If we take $Y = Z = \mathbb{P}^1$, then we have $\kappa = \phi_{\mathrm{III}}(e)$, $\phi_{\mathrm{I}}(e) = \omega_{\mathrm{I}}(e)$ and $\phi_{\mathrm{II}}(e) = \omega_{\mathrm{II}}(e)$, but we have no further relation. In this case, if we take $\eta$ to be the class of the fiber of the projection onto $Z$, then we have an $e$-good splitting. In all the examples, for general $\eta$, there is no $e$-good splitting. A highly non-trivial example, where there is a good splitting is mentioned at the end of the introduction.

# 3 Appendix: a letter from P. Deligne

P. Deligne has sent the author a letter commenting on an earlier draft of this paper. The author is happy to include, with P. Deligne's kind permission, this letter in this appendix as it outlines the simple modifications necessary to obtain the splittings of this note in a Tannakian (tensor product) context.

March 16, 2012

Dear de Cataldo,

Thank you!

Let $\mathscr{A}$ be an Abelian category with a shift functor $A \to A(1)$ as in your text. Let $\mathscr{B}$ be the category of objects of $\mathscr{A}$ given with a finite increasing filtration $F$ and $e : A \to A[2](1)$ verifying HL.

**Corollary.** $\mathscr{B}$ is an abelian category.

*Proof.* (a) for objects $(A, F, e)$ of $\mathscr{B}$ of type $[-r, r]$, with $r > 0$, the "peeling off" 2.2

$$A = F_r \oplus \mathrm{Ker}\left(F_{r-1} \subseteq A \xrightarrow{e^r} A \longrightarrow A/F_{r-1}\right) \oplus e^r\left(F_r\right)$$

is functorial. By induction on $r$, it follows that

(b) the splitting $\omega_{\mathrm{I}}$ is functorial.

If $f : (A, F, e) \longrightarrow (A', F', e')$ is a morphism, (b) implies that $\mathrm{Gr}^F \mathrm{Ker}\,(f) \xrightarrow{\cong} \mathrm{Ker}\,\mathrm{Gr}^F(f)$, and dually for coKer. That $(\mathrm{Ker}\,(f), F, e)$ is in $\mathscr{B}$ follows. It is a kernel. Dually for cokernels. Morphisms are strictly compatible with filtrations, hence $\mathrm{CoIm}(f) \xrightarrow{\cong} \mathrm{Im}\,(f)$.

**Remark.** In your 2.2. you should assume $a < b$.

From now on, all categories are assumed to be $\mathbb{Q}$-linear.

Let $\mathscr{A}', \mathscr{A}''$ and $\mathscr{A}$ be as above and let $\otimes : \mathscr{A}' \times \mathscr{A}'' \longrightarrow \mathscr{A}$ be an exact biadditive functor, compatible with shifts: functorial isomorphisms

$$A'(1) \otimes A'' \xrightarrow{\cong} (A' \otimes A'')(1),$$

and

$$A' \otimes A''(1) \xrightarrow{\cong} (A' \otimes A'')(1)$$

are given, and the two resulting functorial isomorphisms

$$A'(1) \otimes A''(1) \longrightarrow (A' \otimes A'')(2)$$

coincide. Let $\mathscr{B}', \mathscr{B}''$ and $\mathscr{B}$ be the corresponding categories of triples $(A, F, e)$ as above. Given $(A', F', e')$ and $(A'', F'', e'')$, one defines the filtration $F$ (resp. morphism $e$) for $A' \otimes A''$ by

$$F_p = \sum_{p'+p''=p} F'_p \otimes F''_p$$

(so that $\mathrm{Gr}^{F'}(A') \otimes \mathrm{Gr}^{F''}(A'') \xrightarrow{\cong} \mathrm{Gr}^F(A)$), and

$$e = e' \otimes 1 + 1 \otimes e'$$

(so that the same formula holds for the graded $e, e', e''$).

**Proposition.** *This $\otimes$ sends $\mathscr{B}', \mathscr{B}''$ to $\mathscr{B}$.*

One has to check that for graded objects and morphisms of degree 2, $\otimes$ preserves HL. In the graded case, HL for $(A^*, e)$ is equivalent to the existence of $f^* : A^* \longrightarrow A^*(-1)$ of degree $-2$ such that $[e, f]$ id multiplicaiton by $n$ in degree $n$. Stability of this property is proved in the same way that tensor product of representations of Lie algebras are defined.

Suppose now that $\mathscr{A}$ is a Tannakian category and that the twist is the tensor product with an object $\mathbb{Q}(1)$ of rank one. The compatibility between $\otimes$ and shift we required amounts tothe symmetry automorphism of $\mathbb{Q}(1) \otimes \mathbb{Q}(1)$ being the identity.

**Corollary.** *If $\mathscr{A}$ is Tannakian, so is $\mathscr{B}$.*

*Proof.* If $\omega$ is a fiber functor on $\mathscr{A}$, then $(A, F, e) \mapsto \omega(A)$ is a fiber functor on $\mathscr{B}$.

In terms of actions on $SL(2)$, rather in terms of grading and of $e$ and $f$ above, I prefer to state the characteristic property of the (good) splitting $\phi_{\mathrm{III}}$ as follows: it is the filtered isomorphism, with graded the identity:

$$u : \mathrm{Gr}^F(A) \longrightarrow A$$

such that, with $e : \mathrm{Gr}^F(A) \longrightarrow \mathrm{Gr}^F(A)(1)$, and $f$ as above, if $X$ is defiend by

$$u^{-1}eu = e + X,$$

one has $[f, X] = 0$. This characterization makes it clear that this splitting is compatible with tensor products (in the sense of the proposition). It gives an equivalence of the category $\mathscr{B}$ of triples $(A, F, e)$ with the category of graded objects $A^*$, 0 outside of finitely many degrees, given with

$$e : A^* \longrightarrow A^*(1) \text{ and } f : A^* \longrightarrow A(-1)$$

of degree 2, resp. $-2$, with $[e, f] = n$ in degree $n$, and given with $X : (\oplus A^n) \longrightarrow (\oplus A^n)(1)$ such that $[f, X] = 0$.

In the Tannakian case, this equivalence is compatible with $\otimes$ [for $X$'s, $\otimes$ is defined by $X = X' \otimes 1 + 1 \otimes X''$].

Best,

P. Deligne.

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**Journal *of* Singularities**
**Volume 7 (2013), 157-189**

received 6 February 2013
in revised form 2 July 2013
DOI : 10.5427/jsing.2013.7i

# CYCLES ÉVANESCENTS ALGÉBRIQUES ET TOPOLOGIQUES PAR UN MORPHISME SANS PENTE

PH. MAISONOBE

## 1. Introduction

Soit $f : X \to S$ un morphisme d'espaces analytiques complexes réduits. Lorsque $\dim S > 1$, il n'existe en général pas d'analogue de la fibration de Milnor sur un système fondamental de voisinages $\mathcal{V}_x$ d'un point $x$ de $X$ (on peut penser au cas d'un éclatement). De même, l'image directe d'un complexe borné à cohomologie $\mathbf{C}$-constructible sur $X$ par $f_{|V}$ pour $V$ dans $\mathcal{V}_x$ n'est pas nécessairement $\mathbf{C}$-constructible sur $S$.

On comprend mieux la géométrie locale du morphisme $f$ si celui-ci est sans éclatement en codimension zéro au sens de J.-P. Henry, M. Merle et C. Sabbah [H-M-S] et si de plus son discriminant est à croisements normaux. Ainsi que l'a montré C. Sabbah [S4], il existe une suite complète d'éclatements locaux dans $S$ qui permet de se ramener à ce cas.

Dans la suite, nous supposerons que $X$ est une variété analytique complexe, que $S = \mathbf{C}^p$ muni de ses coordonnées canonique. Le morphisme $f$ correspond alors à la donnée de $p$ fonctions holomorphes $f_1, \ldots, f_p$ sur $X$.

Usant de ce stratagème, F. Loeser dans [L] donne ainsi des résultats sur le développement asymptotique des intégrales fibres $\int_{f=t} \phi$ de $f$ et sur les pôles d'intégrales du type

$$\int_X |f_1|^{s_1} \cdots |f_p|^{s_p} \phi \, dx d\overline{x}.$$

Dans [S1], C. Sabbah montre de son côté comment construire des équations fonctionnelles du type Bernstein :

$$b(s_1, \ldots, s_p) m f_1^{s_1} \cdots f_p^{s_p} = P m f_1^{s_1+1} \cdots f_p^{s_p+1},$$

où $m$ est une section d'un $\mathcal{D}_X$-Module holonome régulier et où $b$ est un produit de formes linéaires affines à coefficients entiers positifs. Dans [S5], il utilise ces équations fonctionnelles dans le cas d'un morphisme sans éclatement en codimension zéro et à discriminant à croisement normal pour retrouver le résultat de F. Loeser. Dans sa thèse [R], O. Roualland poursuit dans cette direction et donne dans ce cadre une généralisation des résultats de D. Barlet et H.-M. Maire (voir [B-M]) sur le développement asymptotique d'intégrales fibres à une variable.

Soit $f = (f_1, \ldots, f_p) : X \to \mathbf{C}^p$, où $X$ est une variété analytique complexe. Soit $Y$ un sous-espace analytique de $X$, notons par $F$ le produit $f_1 \cdots f_p$ et désignons par $W^\sharp_{f,Y}$ l'adhérence dans $T^*X \times \mathbf{C}^p$ de

$$\{x, \xi + \sum_{i=1}^{p} s_i \frac{df_i(x)}{f_i(x)}, s_1, \ldots, s_p) \; ; \; (x, \xi) \in T^*_Y X \text{ et } (s_1, \ldots, s_p) \in \mathbf{C}^p\}.$$

Nous observons dans [B-M-M1] que les composantes irréductibles de $W^{\sharp}_{f,Y} \cap F^{-1}(0)$ sont contenues dans une réunion d'hyperplans vectoriels définis par des équations $a_1 s_1 + \cdots + a_p s_p = 0$ à coefficients entiers positifs. Nous appellons ces hyperplans les pentes de $f_{|Y}$. Nous caractérisons dans [B-M-M2] les morphismes sans pente, c'est à dire ceux dont les pentes sont réduites aux hyperplans de coordonnées de $\mathbf{C}^p$.

Les assertions suivantes sont équivalentes :
- $f_{|Y}$ est sans pente,
- $f_{|Y}$ est sans éclatement en codimension zéro et son lieu discriminant est contenu dans les hyperplans de coordonnées de $\mathbf{C}^p$.

Dans [B], J. Briançon avait étudié les germes de morphismes sans pente sur une variété analytique lisse. Il avait montré qu'ils sont caractérisés par une condition $(T)$ dite de transversalité. Il obtenait que si $f : \mathbf{C}^n_{,0} \to \mathbf{C}^p_{,0}$ est sans pente et $f^{-1}(0)$ est lisse, il existe un changement de coordonnées à la source et des entiers $a_1, \ldots, a_p$ tels que $f(x_1, \ldots, x_n) = (x_1^{a_1}, \ldots, x_p^{a_p})$.

**Définition** Si $\Lambda$ est une variété lagrangienne conique de $T^*X$, nous disons que le couple $(f, \Lambda)$ est sans pente si pour tout $T^*_Y X$ composante irréductible de $\Lambda$, le morphisme $(f_{i_1}, \ldots, f_{i_r})_{|Y}$ est sans pente où $\{i_1, \ldots, i_r\}$ est le sous-ensemble des indices $i$ de $\{1, \ldots, p\}$ tels que $f_{i|Y} \neq 0$.

L'objet de cet article est de développer pour un morphisme une théorie des cycles évanescents d'un faisceau à cohomologie $\mathbf{C}$-constructible et d'un système différentiel holonome régulier de variété caractéristique $\Lambda$ sous l'hypothèse que le couple $(f, \Lambda)$ soit sans pente.

**Etude Algébrique :**

Soit $X$ une variété analytique complexe et $M$ un $\mathcal{D}_X$-Module holonome régulier de variété caractéristique $\operatorname{car} M$. Quitte à remplacer $M$ par son image directe sur le graphe de $f$, nous pouvons supposer que les hypersurfaces $H_i = f_i^{-1}(0)$ sont lisses et que leur réunion forme un diviseur à croisements normaux. Nous montrons dans la section 3 que les assertions suivantes sont équivalentes :

- Le couple $(f, \operatorname{car} M)$ est sans pente,
- Le couple $(\mathbf{H} = (H_1, \ldots, H_p), M)$ est sans pente : toute section $m$ de $M$ satisfait pour tout $i \in \{1, \ldots, p\}$ des équations fonctionnelles :

$$b_i(t_i \frac{\partial}{\partial t_i}) m \in V_{0,\ldots,0}(\mathcal{D}_X) t_i m,$$

  où $V_{0,\ldots,0}(\mathcal{D}_X)$ est le terme d'ordre $\mathbf{0} = (0, \ldots, 0)$ de la $V$-multifiltration de $\mathcal{D}_X$ indexée par $\mathbf{Z}^p$ relativement à $\mathbf{H}$ et $(x_1, \ldots, x_n, t_1, \ldots, t_p)$ un système de coordonnées locales de $X$ dans lequel $H_i$ a pour équation $t_i = 0$.

**Supposons que** $(\mathbf{H} = (H_1, \ldots, H_p), M)$ **soit sans pente**. Dans la section 2 sur les systèmes différentiels sans pente nous montrons que :

- $M$ est muni d'une $V^{\mathbf{H}}$-multifiltration canonique notée $V_{\cdot}(M)$ du type Malgrange-Kashiwara.

Cette multifiltration permet de définir par exemple les cycles évanescents de $M$ comme le gradué d'orde $\mathbf{0}$ de la mulfiltration canonique de $M$ :

$$\Psi^{\mathbf{H}} M = \mathrm{gr}^V_{\mathbf{0}} M.$$

C'est un $\mathcal{D}_{\cap_1^p H_i}$-Module cohérent muni de l'action des opérateurs $E_i = t_i \frac{\partial}{\partial t_i}$ $(i \in \mathrm{I})$ qui commutent entre eux. Soit $\mathrm{I} \subset \{1, \ldots, p\}$, notons $\mathrm{I}^c$ son complémentaire et $\mathbf{H}_{\mathrm{I}} = (H_i, i \in \mathrm{I})$. Nous avons de plus :

– Le couple $(\mathbf{H}_{\mathrm{I}}, M)$ est sans pente.
– Pour tout $\mathbf{k}_{\mathrm{I}} \in \mathbf{Z}^{\mathrm{I}}$, le couple $(\mathbf{H}_{\mathrm{I}^c}, \mathrm{gr}^V_{\mathbf{k}_{\mathrm{I}}} M)$ est sans pente.
– Pour tout $(\mathbf{k}_{\mathrm{I}}, \mathbf{k}_{\mathrm{I}^c}) \in \mathbf{Z}^{\mathrm{I}} \times \mathbf{Z}^{\mathrm{I}^c}$ :

$$\mathrm{gr}^{V^{\mathbf{H}_{\mathrm{I}^c}}}_{\mathbf{k}_{\mathrm{I}^c}} \mathrm{gr}^{V^{\mathbf{H}_{\mathrm{I}}}}_{\mathbf{k}_{\mathrm{I}}} M = \mathrm{gr}^V_{\mathbf{k}_{\mathrm{I}}, \mathbf{k}_{\mathrm{I}^c}} M.$$

En particulier, nous obtenons :
– $\Psi^{\mathbf{H}_{\mathrm{I}^c}}(\Psi^{\mathbf{H}_{\mathrm{I}}} M) = \Psi^{\mathbf{H}} M$.
– $\Psi^{\mathbf{H}} M = \Psi^{H_p} \cdots \Psi^{H_2} \Psi^{H_1} M$.
– Les foncteurs $\Psi^{H_p}, \ldots, \Psi^{H_1}$ appliqués à $M$ commutent.

A noter que la régularité du $\mathcal{D}_X$-Module $M$ intervient dans le calcul de la variété caractéristique de $\mathcal{D}_X[s_1, \ldots, s_p] m f_1^{s_1} \ldots f_p^{s_p}$. Ce calcul utilise en effet les résultats de [S2] (théorème 3.2) qui passent par une résolution des singulatités permettant de se ramèner à l'analyse d'un $\mathcal{D}_X$-Module holonome régulier à croisement normal.

Par symétrie avec l'étude topologique qui suit, signalons que suivant [Sc-Sc], si $(\mathbf{H}, M)$ est sans pente, les images directes locales de $M$ par le morphisme $(t_1, \ldots, t_p)$ sont à cohomologie cohérente et ont comme variétés caractéristiques une réunion d'espaces conormaux aux hyperplans de coordonnées.

**Etude Topologique :**

Soit $\mathcal{F}$ un complexe de faisceaux à cohomologie $\mathbf{C}$-constructible. Notons $\operatorname{car} \mathcal{F}$ sa variété caractéristique et **supposons que le couple** $(f, \operatorname{car} \mathcal{F})$ **soit sans pente**.

Un fait remarquable est que l'hypothèse sans pente assure à l'aide de résultats de M. Kashiwara et P. Schapira [K-S] :

– Les images directes locales de $\mathcal{F}$ par le morphisme $f$ sont à cohomologie $\mathbf{C}$-constructible.
– Leurs variétés caractéristiques sont réunions de conormaux à des intersections d'hyperplans de coordonnées de $\mathbf{C}^p$.

On peut alors suivant P. Deligne [D] considérer les diagrammes cartésiens :

$$\begin{array}{ccccccc}
f^{-1}(0) & \xrightarrow{i} & X & \xleftarrow{j} & X^* = X - F^{-1}(0) & \xleftarrow{p} & \tilde{X} \\
\downarrow & & \downarrow f & & \downarrow f^* & & \downarrow \tilde{f} \\
\{0\} & \xrightarrow{i} & \mathbf{C}^p & \xleftarrow{j} & (\mathbf{C}^*)^p & \xleftarrow{p} & \mathbf{C}^p,
\end{array}$$

où $\mathbf{C}^p$ est le revêtement universel de $(\mathbf{C}^*)^p$, $p$ le morphisme :

$$\mathbf{C}^p \longrightarrow (\mathbf{C}^*)^p \ : \ p(z_1, \ldots, z_p) = (e^{2i\pi z_1}, \ldots, e^{2i\pi z_p}),$$

et $i$ , $j$ les morphismes d'inclusion. Nous posons alors :

$$\Psi_f \mathcal{F} = i^{-1} R j_* p_* p^{-1} j^{-1} \mathcal{F},$$

qui est muni d'opérateurs de monodromies $M_1, \ldots, M_p$ qui commutent entre eux.

C. Sabbah avait défini pour $\mathcal{F}$ et $f$ quelconque dans [S3] un analogue de ce complexe sous le nom de complexe d'Alexander de $\mathcal{F}$.

Sous nos hypothèses, il résulte des propriétes des images directes locales de $\mathcal{F}$ par le morphisme $f$ :

– $\Psi_f(\mathcal{F})_0 = R\Gamma(B_\epsilon \cap f^{-1}(t), \mathcal{F})$ où $B_\epsilon$ est une boule assez petite de $X$ centrée à l'origine et $t \in (\mathbf{C}^*)^p$ assez proche de l'origine.

Si $\mathrm{I} = \{i_1, \ldots, i_r\}$ et $\mathrm{I}^c = \{i_{r+1}, \ldots, i_p\}$, notons $f_{\mathrm{I}} = (f_{i_1}, \ldots, f_{i_r})$, nous montrons alors :

– $\Psi_f \mathcal{F}$ est à cohomologie $\mathbf{C}$-constructible.
– La variété caractéristique de $\Psi_f(\mathcal{F})$ n'est autre que la réunion des $W_{f,Y} \cap f^{-1}(0)$ où $T^*_Y X$ décrit les composantes irréductibles de $\operatorname{car} \mathcal{F}$ non contenues dans $F^{-1}(0)$ et $W_{f,Y}$ désigne l'espace conormal relatif au morphisme $f_{|Y}$ qui est l'adhérence des conormaux aux fibres lisses de $f_{|Y}$.
– Le couple $(\Psi_{f_{\mathrm{I}}} \mathcal{F}, \operatorname{car}(\Psi_{f_{\mathrm{I}}} \mathcal{F}))$ est sans pente.
– Les morphismes naturels $\Psi_{f_{\mathrm{I}}}(\Psi_{f_{\mathrm{I}^c}} \mathcal{F}) \to \Psi_f \mathcal{F} \leftarrow \Psi_{f_{\mathrm{I}^c}}(\Psi_{f_{\mathrm{I}}} \mathcal{F})$ sont des isomorphismes compatibles aux monodromies.
– $\Psi_f \mathcal{F} = \Psi_{f_p} \cdots \Psi_{f_2} \Psi_{f_1} \mathcal{F}$.
– Les foncteurs $\Psi_{f_p}, \ldots, \Psi_{f_1}$ appliqués à $\mathcal{F}$ commutent.

**Synthèse :**

Soit $M$ un $\mathcal{D}_X$-Module holonome régulier et $\mathcal{F}$ un complexe à cohomologie $\mathbf{C}$-constructible de variété caractéristique $\Lambda$. Posons :

$$\operatorname{Sol}_X M = R\mathcal{H}om_{\mathcal{D}_X}(M, \mathcal{O}_X).$$

Pour $p = 1$, suite aux travaux de B. Malgrange [M] et M. Kashiwara [K2], de nombreux résultats ont été établis sur $\Psi^H M$. En particulier, $\Psi^H M$ est un $\mathcal{D}_X$-Module holonome régulier supporté par $f^{-1}(0)$, $\Psi^H M$ et $\Psi_f(\operatorname{Sol}_X M)$ se correspondent par la correspondance de Riemann-Hilbert de M. Kashiwara [K3] et Z. Mebkhout [Meb] : $\operatorname{Sol}_H(\Psi^H M) = \Psi_f(\operatorname{Sol}_X M)$ et la monodromie sur $\Psi_f(\operatorname{Sol}_X M)$ se calculent à l'aide de l'action de l'opérateur d'Euler sur $M$ (voir par exemple les notes du cours au CIMPA [M-M]).

Soit $p > 1$. **Suposons le couple** $(f, \Lambda)$ **soit sans pente**. Les formules itératives :

$$\Psi^{\mathbf{H}} M = \Psi^{H_p} \cdots \Psi^{H_2} \Psi^{H_1} M \text{ et } \Psi_f \mathcal{F} = \Psi_{f_p} \cdots \Psi_{f_2} \Psi_{f_1} \mathcal{F}$$

permettent par exemple d'obtenir :

– Le $\mathcal{D}_{\cap_k^p H_k}$-Module est régulier.
– Si $\mathcal{F}$ est pervers, alors $\Psi_f \mathcal{F}$ est pervers.

– $\mathrm{Sol}_{\cap_k^p H_k}(\Psi^{\mathbf{H}} M) = \Psi_f(\mathrm{Sol}_X M)$ et pour tout $1 \leq k \leq p$, la monodromie $M_k$ sur $\Psi_f(\mathrm{Sol}_X M)$ correspond à l'action de $e^{-2i\pi E_k}$.

**Morphismes sans éclatement en codimension zéro et morphismes sans pente**

Terminons cette introduction par quelques remarques sur les morphismes sans éclatement en codimension zéro et les morphismes sans pente qui témoignent de leurs importances.

Nous avons rappelé qu'une suite d'éclatements au but transforme tout morphisme en un morphisme sans éclatement en codimension zéro. Rappelons quelques unes des propriétés des morphismes sans éclatement en codimension zéro. (voir [H-M-S]) :

– Les morphismes sans éclatement en codimension zéro sont stables par changement de base au but.
– Si nous complétons un tel morphisme par une forme linéaire générique, il reste sans éclatement en codimension zéro.
– Les morphismes finis sont sans éclatement en codimension zéro.

Les morphismes sans pente apparaissent alors naturellement par un changement de base résolvant le discriminant d'un morphisme sans éclatement en codimension zéro.

Les singularités $S$ quasi-ordinaires de dimension $d$ sont les singularités qui admettent une projection finie sur $\mathbf{C}^d$ non ramifiée en dehors d'un diviseur à croisement normal. Elles donnent des familles d'exemples de morphismes sans pente. Ainsi si $S$ est une hypersurface à singularité quasi-ordinaire de $\mathbf{C}^n$ définie par une fonction analytique $f$ et $\pi$ la projection quasi-ordinaire associée, le couple $(\pi, \mathrm{car}\, \Psi_f \mathbf{C})$ est sans pente. Dans [G-G], P. Gonzalez Perez et M. Gonzalez Villa étudie la fibre de Milnor d'une telle fonction $f$. Notre travail donne dans ce cas des informations sur $\Psi_f \mathbf{C}$ ou sur le module holonome régulier qui lui correspond. Les morphismes quasi-ordinaires sont obtenus par changements de base au but de morphismes finis. Ils sont à la source de la méthode de Jung de résolution des singularités (voir [Li]).

**Remerciements**

J'exprime tous mes remerciements à M. Merle et C. Sabbah. Ils sont autant par leurs travaux que par de nombreuses discussions à l'origine et à la conclusion de ce travail.

## 2. Cycles évanescents des systèmes différentiels sans pente

Soit $X$ une variété analytique complexe de dimension $n+p$. Nous désignons par $\mathcal{O}_X$ le faisceau des fonctions holomorphes sur $X$ et par $\mathcal{D}_X$ celui des opérateurs différentiels holomorphes.

Dans la suite, nous fixons $p$ hypersurfaces lisses $H_1, \ldots, H_p$ de $X$ d'idéaux $\mathcal{J}_1, \ldots, \mathcal{J}_p$ dont la réunion forme un diviseur à croisements normaux. Notons $\mathbf{H}$ l'ensemble $\{H_1, \ldots, H_p\}$ et pour $\mathbf{k} = (k_1, \ldots, k_p) \in \mathbf{Z}^p$, nous poserons $\mathcal{J}^{\mathbf{k}} := \prod_{i=1}^p \mathcal{J}_i^{k_i}$, avec la convention que $\mathcal{J}_i^{k_i} = \mathcal{O}_X$ pour $k_i \leq 0$. Ci-dessous, $\mathbf{Z}^p$ sera muni de son ordre partiel naturel , et on notera $\mathbf{k} < \mathbf{l} \Leftrightarrow \mathbf{k} \leq \mathbf{l}$ et $\mathbf{k} \neq \mathbf{l}$ et $\mathbf{1}_i = (0, \ldots, 1, 0, \ldots, 0)$ où le 1 est plaçé à la i-ème place.

**Définition 1.** *Pour* $\mathbf{k} \in \mathbf{Z}^p$*, et* $x \in X$*, nous posons :*

$$(V_{\mathbf{k}}^{\mathbf{H}}\mathcal{D}_X)_x := \{P \in \mathcal{D}_{X,x} \mid \forall\, \mathbf{m} \in \mathbf{Z}^p\ ,\ P(\mathcal{J}_x^{\mathbf{k}+\mathbf{m}}) \subset \mathcal{J}_x^{\mathbf{m}}\},$$

*que nous noterons* $(V_{\mathbf{k}}\mathcal{D}_X)_x$ *si aucune confusion n'est à craindre.*

De manière tout à fait analogue au cas où $p = 1$ ([K2], voir aussi [M-M], [S1]), nous définissons alors une filtration croissante $V_{\mathbf{k}}\mathcal{D}_X$ de $\mathcal{D}_X$ indexée par $\mathbf{Z}^p$, qui satisfait à

$$V_{\mathbf{k}}(\mathcal{D}_X)V_{\mathbf{m}}(\mathcal{D}_X) \subset V_{\mathbf{k}+\mathbf{m}}(\mathcal{D}_X),$$

avec égalité si les composantes de $\mathbf{k}$ et $\mathbf{m}$ ont même signe.

Soit $\mathrm{I} \subset \{1,\ldots,p\}$ et $\mathrm{I}^c$ son complémentaire. En notant $\mathbf{H}_{\mathrm{I}} = \{H_i\}_{i\in\mathrm{I}}$, nous avons pour tout $\mathbf{k}_{\mathrm{I}} \in \mathbf{Z}^{\mathrm{I}}$ :

$$V_{\mathbf{k}_{\mathrm{I}}}^{\mathbf{H}_{\mathrm{I}}}\mathcal{D}_X = \sum_{\mathbf{k}_{\mathrm{I}^c}\in\mathbf{Z}^{\mathrm{I}^c}} V_{\mathbf{k}_{\mathrm{I}},\mathbf{k}_{\mathrm{I}^c}}\mathcal{D}_X = \bigcup_{\mathbf{k}_{\mathrm{I}^c}\in\mathbf{Z}^{\mathrm{I}^c}} V_{\mathbf{k}_{\mathrm{I}},\mathbf{k}_{\mathrm{I}^c}}\mathcal{D}_X.$$

Pour $\mathrm{I} = \{i\}$ et $k \in \mathbf{Z}$, $V_k^{\mathbf{H}_{\{i\}}}\mathcal{D}_X$ noté $V_k^{H_i}\mathcal{D}_X$, n'est autre que le terme d'ordre $k$ de la $V$-filtration de $\mathcal{D}_X$ le long de l'hypersurface $H_i$ et nous avons :

$$V_{\mathbf{k}_{\mathrm{I}}}^{\mathbf{H}_{\mathrm{I}}}\mathcal{D}_X = \bigcap_{i\in\mathrm{I}} V_{k_i}^{H_i}\mathcal{D}_X.$$

Soit $\mathrm{I} \subset \{1,\ldots,p\}$, sur chaque anneau gradué :

$$\mathrm{gr}_{\mathbf{0}_{\mathrm{I}}}^{V^{\mathbf{H}_{\mathrm{I}}}}(V_{\mathbf{0}}\mathcal{D}_X) = \frac{V_{\mathbf{0}}\mathcal{D}_X}{\sum_{i\in\mathrm{I}} V_{\mathbf{0}-\mathbf{1}_i}\mathcal{D}_X}$$

sont définis les champs d'Euler $E_i$ $(i \in I)$ induits par l'action de $t_i\frac{\partial}{\partial t_i}$ dans un système de coordonnées locales $(x_1,\ldots,x_n,t_1,\ldots,t_p)$ tel que $\mathcal{J}_i = (t_i)$. Ces opérateurs commutent deux à deux.

La notion de bonne $V$-multifiltration pour un $\mathcal{D}_X$-Module cohérent $M$ a été introduite dans [S1]. Une telle multifiltration croissante $U_{\cdot}M$ est indexée par $\mathbf{Z}^p$. Pour $\mathbf{k} \in \mathbf{Z}^p$, nous utilisons la notation $U_{<\mathbf{k}}M$ pour $\sum_{\mathbf{k}'<\mathbf{k}} U_{\mathbf{k}'}M$. Si nous nous donnons une partition $\{1,\ldots,p\} = \mathrm{I}\cup\mathrm{I}^c$, nous posons :

$$U_{<\mathbf{k}_{\mathrm{I}},\mathbf{k}_{\mathrm{I}^c}}M = \sum_{\mathbf{k}'_{\mathrm{I}}<\mathbf{k}_{\mathrm{I}}} U_{\mathbf{k}'_{\mathrm{I}},\mathbf{k}_{\mathrm{I}^c}}M.$$

Nous avons des propriétés complétement analogues à celles des bonnes V-filtrations lorsque $p = 1$ ([K2], voir aussi [M-M], [S1]). Par exemple, nous avons :

- Une $V$-multifiltration $U_{\cdot}M$ de $M$ (i.e. qui satisfait à $V_{\mathbf{k}}\mathcal{D}_X.U_{\mathbf{k}'}M \subset U_{\mathbf{k}+\mathbf{k}'}M$) est bonne si et seulement si, localement, elle est engendrée par un nombre fini de sections locales $(m_j)_{j\in J}$ : pour tout $j \in J$, il existe $\mathbf{k_j} \in \mathbf{Z}^p$ tel que $U_{\mathbf{k}}M = \sum_{j\in J} V_{\mathbf{k}+\mathbf{k_j}}\mathcal{D}_X.m_j$ et cela pour tout $\mathbf{k} \in \mathbf{Z}^p$.
- Deux bonnes $V$-multifiltrations $U_{\cdot}M$ et $U'_{\cdot}M$ sont comparables localement, c'est à dire que localement il existe $\mathbf{k_0} \in \mathbf{N}^p$ tel que pour tout $\mathbf{k} \in \mathbf{Z}^p$ on ait :

$$U'_{\mathbf{k}-\mathbf{k_0}}M \subset U_{\mathbf{k}}M \subset U'_{\mathbf{k}+\mathbf{k_0}}M \subset U_{\mathbf{k}+2\mathbf{k_0}}M.$$

- Dans une suite exacte courte de $\mathcal{D}_X$-Modules cohérents, une bonne $V$-multifiltration du terme central induit une bonne $V$-multifiltration des termes extrêmes.

Considérons une bonne $V$-multifiltration $U_{\cdot}M$ d'un $\mathcal{D}_X$-Module cohérent $M$. Elle engendre une bonne $V^{\mathbf{H}_{\mathrm{I}}}$-multifiltration $U_{\cdot}^{\mathbf{H}_{\mathrm{I}}}M$ de $M$ défini pour tout $\mathbf{k}_{\mathrm{I}} \in \mathbf{Z}^{\mathrm{I}}$ par

$$U_{\mathbf{k}_{\mathrm{I}}}^{\mathbf{H}_{\mathrm{I}}}M = \sum_{\mathbf{k}_{\mathrm{I}^c} \in \mathbf{Z}^{\mathrm{I}^c}} U_{\mathbf{k}_{\mathrm{I}},\mathbf{k}_{\mathrm{I}^c}}M = \bigcup_{\mathbf{k}_{\mathrm{I}^c} \in \mathbf{Z}^{\mathrm{I}^c}} U_{\mathbf{k}_{\mathrm{I}},\mathbf{k}_{\mathrm{I}^c}}M.$$

Chaque $U_{\mathbf{k}_{\mathrm{I}}}^{\mathbf{H}_{\mathrm{I}}}M$ est un $V_{\mathbf{0}_{\mathrm{I}}}^{\mathbf{H}_{\mathrm{I}}}\mathcal{D}_X$-Module cohérent. Pour tout $J \subset \{1,\dots,p\}$ disjoint de $I$, l'anneau $V_{\mathbf{0}_{\mathrm{I}}}^{\mathbf{H}_{\mathrm{I}}}\mathcal{D}_X$ est muni d'une $V^{\mathbf{H}_{\mathrm{J}}}$-multifiltration indexée par $\mathbf{Z}^{\mathrm{J}}$ dont le terme d'ordre $\mathbf{k}_{\mathrm{J}}$ est $V_{\mathbf{0}_{\mathrm{I}},\mathbf{k}_{\mathrm{J}}}^{\mathbf{H}_{\mathrm{I}}\cup\mathbf{H}_{\mathrm{J}}}\mathcal{D}_X$. La notion de bonne $V^{\mathbf{H}_{\mathrm{J}}}$-multifiltration d'un $V_{\mathbf{0}_{\mathrm{I}}}^{\mathbf{H}_{\mathrm{I}}}\mathcal{D}_X$-Module cohérent se définit de manière analogue à ce qui a été vu ci-dessus, et satisfait à des propriétés similaires. Le Module $U_{\mathbf{k}_{\mathrm{I}}}^{\mathbf{H}_{\mathrm{I}}}M$ est alors muni de la bonne $V^{\mathbf{H}_{\mathrm{J}}}$-multifiltration dont le terme d'ordre $\mathbf{k}_{\mathrm{J}} \in \mathbf{Z}^{\mathrm{J}}$ est

$$U_{\mathbf{k}_{\mathrm{J}}}^{\mathbf{H}_{\mathrm{J}}}(U_{\mathbf{k}_{\mathrm{I}}}^{\mathbf{H}_{\mathrm{I}}}M) = U_{\mathbf{k}_{\mathrm{I}},\mathbf{k}_{\mathrm{J}}}^{\mathbf{H}_{\mathrm{I}}\cup\mathbf{H}_{\mathrm{J}}}M.$$

De même, chaque I-multigradué $\mathrm{gr}_{\mathbf{k}_{\mathrm{I}}}^{U_{\cdot}^{\mathbf{H}_{\mathrm{I}}}}M = U_{\mathbf{k}_{\mathrm{I}}}^{\mathbf{H}_{\mathrm{I}}}M/U_{<\mathbf{k}_{\mathrm{I}}}^{\mathbf{H}_{\mathrm{I}}}M$ est $\mathrm{gr}_{\mathbf{0}_{\mathrm{I}}}^{V^{\mathbf{H}_{\mathrm{I}}}}\mathcal{D}_X$-cohérent et muni d'une bonne $V^{H_{\mathrm{J}}}$-multifiltration préservée par les champs d'Euler $E_i$ $(i \in \mathrm{I})$ :

$$U_{\mathbf{k}_{\mathrm{J}}}^{\mathbf{H}_{\mathrm{J}}}(\mathrm{gr}_{\mathbf{k}_{\mathrm{I}}}^{U_{\cdot}^{\mathbf{H}_{\mathrm{I}}}}M) = \frac{U_{\mathbf{k}_{\mathrm{I}},\mathbf{k}_{\mathrm{J}}}^{\mathbf{H}_{\mathrm{I}}\cup\mathbf{H}_{\mathrm{J}}}M + U_{<\mathbf{k}_{\mathrm{I}}}^{\mathbf{H}_{\mathrm{I}}}M}{U_{<\mathbf{k}_{\mathrm{I}}}^{\mathbf{H}_{\mathrm{I}}}M} = \frac{U_{\mathbf{k}_{\mathrm{I}},\mathbf{k}_{\mathrm{J}}}^{\mathbf{H}_{\mathrm{I}}\cup\mathbf{H}_{\mathrm{J}}}M}{U_{\mathbf{k}_{\mathrm{I}},\mathbf{k}_{\mathrm{J}}}^{\mathbf{H}_{\mathrm{I}}\cup\mathbf{H}_{\mathrm{J}}}M \cap U_{<\mathbf{k}_{\mathrm{I}}}^{\mathbf{H}_{\mathrm{I}}}M}.$$

Le gradué d'ordre $\mathbf{k}_{\mathrm{J}}$ est :

$$\begin{aligned}\mathrm{gr}_{\mathbf{k}_{\mathrm{J}}}^{U_{\cdot}^{H_{\mathrm{J}}}}(\mathrm{gr}_{\mathbf{k}_{\mathrm{I}}}^{U_{\cdot}^{\mathbf{H}_{\mathrm{I}}}}M) &= \frac{U_{\mathbf{k}_{\mathrm{I}},\mathbf{k}_{\mathrm{J}}}^{\mathbf{H}_{\mathrm{I}}\cup\mathbf{H}_{\mathrm{J}}}M/U_{\mathbf{k}_{\mathrm{I}},\mathbf{k}_{\mathrm{J}}}^{\mathbf{H}_{\mathrm{I}}\cup\mathbf{H}_{\mathrm{J}}} \cap U_{<\mathbf{k}_{\mathrm{I}}}^{\mathbf{H}_{\mathrm{I}}}M}{U_{\mathbf{k}_{\mathrm{I}},<\mathbf{k}_{\mathrm{J}}}^{\mathbf{H}_{\mathrm{I}}\cup\mathbf{H}_{\mathrm{J}}}M/U_{\mathbf{k}_{\mathrm{I}},<\mathbf{k}_{\mathrm{J}}}^{\mathbf{H}_{\mathrm{I}}\cup\mathbf{H}_{\mathrm{J}}}M \cap U_{<\mathbf{k}_{\mathrm{I}}}^{\mathbf{H}_{\mathrm{I}}}M} \\ &= \frac{U_{\mathbf{k}_{\mathrm{I}},\mathbf{k}_{\mathrm{J}}}^{\mathbf{H}_{\mathrm{I}}\cup\mathbf{H}_{\mathrm{J}}}M}{U_{\mathbf{k}_{\mathrm{I}},<\mathbf{k}_{\mathrm{J}}}^{\mathbf{H}_{\mathrm{I}}\cup\mathbf{H}_{\mathrm{J}}}M + U_{\mathbf{k}_{\mathrm{I}},\mathbf{k}_{\mathrm{J}}}^{\mathbf{H}_{\mathrm{I}}\cup\mathbf{H}_{\mathrm{J}}}M \cap U_{<\mathbf{k}_{\mathrm{I}}}^{\mathbf{H}_{\mathrm{I}}}M}.\end{aligned}$$

**Définition 2.** *Soit $M$ un $\mathcal{D}_X$-Module cohérent.*

(1) *On dit que le couple $(\mathbf{H}, M)$ est multispécialisable sans pente si au voisinage de tout point de $X$, il existe une bonne $V$-multifiltration $U_{\cdot}(M)$ de $M$ et des polynômes $b_i(s) \in \mathbf{C}[s]$ $(i \in \{1,\dots,p\})$ tels que pour tout $\mathbf{k} \in \mathbf{Z}^p$, $b_i(E_i + k_i)U_{\mathbf{k}}M \subset U_{\mathbf{k}-\mathbf{1}_i}M$.*

(2) *On dit que le couple $(\mathbf{H}, M)$ est multispécialisable sans pente par section si, pour toute section locale $m$ de $M$, il existe des polynômes $b_i(s) \in \mathbf{C}[s]$ $(i \in \{1,\dots,p\})$ tels que $b_i(E_i)m \in V_{\mathbf{0}-\mathbf{1}_i}\mathcal{D}_X.m$.*

**Proposition 1.** *Les deux définitions 2.1 et 2.2 sont équivalentes et si la condition 2.1 est satisfaite pour une bonne $V$-multifiltration de $M$, elle l'est pour toute.*

On dira simplement que $(\mathbf{H}, M)$ est sans pente lorsque ces propriétés sont satisfaites. Pour toute section $m$ de $M$, on notera $b_{i,m}$ le polynôme unitaire de plus bas degré tel que $b_i(E_i)m \in V_{\mathbf{0}-\mathbf{1}_i}\mathcal{D}_X.m$ et $b_{i,U_{\cdot}(M)}$ le polynôme unitaire de plus bas degré tel que, pour tout $\mathbf{k} \in \mathbf{Z}^p$, $b_{i,U_{\cdot}(M)}(E_i + k_i)U_{\mathbf{k}}M \subset U_{\mathbf{k}-\mathbf{1}_i}M$.

**Preuve :** Supposons $(\mathbf{H}, M)$ multispécialisable sans pente par section. Soit une $V$-multifiltration $U_{\cdot}(M)$ de $M$. Localement, cette multifiltration est engendrée par un nombre fini de sections locales $(m_j)_{j\in J}$ de poids $\mathbf{k_j} \in \mathbf{Z}^p$. On en déduit alors pour tout $\mathbf{k} \in \mathbf{Z}^p$ et $i \in \{1,\dots,p\}$ :

$$\prod_{j\in J} b_{i,m_j}(E_i + k_{j,i} + k_i)U_{\mathbf{k}}M \subset U_{\mathbf{k}-\mathbf{1}_i}M,$$

où $\mathbf{k_j} = (k_{j,1},\dots,k_{j,p})$. Ainsi, toute bonne $V$-multifiltration de $M$ satisfait la définition 2.1.

Soit une $V$-multifiltration $U_{\cdot}(M)$ de $M$ vérifiant 2.1. Soit $\mathrm{I} \subset \{1, \ldots, p\}$ de cardinal $p-1$ et $\{i\} = \mathrm{I}^c$. Pour tout $m \in M$ , il existe $l_i \in \mathbf{Z}$ et $\mathbf{l}_{\mathrm{I}} \in \mathbf{Z}^{\mathrm{I}}$ tel que $m \in U_{l_i, \mathbf{l}_{\mathrm{I}}} = U^{H_i}_{\mathbf{l}_i}(U^{\mathbf{H}_{\mathrm{I}}}_{l_{\mathrm{I}}} M)$.

Considérons le $V^{\mathbf{H}_{\mathrm{I}}}_{\mathbf{0}_{\mathrm{I}}}\mathcal{D}_X$-sous-module cohérent $V^{\mathbf{H}_{\mathrm{I}}}_{\mathbf{0}_{\mathrm{I}}}\mathcal{D}_X \cdot m$ de $U^{\mathbf{H}_{\mathrm{I}}}_{\mathbf{l}_{\mathrm{I}}}M$. Il est muni de deux $V^{H_i}V^{\mathbf{H}_{\mathrm{I}}}_{\mathbf{0}_{\mathrm{I}}}\mathcal{D}_X$-multifiltrations.

– La première, $(V^{H_i}_{\cdot}V^{\mathbf{H}_{\mathrm{I}}}_{\mathbf{0}_{\mathrm{I}}}\mathcal{D}_X) \cdot m$, est celle engendrée par $m$. Elle est bonne par définition.
– La seconde est la filtration du $V^{\mathbf{H}_{\mathrm{I}}}_{\mathbf{0}_{\mathrm{I}}}\mathcal{D}_X$-sous-module $V^{\mathbf{H}_{\mathrm{I}}}_{\mathbf{0}_{\mathrm{I}}}\mathcal{D}_X \cdot m$ induite par la bonne $V^{H_i}V^{\mathbf{H}_{\mathrm{I}}}_{\mathbf{0}_{\mathrm{I}}}\mathcal{D}_X$ multifiltration $U^{H_i}_{\cdot}(U^{\mathbf{H}_{\mathrm{I}}}_{\mathbf{l}_{\mathrm{I}}}M)$. Elle est aussi bonne d'après les propriétés vues plus haut.

Elles sont donc comparables et il existe donc un entier $r \geq 0$ tel que :

$$U^{H_i}_{l_i-r}(U^{\mathbf{H}_{\mathrm{I}}}_{\mathbf{l}_{\mathrm{I}}}M) \cap V^{\mathbf{H}_{\mathrm{I}}}_{\mathbf{0}_{\mathrm{I}}}\mathcal{D}_X m \subset V^{H_i}_{-1}(V^{\mathbf{H}_{\mathrm{I}}}_{\mathbf{0}_{\mathrm{I}}}\mathcal{D}_X m).$$

Il en résulte :

$$b_{i,U_{\cdot}(M)}(E_i + l_i - r + 1) \cdots b_{i,U_{\cdot}(M)}(E_i + l_i)m \in V_{\mathbf{0}-\mathbf{1}_i}\mathcal{D}_X m.$$

Cela montre que $(\mathbf{H}, M)$ est multispécialisable sans pente par section.

**Théorème 1.** *Si le couple $(\mathbf{H}, M)$ est sans pente, il existe une unique $V$-multifiltration globale note $V_{\cdot}(M)$ telle que les parties réelles des racines des polynômes $b_{i,V_{\cdot}(M)}$ soient dans l'intervalle $[-1, 0[$. De plus, nous avons pour tout $x \in X$ et $\mathbf{k} \in \mathbf{Z}^p$ :*

$$(V_{\mathbf{k}}M)_x = \{m \in M_x \; ; \; \operatorname{Re}\alpha \geq -k_i - 1 \; , \; \forall \; \alpha \in b^{-1}_{i,m}(0) \text{ et } 1 \leq i \leq p\},$$

*où $\operatorname{Re}\alpha$ désigne la partie réelle du nombre $\alpha$.*

On appelle cette bonne $V$-multifiltration, la $V$-multifiltration de Malgrange-Kashiwara (ou multifiltration canonique de $M$).

**Preuve :** Supposons $(\mathbf{H}, M)$ sans pente. On montre comme pour $p = 1$, en utilisant des opérations élémentaires de décalage d'entiers, l'existence d'une bonne $V$-multifiltration de $M$ comme annoncée dans le théorème. Son unicité se prouve en utilisant un argument de E. Bézout. Soit $m \in (V_{\mathbf{k}}M)_x$. Suivant la preuve de la proposition 1, le polynôme $b_{i,m}$ divise $b_{i,V_{\cdot}(M)}(s + k_i - r + 1) \cdots b_{i,V_{\cdot}(M)}(s + k_i)$ pour $r$ entier assez grand. Donc les parties réelles des racines des $b_{i,m}$ sont supérieures ou égales à $-k_i - 1$. Inversement, soit $m \in M_x$ tel que les parties réelles des racines des $b_{i,m}$ sont supérieures ou égales à $-k_i - 1$. Il existe $\mathbf{r} \in \mathbf{N}^p$ tel que $m \in V_{\mathbf{k}+\mathbf{r}}M$. On a :

$$\begin{gathered} b_{i,m}(E_i)m \in V_{\mathbf{0}-\mathbf{1}_i}\mathcal{D}_X.m \in V_{\mathbf{k}+\mathbf{r}-\mathbf{1}_i}M, \\ b_{i,V_{\cdot}(M)}(E_i + k_i + r_i)m \in V_{\mathbf{k}+\mathbf{r}-\mathbf{1}_i}M. \end{gathered}$$

Si $r_i > 0$, les polynômes $b_{i,V_{\cdot}(M)}(s + k_i + r + 1)$ et $b_{i,m}(s)$ sont premiers entre eux. En utilisant une identité de E. Bézout, on obtient $m \in V_{\mathbf{k}+\mathbf{r}-\mathbf{1}_i}M$. Il reste à itérer pour obtenir $m \in (V_{\mathbf{k}}M)_x$.

**Corollaire 1.** *Supposons $(\mathbf{H}, M)$ sans pente.*

– *Alors pour tout $\mathrm{I} \subset \{1, \ldots, p\}$, le couple $(\mathbf{H}_{\mathrm{I}}, M)$ est sans pente.*
– *La $V^{\mathbf{H}_{\mathrm{I}}}\mathcal{D}_X$-multifiltration canonique de $M$ est la multifiltration $V^{\mathbf{H}_{\mathrm{I}}}_{\cdot}M$ associée à la multifiltration canonique de $M$ : $V^{\mathbf{H}_{\mathrm{I}}}_{\mathbf{k}_{\mathrm{I}}}M = \sum_{\mathbf{k}_{\mathrm{I}^c} \in \mathbf{Z}^{\mathrm{I}^c}} V_{\mathbf{k}_{\mathrm{I}}, \mathbf{k}_{\mathrm{I}^c}}M$.*

*Nous avons de plus :*

(1) *Pour tout $x \in X$ :*

$$(V^{\mathbf{H}_{\mathrm{I}}}_{\mathbf{k}_{\mathrm{I}}}M)_x = \{m \in M_x \; ; \; \forall \alpha \in b^{-1}_{i,m}(0) \text{ et } \forall \, i \in \mathrm{I} \; : \; \operatorname{Re}\alpha \geq -k_i - 1\}.$$

(2) *Pour tout $I, J \subset \{1, \ldots, p\}$ d'intersection vide et $(\mathbf{k}_{\mathrm{I}}, \mathbf{k}_{\mathrm{J}}) \in \mathbf{Z}^{\mathrm{I}} \times \mathbf{Z}^{\mathrm{J}}$ :*

$$V^{\mathbf{H}_{\mathrm{I}} \cup \mathbf{H}_{\mathrm{J}}}_{\mathbf{k}_{\mathrm{I}}, \mathbf{k}_{\mathrm{J}}}M = V^{\mathbf{H}_{\mathrm{I}}}_{\mathbf{k}_{\mathrm{I}}}M \cap V^{\mathbf{H}_{\mathrm{J}}}_{\mathbf{k}_{\mathrm{J}}}M.$$

(3) *Pour tout* $\mathbf{k} \in \mathbf{Z}^p$, $V_{\mathbf{k}}M = \cap_{i=1}^{p} V_{k_i}^{H_i} M$.

Nous voyons ainsi que si $(\mathbf{H}, M)$ est sans pente, la bonne $V$-multifiltration canonique $V_{\cdot}M$ est sans pente au sens de [S1] p.309.

**Preuve** : Il résulte de la définition de $V_{\cdot}^{\mathbf{H}_{\mathrm{I}}}$ que pour tout $i \in \mathrm{I}$ et $\mathbf{k}_{\mathrm{I}} \in \mathbf{Z}^{\mathrm{I}}$ :

$$b_{i,V_{\cdot}(M)}(t_i \frac{\partial}{\partial t_i}) V_{\mathbf{k}_{\mathrm{I}}}^{\mathbf{H}_{\mathrm{I}}} M \subset V_{\mathbf{k}_{\mathrm{I}}-\mathbf{1}_i}^{\mathbf{H}_{\mathrm{I}}} M.$$

Ainsi, $(\mathbf{H}_{\mathrm{I}}, M)$ est sans pente de filtration canonique $V_{\cdot}^{\mathbf{H}_{\mathrm{I}}}M$. Une section $m$ appartient à $V_{\mathbf{k}_{\mathrm{I}}}^{\mathbf{H}_{\mathrm{I}}}M$ si et seulement s'il existe $\mathbf{l}_{\mathrm{I}^c} \in \mathbf{Z}^{\mathrm{I}^c}$ tel que $m \in V_{\mathbf{k}_{\mathrm{I}},\mathbf{l}_{\mathrm{I}^c}}M$. Donc, si et seulement si les parties réelles des racine des $b_{i,m}$ pour $i \in \mathrm{I}$ sont supérieures ou égales à $-k_i - 1$. Cela montre l'assertion 1. Les deux autres assertions s'en déduisent.

**Proposition 2.** *Supposons* $(\mathbf{H}, M)$ *sans pente. Pour tout* $I, J \subset \{1, \dots, p\}$ *d'intersection vide, nous avons les isomorphismes naturels de* $\mathrm{gr}_{\mathbf{0}_{\mathrm{I}\cup\mathrm{J}}}^{V^{\mathbf{H}_{\mathrm{I}}\cup\mathbf{H}_{\mathrm{J}}}}(V_{\mathbf{0}}\mathcal{D}_X)$*-Modules :*

$$\mathrm{gr}_{\mathbf{k}_{\mathrm{J}}}^{V_{\cdot}^{\mathbf{H}_{\mathrm{J}}}}(\mathrm{gr}_{\mathbf{k}_{\mathrm{I}}}^{V_{\cdot}^{\mathbf{H}_{\mathrm{I}}}} M) = \frac{V_{\mathbf{k}_{\mathrm{I}},\mathbf{k}_{\mathrm{J}}}^{\mathbf{H}_{\mathrm{I}}\cup\mathbf{H}_{\mathrm{J}}} M}{V_{<(\mathbf{k}_{\mathrm{I}},\mathbf{k}_{\mathrm{J}})}^{\mathbf{H}_{\mathrm{I}}\cup\mathbf{H}_{\mathrm{J}}} M} = \mathrm{gr}_{\mathbf{k}_{\mathrm{I}},\mathbf{k}_{\mathrm{J}}}^{V_{\cdot}^{\mathbf{H}_{\mathrm{I}}\cup\mathbf{H}_{\mathrm{J}}}} M.$$

**Preuve** : La proposition résulte directement du lemme suivant.

**Lemme 1.** *Sous les hypothèses de la proposition 2, nous avons :*

$$V_{\mathbf{k}_{\mathrm{I}},<\mathbf{k}_{\mathrm{J}}}^{\mathbf{H}_{\mathrm{I}}\cup\mathbf{H}_{\mathrm{J}}} M + (V_{\mathbf{k}_{\mathrm{I}},\mathbf{k}_{\mathrm{J}}}^{\mathbf{H}_{\mathrm{I}}\cup\mathbf{H}_{\mathrm{J}}} M \cap V_{<\mathbf{k}_{\mathrm{I}}}^{\mathbf{H}_{\mathrm{I}}} M) = V_{<(\mathbf{k}_{\mathrm{I}},\mathbf{k}_{\mathrm{J}})}^{\mathbf{H}_{\mathrm{I}}\cup\mathbf{H}_{\mathrm{J}}} M.$$

**Preuve** : Le point clef est de montrer l'inclusion :

$$V_{\mathbf{k}_{\mathrm{I}},\mathbf{k}_{\mathrm{J}}}^{\mathbf{H}_{\mathrm{I}}\cup\mathbf{H}_{\mathrm{J}}} M \cap V_{<\mathbf{k}_{\mathrm{I}}}^{\mathbf{H}_{\mathrm{I}}} M \subset V_{<(\mathbf{k}_{\mathrm{I}},\mathbf{k}_{\mathrm{J}})}^{\mathbf{H}_{\mathrm{I}}\cup\mathbf{H}_{\mathrm{J}}} M.$$

Soit $m \in V_{\mathbf{k}_{\mathrm{I}},\mathbf{k}_{\mathrm{J}}}^{\mathbf{H}_{\mathrm{I}}\cup\mathbf{H}_{\mathrm{J}}} M \cap V_{<\mathbf{k}_{\mathrm{I}}}^{\mathbf{H}_{\mathrm{I}}} M$. Soit $j \in J$. Pour $a \geq 0$ entier assez grand :

$$m \in V_{\mathbf{k}_{\mathrm{I}},\mathbf{k}_{\mathrm{J}}}^{\mathbf{H}_{\mathrm{I}}\cup\mathbf{H}_{\mathrm{J}}} M \cap V_{<\mathbf{k}_{\mathrm{I}},k_j+a}^{\mathbf{H}_{\mathrm{I}\cup\{j\}}} M.$$

Nous avons :

$$b_{j,V_{\cdot}(M)}(E_j + k_j + 1) \cdots b_{j,V_{\cdot}(M)}(E_j + k_j + a) m \in V_{\mathbf{k}_{\mathrm{I}},\mathbf{k}_{\mathrm{J}}}^{\mathbf{H}_{\mathrm{I}}\cup\mathbf{H}_{\mathrm{J}}} M \cap V_{<\mathbf{k}_{\mathrm{I}},k_j}^{\mathbf{H}_{\mathrm{I}\cup\{j\}}} M.$$

$$b_{j,m}(E_j + k_j) m \in V_{(\mathbf{k}_{\mathrm{I}},\mathbf{k}_{\mathrm{J}})-\mathbf{1}_j}^{\mathbf{H}_{\mathrm{I}}\cup\mathbf{H}_{\mathrm{J}}} M.$$

Si $a > 0$, les polynômes $b_{j,V_{\cdot}(M)}(s + k_j + 1) \cdots b_{j,V_{\cdot}(M)}(s + k_j + a)$ et $b_{j,m}(s + k_j)$ sont premiers entre eux, nous déduisons d'une relation de E. Bézout :

$$m \in V_{<(\mathbf{k}_{\mathrm{I}},\mathbf{k}_{\mathrm{J}})}^{\mathbf{H}_{\mathrm{I}}\cup\mathbf{H}_{\mathrm{J}}} M + (V_{\mathbf{k}_{\mathrm{I}},\mathbf{k}_{\mathrm{J}}}^{\mathbf{H}_{\mathrm{I}}\cup\mathbf{H}_{\mathrm{J}}} M \cap V_{<(\mathbf{k}_{\mathrm{I}},k_j)}^{\mathbf{H}_{\mathrm{I}\cup\{j\}}} M).$$

Le point clef en résulte par récurrence sur le cardinal de J.

Il résulte de la proposition 2 que pour $\mathrm{I} = \{i_1, \dots, i_r\}$ et tout $\mathbf{k}_{\mathrm{I}} \in \mathbf{Z}^{\mathrm{I}}$, le gradué $\mathrm{gr}_{\mathbf{k}_{\mathrm{I}}}^{V_{\cdot}^{\mathbf{H}_{\mathrm{I}}}} M$ est obtenu comme le gradué pour les filtrations induites par les $V_{\cdot}^{H_{i_s}} M$ sur les gradués précédents $\mathrm{gr}_{k_{i_{s-1}}}^{V_{\cdot}^{\mathbf{H}_{i_{s-1}}}} \cdots \mathrm{gr}_{k_{i_1}}^{V_{\cdot}^{\mathbf{H}_{i_1}}} M$.

**Définition 3.** *Si* $(\mathbf{H}, M)$ *est sans pente, on appelle cycles* $\mathbf{H}_{\mathrm{I}}$*-proches et* $\mathbf{H}_{\mathrm{I}^c}$*-évanescents de* $M$ *le* $\mathrm{gr}_{\mathbf{0}}^{V}\mathcal{D}_X$*-Module :*

$$\Psi_{\mathbf{H}_{\mathrm{I}}} \Phi_{\mathbf{H}_{\mathrm{I}^c}} M = \Phi_{\mathbf{H}_{\mathrm{I}^c}} \Psi_{\mathbf{H}_{\mathrm{I}}} M := \mathrm{gr}_{-\mathbf{1}_{\mathrm{I}},\mathbf{0}_{\mathrm{I}^c}}^{V} M.$$

*On appelle cycles* $\mathbf{H}$*-proches de* $M$ *et cycles* $\mathbf{H}$*-évanescents de* $M$ *respectivement :*

$$\Psi_{\mathbf{H}} M = \mathrm{gr}_{-\mathbf{1}}^{V} M \quad \text{et} \quad \Phi_{\mathbf{H}} M = \mathrm{gr}_{\mathbf{0}}^{V} M.$$

Rappelons que $b_{i,V_{\cdot}(M)}(E_i+k_i)$ annule $\mathrm{gr}^V_{\mathbf{k}}M$ et que les parties réelles des racines des polynômes $b_{i,V_{\cdot}(M)}(s+k_i)$ appartiennent à l'intervalle $[-k_i-1,-k_i[$. Soit, $\alpha\in\mathbf{C}^p$ et

$$\lceil\alpha\rceil=(\lceil\alpha_1\rceil,\ldots,\lceil\alpha_p\rceil),$$

où $\lceil\alpha_i\rceil$ est le plus grand entier inférieur ou égal à Re $\alpha_i$. On note alors :

$$\Psi_{\mathbf{H},\alpha}M=\cap_{i=1}^p(\cup_{N\in\mathbf{N}}\mathrm{Ker}\,[(E_i+\alpha_i)^N:\mathrm{gr}^V_{\lceil\alpha\rceil}M\to\mathrm{gr}^V_{\lceil\alpha\rceil}M]).$$

Les intersections et réunions commutent, car les réunions sont localement finis comme les multiplicités de la racine $-\alpha_i$ de $b_{i,V_{\cdot}(M)}(s)$. Comme les opérateurs $E_i$ commutent, nous avons pour tout $\mathbf{k}\in\mathbf{Z}^p$ :

$$\mathrm{gr}^V_{\mathbf{k}}M=\bigoplus_{\alpha;\lceil\alpha\rceil=\mathbf{k}}\Psi_{\mathbf{H},\alpha}M.$$

En particulier :

$$\Psi_{\mathbf{H}_\mathrm{I}}\Phi_{\mathbf{H}_{\mathrm{I}^c}}M=\bigoplus_{\lceil\alpha_\mathrm{I}\rceil=-\mathbf{1}_\mathrm{I};\lceil\alpha_{\mathrm{I}^c}\rceil=\mathbf{0}_{\mathrm{I}^c}}\Psi_{\mathbf{H},\alpha}M.$$

Nous notons que les $\alpha$ pour lesquels $\Psi_{\mathbf{H},\alpha}M\neq 0$ sont dans un réseau défini à l'aide des racines des polynômes $b_{i,V_{\cdot}(M)}$.

Supposons $H_1,\ldots,H_p$ munis d'équations globales. Le gradué $\mathrm{gr}^{V^{\mathbf{H}_\mathrm{I}}}_{\mathbf{0}_\mathrm{I}}\mathcal{D}_X$ s'identifie à

$$\mathcal{D}_{\cap_{i\in\mathrm{I}}H_i}[E_1,\ldots,E_p].$$

Pour $j\in\mathrm{I}^c$, nous continuons à noter $H_j$ l'hypersurface $H_j\cap_{i\in\mathrm{I}}H_i$ de $\cap_{i\in\mathrm{I}}H_i$.

**Proposition 3.** *Supposons $H_1,\ldots,H_p$ munis d'équations globales et le couple $(\mathbf{H},M)$ sans pente. Soit $I=\{i_1,\ldots,i_r\}$, $I^c=\{i_{r+1},\ldots,i_p\}$ et $\mathbf{k}_\mathrm{I}\in\mathbf{Z}^\mathrm{I}$.*

- *Le $\mathcal{D}_{\cap_{i\in\mathrm{I}}H_i}[(E_i)_{i\in\mathrm{I}}]$-Module $\mathrm{gr}^{V^{\mathbf{H}_\mathrm{I}}_{\cdot}}_{\mathbf{k}_\mathrm{I}}M$ est un $\mathcal{D}_{\cap_{i\in\mathrm{I}}H_i}$-Module cohérent.*
- *Le couple $(\mathbf{H}_{\mathrm{I}^c},\mathrm{gr}^{V^{\mathbf{H}_\mathrm{I}}_{\cdot}}_{\mathbf{k}_\mathrm{I}}M)$ est sans pente.*
- *Sa filtration canonique est la filtration $V^{\mathbf{H}_{\mathrm{I}^c}}_{\cdot}(\mathrm{gr}^{V^{\mathbf{H}_\mathrm{I}}_{\cdot}}_{\mathbf{k}_\mathrm{I}}M)$ induite par la filtration canonique de $M$.*
- *Nous avons l'égalité de $\mathcal{D}_{\cap_{i=1}^pH_i}[(E_i)_{i\in\mathrm{I}}]$-Module :*
$$\Psi_{\mathbf{H}_\mathrm{I}}\Phi_{\mathbf{H}_{\mathrm{I}^c}}M=\Psi_{H_{i_r}}\cdots\Psi_{H_{i_1}}\Phi_{H_{i_p}}\cdots\Phi_{H_{i_{r+1}}}M.$$
- *Les foncteurs $\Phi_{H_{i_{r+1}}},\cdots,\Phi_{H_{i_p}},\Psi_{H_{i_1}},\cdots,\Psi_{H_{i_r}}$ commutent quand ils sont appliqués à $M$.*
- *Pour tout $\alpha\in\mathbf{C}^p$, $\Psi_{\mathbf{H},\alpha}M=\Psi_{H_1,\alpha_1}\ldots\Psi_{H_p,\alpha_pM}$.*

**Preuve** : C'est une reformulation de la proposition 2. La cohérence de $\mathrm{gr}^{V^{\mathbf{H}_\mathrm{I}}_{\cdot}}_{\mathbf{k}_\mathrm{I}}M$ comme $\mathcal{D}_{\cap_{i\in\mathrm{I}}H_i}$-Module provient du fait que ce module est annulé par les opérateurs $b_{i,V_{\cdot}(M)}(E_i+k_i)$ pour $i\in\mathrm{I}$. L'assertion sur les $\Psi_{\mathbf{H},\alpha}M$ provient du fait que les opérateurs d'Euler commutent et sont d'ordre $\mathbf{0}$.

Nous notons que $\Psi_{\mathbf{H}_\mathrm{I}}\Phi_{\mathbf{H}_{\mathrm{I}^c}}M=\Psi_{\mathbf{H}_\mathrm{I}}(\Phi_{\mathbf{H}_{\mathrm{I}^c}}M)=\Phi_{\mathbf{H}_{\mathrm{I}^c}}(\Psi_{\mathbf{H}_\mathrm{I}}M)$.

## 3. Morphisme sans pente et systèmes différentiels holonomes réguliers

3.1. **Morphisme sans pente.** Soit $X$ un germe de variété analytique de dimension $n$ et $Y$ un sous-espace irréductible de $X$. Soit $f_1,\ldots,f_p$, $p$ fonctions analytiques sur $X$ nulles à l'origine. Notons par $F$ leur produit $f_1f_2\cdots f_p$, désignons par $f$ l'application :

$$f:X\longrightarrow\mathbf{C}^p\quad,\quad(x_1,\ldots,x_n)\longmapsto(f_1(x_1,\ldots,x_n),\ldots,f_p(x_1,\ldots,x_n))$$

et par $f_{|Y}$ sa restriction à $Y$.

On désigne par $T^*X$ le fibré cotangent à $X$ et $\pi : T^*X \to X$ sa projection canonique. L'adhérence du fibré conormal à la partie lisse de $Y$ est notée $T^*_Y X$ et appelée espace conormal à $Y$ dans $X$.

Dans ce sous-paragraphe 3.1, sauf mention du contraire, on supposera que $F$ n'est pas identiquement nulle sur $Y$.

Nous désignons par $W^\sharp_{f_1,\dots,f_p,Y}$ l'adhérence dans $T^*X \times \mathbf{C}^p$ de

$$\{(x, \xi + \sum_{i=1}^p s_i \frac{df_i(x)}{f_i(x)}, s_1, \dots, s_p) \ ; \ s_i \in \mathbf{C} \ , \ (x,\xi) \in T^*_Y X \text{ et } F(x) \neq 0\}.$$

C'est un sous-espace irréductible de $T^*X \times \mathbf{C}^p$ de dimension $n+p$.

Notons $\pi_2 : T^*X \times \mathbf{C}^p \to \mathbf{C}^p$ la projection canonique $(x,\xi,s) \mapsto s$ et par $\pi_1 : T^*X \times \mathbf{C}^p \to T^*X$ la projection canonique $(x,\xi,s) \mapsto (x,\xi)$.

**Remarque 1.** *Dans* [B-M-M1]*, nous avions établi les résultats suivants :*

(1) *Les fibres réduites de la restriction de $\pi_2$ à $W^\sharp_{f_1,\dots,f_p,Y}$ sont des sous-espaces lagrangiens de $T^*X$. La fibre au-dessus de l'origine est un sous-espace lagrangien conique que nous noterons $W^\sharp_{f_1,\dots,f_p,Y}(0)$.*

(2) *La projection par $\pi_2$ de $W^\sharp_{f_1,\dots,f_p,Y} \cap F^{-1}(0)$ est une réunion d'hyperplans vectoriels de $\mathbf{C}^p$ dont les équations sont des formes linéaires à coefficients entiers positifs ou nuls. Plus précisément, soit $G$ une composante irréductible de $W^\sharp_{f_1,\dots,f_p,Y} \cap F^{-1}(0)$, si $n_j$ désigne la multiplicité de $f_j$ le long de $G$, $\pi_2(G)$ est l'hyperplan de $\mathbf{C}^p$ d'équations : $n_1 s_1 + \cdots + n_p s_p = 0$.*

(3) *La partie de $W^\sharp_{f_1,\dots,f_p,Y}$ au-dessus de la droite vectorielle $s_1 = \cdots = s_p$ s'identifie à $W^\sharp_{F,Y}$.*

**Définition 4.** *Suivant* [B-M-M2]*, nous dirons que le morphisme $f_{|Y}$ est sans pente si la projection $\pi_2(W^\sharp_{f_1,\dots,f_p,Y} \cap F^{-1}(0))$ est la réunion des hyperplans de coordonnées.*

Nous désignerons par $W_{f,Y}$ l'adhérence dans $T^*X$ des espaces conormaux aux fibres de $f_{|Y}$. Cet espace $W_{f,Y}$ est donc l'adhérence dans $T^*X$ de

$$\{(x, \xi + \sum_{i=1}^p \lambda_i df_i(x) \ ; \ \lambda_i \in \mathbf{C} \text{ et } (x,\xi) \in T^*_Y X\}.$$

C'est un sous-espace irréductible de $T^*X$ de dimension $n+r$ où $r$ est le rang de l'application $f_{|Y} : Y \to \mathbf{C}^p$.

Si l'on considère le diagramme entre espaces cotangents associé à $f$ :

$$T^*X \xleftarrow{{}^t f'} X \times_{\mathbf{C}^p} T^*\mathbf{C}^p \xrightarrow{f_\pi} T^*\mathbf{C}^p,$$

l'espace $W_{f,Y}$ n'est autre que l'adhérence de

$$T^*_Y X + {}^t f'(X \times_{\mathbf{C}^p} T^*\mathbf{C}^p).$$

Suivant J.-P. Henry, M. Merle et C. Sabbah dans [H-M-S] , donnons la définition d'un morphisme sans éclatement en codimension zéro.

**Définition 5.** *([H-M-S]) Le morphisme $f_{|Y}$ est dit sans éclatement en codimension zéro si $W_{f,Y} \cap (f=0)$ est de dimension inférieure ou égale à $n$. Cette intersection est alors une sous variété lagrangienne de $T^*X$ que nous noterons $W^0_{f,Y}$.*

Nous noterons que la définition de $W_{f,Y}$ et de $f_{|Y}$ sans éclatement en codimension zéro ne demande aucune hypothèse sur la restriction de $F$ à $Y$ (certains $f_i$ peuvent s'annuler sur $Y$).

Nous notons $Crit^0(f_1,\dots,f_p,Y)$ l'adhérence de l'ensemble des $y \in Y$ , $F(y) \neq 0$ pour lesquels il existe une solution $(s_1,\dots,s_p) \neq 0$ de l'équation :

$$\sum_{i=1}^{p} s_i \frac{df_i(x)}{f_i(x)} \in T^*_Y X.$$

**Théorème 2.** *([B-M-M2], théorème 2.7) $f_{|Y}$ est sans pente si et seulement si $f_{|Y}$ est sans éclatement en codimension zéro et $Crit^0(f_1,\dots,f_p,Y)$ est vide.*

**Remarque 2.** *Rappelons ([B-M-M2], preuve du théorème 2.7) que si $f_{|Y}$ est sans pente :*
- *$\forall i \in \{1,\dots,p\}$ , $W^\sharp_{f,Y} \cap f_i^{-1}(0) \subset W^\sharp_{f,Y} \cap s_i^{-1}(0)$.*
- *Le morhisme $\pi_1$ : $W^\sharp_{f,Y} \to W_{f,Y}$ est fini et propre.*
- *L'espace $W^\sharp_{f,Y} \cap f^{-1}(0)$ est contenue dans $s_1 = \cdots = s_p = 0$ et s'identifie à la variété lagrangienne $W_{f,Y} \cap f^{-1}(0)$.*

Notons $i$ l'immersion fermée :

$$i : X \to X \times \mathbf{C}^p \ , \ x \longmapsto (x, t_1 = f_1(x), \dots, t_p = f_p(x)).$$

Considérons le diagramme entre espaces cotangents associé à $i$ :

$$T^*X \xleftarrow{{}^t i'} X \times_{X\times\mathbf{C}^p} T^*(X\times\mathbf{C}^p) \xrightarrow{i_\pi} (X \times \mathbf{C}^p).$$

Nous pouvons vérifier :

$$i_\pi({}^t i')^{-1}(T^*_Y X) = T^*_{i(Y)}(X \times \mathbf{C}^p).$$

Les espaces $W^\sharp_{(t_1,\dots,t_p),i(Y)}$ et $W^\sharp_{f,Y}$ s'identifient. On en déduit :

**Remarque 3.** *$f_{|Y}$ est sans pente si et seulement si $(t_1,\dots,t_p)_{|i(Y)}$ est sans pente.*

**Proposition 4.** *Supposons que $f_{|Y}$ soit sans pente, alors pour tout sous-ensemble $\{i_1,\dots,i_r\}$ de $\{1,\dots,p\}$ le morphisme $(f_{i_1},\dots,f_{i_r})_{|Y}$ est sans pente.*

**Preuve :** On peut supposer $\{i_1,\dots,i_r\} = \{1,\dots,r\}$, posons $f' = (f_1,\dots,f_r)$ et $F' = f_1 f_2 \dots f_r$. Comme $F$ est non nulle sur l'espace irréductible $Y$, tout $(x,\xi) \in T^*_Y X$ est limite de points n'appartenant pas à $F^{-1}(0)$. Ainsi, l'adhérence de

$$\{(x, \xi + \sum_{i=1}^{r} s_i \frac{df_i(x)}{f_i(x)}, s_1,\dots,s_r,0,\dots,0) \ ; \ s_i \in \mathbf{C}\,,\ (x,\xi) \in T^*_Y X \text{ et } F'(x) \neq 0\}$$

est contenu dans $W^\sharp_{f,Y} \cap (s_{r+1} = \ldots = s_p = 0)$. Comme cette adhérence s'identifie à $W^\sharp_{f',Y}$ :

$$W^\sharp_{f',Y} \subset W^\sharp_{f,Y} \cap (s_{r+1} = \cdots = s_p = 0).$$

Si $G$ est une composante irréductible de $W^\sharp_{f',Y} \cap F'^{-1}(0)$, $G$ est contenue dans un $f_i = 0$ pour $i \in \{1,\dots,r\}$. Ainsi, $G \subset W^\sharp_{f,Y} \cap f_i^{-1}(0)$ est d'après la remarque 2 contenue dans $s_i = 0$.

**Proposition 5.** *Supposons $f_{|Y}$ sans pente. Posons, $f' = (f_1, \dots, f_r)$, $f'' = (f_{r+1}, \dots, f_p)$. Si $T^*_Z X$ est une composante irréductible de $W^\sharp_{f'',Y}(0)$ et que $F' = f_1 \dots f_r$, est non identiquement nul sur $Z$, alors $f'_{|Z}$ est sans pente.*

**Preuve :** Posons : $F'' = f_{r+1} \dots f_p$. Nous allons montrer que :

$$W^\sharp_{f',Z} \subset W^\sharp_{f,Y} \cap (s_{r+1} = \dots = s_p = 0).$$

Soit $(x, \xi, s') \in W^\sharp_{f',Z}$, par définition ce point est limite de points :

$$(y, \eta + \sum_{i=1}^{r} s'_i \frac{df_i(y)}{f_i(y)}, s'_1, \dots, s'_r)\ ;\ s'_i \in \mathbf{C}\ ,\ (y,\eta) \in T^*_Z X \text{ et } F'(y) \neq 0).$$

D'autre part, les $(y, \eta, 0) \in W^\sharp_{f'',Y}(0)$ sont par définition limites de points :

$$(z, \lambda + \sum_{i=r+1}^{p} s''_i \frac{df_i(z)}{f_i(z)}, s''_{r+1}, \dots, s''_p)\ ;\ s''_i \in \mathbf{C}\ ,\ (z,\lambda) \in T^*_Y X \text{ et } F''(z) \neq 0).$$

Pour $z$ assez proche de $y$, $F'(z) \neq 0$ et $(x, \xi, s', 0)$ est limite de points :

$$(z, \lambda + \sum_{i=1}^{r} s'_i \frac{df_i(z)}{f_i(z)} + \sum_{i=r+1}^{p} s''_i \frac{df_i(z)}{f_i(z)}, s'_1, \dots, s'_r, s''_{r+1}, \dots, s''_p),$$

où $s'_i$ , $s''_i \in \mathbf{C}$, $(z, \lambda) \in T^*_Y X$ et $F(z) \neq 0$. Il en résulte l'inclusion attendue. La preuve se termine comme la preuve de la proposition 4.

**Proposition 6.** *Supposons $f_{|Y}$ sans pente. Posons, $f' = (f_1, \dots, f_r)$ et $F' = f_1 \dots f_r$. Soit $T^*_Z X$ une composante irréductible de $W^\sharp_{F,Y}(0)$, si $F'$ est non identiquement nul sur $Z$, alors $f'_{|Z}$ est sans pente.*

**Preuve :** Nous allons montrer que

$$W^\sharp_{f',Z} \subset W^\sharp_{f,Y} \cap (s_{r+1} = \dots = s_p = 0).$$

Soit $(x, \xi, s') \in W^\sharp_{f',Z}$, par définition ce point est limite de points :

$$(y, \eta + \sum_{i=1}^{r} s'_i \frac{df_i(y)}{f_i(y)}, s'_1, \dots, s'_r)\ ;\ s'_i \in \mathbf{C}\ ,\ (y,\eta) \in T^*_Z X \text{ et } F'(y) \neq 0).$$

D'autre part, les $(y, \eta, 0) \in W^\sharp_{F,Y}(0)$ sont par définition limites de points :

$$(z, \lambda + \sum_{i=1}^{p} s \frac{df_i(z)}{f_i(z)}, s)\ ;\ s \in \mathbf{C}\ ,\ (z,\lambda) \in T^*_Y X \text{ et } F(z) \neq 0).$$

Ainsi, $(x, \xi, s', 0)$ est limite de points :

$$(z, \lambda + \sum_{i=1}^{r} (s + s'_i) \frac{df_i(z)}{f_i(z)} + \sum_{i=r+1}^{p} s \frac{df_i(z)}{f_i(z)}, s + s'_1, \dots, s + s'_r, s, \dots, s),$$

où $s'_i$, $s \in \mathbf{C}$, $(z, \lambda) \in T^*_Y X$ et $F(z) \neq 0$. Il en résulte l'inclusion attendue. La preuve se termine comme la preuve de la proposition 4.

**Proposition 7.** *Si $T^*_Z X$ est une composante irréductible de $W^\sharp_{f'',Y}(0)$, alors :*

$$W^\sharp_{f',Z}(0) \subset W^\sharp_{f,Y}(0).$$

**Preuve :** Si $(x,\xi) \in W^{\sharp}_{f',Z}(0)$, $(x,\xi,0)$ est limite d'une suite :

$$(x_n, \eta_n + \sum_{i=1}^{r} s_i(n)\frac{df_i(x_n)}{f_i(x_n)}, s_1(n), \ldots, s_r(n)),$$

où $(x_n,\eta_n) \in T^*_Z X$ et $F'(x_n) \neq 0$. De la définition de $T^*_Z(X)$, il résulte que $(x_n,\eta_n,0)$ est lui même limite d'une suite :

$$(y_{n,m}, \eta_{n,m} + \sum_{i=r+1}^{p} s_i(n,m)\frac{df_i(y_{n,m})}{f_i(y_{n,m})}, s_{r+1}(n,m), \ldots, s_p(n,m)),$$

où $y_{n,m}, \eta_{n,m} \in T^*_Y X$ et $F''(y_{n,m}) \neq 0$. Il en résulte que $(x,\xi,0)$ est alors limite d'une sous-suite :

$$(y_{n,m}, \eta_{n,m} + \sum_{i=1}^{r} s_i(n)\frac{df_i(y_{n,m})}{f_i(y_{n,m})} + \sum_{i=r+1}^{p} s_i(n,m)\frac{df_i(y_{n,m})}{f_i(y_{n,m})},$$

$$s_1(n), \ldots, s_r(n), s_{r+1}(n,m), \ldots, s_p(n,m)),$$

On obtient : $(x,\xi,0) \in W^{\sharp}_{f,Y}(0)$.

**Proposition 8.** *Si $f_{|Y}$ est sans pente, $f_\pi({}^t f')^{-1}(T^*_Y X)$ est contenu dans la réunion des conormaux aux intersections des hyperplans de coordonnées de $\mathbf{C}^p$.*

**Preuve :** Soit $(t,\eta) = (t_1,\ldots,t_p,\eta_1,\ldots,\eta_p) \in f_\pi({}^t f')^{-1}(T^*_Y X)$. Supposons :

$$t_1 \neq 0, \ldots, t_r \neq 0 \quad \text{et} \quad t_{r+1} = \cdots = t_p = 0.$$

Par hypothèse, il existe $(x,\xi) \in T^*_Y X$ avec $t_1 = f_1(x), \ldots, t_p = f_p(x)$, tel que :

$$\sum_{i=1}^{p} \eta_i df_i(x) = \xi.$$

Prenons une suite $(x_n,\xi_n) \in T^*_Y X$ tendant vers $(x,\xi)$ avec $F(x_n) \neq 0$. On observe que la suite de $W^{\sharp}_{f,Y}$ :

$$(x_n, \sum_{i=1}^{p} \eta_i f_i(x_n)\frac{df_i(x_n)}{f_i(x_n)} - \xi_n, \eta_1 f_1(x_n), \ldots, \eta_p f_p(x_n))$$

converge vers $((x,0), \eta_1 f_1(x), \ldots, \eta_r f_r(x), 0, \ldots, 0)$ qui appartient donc à $W^{\sharp}_{f,Y}$. Comme sous l'hypothèse sans pente (remarque 2), le morphisme $W^{\sharp}_{f,Y} \to W_{f,Y}$ est fini et que les fibres de $(x,0)$ sont homogènes, il en résulte :

$$\eta_1 f_1(x) = \cdots = \eta_r f_r(x) = 0 \quad \text{et donc} \quad \eta_1 = \cdots = \eta_r = 0.$$

Ainsi : $(t,\eta) \in T^*_{t_{r+1}=\cdots=t_p=0}\mathbf{C}^p$.

Pour prendre en compte les composantes irréductibles d'une variété lagrangienne conique de $T^*X$, nous énonçons :

**Définition 6.** *Soit $\Lambda$ une variété lagrangienne conique de $T^*X$. Nous disons que $(f,\Lambda)$ est sans pente si pour toute composante irréductible $T^*_Z X$ de $\Lambda$, le morphisme $(f_{i_1},\ldots,f_{i_r})_{|Z}$ est sans pente où $\{i_1,\ldots,i_r\}$ est l'ensemble des indices $i$ entre $1$ et $p$ tels $f_{i|Z} \neq 0$.*

3.2. **Systèmes différentiels holonomes réguliers sans pente.** Nous conservons les notations du paragraphe précédent.

Soit $M$ un $\mathcal{D}_X$-Module holonome régulier de variété caractéristique :

$$\mathrm{car}(M) = \bigcup_{l\in L} T^*_{Y_l}X.$$

Désignons par $\mathcal{O}_X[s_1,\ldots,s_p,1/F]f_1^{s_1}\ldots f_p^{s_p}$ le $\mathcal{O}_X[s_1,\ldots,s_p,1/F]$-Module libre de rang 1 de base $f_1^{s_1}\ldots f_p^{s_p}$. Le produit tensoriel

$$M\otimes_{\mathcal{O}_X}\mathcal{O}_X[s_1,\ldots,s_p,1/F]f_1^{s_1}\ldots f_p^{s_p}$$

est muni de la structure naturelle de $\mathcal{D}_X[s_1,\ldots,s_p]$-Module définie pour toute section $m$ de $M$ et $a$ de $\mathcal{O}_X[s_1,\ldots,s_p,1/F]$ par :

$$\frac{\partial}{\partial x_i}(m\otimes af_1^{s_1}\ldots f_p^{s_p}) =$$

$$\frac{\partial}{\partial x_i}m\otimes af_1^{s_1}\ldots f_p^{s_p} + m\otimes\frac{\partial a}{\partial x_i}f_1^{s_1}\ldots f_p^{s_p} + \sum_{j=1}^{p} m\otimes s_j a\frac{\frac{\partial f_j}{\partial x_i}}{f_j}f_1^{s_1}\ldots f_p^{s_p}.$$

Pour toute section $m$ de $M$, $\mathcal{D}_X mf_1^{s_1}\ldots f_p^{s_p}$ est un $\mathcal{D}_X$-Module cohérent et $\mathcal{D}_X[s_1,\ldots,s_p]mf_1^{s_1}\ldots f_p^{s_p}$ est un $\mathcal{D}_X[s_1,\ldots,s_p]$-Module cohérent.

L'anneau $\mathcal{D}_X$ est filtré naturellement par l'ordre naturel des opérateurs différentiels. L'anneau $\mathcal{D}_X[s_1,\ldots,s_p]$ est filtré en donnant de plus à $s_i$ le poids 1.

**Proposition 9.** *(voir théorème 3.3 [B-M-M3]) Soit $M$ un $\mathcal{D}_X$-Module holonome régulier $M$ de variété caractéristique $\bigcup_{l\in L} T^*_{Y_l}X$ et $m$ une section de $M$ engendrant $M$. Nous avons :*

*a) $\mathcal{D}_X mf_1^{s_1}\ldots f_p^{s_p}$ est un $\mathcal{D}_X$-Module cohérent de variété caractéristique :*

$$\mathrm{car}_{\mathcal{D}_X}(\mathcal{D}_X mf_1^{s_1}\ldots f_p^{s_p}) = \bigcup_{F_{|Y_l}\neq 0} W_{f_1,\ldots,f_p,Y_l}.$$

*b) $\mathcal{D}_X[s_1,\ldots,s_p]mf_1^{s_1}\ldots f_p^{s_p}$ est un $\mathcal{D}_X[s_1,\ldots,s_p]$-Module cohérent de variété caractéristique :*

$$\mathrm{car}_{\mathcal{D}_X[s_1,\ldots,s_p]}(\mathcal{D}_X[s_1,\ldots,s_p]mf_1^{s_1}\ldots f_p^{s_p}) = \bigcup_{F_{|Y_l}\neq 0} W^\sharp_{f_1,\ldots,f_p,Y_l}.$$

**Proposition 10.** *Soit $M$ un $\mathcal{D}_X$-Module holonome régulier $M$ de variété caractéristique $\bigcup_{l\in L} T^*_{Y_l}X$ et $m$ une section de $M$ engendrant $M$. Soit $x\in X$ et $L'$ l'ensemble des $l\in L$ tels que $x\in Y_l$ et $F_{|Y_l}\neq 0$ au voisinage de $x$. Les conditions locales au voisinage de $x$ sont équivalentes :*

(1) *Pour tout $l\in L'$ , $f_{|Y_l}$ est sans pente.*

(2) *Il existe $p$ polynômes $b_i(s)\in\mathbf{C}[s]$ non nuls tels que :*

$$b_1(s_1)\cdots b_p(s_p)mf_1^{s_1}\ldots f_p^{s_p}\in\mathcal{D}_X[s_1,\ldots,s_p]mf_1^{s_1+1}\ldots f_p^{s_p+1}.$$

(3) *Il existe $p$ polynômes $b_i(s)\in\mathbf{C}[s]$ non nuls tels que :*

$$b_i(s_i)mf_1^{s_1}\ldots f_p^{s_p}\in\mathcal{D}_X[s_1,\ldots,s_p]mf_if_1^{s_1}\ldots f_p^{s_p}.$$

**Preuve :** L'équivalence entre les propriétés 1 et 2 est l'objet du théorème 3.3 [B-M-M3]. Comme la propriété 3 implique clairement la propriété 2, il reste à montrer que 1 et 2 impliquent 3.

On commence par remplacer $M$ par $M[1/F]$ qui a pour variété caractéristique (voir [G] théorème 5.5) :

$$\bigcup_{l\in L'} W^{\sharp}_{F,Y_l}(0) = \bigcup_{l\in L'} T^*_{Y_l}X \bigcup_{l\in L'} (W_{F,Y_l} \cap (F=0)).$$

L'avantage est que, sous l'hypothèse 1, si $T^*_Z X$ est une composante de la variété caractéristique de $M[1/F]$ non nul sur le produit $f_1\cdots f_r$, le morphisme $(f_1\cdots f_r)_{|Z}$ est sans pente (voir proposition 6). Par récurrence sur $p$, on peut ainsi supposer que la condition 3 est satisfaite pour toute famille de $p-1$ fonctions choisies dans la famille $f_1,\ldots,f_p$. Considérons alors une équation fonctionnelle non triviale :

$$(*) \quad c(s_1)mf_1^{s_1}\ldots f_p^{s_p} \in \sum_{i=1}^{p} \mathcal{D}_X[s_1,\ldots,s_p]mf_i f_1^{s_1}\ldots f_p^{s_p}.$$

L'existence d'une telle équation provient sous l'hypothèse 1 de l'holonomie du $\mathcal{D}_X$-Module :

$$L = \frac{\mathcal{D}_X[s_1,\ldots,s_p]mf_1^{s_1}\ldots f_p^{s_p}}{\sum_{i=1}^{p}\mathcal{D}_X[s_1,\ldots,s_p]mf_i f_1^{s_1}\ldots f_p^{s_p}}.$$

Pour la preuve de l'holonomie, on pourra se reporter à la preuve du théorème 3.3 [B-M-M3]. Les arguments sont les suivants : sous l'hypothèse sans pente la projection $W^{\sharp}_{f,Y} \to W_{f,Y}$ est finie, de la proposition 9 il résulte alors la cohérence de $L$ comme $\mathcal{D}_X$-Module, enfin sous l'hypothèse sans pente la proposition 9 donne une majoration de la variété caractéristique de $L$ par une variété lagrangienne.
En itérant l'équation fonctionnelle $*$, on trouve pour tout entier $k$ des équations fonctionnelles :

$$c(s_1)mf_1^{s_1}\ldots f_p^{s_p} \in$$

$$\mathcal{D}_X[s_1,\ldots,s_p]mf_1 f_1^{s_1}\ldots f_p^{s_p} + \sum_{i=2}^{p}\mathcal{D}_X[s_1,\ldots,s_p]mf_i^k f_1^{s_1}\ldots f_p^{s_p}.$$

D'après l'hypothèse de récurence, nous avons pour tout $j\in\{2,\ldots,p\}$ des équations non triviales :

$$b_(s_1)mf_1^{s_1}\ldots f_{j-1}^{s_{j-1}} f_{j+1}^{s_{j+1}} f_p^{s_p} \in \mathcal{D}_X[s_1,\ldots,s_p]mf_1 f_1^{s_1}\ldots f_{j-1}^{s_{j-1}} f_{j+1}^{s_{j+1}} f_p^{s_p}.$$

En multipliant cette équation par $f_j^{s_j+k}$ pour un entier $k$ assez grand (pour $U\in\mathcal{D}_X[s_1,\ldots,s_p]$ de degré inférieur à $k$, $f_j^{s_j+k}U = Vf_j^{s_j}$ où $V\in\mathcal{D}_X[s_1,\ldots,s_p]$), nous obtenons :

$$b_(s_1)mf_1 f_1^{s_1}\ldots f_{j-1}^{s_{j-1}} f_j^{s_j+k} f_{j+1}^{s_{j+1}}\ldots f_p^{s_p} \in \mathcal{D}_X[s_1,\ldots,s_p]mf_1 f_1^{s_1}\ldots f_p^{s_p}.$$

Nous en déduisons une équation non triviale :

$$b_i(s_i)mf_1^{s_1}\ldots f_p^{s_p} \in \mathcal{D}_X[s_1,\ldots,s_p]mf_i f_1^{s_1}\ldots f_p^{s_p},$$

puis par symétrie que 4 est vérifiée. Le théorème est démontré. Cette méthode avait été utilisée dans le corollaire 3 page 131 de [B-B-M-M] pour montrer l'existence de telles équations dans le cas particulier $M=\mathcal{O}_X$.

Notons que $M$ vérifie les hypothèses de cette proposition 10 si et seulement si $M[1/F]$ les vérifient.

Revenons maintenant à la situation où $H_1,\ldots H_p$ sont $p$ hypersurfaces lisses dont la réunion forme un diviseur à croisements normaux. Notons toujours $\mathbf{H}$ l'ensemble $\{H_1,\ldots,H_p\}$. Soit $(x_1,\ldots,x_n,t_1,\ldots,t_p)$ un système de coordonnées dans lequel $H_i$ a pour équation $t_i=0$.

**Corollaire 2.** *Soit $M$ un $\mathcal{D}_X$-Module holonome régulier $M$ de variété caractéristique $\bigcup_{l\in L} T^*_{Y_l}X$. Soit $(x_1,\ldots,x_n,t_1,\ldots,t_p)$ un système de coordonnées dans lequel $H_i$ a pour équation $t_i=0$. Soit $L'$ l'ensemble des $l\in L$ tels que $F_{|Y_l}\neq 0$ au voisinage de l'origine. Les conditions locales au voisinage de l'origine sont équivalentes :*

(1) *Pour tout $l\in L'$ , $(t_1,\ldots,t_p)_{|Y_l}$ est sans pente.*

(2) *Le couple $((t_1,\ldots,t_p),\operatorname{car} M[1/t_1\ldots t_p])$ est sans pente.*

(3) *Le couple $(\mathbf{H}, M[\star(H_1\cup\ldots\cup H_p)])$ est sans pente.*

**Preuve :** Soit $m$ une section d'un $\mathcal{D}_X$-Module $M$. Soit $b\in\mathbf{C}[s]$ un polynôme d'une variable, en suivant la même preuve que celle du lemme 4.4-1 [M-M], on montre que les conditions suivantes sont équivalentes :

a) $b(s_1)mt_1^{s_1}\ldots t_p^{s_p}\in\mathcal{D}_X[s_1,\ldots,s_p]mt_1t_1^{s_1}\ldots t_p^{s_p}$ ,

b) il existe $A\in V_{0,0,\ldots,0}(\mathcal{D}_X)$ tel que dans $M[\dfrac{1}{t_1\ldots t_p}]$ :

$$b(t_1\frac{\partial}{\partial t_1})m = At_1m.$$

L'équivalence entre les conditions 1 et 3 résulte alors directement de la proposition 10. L'équivalence entre les conditions 1 et 2 résulte de la proposition 6.

**Théorème 3.** *Soit $M$ un $\mathcal{D}_X$-Module holonome régulier. Soit un système de coordonnées $(x_1,\ldots,x_n,t_1,\ldots,t_p)$ de $X$ dans lequel $H_i$ a pour équation $t_i=0$. Les conditions locales au voisinage de l'origine sont équivalentes :*

(1) *Le couple $((t_1,\ldots,t_p),\operatorname{car} M)$ est sans pente.*

(2) *Le couple $(\mathbf{H},M)$ est sans pente.*

**Preuve :** Supposons 1, d'après le corollaire 2, le couple $(\mathbf{H}, M[\dfrac{1}{t_1\ldots t_p}])$ est sans pente. Soit $m\in M$, il existe donc $c(s)\in\mathbf{C}[s]$ et $C\in V_{0,0,\ldots,0}(\mathcal{D}_X)$ tel que la section $m'=(c(t_1\frac{\partial}{\partial t_1})-Ct_1)m$ soit nulle dans le localisé $M[\dfrac{1}{t_1\ldots t_p}]$. La classe $\dot{m}'$ de la section $m'$ dans $M[\dfrac{1}{t_1t_3\ldots t_p}]$ est donc supportée par $t_2=0$. Les composantes irréductibles $T^*_YX$ de la variété caractéristique de $\mathcal{D}_X\dot{m}'$ sont ainsi contenues dans $t_2=0$. Ceux sont des composantes irréductibles des $W^\sharp_{t_1,t_3,\ldots t_p,Z}(0)$ où $T^*_ZX$ décrit les composantes irréductibles de $\operatorname{car} M$. Il résulte de la proposition 6 que si le produit $t_1t_3\ldots t_p$, est non nul sur $Y$, le morphisme $(t_1,t_3,\ldots,t_p)_{|Y}$ est sans pente. Ainsi, $\mathcal{D}_X\dot{m}'$ est supporté par $t_2=0$ et le couple $((t_1,t_3,\ldots,t_p),\operatorname{car}\mathcal{D}_X\dot{m}')$ est sans pente. Par l'équivalence entre $\mathcal{D}_X$-module supporté par une hypersurface $H$ lisse et $\mathcal{D}_H$-module, il existe suivant le corollaire 2 : $c'(s)\in\mathbf{C}[s]$ et $C'\in V_{0,0,\ldots,0}(\mathcal{D}_X)$ indépendants de $t_2$ et $\dfrac{\partial}{\partial t_2}$ tels que la section $(c'(t_1\frac{\partial}{\partial t_1})-C't_1)m'$ soit nulle dans le localisé $M[\dfrac{1}{t_1t_3\ldots t_p}]$. Itérons, on obtient $c''(s)\in\mathbf{C}[s]$ et $C''\in V_{0,0,\ldots,0}(\mathcal{D}_X)$ tels que la section $(c''(t_1\frac{\partial}{\partial t_1})-C''t_1)m$ soit nulle dans le localisé $M[\dfrac{1}{t_1}]$. D'où, pour un certain entier $r$ :

$$(\frac{\partial}{\partial t_1})^r t_1^r(c''(t_1\frac{\partial}{\partial t_1})-C''t_1)m=0.$$

Ainsi, $m$ satisfait une équation fonctionnelle :

$$b(t_1\frac{\partial}{\partial t_1})m=-At_1m,$$

où $b(s) \in \mathbf{C}[s]$ non nul et $A \in V_{0,0,\ldots,0}(\mathcal{D}_X)$. Nous avons ainsi montré que $(\mathbf{H}, M)$ est sans pente.

Inversement, si $(\mathbf{H}, M)$ est sans pente, d'après le corollaire 1 pour tout $\mathrm{I} \subset \{1, \ldots, p\}$, $(\mathbf{H}_{\mathrm{I}}, M)$ est sans pente. La condition 1 résulte alors du corollaire 2.

Examinons enfin un cas particulier étudié dans [D-M-S-T] et [M-T2] que nous généraliserons de façon géométrique à la section 4.3.2.

**Remarque 4.** *Soit $M$ un $\mathcal{D}_X$-Module holonome régulier. Nous supposons que $M$ et $M[*H_1]$ sont non caratéristiques pour $H_2 \cap \ldots \cap H_p$. Alors $(\mathbf{H}, M)$ est sans pente.*

**Preuve** Considérons un système de coordonnées $(x_1, \ldots, x_n, t_1, \ldots, t_p)$ dans lequel $H_i$ a pour équation $t_i = 0$. Nous avons montré (proposition 3.0.5 [M-T2] ou lemme 3.2 [D-M-S-T]) que toute section $m$ de $M$ vérifie pour tout $i \in \{2, \ldots, p\}$ des équations fonctionnelles :

$$(\frac{\partial}{\partial t_i}^k + A_{i,1}\frac{\partial}{\partial t_i}^{k-1} + \cdots + A_{i,k})m = 0,$$

où les $A_{i,j}$ sont des opérateurs différentiels indépendants de $\dfrac{\partial}{\partial t_1}, \ldots, \dfrac{\partial}{\partial t_p}$. En multipliant par $t_i^k$, nous obtenons pour $i \in \{2, \ldots, p\}$ :

$$(t_i)^k(\frac{\partial}{\partial t_i}^k)m \in V_{0,\ldots,0}(\mathcal{D}_X)\, t_i m.$$

De ces équations, nous avions déduit que $M$ est relativement spécialisable par rapport à $H_1$ (proposition 3.0.5 [M-T2] ou proposition 3.1 [D-M-S-T]. Autrement dit, toute section $m$ de $M$ vérifie une équation fonctionnelle non triviale :

$$b(t_1\frac{\partial}{\partial t_1})m = At_1 m,$$

où $A$ est un opérateur de $V_0^{H_1}(\mathcal{D}_X)$ indépendant de $\dfrac{\partial}{\partial t_2}, \ldots, \dfrac{\partial}{\partial t_p}$. On obtient donc une équation non triviale :

$$b(t_1\frac{\partial}{\partial t_1})m \in V_{-1,0,\ldots,0}(\mathcal{D}_X).$$

Cela montre bien que $(\mathbf{H}, M)$ est sans pente.

## 4. Cycles évanescents d'un faisceau construtible par un morphisme sans pente

Soit $D_c^b(\mathbf{C}_X)$ la catégorie des complexes bornés de faisceaux de $\mathbf{C}$-espaces vectoriels dont les groupes de cohomologie sont des faisceaux $\mathbf{C}$-constructibles.

### 4.1. Images directes locales d'un faisceau constructible et complexe d'Alexander d'un faisceau constructible.

4.1.1. *Images directes locales d'un faisceau constructible.* Soit $X = \mathbf{C}^n$, un germe de variété analytique de dimension $n$. Soit $f = (f_1, \ldots, f_p) : X \to \mathbf{C}^p$ un morphisme analytique et $\mathcal{F} \in D_c^b(\mathbf{C}_X)$.

Suivant [K-S] proposition 8.6.4 , [Bry], [L-M], on sait associer à $\mathcal{F}$ sa variété caractéristique $\mathrm{car}\,\mathcal{F}$ qui est un sous-espace analytique conique lagrangien du fibré cotangent à $X$ : $\mathrm{car}\,\mathcal{F}$ est la réunion de l'image du support de $\mathcal{F}$ dans la section nulle de $T^*X$ et de l'adhérence des points $(x, \xi)$ pour lesquels il existe $g : X, x \to \mathbf{C}$ tel que $dg(x) = \xi$ et $\phi_g(\mathcal{F}) \neq 0$.

**Définition 7.** *Soit $\mathcal{F} \in D^b_c(\mathbf{C}_X)$ de variété caractéristique* $\operatorname{car}\mathcal{F} = \bigcup_{l\in L} T^*_{Y_l}X$. *Nous dirons que $(f, \operatorname{car}\mathcal{F})$ est sans éclatement en codimension zéro si pour tout $l \in L$, $f_{|Y_l}$ est sans éclatement en codimension zéro.*

Cela revient à demander que l'intersection par $f = 0$ de l'adhérence de $\operatorname{car}\mathcal{F} +^t f'(X \times_{\mathbf{C}^p} \mathbf{C}^p)$ dans $T^*X$ soit une variété lagrangienne de $T^*X$.

Notons :
- $B_\epsilon = \{x \in \mathbf{C}^n \ ; \ \sum_{i=1}^n \mid x_i \mid^2 < \epsilon^2\}$,
- $\overline{B_\epsilon}$ l'adhérence de $B_\epsilon$ dans $\mathbf{C}^n$ ,
- $D_\eta = \{t = (t_1, \dots, t_p) \in \mathbf{C}^p \ ; \ \mid t_i \mid < \eta\}$,
- $\overline{B_{\epsilon,\eta}} = \overline{B_\epsilon} \cap f^{-1}(D_\eta)$ et $B_{\epsilon,\eta} = B_\epsilon \cap f^{-1}(D_\eta)$,
- $\overline{f}_{\epsilon,\eta} : \overline{B_{\epsilon,\eta}} \to D_\eta$ et $f_{\epsilon,\eta} : B_{\epsilon,\eta} \to D_\eta$ respectivement les restrictions de $f$ à $\overline{B_{\epsilon,\eta}}$ et $B_{\epsilon,\eta}$.

**Proposition 11.** *Soit $\mathcal{F} \in D^b_c(\mathbf{C}_X)$. Supposons que $(f, \operatorname{car}\mathcal{F})$ soit sans éclatement en codimension zéro. Il existe alors $\epsilon_0 > 0$ et une fonction décroissante $\eta : ]0, \epsilon_0] \to \mathbf{R}^+ - \{0\}$ telle que pour tout $0 < \epsilon' \leq \epsilon \leq \epsilon_0$ :*

$$R(\overline{f}_{\epsilon,\eta(\epsilon')})_*\mathcal{F} = R(f_{\epsilon,\eta(\epsilon')})_*\mathcal{F}.$$

*Ces images directes sont à cohomologie $\mathbf{C}$-constructible et indépendantes de $\epsilon \in [\epsilon', \epsilon_0]$. Leurs variétés caractéristiques sont contenues dans :*

$$(f_{\epsilon,\eta(\epsilon')})_\pi({}^t f'_{\epsilon,\eta(\epsilon')})^{-1}(\operatorname{car}\mathcal{F}).$$

*Nous les appelerons les images directes locales de $\mathcal{F}$.*

Remarque sur la preuve : c'est un résultat bien connu sur les morphismes sans éclatement en codimension 0 qui s'appuie sur les propositions 5.4.17 et 8.5.8 de [K-S]. Par hypothèse,

$$\Lambda = \overline{\operatorname{car}\mathcal{F} +^t f'(X \times_{\mathbf{C}^p} \mathbf{C}^p)} \cap f^{-1}(0)$$

est une variété lagrangienne conique de $T^*X$. Si l'on considère la fonction anlytique $\phi(x) = \sum_{i=1}^n \mid x_i \mid^2$ sur $\Lambda$, les valeurs réeles telles qu'il existe $x \in \mathbf{C}^n$ vérifiant $t = \phi(x)$ et $d\phi(x) \in \Lambda$ sont discrètes (proposition 8.3.12 [K-S]). La plus petite de ses valeurs strictement positives est la valeur $\epsilon_0$ attendue dans la proposition (voir preuve de proposition 8.5.8).

4.1.2. *Complexe d'Alexander d'un faisceau constructible.* Suivant C. Sabbah ([S3] définition 2.2.7) à une image directe propre près, définissons le complexe d'Alexander de $\mathcal{F}$. Pour ce faire, considérons le diagramme :

$$\begin{array}{ccccccc}
f^{-1}(0) & \xrightarrow{i} & X & \xleftarrow{j} & X^* = X - F^{-1}(0) & \xleftarrow{p} & \tilde{X} \\
\downarrow & & \downarrow f & & \downarrow f^* & & \downarrow \tilde{f} \\
\{0\} & \xrightarrow{i} & \mathbf{C}^p & \xleftarrow{j} & (\mathbf{C}^*)^p & \xleftarrow{p} & \mathbf{C}^p,
\end{array}$$

où $\mathbf{C}^p$ est le revêtement universel de $(\mathbf{C}^*)^p$ et $p$ le morphisme :

$$\mathbf{C}^p \longrightarrow (\mathbf{C}^*)^p \ : \ p(z_1, \dots, z_p) = (e^{2i\pi z_1}, \dots, e^{2i\pi z_p}).$$

**Définition 8.** *On appelle complexe d'Alexander de $\mathcal{F} \in D^b_c(\mathbf{C}_X)$, le complexe :*

$${}^A\Psi_f\mathcal{F} = i^{-1}Rj_*p_*p^{-1}j^{-1}\mathcal{F}.$$

La i$^{éme}$ monodromie sur ${}^A\Psi_f\mathcal{F}$ est l'isomorphisme :

$$M_i : {}^A\Psi_f\mathcal{F} \longrightarrow {}^A\Psi_f\mathcal{F}$$

induit par le morphisme d'adjonction :

$$(jp)_*(jp)^{-1}\mathcal{F} \longrightarrow (jp)_*(T_i)_*T_i^{-1}(jp)^{-1}\mathcal{F} = (jp)_*(jp)^{-1}\mathcal{F},$$

où $T_i$ est le morphisme de translation :

$$T_i \ : \ \mathbf{C}^p \longrightarrow \mathbf{C}^p \ ; \ (z_1, \ldots, z_p) \longmapsto (z_1, \ldots z_{i-1}, z_i + 1, z_{i+1}, \ldots, z_p).$$

Le morphisme $can : i^{-1}\mathcal{F} \longrightarrow {}^A\Psi_f(\mathcal{F})$ désigne le morphisme induit par le morphisme d'adjonction associé à $j \circ p$ :

$$\mathcal{F} \longrightarrow j_*p_*p^{-1}j^{-1}\mathcal{F} \ : \ s \longmapsto s \circ j \circ p.$$

Pour tout $i \in \{1, 2, \ldots, p\}$, il résulte de $j \circ p \circ T_i = j \circ p$ que $M_i \circ \mathrm{can} = \mathrm{can}$.

Soit $f' = (f_1, \ldots, f_r)$, $f'' = (f_{r+1}, \ldots, f_p)$, $F' = f_1 \ldots f_r$ et $F'' = f_{r+1} \ldots f_p$. Considérons les complexes d'Alexander ${}^A\Psi_{f'}\mathcal{F}$ et ${}^A\Psi_{f''}\mathcal{F}$ associés aux diagrammes :

$$\begin{array}{ccccccc}
f'^{-1}(0) & \xrightarrow{i'} & X & \xleftarrow{j'} & X'^* = X - F'^{-1}(0) & \xleftarrow{p'} & \tilde{X}' \\
\downarrow & & \downarrow f' & & \downarrow f'^* & & \downarrow \tilde{f}' \\
\{0\} & \xrightarrow{i'} & \mathbf{C}^r & \xleftarrow{j'} & (\mathbf{C}^*)^r & \xleftarrow{p'} & \mathbf{C}^r,
\end{array}$$

$$\begin{array}{ccccccc}
f''^{-1}(0) & \xrightarrow{i''} & X & \xleftarrow{j''} & X''^* = X - F''^{-1}(0) & \xleftarrow{p''} & \tilde{X}'' \\
\downarrow & & \downarrow f'' & & \downarrow f''^* & & \downarrow \tilde{f}'' \\
\{0\} & \xrightarrow{i''} & \mathbf{C}^{p-r} & \xleftarrow{j''} & (\mathbf{C}^*)^{p-r} & \xleftarrow{p''} & \mathbf{C}^{p-r}.
\end{array}$$

Considérons alors les diagrammes de carrés cartésiens :

$$\begin{array}{lllll}
f^{-1}(0) & \xrightarrow{i'} & f''^{-1}(0) & \xleftarrow{j'p'} & \tilde{X}' \cap f''^{-1}(0) \\
\downarrow i'' & \searrow i & \downarrow i'' & & \downarrow i'' \\
f'^{-1}(0) & \xrightarrow{i'} & X & \xleftarrow{j'p'} & \tilde{X}' \\
\uparrow j''p'' & & \uparrow j''p'' & \nwarrow jp & \uparrow \pi'' \\
\tilde{X}'' \cap f'^{-1}(0) & \xrightarrow{i'} & \tilde{X}'' & \xleftarrow{\pi'} & \tilde{X}.
\end{array}$$

Pour simplifier, posons $a = j'p'$ et $b = j''p''$. On a les morphismes :

$$
\begin{array}{lcl}
{}^A\Psi_{f'}({}^A\Psi_{f''}\mathcal{F}) & & {}^A\Psi_{f'}(i''^{-1}\mathcal{F}) \\
\quad \| & & \quad \| \\
i'^{-1}Ra_*a^{-1}i''^{-1}Rb_*b^{-1}\mathcal{F} & \xleftarrow{adj.(b)} & i'^{-1}Ra_*a^{-1}i''^{-1}\mathcal{F} \\
\quad \| & & \quad \| \\
i'^{-1}Ra_*i''^{-1}a^{-1}b_*b^{-1}\mathcal{F} & \xleftarrow{adj.(b)} & i'^{-1}Ra_*i''^{-1}a^{-1}\mathcal{F} \\
\quad \uparrow chgt.base/i'' & & \quad \uparrow chgt.base/i'' \\
i'^{-1}i''^{-1}Ra_*a^{-1}b_*b^{-1}\mathcal{F} & \xleftarrow{adj.(b)} & i'^{-1}i''^{-1}Ra_*a^{-1}\mathcal{F} \\
\quad \simeq\downarrow chgt.base/a & & \quad \| \\
{}^A\Psi_f\mathcal{F} & \xleftarrow{adj.(\pi'')} & i''^{-1}({}^A\Psi_{f'}\mathcal{F}) \\
\quad \simeq\uparrow chgt.base/b & & \quad \| \\
i''^{-1}i'^{-1}Rb_*b^{-1}a_*a^{-1}\mathcal{F} & \xleftarrow{adj.(b)} & i''^{-1}i'^{-1}Ra_*a^{-1}\mathcal{F} \\
\quad \downarrow chgt.base/i' & & \quad \| \\
i''^{-1}Rb_*i'^{-1}b^{-1}a_*a^{-1}\mathcal{F} & & i''^{-1}i'^{-1}Ra_*a^{-1}\mathcal{F} \\
\quad \| & & \quad \| \\
i''^{-1}Rb_*b^{-1}i'^{-1}a_*a^{-1}\mathcal{F} & \xleftarrow{adj.(b)} & i''^{-1}i'^{-1}Ra_*a^{-1}\mathcal{F} \\
\quad \| & & \quad \| \\
{}^A\Psi_{f''}({}^A\Psi_{f'}\mathcal{F}) & & i''^{-1}({}^A\Psi_{f'}\mathcal{F}).
\end{array}
$$

Ces diagrammes commutent puisque l'adjonction commute au changement de base.

Pour $i \leq r$, les isomorphismes de translation $T_i$ sur $\tilde{X'}, \tilde{X}, \tilde{X'} \cap f''^{-1}(0)$ commutent aux mophismes $\pi''$ et $i''$. On en déduit la commutativité du diagramme :

$$
\begin{array}{ccc}
{}^A\Psi_f\mathcal{F} & \xleftarrow{M_i({}^A\Psi_f\mathcal{F})} & {}^A\Psi_f\mathcal{F} \\
\downarrow & & \downarrow \\
{}^A\Psi_{f'}({}^A\Psi_{f''}\mathcal{F}) & \xleftarrow{M_i\left({}^A\Psi_{f'}({}^A\Psi_{f''}\mathcal{F})\right)} & {}^A\Psi_{f'}(i''^{-1}\mathcal{F}).
\end{array}
$$

De même pour $i > r$, les isomorphismes de translation $T_i$ sur $\tilde{X''}, \tilde{X}, \tilde{X''} \cap f'^{-1}(0)$ commutent aux mophismes $\pi''$ et $i'$. On en déduit la commutativité du diagramme :

$$
\begin{array}{ccc}
{}^A\Psi_f\mathcal{F} & \xleftarrow{M_i({}^A\Psi_f\mathcal{F})} & {}^A\Psi_f\mathcal{F} \\
\downarrow & & \downarrow \\
{}^A\Psi_{f'}({}^A\Psi_{f''}\mathcal{F}) & \xleftarrow{{}^A\Psi_{f'}\left(M_i({}^A\Psi_{f''}\mathcal{F})\right)} & {}^A\Psi_{f'}(i''^{-1}\mathcal{F}).
\end{array}
$$

En résumé :

**Proposition 12.** *Il existe un diagrame canonique commutatif compatible aux morphismes de monodromies :*

$$
\begin{array}{ccc}
{}^A\Psi_{f'}({}^A\Psi_{f''}\mathcal{F}) & \longleftarrow & {}^A\Psi_{f'}(i''^{-1}\mathcal{F}) \\
\uparrow & & \uparrow \\
{}^A\Psi_f\mathcal{F} & \longleftarrow & i''^{-1}({}^A\Psi_{f'}\mathcal{F}) \\
\downarrow & \swarrow & \\
{}^A\Psi_{f''}({}^A\Psi_{f'}\mathcal{F}). & &
\end{array}
$$

Pour $p > 1$, suivant C. Sabbah ([S3] définition 2.2.7), ${}^A\Psi_f(\mathcal{F})$ n'est en général pas **C**-constructible, par contre c'est un complexe de faisceaux à cohomologie $\mathbf{C}[\mathbf{Z}^p]$ constructible.

4.2. **Cycles évanescents d'un faisceau construtible par un morphisme sans pente.** Soit $X$ un germe de variété analytique de dimension $n$ et $Y$ un sous-espace irréductible de $X$. Soit $f_1, \dots, f_p$ des fonctions analytiques sur $X$ nulles à l'origine. Notons $F = f_1 f_2 \cdots f_p$ leur produit et désignons par $f$ l'application :

$$f : X \longrightarrow \mathbf{C}^p \quad , \quad (x_1, \dots, x_n) \longmapsto (f_1(x_1, \dots, x_n), \dots, f_p(x_1, \dots, x_n))$$

et par $f_{|Y}$ sa restriction à $Y$.

Soit $\mathcal{F} \in D_c^b(\mathbf{C}_X)$. Rappelons la définition 6 : le couple $((f_1, \dots, f_p), \operatorname{car}\mathcal{F})$ est sans pente si pour toute composante irréductible $T_Z^*X$ de la variété caractéristique de $\mathcal{F}$, on ait $(f_{i_1}, \dots, f_{i_r})_{|Z}$ sans pente où $\{i_1, \dots, i_r\} = \{0 \leq i \leq p\ ;\ f_{i|Z} \neq 0\}$.

Si $((f_1, \dots, f_p), \operatorname{car}\mathcal{F})$ est sans pente, $((f_1, \dots, f_p), \operatorname{car}\mathcal{F})$ est sans éclatement en codimension 0 et nous avons rappelé l'existence d'images directes locales $R(f_{\epsilon,\eta(\alpha)})_*\mathcal{F}$ dont la variété caractéristique est majorée par :

$$(f_{\epsilon,\eta(\alpha)})_\pi({}^t f'_{\epsilon,\eta(\alpha)})^{-1} \operatorname{car} F.$$

Il résulte de la proposition 8 que le sous-espace $(f_{\epsilon,\eta(\alpha)})_\pi({}^t f'_{\epsilon,\eta(\epsilon)})^{-1} \operatorname{car} F$ est réunion de conormaux à des intersections d'hyperplans de coodonnées de $\mathbf{C}^p$. Il résulte alors de la proposition 11 :

**Proposition 13.** *Si $((f_1, \dots, f_p), \operatorname{car}\mathcal{F})$ est sans pente, les images directes locales $R(f_{\epsilon,\eta(\alpha)})_*\mathcal{F}$ sont constuctibles et leurs variétés caractéristiques sont réunion de conormaux à des intersections d'hyperplans de coodonnées de $\mathbf{C}^p$.*

Les groupes de cohomologie de $R(f_{\epsilon,\eta(\alpha)})_*\mathcal{F}$ sont alors localement constants sur les strates de la stratification induite par les intersections des hyperplans de coordonnées de $\mathbf{C}^p$. Le complexe de faisceaux ${}^A\Psi_f\mathcal{F}$ est alors la généralisation naturelle pour un morphisme sans pente du foncteur des cycles évanescents défini par P. Deligne dans [D].

**Définition 9.** *Si $((f_1, \dots, f_p), \operatorname{car}\mathcal{F})$ est sans pente, nous appelons ${}^A\Psi_f(\mathcal{F})$ le complexe des cycles évanescents de $\mathcal{F}$ par $f$ et le notons $\Psi_f(\mathcal{F})$.*

**Théorème 4.** *Soit $\mathcal{F} \in D_c^b(\mathbf{C}_X)$, si $((f_1, \dots, f_p), \operatorname{car}\mathcal{F})$ est sans pente, nous avons :*

(1) $\Psi_f\mathcal{F} \in D_c^b(\mathbf{C}_X \cap f^{-1}(0))$ *et* $\operatorname{car}\Psi_f\mathcal{F} \subset \bigcup_{\alpha\in A} W^\sharp_{f_1,\dots,f_p,Z_\alpha} \cap f^{-1}(0)$, *où* $(T^*_{Z_\alpha}X)_{\alpha\in A}$ *décrit les composantes irréductibles de* $\operatorname{car}\mathcal{F}$ *sur lesquelles* $F$ *est non identiquement nulle.*

(2) *Pour tout $r$ compris entre 1 et $p$, les morphismes canoniques :*

$$\Psi_{(f_1,\dots,f_p)}\mathcal{F} \longrightarrow \Psi_{(f_1,\dots,f_r)}(\,\Psi_{(f_{r+1},\dots,f_p)}\mathcal{F}\,)$$

*sont des isomorphismes.*

**Lemme 2.** *Si $((f_1, \dots, f_p), \operatorname{car}\mathcal{F})$ est sans pente, soit le réel positif $\epsilon_0$ et la fonction $\eta$ associés aux images directes locales de $\mathcal{F}$ par $f$. Alors :*

– *Pour tout $0 < \epsilon \leq \epsilon_0$ :*

$$(i^{-1}\mathcal{F})_0 \simeq R\Gamma(\overline{B}_{\epsilon,\eta(\epsilon)} \cap f^{-1}(0), \mathcal{F}).$$

– *Pour tout $0 < \epsilon \leq \epsilon_0$, $t = (t_1, \dots, t_p) \in D^*_{\eta(\epsilon)} = D_{\eta(\epsilon)} - \{t_1 \dots t_p = 0\}$ :*

$$(\Psi_f\mathcal{F})_0 \simeq R\Gamma(\overline{B}_{\epsilon,\eta(\epsilon)} \cap f^{-1}(t), \mathcal{F}).$$

*On rappelle que $i$ désigne l'inclusion de $f^{-1}(0)$ dans $X$ et $\overline{B}_{\epsilon,\eta(\epsilon)}$ désigne $\overline{B}_\epsilon \cap f^{-1}(D_{\eta(\epsilon)})$.*

**Preuve du lemme :** Nous avons le diagramme commutatif suivant :

$$\begin{array}{ccc} R\Gamma(\overline{B}_{\epsilon,\eta(\epsilon)},\mathcal{F}) & \xrightarrow{j_\epsilon} & R\Gamma(\overline{B}_{\epsilon,\eta(\epsilon)}\cap f^{-1}(0),\mathcal{F}) \\ 1\downarrow & & \downarrow 3 \\ R\Gamma(D_{\eta(\epsilon)},Rf_*\,\mathcal{F}_{|\overline{B}_\epsilon}) & \xrightarrow{2} & (Rf_*\,\mathcal{F}_{|\overline{B}_\epsilon})_0. \end{array}$$

La flêche verticale 1 est un isomorphisme depuis que l'image directe d'un faisceau flasque est flasque. La flêche horizontale 2 est un isomorphisme par constructibilité de $Rf_*(\mathcal{F}_{|\overline{B}_{\epsilon,\eta(\epsilon)}})$ relativement au croisement normal de $D_{\eta(\epsilon)}$. La flêche verticale 3 est un isomorphisme, car la restriction de $f$ à $\overline{B}_\epsilon$ est propre. Il en résulte que $j_\epsilon$ est un isomorphisme.

On a d'autre part, pour tout $0<\epsilon'\leq\epsilon\leq\epsilon_0$, le diagramme commutatif :

$$\begin{array}{ccc} R\Gamma(\overline{B}_{\epsilon,\eta(\epsilon)},\mathcal{F}) & \xrightarrow{j_\epsilon\ (iso)} & R\Gamma(\overline{B}_{\epsilon,\eta(\epsilon)}\cap f^{-1}(0),\mathcal{F}) \\ r_{\epsilon',\epsilon}\downarrow & & \downarrow 4 \\ R\Gamma(\overline{B}_{\epsilon',\eta(\epsilon')},\mathcal{F}) & \xrightarrow{j_{\epsilon'}\ (iso)} & R\Gamma(\overline{B}_{\epsilon',\eta(\epsilon')}\cap f^{-1}(0),\mathcal{F}) \\ \downarrow & & \downarrow \\ (i^{-1}\mathcal{F})_0 & = & (i^{-1}\mathcal{F})_0. \end{array}$$

La flêche 4 est un isomorphisme, car : $R(\overline{f}_{\epsilon,\eta(\epsilon')})_*\mathcal{F}=R(\overline{f}_{\epsilon',\eta(\epsilon')})_*\mathcal{F}$ (voir proposition 11). La première partie du lemme s'en déduit, puisque les $\overline{B}_{\epsilon,\eta(\epsilon)}$ forment un système fondamental de voisinages de l'origine.

Montrons la deuxième partie du lemme. Reprenons pour cela le diagramme des cycles évanescents de $f$ en restriction à $\overline{B}_{\epsilon,\eta(\epsilon')}$ :

$$\begin{array}{ccccccc} \overline{B}_{\epsilon,\eta(\epsilon')}\cap f^{-1}(0) & \xrightarrow{i} & \overline{B}_{\epsilon,\eta(\epsilon')} & \xleftarrow{j} & \overline{B}^*_{\epsilon,\eta(\epsilon')}=\overline{B}_{\epsilon,\eta(\epsilon')}-F^{-1}(0) & \xleftarrow{p} & \tilde{B}_{\epsilon,\eta(\epsilon')} \\ \downarrow & & \downarrow\overline{f}_{\epsilon,\eta(\epsilon')} & & \downarrow\overline{f^*}_{\epsilon,\eta(\epsilon')} & & \downarrow\tilde{f}_{\epsilon,\eta(\epsilon')} \\ \{0\} & \xrightarrow{i} & D_{\eta(\epsilon')} & \xleftarrow{j} & D^*_{\eta(\epsilon')}=D_{\eta(\epsilon')}-\{t_1\dots t_p=0\} & \xleftarrow{p} & \tilde{D}_{\eta(\epsilon')} \end{array}$$

où $\tilde{D}_{\eta(\epsilon')}$ est le revêtement universel de $D^*_{\eta(\epsilon')}$. L'application $p$ étant un revêtement :

$$p^{-1}R(\overline{f^*}_{\epsilon,\eta(\epsilon')})_*\mathcal{F}=R(\tilde{f}_{\epsilon,\eta(\epsilon')})_*(p^{-1}\mathcal{F}).$$

Pour tout $\tilde{w}\in\tilde{D}_{\eta(\epsilon')}$ et $w=p(\tilde{w})$, l'égalité des fibres donnent :

$$R\Gamma(\overline{B}_{\epsilon,\eta(\epsilon')}\cap f^{-1}(w),\mathcal{F})=R\Gamma(\tilde{B}_{\epsilon,\eta(\epsilon')}\cap\tilde{f}^{-1}(w),p^{-1}\mathcal{F}).$$

D'autre part, comme $R(\overline{f}_{\epsilon,\eta(\epsilon')})_*\mathcal{F}$ est constructible relativement au croisement normal de $D_{\eta(\epsilon')}$, $R(\overline{f^*}_{\epsilon,\eta(\epsilon')})_*\mathcal{F}$ est à cohomologie localement constante. Ainsi, $R(\tilde{f}_{\epsilon,\eta(\epsilon')})_*(p^{-1}\mathcal{F})$ est à cohomologie localement constante sur $\tilde{D}_{\eta(\epsilon')}$, donc constante, car $\tilde{D}_{\eta(\epsilon')}$ est isomorphe à $\mathbf{C}^p$.

On obtient, pour tout $0<\epsilon'\leq\epsilon\leq\epsilon_0$ et $w\in\tilde{D}_{\eta(\epsilon')}$ le diagramme commutatif :

$$\begin{array}{ccc} R\Gamma(\tilde{D}_{\eta(\epsilon')},R(\tilde{f}_{\epsilon,\eta(\epsilon')})_*(p^{-1}\mathcal{F})) & \xrightarrow{\simeq} & R(\tilde{f}_{\epsilon,\eta(\epsilon')})_*(p^{-1}\mathcal{F})_{\tilde{w}}=R(\overline{f}_{\epsilon,\eta(\epsilon')})_*(p^{-1}\mathcal{F})_w \\ \downarrow & & \downarrow 5 \\ R\Gamma(\tilde{B}_{\epsilon,\eta(\epsilon)},p^{-1}\mathcal{F}) & \xrightarrow{\simeq} & R\Gamma(\overline{B}_\epsilon\cap f^{-1}(w),\mathcal{F}) \\ \downarrow & & \downarrow 6 \\ R\Gamma(\tilde{B}_{\epsilon',\eta(\epsilon')},p^{-1}\mathcal{F}) & \xrightarrow{\simeq} & R\Gamma(\overline{B}_{\epsilon'}\cap f^{-1}(w),\mathcal{F}). \end{array}$$

La flêche 5 est un isomorphisme, puisque $\overline{f}_{\epsilon,\eta(\epsilon')}$ est propre. La flêche 6 est un isomorphisme puisque $R(\overline{f}_{\epsilon,\eta(\epsilon')})_*\mathcal{F} = R(\overline{f}_{\epsilon',\eta(\epsilon')})_*\mathcal{F}$.

Par définition du foncteur $\Psi_f$ :

$$(\Psi_{(f_1,\ldots,f_p)}\mathcal{F})_0 = \lim_{\overrightarrow{\epsilon}} R\Gamma(\tilde{B}_{\epsilon,\eta(\epsilon)}, p^{-1}\mathcal{F}).$$

On obtient donc pour tout $0 < \epsilon' \leq \epsilon \leq \epsilon_0$ et $w \in \tilde{D}_{\eta(\epsilon')}$ :

$$(\Psi_{(f_1,\ldots,f_p)}\mathcal{F})_0 = R\Gamma(\overline{B}_\epsilon \cap f^{-1}(w), \mathcal{F}).$$

ce qui démontre la deuxième partie du lemme.

**Preuve du théorème :** Nous procédons par récurrence sur $p$. Posons $f' = (f_1, \ldots, f_r)$ et $f'' = (f_{r+1}, \ldots, f_p)$.

Pour $p = 1$, $\Psi_f\mathcal{F}$ est le complexe des cycles évanescents de P. Deligne [D]. Ce complexe est constructible (voir par exemple [K-S] proposition 8.6.3.), sa variété caratéristique (voir [G] théorème 5.5) est :

$$\bigcup_{\alpha\in A} W^\sharp_{f,Z_\alpha} \cap f^{-1}(0),$$

où $(T^*_{Z_\alpha}X)_{\alpha\in A}$ décrit les composantes irréductibles de $\operatorname{car}\mathcal{F}$ non identiquement nulles sur $F$.

Supposons $p > 1$. Par hypothèse de récurrence, $\Psi_{(f_{r+1},\ldots,f_p)}\mathcal{F}$ est à cohomologie constructible et sa variété caractéristique est contenue dans la réunion des $W^\sharp_{f'',Z_\alpha}(0)$ où $T^*_{Z_\alpha}X$ est une composante irréductible de $\operatorname{car}\mathcal{F}$ non identiquement nulles sur $F''$. D'après la proposition 5, si $T^*_Z Y$ est une composante irréductible de $W^\sharp_{f'',Z_\alpha}(0)$ et que $Z$ est non identiquement nul sur $F' = f_1 \ldots f_r$, alors $f'_{|Z}$ est sans pente. Il en résulte que $(f', \operatorname{car}\Psi_{f''}\mathcal{F})$ est sans pente.

Posons :
- $D'_\eta = \{t = (t_1, \ldots, t_r) \in \mathbf{C}^r \ ; \ | \ t_i \ |< \eta\}$ ,
- $D''_\eta = \{t = (t_{r+1}, \ldots, t_p) \in \mathbf{C}^{p-r} \ ; \ | \ t_i \ |< \eta\}$ ,
- $D'^*_\eta = D'_\eta - \{t_1 \ldots t_r = 0\}$ et $D''^*_\eta = D''_\eta - \{t_{r+1} \ldots t_p = 0\}$.

Quitte à les diminuer, on peut supposer que $\epsilon_0$ et les fonctions $\eta$ associés aux images directes locales de $\Psi_{f''}(\mathcal{F})$ par $f'$ et de $\mathcal{F}$ par $f$ coïncident. D'après le lemme 2, pour tout $0 < \alpha < \epsilon_0$ et tout $w' \in D'^*_{\eta(\alpha)}$ :

$$(\Psi_{f'}(\Psi_{f''}\mathcal{F}))_0 = R\Gamma(\overline{B}_\alpha \cap f'^{-1}(w'), \Psi_{f''}\mathcal{F}).$$

Nous allons donc calculer : $R\Gamma(\overline{B}_\alpha \cap f'^{-1}(w'), \Psi_{f''}\mathcal{F})$. Fixons $w' \in D'^*_{\eta(\alpha)}$ et considérons $\beta$ et $\gamma$ tel que :

$$0 < \alpha < \beta \leq \epsilon_0 \quad \text{et} \quad 0 < \gamma < \eta(\beta) - \max\{| \ w'_1 \ | \ldots | \ w'_r \ |= \gamma(\beta)\}.$$

Posons :

$$\begin{aligned} D_{\eta(\beta),\gamma,w'} &= \left\{ t \in \mathbf{C}^p \ ; \ \begin{pmatrix} | \ t_i - w'_i \ |< \gamma & \text{pour } i \in \{1,\ldots,r\} \\ | \ t_j \ |< \eta(\beta) & \text{pour } j \in \{r+1,\ldots,p\} \end{pmatrix} \right\}. \\ T_{\beta,\gamma,w'} &= \{x \in \overline{B}_\beta \ ; \ f(x) \in D_{\eta(\beta),\gamma,w'}\}. \end{aligned}$$

Considérons le diagramme :

$$\begin{array}{ccccc} T_{\beta,\gamma,w'} & \overset{\overline{f}_{\beta,\gamma,w'}}{\longrightarrow} & D_{\eta(\beta),\gamma,w'} & \overset{\pi''}{\longrightarrow} & D''_{\eta(\beta)} \\ || & & & & || \\ T_{\beta,\gamma,w'} & & \overset{\overline{f''}_{\beta,\gamma,w'}}{\longrightarrow} & & D''_{\eta(\beta)}, \end{array}$$

où $\overline{f}_{\beta,\gamma,w'}$ et $\overline{f''}_{\beta,\gamma,w'}$ désignent les restrictions de $f$ et $f''$ et où $\pi''$ désigne la projection $(t', t'') \mapsto t''$.

Le complexe $R(\overline{f}_{\beta,\gamma,w'})_*\mathcal{F}$ est à cohomologie constructible relativement au croisement normal de $D_{\eta(\beta),\gamma,w'}$. Il en résulte que $R(\overline{f''}_{\beta,\gamma,w'})_*\mathcal{F}$ qui est égal à $R(\pi'')_*R(\overline{f}_{\beta,\gamma,w'})_*\mathcal{F}$ est à cohomologie constructible relativement à ce même croisement normal.

Reprenons une partie du diagramme des cycles évanescents de $f''$ :

$$\begin{array}{ccccc} T_{\beta,\gamma,w'} & \longleftarrow & T^*_{\beta,\gamma,w'} & \overset{p''}{\longleftarrow} & \tilde{T}_{\beta,\gamma,w'} \\ \downarrow & & \downarrow \overline{f''}_{\beta,\gamma,w'} & & \downarrow \tilde{f''}_{\beta,\gamma,w'} \\ D''_{\eta(\beta)} & \longleftarrow & D''^*_{\eta(\beta)} & \overset{p''}{\longleftarrow} & \tilde{D}''_{\eta(\beta)}, \end{array}$$

où $T^*_{\beta,\gamma,w'} = T_{\beta,\gamma,w'} - \{f_{r+1} \dots f_p = 0\}$. L'application $p$ étant un revêtement :

$$R(\tilde{f''}_{\beta,\gamma,w'})_*(p''^{-1}\mathcal{F}) = p''^{-1}(R(\overline{f''}_{\beta,\gamma,w'})_*\mathcal{F})$$

est à cohomologie localement constante sur $\tilde{D}''_{\eta(\beta)}$. Pour tout $\tilde{w''} \in \tilde{D}''_{\eta(\alpha)}$, si $w'' = p''(\tilde{w''})$, on a les isomorphismes :

$$\begin{array}{rcl} R\Gamma(\tilde{T}_{\beta,\gamma,w'}, p''^{-1}\mathcal{F})) & \simeq & R\Gamma(\tilde{D}''_{\eta(\beta)}, R(\tilde{f''}_{\beta,\gamma,w'})_*(p''^{-1}\mathcal{F}) \\ & \simeq & R(\tilde{f''}_{\beta,\gamma,w'})_*(p''^{-1}\mathcal{F})_{\tilde{w''}} \\ & \simeq & R(\overline{f''}_{\beta,\gamma,w'})_*\mathcal{F})_{w''} \\ & \simeq & R\Gamma(\cap_{i=1}^{r}\{\mid t_i - w'_i \mid < \gamma\} \cap_{i=r+1}^{p} \{t_i = w''_i\}, R(\overline{f}_{\beta,\gamma,w'})_*\mathcal{F})) \end{array} .$$

Par constructibilité de $R(\overline{f}_{\beta,\gamma,w'})_*\mathcal{F})$ relativement au croisement normal de $D_{\eta(\beta),\gamma,w'}$, nous obtenons :

$$\begin{array}{rcl} R\Gamma(\tilde{T}_{\beta,\gamma,w'}, p''^{-1}\mathcal{F}) & \simeq & R\Gamma(\cap_{i=1}^{r}\{t_i = w'_i\} \cap_{i=r+1}^{p} \{t_i = w''_i\}, R(\overline{f}_{\beta,\gamma,w'})_*\mathcal{F})) \\ & \simeq & R\Gamma(\overline{B}_\beta \cap f^{-1}(w', w''), \mathcal{F}). \end{array}$$

Ainsi pour tout $\beta$ tel que $0 < \alpha < \beta < \epsilon_0$ et tout $(w', w'') \in D^*_{\eta(\alpha)}$ :

$$R\Gamma(\tilde{T}_{\beta,\gamma,w'}, p''^{-1}\mathcal{F}) \simeq R\Gamma(\overline{B}_\beta \cap f^{-1}(w', w''), \mathcal{F})$$

qui sont de plus indépendants de $\beta$.

Comme la restriction de $f$ à $\overline{B}_\alpha$ est propre, les $T_{\beta,\gamma,w'}$ forment un système fondamental de voisinage de $\overline{B}_\alpha \cap f'^{-1}(w') \cap f''^{-1}(0)$. Ainsi :

$$\begin{array}{rcl} (\Psi_{f'}(\Psi_{f''}\mathcal{F}))_0 & \simeq & R\Gamma(\overline{B}_\alpha \cap f'^{-1}(w'), \Psi_{f''}\mathcal{F}) \\ & \simeq & \varinjlim_{\beta,\gamma} R\Gamma(\tilde{T}_{\beta,\gamma,w'}, p''^{-1}\mathcal{F}) \\ & \simeq & R\Gamma(\overline{B}_\beta \cap f^{-1}(w', w''), \mathcal{F}) \\ & \simeq & (\Psi_{(f_1,\dots,f_p)}\mathcal{F})_0. \end{array}$$

On obtient enfin, par récurrence, la majoration de la variété caractéristique de $\Psi_f(\mathcal{F})$ à l'aide de la proposition 7.

**Théorème 5.** *Si* $((f_1,\dots,f_p),\operatorname{car}\mathcal{F})$ *est sans pente, les isomorphismes canoniques :*

$$i'^{-1}(\Psi_{f''}\mathcal{F}) \longrightarrow \Psi_{f''}(i'^{-1}\mathcal{F})$$

*sont des isomorphismes de* $D^b_c(\mathbf{C}_X)$*. De plus,*

$$\operatorname{car}(i'^{-1}\Psi_{f''}\mathcal{F}) \subset \bigcup_{\alpha\in A} W^{\sharp}_{f_1,\dots,f_p,Z_\alpha} \cap f^{-1}(0),$$

*où* $(T^*_{Z_\alpha}X)_{\alpha\in A}$ *décrit les composantes irréductibles de* $\operatorname{car}\mathcal{F}$ *sur lesquelles* $F''$ *est non identiquement nulle.*

**Preuve du théorème :** La constructibilité étant conservée par image inverse, il résulte du théorème 4 que les complexes $i'^{-1}(\Psi_{f''}\mathcal{F})$ et $\Psi_{f''}(i'^{-1}\mathcal{F})$ sont à cohomologie constructible. Soit $\mathcal{G}\in D^b_c(\mathbf{C}_X)$, il résulte du corollaire 6.4.4 , de la proposition 6.2.4 et du lemme 6.2.1 de [K-S] que :

$$\operatorname{car}(i'^{-1}(\mathcal{G})) \subset \cup W^{\sharp}_{f',Z_\alpha}(0),$$

où $T^*_{Z_\alpha}X$ décrit les composantes irréductibles de $\operatorname{car}\mathcal{G}$. La majoration attendue résulte alors de celle du théorème 4 et de la proposition 7. On peut aussi par récurrence traiter le cas où le but de $f'$ est de dimension 1 et utiliser le calcul de la variété caractéristique de la restriction à une hypersurface donnée dans [G] ou [B-M-M4].

Il nous reste à montrer que $i'^{-1}$ et $\Psi_{f''}$ commutent. Cette preuve est identique à celle du théoréme 4. On peut supposer que $\epsilon_0$ et les fonctions $\eta$ associées aux images directes locales de $\Psi_{f''}\mathcal{F}$ par $f'$ et de $\mathcal{F}$ par $f$ coïncident.

D'après le lemme 2, pour tout $0<\alpha<\epsilon_0$ :

$$(i'^{-1}\Psi_{f''}\mathcal{F})_0 = R\Gamma(\overline{B}_\alpha \cap f'^{-1}(0), \Psi_{f''}\mathcal{F}).$$

Nous allons donc calculer : $R\Gamma(\overline{B}_\alpha\cap f'^{-1}(0),\Psi_{f''}\mathcal{F})$. Pour tout $\beta$ tel que $0<\alpha<\beta\le\epsilon_0$ et $0<\gamma<\eta(\beta)$, posons :

$$\begin{array}{rcl} D_{\eta(\beta),\gamma} & = & \left\{ t\in\mathbf{C}^p \; ; \; \left[\begin{array}{ll} \mid t_i \mid < \gamma & \text{pour } i\in\{1,\dots,r\} \\ \mid t_j \mid < \eta(\beta) & \text{pour } j\in\{r+1,\dots,p\} \end{array}\right.\right\}. \\ T_{\beta,\gamma} & = & \{x\in\overline{B}_\beta \; ; \; f(x)\in D_{\eta(\beta),\gamma}\}. \end{array}$$

Considérons le diagramme :

$$\begin{array}{ccccc} T_{\beta,\gamma} & \xrightarrow{\overline{f}_{\beta,\gamma}} & D_{\eta(\beta),\gamma} & \xrightarrow{\pi''} & D''_{\eta(\beta)} \\ || & & & & || \\ T_{\beta,\gamma} & & \xrightarrow{\overline{f''}_{\beta,\gamma}} & & D''_{\eta(\beta)}, \end{array}$$

où $\overline{f}_{\beta,\gamma}$ et $\overline{f''}_{\beta,\gamma}$ désignent les restritions de $f$ et $f''$.

Le complexe $R(\overline{f}_{\beta,\gamma})_*\mathcal{F}$ est à cohomologie constructible relativement au croisement normal de $D_{\eta(\beta),\gamma}$. Il en résulte que $R(\overline{f''}_{\beta,\gamma})_*\mathcal{F}$ qui est égal à $R(\pi'')_*R(\overline{f}_{\beta,\gamma})_*\mathcal{F}$ est à cohomologie constructible relativement au croisement normal de $D''_{\eta(\beta)}$.

Reprenons une partie du diagramme des cycles évanescents de $f''$ :

$$\begin{array}{ccccc}
T_{\beta,\gamma} & \longleftarrow & T^*_{\beta,\gamma} & \xleftarrow{p''} & \tilde{T}_{\beta,\gamma} \\
\downarrow & & \downarrow \overline{f''}_{\beta,\gamma} & & \downarrow \tilde{f''}_{\beta,\gamma} \\
D''_{\eta(\beta)} & \longleftarrow & D''^*_{\eta(\beta)} & \xleftarrow{p''} & \tilde{D}''_{\eta(\beta)},
\end{array}$$

où $T^*_{\beta,\gamma} = T_{\beta,\gamma} - \{f_{r+1} \dots f_p = 0\}$. L'application $p$ étant un revêtement :

$$R(\tilde{f''}_{\beta,\gamma})_*(p''^{-1}\mathcal{F}) = p''^{-1}(R(\overline{f''}_{\beta,\gamma})_*\mathcal{F})$$

est à cohomologie localement constante sur $\tilde{D}''_{\eta(\beta)}$. Pour tout $\tilde{w''} \in \tilde{D}''_{\eta(\alpha)}$, si $w'' = p''(\tilde{w''})$, on a les isomorphismes :

$$\begin{array}{rcl}
R\Gamma(\tilde{T}_{\beta,\gamma}, p''^{-1}\mathcal{F}) & \simeq & R\Gamma(\tilde{D}''_{\eta(\beta)}, R(\tilde{f''}_{\beta,\gamma})_*(p''^{-1}\mathcal{F}) \\
& \simeq & R(\tilde{f''}_{\beta,\gamma})_*(p''^{-1}\mathcal{F})_{\tilde{w''}} \\
& \simeq & R(\overline{f''}_{\beta,\gamma})_*\mathcal{F})_{w''} \\
& \simeq & R\Gamma(\cap_{i=1}^r \{\mid t_i \mid < \gamma\} \cap_{i=r+1}^p \{t_i = w''_i\}, R(\overline{f}_{\beta,\gamma,w'})_*\mathcal{F}))
\end{array}$$

Par constructibilité de $R(\overline{f}_{\beta,\gamma})_*\mathcal{F}$ relativement au croisement normal de $D_{\eta(\beta),\gamma}$, nous obtenons :

$$\begin{array}{rcl}
R\Gamma(\tilde{T}_{\beta,\gamma,w'}, p''^{-1}\mathcal{F}) & \simeq & R\Gamma(\cap_{i=1}^r \{t_i = 0\} \cap_{i=r+1}^p \{t_i = w''_i\}, R(\overline{f}_{\beta,\gamma,w'})_*\mathcal{F}) \\
& \simeq & R\Gamma(\overline{B}_\beta \cap f^{-1}(0, w''), \mathcal{F}).
\end{array}$$

Ainsi que pour tout $\beta$ tel que $0 < \alpha < \beta < \epsilon_0$ et tout $w'' \in D^*_{\eta(\alpha)}$ :

$$R\Gamma(\tilde{T}_{\beta,\gamma}, p''^{-1}\mathcal{F}) \simeq R\Gamma(\overline{B}_\beta \cap f^{-1}(0, w''), \mathcal{F})$$

qui sont de plus indépendants de $\beta$.

Comme la restriction de $f$ à $\overline{B}_\alpha$ est propre, les $T_{\beta,\gamma}$ forment un système fondamental de voisinage de $\overline{B}_\alpha \cap f'^{-1}(0) \cap f''^{-1}(0)$. Ainsi :

$$\begin{array}{rcl}
(\, i'^{-1}(\Psi_{f''}\mathcal{F})\,)_0 & \simeq & R\Gamma(\overline{B}_\alpha \cap f'^{-1}(0), \Psi_{f''}\mathcal{F}) \\
& \simeq & \lim\limits_{\substack{\rightarrow \\ \beta,\gamma}} R\Gamma(\tilde{T}_{\beta,\gamma}, p''^{-1}\mathcal{F}) \\
& \simeq & R\Gamma(\overline{B}_\beta \cap f^{-1}(0, w''), \mathcal{F}) \\
& \simeq & (\Psi_{f''}(i'^{-1}\mathcal{F}))_0.
\end{array}$$

Cela montre le résultat attendu.

**Proposition 14.** *Si* $((f_1, \dots, f_p), \operatorname{car}\mathcal{F})$ *est sans pente, soit* $(T^*_{Y_\alpha}X)_{\alpha\in A}$ *les composantes irréductibles de* $\operatorname{car}\mathcal{F}$ *non contenues dans* $F^{-1}(0)$*, alors :*

$$\operatorname{car}(\Psi_f\mathcal{F}) = \bigcup_{\alpha\in A} W^\sharp_{f,Y_\alpha} \cap f^{-1}(0) = \bigcup_{\alpha\in A} W_{f,Y_\alpha} \cap f^{-1}(0) \quad .$$

**Preuve :** Comme $f_{|Y_\alpha}$ est sans pente pour $\alpha \in A$, on a (voir remarque 2) :

$$W^\sharp_{f_1,\dots,f_p,Y_\alpha} \cap f^{-1}(0) = W_{f_1,\dots,f_p,Y_\alpha} \cap f^{-1}(0).$$

Cela montre la dernière égalité dans la proposition. Pour le reste procédons par récurrence sur $p$.

Pour $p = 1$, soit $\Sigma_1$ les projections des composantes irréductibles $T^*_Y X$ de $\mathrm{car}\,\mathcal{F}$ non contenues dans $f_1 = 0$. Suivant [G] théorème 5.5 :

$$\mathrm{car}\,(\Psi_{f_1}(\mathcal{F})) = \bigcup_{Y \in \Sigma_1} W^\sharp_{f_1,Y} \cap f_1^{-1}(0).$$

C'est exactement la proposition pour $p = 1$.

Pour $Y \in \Sigma_1$, soit $\Sigma'_Y$ les projections des composantes irréductibles $T^*_Z X$ de $W^\sharp_{f_1,Y} \cap f_1^{-1}(0)$ non contenues dans $f_2 \dots f_p = 0$. Comme $\Psi_f \mathcal{F} = \Psi_{f_2,\dots,f_p}(\Psi_{f_1}\mathcal{F})$, par hypothèse de récurrence :

$$\mathrm{car}\,(\Psi_f \mathcal{F}) = \bigcup_{Y \in \Sigma_1} \bigcup_{Z \in \Sigma'_Y} W^\sharp_{f_2,\dots,f_p,Z} \cap (f_2 = \dots = f_p = 0).$$

Comme $Y \subset (f_2 \dots f_p = 0)$ implique $W^\sharp_{f_1,Y} \subset (f_2 \dots f_p = 0)$, on a encore :

$$\mathrm{car}\,(\Psi_f \mathcal{F}) = \bigcup_{Y \in \Sigma'_1} \bigcup_{Z \in \Sigma'_Y} W^\sharp_{f_2,\dots,f_p,Z} \cap (f_2 = \dots = f_p = 0),$$

oú $\Sigma'_1$ désignent l'ensemble des projections des composantes irréductibles de $\mathrm{car}\,\mathcal{F}$ non contenues dans $F = 0$.

La proposition résultera alors du lemme suivant :

**Lemme 3.** *Soit $Y$ non contenue dans $F = 0$ et $\Sigma'_Y$ les projections des composantes irréductibles de $W^\sharp_{f_1,Y} \cap f_1^{-1}(0)$ non contenues dans $f_2 \dots f_p = 0$. Supposons $f_{|Y}$ sans pente, alors :*

$$W^\sharp_{f,Y} \cap f_1^{-1}(0) = \bigcup_{Z \in \Sigma'_Y} W^\sharp_{f_2,\dots,f_p,Z} \subset T^*X \times \mathbf{C}^{p-1}.$$

L'identification faite dans ce lemme est justifiée par le fait que sous l'hypothèse sans pente $W^\sharp_{f,Y} \cap f_1^{-1}(0)$ est contenue dans $s_1 = 0$ (voir remarque 1).

**Preuve du lemme :** On rappelle de plus que sous l'hypothèse sans pente (voir remarque 1) les composantes irréductibles de $W^\sharp_{f,Y} \cap f_1^{-1}(0)$ (qui sont de dimension $n + p - 1$) ne sont pas contenues dans l'hypersurface $f_2 \dots f_p = 0$. Il en résulte :

$$W^\sharp_{f,Y} \cap f_1^{-1}(0) = \overline{W^\sharp_{f,Y} \cap f_1^{-1}(0) \cap (f_2 \dots f_p \neq 0)}.$$

Si $(x, \xi, 0, s_2, \dots, s_p) \in W^\sharp_{f,Y} \cap f_1^{-1}(0) \cap (f_2 \dots f_p \neq 0)$, il est limite de points :

$$(x_n, \xi_n + s_1(n)\frac{df_1(x_n)}{f_1(x_n)} + \sum_{j=2}^{p} s_j(n)\frac{df_j(x_n)}{f_j(x_n)}, s_1(n), \dots, s_p(n)).$$

Le point $(x_n, \sum_{j=2}^{p} s_j(n)\frac{df_j(x_n)}{f_j(x_n)}, s_2(n), \dots, s_p(n))$ tend vers

$$(x, \sum_{j=2}^{p} s_j \frac{df_j(x)}{f_j(x)}, s_2, \dots, s_p).$$

Il en résulte que le point $(x_n, \xi_n + s_1(n)\frac{df_1(x_n)}{f_1(x_n)}, s_1(n))$ tend vers une limite :

$$(x, \alpha, 0) \in W^\sharp_{f_1,Y} \cap f_1^{-1}(0) \cap (f_2 \dots f_p \neq 0) \quad .$$

Cela montre l'inclusion :

$$W^{\sharp}_{f,Y} \cap f_1^{-1}(0) \subset \bigcup_{Z \in \Sigma'_Y} W^{\sharp}_{f_2,\ldots,f_p,Z}.$$

Inversement, si $(x, \xi, s_2, \ldots, s_p) \in W^{\sharp}_{f_2,\ldots,f_p,Z}$ pour un $Z \in \Sigma'_Y$, ce point est limite de

$$(x_n, \eta_n + \sum_{j=2}^{p} s_j(n) \frac{df_j(x_n)}{f_j(x_n)}, s_2(n), \ldots, s_p(n)) \ ,$$

où $(x_n, \eta_n) \in T^*_Z X$. Chaque $(x_n, \eta_n, 0)$ est alors limite de

$$(x_{n,m}, \eta_{n,m} + s_1(n,m) \frac{df_1(x_{n,m})}{f_1(x_{n,m})}, s_1(n,m),$$

où $(x_{n,m}, \eta_{n,m}) \in T^*_Y X$. Il en résulte :

$$(x, \xi, s_2, \ldots, s_p) \in W^{\sharp}_{f,Y} \cap f_1^{-1}(0).$$

**Remarque 5.** *Soit $\mathcal{F} \in D^b_c(\mathbf{C}_X)$ de variété caratéristique $\bigcup T^*_Y X$. Désignons par $j$ l'inclusion ouverte $j : F \neq 0 \to X$. Si $f_{|Y}$ est sans pente pour les $Y$ non contenues dans $F^{-1}(0)$, alors le couple $(f, \operatorname{car} j_! j^{-1}\mathcal{F})$ est sans pente pour $j_! j^{-1}\mathcal{F}$.*

**Preuve :** En effet (voir [G]) :

$$\operatorname{car} j_! j^{-1}\mathcal{F} = \bigcup_{F_{|Y} \neq 0} W^{\sharp}_{F,Y} \cap F^{-1}(0).$$

Il reste à utiliser la proposition 6. Cette remarque est l'analogue géométrique de l'équivalence entre les propriétés 1 et 2 du corollaire 2.

Cette remarque est utile puisque pour tout $f$ : $\Psi_f \mathcal{F} = \Psi_f(j_! j^{-1}\mathcal{F})$.

### 4.3. Restriction non caractéristique et morphisme sans pente.

4.3.1. *Restriction non caractéristique.* Considérons $X$ un voisinage de l'origine dans $\mathbf{C}^n$ et $Y$ un germe de sous-espace analytique de $X$. Soit $g = (g_1, \ldots, g_p) : X \to \mathbf{C}^p$ une submersion. Nous dirons que $g$ est à fibres non caractéristiques pour $T^*_Y X$ si

$$T^*_Y X \cap T^*_{g^{-1}(0)} X \subset T^*_X X.$$

Cela implique pour $t \in \mathbf{C}^p$ voisin de l'origine :

$$T^*_Y X \cap T^*_{g^{-1}(t)} X \subset T^*_X X.$$

On notera également que, sous cette hypothèse, les $g_i$ sont non identiquement nulles sur $Y$.

Si $A$ et $B$ désignent deux sous-ensembles de $T^*X$, rappelons la notation :

$$A + B = \{(x, a+b) \ ; \ (x,a) \in A \text{ et } (x,b) \in B\}.$$

**Lemme 4.** *Si $g : \mathbf{C}^n \to \mathbf{C}^p$ est une submersion à fibres non caractéristiques pour $T^*_Y X$, nous avons :*

(1) $g_\pi({}^t g')^{-1}(T^*_Y X) \subset T^*_{\mathbf{C}^p} \mathbf{C}^p$,

(2) $W_{g,Y} \bigcap g^{-1}(0) = T^*_Y X + T^*_{g^{-1}(0)} X$ *(qui est donc sous-variété lagrangienne de $T^*X$).*

**Preuve du lemme :** Comme le covecteur $\sum_{i=1}^p \eta_i dg_i(x)$ est conormal aux fibres de $g$, le premier point résulte du fait que sous nos hypothèses, la relation :

$$\sum_{i=1}^p \eta_i dg_i(x) \in T^*_Y X \quad \Longrightarrow \quad \eta_1 = \cdots = \eta_p = 0.$$

Suivant [K-S] lemme 5.4.7 et corollaire 8.3.18, l'hypothèse non caractéristique implique que $T^*_Y X + T^*_{g^{-1}(0)} X$ est un sous-ensemble analytique lagrangien fermé de $T^*X$.

Il reste à montrer que $W_{g,Y} \bigcap g^{-1}(0) = T^*_Y X + T^*_{g^{-1}(0)} X$.

$$T^*_Y X + T^*_{g^{-1}(0)} X = \{(x, \xi + \sum_{i=1}^p \eta_i dg_i(x))\ ;\ (x,\xi) \in T^*_Y X \text{ et } g(x) = 0\}.$$

De la définition de $W_{g,Y}$, il résulte :

$$T^*_Y X + T^*_{g^{-1}(0)} X \subset W_{g,Y} \cap g^{-1}(0).$$

Nous avons puisque $g$ est une submersion :

$$W_g = \{(x, \sum_{i=1}^p \eta_i dg_i(x))\ ;\ x \in X \text{ et } (\eta_1, \ldots \eta_p) \in \mathbf{C}^p\} \subset T^*X.$$

Il résulte de l'hypothèse non caractéristique que l'intersection de $W_g$ et de $T^*_Y X$ est contenue dans la section nulle de $X$. Suivant [K-S] lemme 5.4.7 $T^*_Y X + W_g$ est un sous-ensemble analytique fermé de $T^*X$. Il en résulte :

$$W_{g,Y} \subset T^*_Y X + W_g.$$

D'où, l'inclusion qui manque, puisque $T^*_{g^{-1}(0)} X = W_g \cap g^{-1}(0)$.

Il résulte des lemmes 4 et 2, des propositions 11 et 14 la proposition suivante :

**Proposition 15.** *Soit $\mathcal{F} \in D^b_c(\mathbf{C}_X)$ et $g : \mathbf{C}^n \to \mathbf{C}^p$ une submersion à fibres non caractéristiques pour toute composante de la variété caractéristique de $\mathcal{F}$. Alors, nous avons :*
- *Le couple $(g, \mathrm{Car}\,\mathcal{F})$ est sans pente.*
- *Les images directes locales $Rg_*\mathcal{F}$ sont à cohomologie localement constante.*
- *Le morphime canonique $i^{-1}\mathcal{F} \to \Psi_g\mathcal{F}$ est un isomorphisme.*
- $\mathrm{Car}\,(i^{-1}\mathcal{F}) = \mathrm{Car}\,(\Psi_g\mathcal{F}) = \mathrm{Car}\,\mathcal{F} + T^*_{g^{-1}(0)} X$.

4.3.2. *Restriction non caractéristique d'un morphisme sans pente.* Considérons $X$ un voisinage de l'origine dans $\mathbf{C}^n$ et $Y$ un germe de sous-espaces analytique de $X$. On considère une submersion $g = (g_1, \ldots, g_r) : \mathbf{C}^n \to \mathbf{C}^r$ et $f = (f_{r+1}, \ldots, f_p) : \mathbf{C}^n \to \mathbf{C}^{p-r}$. On note $(g, f)$ l'application :

$$(g, f) : \mathbf{C}^n \longrightarrow \mathbf{C}^p\ :\ x \mapsto (g(x), f(x)).$$

**Lemme 5.** *On suppose que $f_{|Y}$ est sans pente et que la submersion $g$ est à fibres non caractéristiques pour $W_{f,Y} \cap f^{-1}(0)$. Alors :*

(1) *Au voisinage de $(g, f)^{-1}(0)$, le sous-espace $(g, f)_\pi({}^t(g, f)')^{-1}(T^*_Y X)$ s'identifie au croisement normal $T_{\mathbf{C}^r}\mathbf{C}^r \times f_\pi({}^t f')^{-1}(T^*_Y X)$ de $\mathbf{C}^p$.*

(2) $W_{(g,f),Y} \cap (g, f)^{-1}(0) = (W_{f,Y} \cap f^{-1}(0)) + T^*_{g^{-1}(0)} X$.

(3) $(g, f)_{|Y} : Y \to \mathbf{C}^p$ *est sans pente.*

**Preuve du lemme :** Supposons $\sum_{i=1}^{r}\lambda_i dg_i(x)+\sum_{i=r+1}^{p}\lambda_i df_i(x)\in T^*_YX$. Cela implique au voisinage de $(g,f)^{-1}(0)$ que $\lambda_i=0$ pour $i\in\{1,\dots,r\}$. On obtiendrait sinon un covecteur non nul dans $W_{f,Y}\cap f^{-1}(0)\cap T^*_{g^{-1}(0)}X$ ce qui contredirait l'hypothèse que $g$ est à fibres non caractéristiques. Il en résulte : $\sum_{i=r+1}^{p}\lambda_i df_i(x)\in T^*_YX$ et donc $(x,\lambda_{r+1},\dots,\lambda_p)$ appartient à $f_\pi({}^tf')^{-1}(T^*_YX)$. Cela établit le point 1 de la proposition.

Soit $(x,\xi)\in W_{f,Y}\cap f^{-1}(0)+T^*_{g^{-1}(0)}X$, il s'écrit :

$$(x,\xi)=(x,\alpha+\sum_{i=1}^{r}\lambda_i dg_i(x)) \text{ où } (x,\alpha)\in W_{f,Y} \text{ et } g(x)=f(x)=0.$$

Le point $(x,\alpha)$ est donc limite de $(x_n,\beta_n+\sum_{i=r+1}^{p}\lambda_i(n)df_i(x_n))$ où $(x_n,\beta_n)\in T^*_YX$. Il en résulte que $(x,\xi)$ est limite de la suite :

$$(x_n,\beta_n+\sum_{i=1}^{r}\lambda_i dg_i(x_n))+\sum_{i=r+1}^{p}\lambda_i(n)df_i(x_n)).$$

Ainsi, $(x,\xi)\in W_{(g,f),Y}\cap(g,f)^{-1}(0)$.

Inversement, supposons $(x,\xi)\in W_{(g,f),Y}\cap(g,f)^{-1}(0)$. Le point $(x,\xi)$ est alors limite de

$$(x_n,\beta_n+\sum_{i=1}^{r}\lambda_i dg_i(x_n))+\sum_{i=r+1}^{p}\lambda_i(n)df_i(x_n)),$$

où $(x_n,\beta_n)\in T^*_YX$. Or, par hypothèse, au vosinage de $(g,f)^{-1}(0)$ :

$$W_g\cap W_{f,Y}\subset T^*_XX.$$

Il en résulte $W_g+W_{f,Y}$ fermé et $(x,\xi)\in W_g+W_{f,Y}$. D'où :

$$(x,\xi)\in(W_{f,Y}\cap f^{-1}(0))+T^*_{g^{-1}(0)}X.$$

Ainsi, on obtient l'égalité attendue au point 2.

Enfin, $(W_{f,Y}\cap f^{-1}(0))\cap T^*_{g^{-1}(0)}X$ est par hypothèse contenue dans la section nulle de $T^*X$. Ces deux variétés étant lagrangiennes coniques, il en résulte que $(W_{f,Y}\cap f^{-1}(0))+T^*_{g^{-1}(0)}X$ est elle même lagrangienne conique. La troisième assertion résulte alors des deux premières.

Nous déduisons de ce lemme :

**Proposition 16.** *Soit $\mathcal{F}\in D^b_c(\mathbf{C}_X)$ et $\cup_{\alpha\in A}T^*_{Y_\alpha}X$ sa variété caractéristique. Notons $I_\alpha=\{i\in\{r+1,\dots,p\}\ ;\ f_{i|Y_\alpha}\neq 0\}$ et $f_\alpha=(f_j)_{j\in_\alpha}$. Supposons que le couple $(f,\operatorname{Car}\mathcal{F})$ soit sans pente et et que la submersion $g$ soit à fibres non caractéristiques pour les $W_{f_\alpha,Y_\alpha}\cap f_\alpha^{-1}(0)$, alors :*

- *$((g,f),\operatorname{Car}\mathcal{F})$ est sans pente.*
- *Les images directes locales $R(g,f)_*\mathcal{F}$ sont à cohomologie constructible relativement au croisement normal $T_{\mathbf{C}^r}\mathbf{C}^r\times f_\pi({}^tf')^{-1}(\operatorname{car}F)$.*
- *Si $i$ désigne l'inclusion de $g^{-1}(0)$ dans $X$, nous avons les isomorphismes canoniques :*

$$\Psi_f(i^{-1}\mathcal{F})\simeq i^{-1}(\Psi_f\mathcal{F})\simeq\Psi_{g,f}(\mathcal{F}).$$

- $\operatorname{car}\Psi_{g,f}\mathcal{F})=\cup_{Y_\beta\in B}(W_{f,Y_\beta}\cap g^{-1}(0))+T^*_{g^{-1}(0)}X$ *, où $Y_\beta$ décrit les projections des composantes de* $\operatorname{car}\mathcal{F}$ *non identiquement nulles sur le produit $F=f_{r+1}\dots f_p$.*

Si dans la proposition, nous supposons seulement que la submersion $g$ est à fibres non caractéristiques sur les $W_{f,Y} \cap f^{-1}(0)$ où $T^*_Y X$ est une composante de car $\mathcal{F}$ sur laquelle $F$ est non identiquement nulle, alors $j_! j^{-1}\mathcal{F}$ vérifie les hypothèses de la propoposition. En particulier, nous aurons toujours :

$$\Psi_f(i^{-1}\mathcal{F}) \simeq i^{-1}\Psi_f(\mathcal{F}) \simeq \Psi_{g,f}(\mathcal{F}).$$

Dans nos articles [M-T2] et [D-M-S-T], nous avions étudié cette situation d'un point de vue algébrique lorque $f$ est constituée d'une seule fonction ($p - r = 1$). La proposition 16 apporte un complément géométrique à cette étude.

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Univ. Nice Sophia Antipolis, CNRS, LJAD, UMR 7351, 06100 Nice, France

**Journal *of* Singularities**
**Volume 7 (2013), 190-204**

received 20 January 2013
in revised form 8 July 2013
DOI: 10.5427/jsing.2013.7j

# ON A SINGULAR VARIETY ASSOCIATED TO A POLYNOMIAL MAPPING

NGUYEN THI BICH THUY, ANNA VALETTE, AND GUILLAUME VALETTE

Abstract. In the paper, "*Geometry of polynomial mapping at infinity via intersection homology*", the second and third authors associated to a given polynomial mapping $F : \mathbb{C}^2 \to \mathbb{C}^2$ with nonvanishing Jacobian a variety whose homology or intersection homology describes the geometry of singularities at infinity of the mapping. We generalize that result.

## 1. INTRODUCTION

In 1939, O. H. Keller [9] stated the famous Jacobian conjecture: any polynomial mapping $F : \mathbb{C}^n \to \mathbb{C}^n$ with nowhere vanishing Jacobian is a polynomial automorphism. The problem remains open today even for dimension 2. We call the smallest set $S_F$ such that the mapping $F : X \setminus F^{-1}(S_F) \to Y \setminus S_F$ is proper, the asymptotic set of $F$. The Jacobian conjecture reduces to show that the asymptotic set of a complex polynomial mapping with nonzero constant Jacobian is empty. So the set of points at which a polynomial mapping fails to be proper plays an important role.

The second and third authors gave in [14] a new approach to study the Jacobian conjecture in the case of dimension 2: they constructed a real pseudomanifold denoted $N_F \subset \mathbb{R}^\nu$, where $\nu > 2n$, associated to a given polynomial mapping $F : \mathbb{C}^n \to \mathbb{C}^n$, such that the singular part of the variety $N_F$ is contained in $(S_F \times K_0(F)) \times \{0_{\mathbb{R}^{\nu-2n}}\}$ where $K_0(F)$ is the set of critical values of $F$. In the case of dimension 2, the homology or intersection homology of $N_F$ describes the geometry of the singularities at infinity of the mapping $F$.

Our aim is to improve this result in the general case of dimension $n > 2$ and compute the intersection homology of the associated pseudomanifold $N_F$. Let $\hat{F}_i$ be the leading forms of the components $F_i$ of the polynomial mapping $F = (F_1, \dots, F_n) : \mathbb{C}^n \to \mathbb{C}^n$. We show (Theorem 4.5) that if the polynomial mapping $F : \mathbb{C}^n \to \mathbb{C}^n$ has nowhere vanishing Jacobian and if $rank(D\hat{F}_i)_{i=1,\dots,n} > n - 2$, then the condition of properness of $F$ is equivalent to the condition of vanishing homology or intersection homology of $N_F$. Moreover, it is indeed more precise to compute intersection homology rather than homology. In order to compute the intersection homology of the variety $N_F$, we show that it admits a stratification which is locally topologically trivial along the strata. The main feature of intersection homology is to satisfy Poincaré duality that is more interesting in the case where the stratification has no stratum of odd real dimension. We show that the variety $N_F$ admits a Whitney stratification with only even dimensional strata. It is well known that Whitney stratification are locally topologically trivial along the strata.

Research partially supported by the NCN grant 2011/01/B/ST1/03875 and by APO, no 2010-80, Région Provence-Alpes-Côte d'Azur.

**Acknowledgements.** Research partially supported by the NCN grant 2011/01/B/ST1/03875. The first author was supported by the Région Provence-Alpes-Côte d'Azur, doctoral contract APO, n$^o$ 2010-80.

This article was written in Kraków, Poland, when the first author was invited in Kraków at the Polish Academy of Science. It is our pleasure to thank this Institute for its kind hospitality and very good working conditions.

It is our pleasure to thank Jean-Paul Brasselet for his interest and encouragements.

## 2. PRELIMINARIES AND BASIC DEFINITION

In this section we set-up our framework. All the considered sets in this article are semi-algebraic.

2.1. **Notations and conventions.** Given a topological space $X$, singular simplices of $X$ will be semi-algebraic continuous mappings $\sigma : T_i \to X$, where $T_i$ is the standard $i$-simplex in $\mathbb{R}^{i+1}$. Given a subset $X$ of $\mathbb{R}^n$ we denote by $C_i(X)$ the group of $i$-dimensional singular chains (linear combinations of singular simplices with coefficients in $\mathbb{R}$); if $c$ is an element of $C_i(X)$, we denote by $|c|$ its support. By $Reg(X)$ and $Sing(X)$ we denote respectively the regular and singular locus of the set $X$. Given $X \subset \mathbb{R}^n$, $\overline{X}$ will stand for the topological closure of $X$. Given a point $x \in \mathbb{R}^n$ and $\alpha > 0$, we write $\mathbb{B}(x, \alpha)$ for the ball of radius $\alpha$ centered at $x$ and $\mathbb{S}(x, \alpha)$ for the corresponding sphere, boundary of $\mathbb{B}(x, \alpha)$.

2.2. **Intersection homology.** We briefly recall the definition of intersection homology; for details, we refer to the fundamental work of M. Goresky and R. MacPherson [3] (see also [2]).

**Definition 2.1.** Let $X$ be a $m$-dimensional semi-algebraic set. A **semi-algebraic stratification of** $X$ is the data of a finite semi-algebraic filtration

$$X = X_m \supset X_{m-1} \supset \cdots \supset X_0 \supset X_{-1} = \emptyset \tag{2.2}$$

such that for every $i$, the set $S_i = X_i \setminus X_{i-1}$ is either empty or a topological manifold of dimension $i$. A connected component of $S_i$ is called **a stratum** of $X$.

**Definition 2.3** ([16])**.** One says that the **Whitney** $(b)$ **condition** is realized for a stratification if for each pair of strata $(S, S')$ and for any $y \in S$ one has: Let $\{x_n\}$ be a sequence of points in $S'$ with limit $y$ and let $\{y_n\}$ be a sequence of points in $S$ tending to $y$, assume that the sequence of tangent spaces $\{T_{x_n} S'\}$ admits a limit $T$ for $n$ tending to $+\infty$ (in a suitable Grassmanian manifold) and that the sequence of directions $x_n y_n$ admits a limit $\lambda$ for $n$ tending to $+\infty$ (in the corresponding projective manifold), then $\lambda \in T$.

We denote by $cL$ the open cone on the space $L$, the cone on the empty set being a point. Observe that if $L$ is a stratified set then $cL$ is stratified by the cones over the strata of $L$ and a 0-dimensional stratum (the vertex of the cone).

**Definition 2.4.** A stratification of $X$ is said to be **locally topologically trivial** if for every $x \in X_i \setminus X_{i-1}$, $i \geq 0$, there is an open neighborhood $U_x$ of $x$ in $X$, a stratified set $L$ and a semi-algebraic homeomorphism

$$h : U_x \to (0;1)^i \times cL,$$

such that $h$ maps the strata of $U_x$ (induced stratification) onto the strata of $(0;1)^i \times cL$ (product stratification).

We will use the following definition of a semi-algebraic pseudomanifold :

**Definition 2.5.** A **(semi-algebraic) pseudomanifold** in $\mathbb{R}^n$ is a subset $X \subset \mathbb{R}^n$ whose singular locus is of codimension at least 2 in $X$ and whose regular locus is dense in X.

A **stratified pseudomanifold** (of dimension $m$) is the data of an $m$-dimensional pseudomanifold $X$ together with a semi-algebraic filtration:

$$X = X_m \supset X_{m-1} \supset X_{m-2} \supset \cdots \supset X_0 \supset X_{-1} = \emptyset,$$

which constitutes a locally topologically trivial stratification of $X$.

**Definition 2.6.** A **stratified pseudomanifold with boundary** is a semi-algebraic couple $(X, \partial X)$ together with a semi-algebraic filtration

$$X = X_m \supset X_{m-1} \supset X_{m-2} \supset \cdots \supset X_0 \supset X_{-1} = \emptyset,$$

such that:

(1) $X \setminus \partial X$ is an $m$-dimensional stratified pseudomanifold (with the filtration $X_j \setminus \partial X$),
(2) $\partial X$ is a stratified pseudomanifold (with the filtration $X'_j := X_{j+1} \cap \partial X$),
(3) $\partial X$ has a **stratified collared neighborhood**: there exist a neighborhood $U$ of $\partial X$ in $X$ and a semi-algebraic homeomorphism $\phi : \partial X \times [0,1] \to U$ such that

$$\phi(X'_{j-1} \times [0,1]) = U \cap X_j \text{ and } \phi(\partial X \times \{0\}) = \partial X.$$

**Definition 2.7.** A **perversity** is an $(m-1)$-uple of integers $\bar{p} = (p_2, p_3, \dots, p_m)$ such that $p_2 = 0$ and $p_{k+1} \in \{p_k, p_k + 1\}$.

Traditionally we denote the zero perversity by $\overline{0} = (0, \dots, 0)$, the maximal perversity by $\overline{t} = (0, 1, \dots, m-2)$, and the middle perversities by $\overline{m} = (0, 0, 1, 1, \dots, [\frac{m-2}{2}])$ (lower middle) and $\overline{n} = (0, 1, 1, 2, 2, \dots, [\frac{m-1}{2}])$ (upper middle). We say that the perversities $\overline{p}$ and $\overline{q}$ are **complementary** if $\overline{p} + \overline{q} = \overline{t}$.

Given a stratified pseudomanifold $X$, we say that a semi-algebraic subset $Y \subset X$ is $(\bar{p}, i)$-**allowable** if $\dim(Y \cap X_{m-k}) \leq i - k + p_k$ for all $k \geq 2$.

In particular, a subset $Y \subset X$ is $(\overline{t}, i)$-allowable if $\dim(Y \cap Sing(X)) < i - 1$.

Define $IC_i^{\overline{p}}(X)$ to be the $\mathbb{R}$-vector subspace of $C_i(X)$ consisting of those chains $\xi$ such that $|\xi|$ is $(\overline{p}, i)$-allowable and $|\partial \xi|$ is $(\overline{p}, i-1)$-allowable.

**Definition 2.8.** The $i^{th}$ **intersection homology group with perversity** $\overline{p}$, denoted by $IH_i^{\overline{p}}(X)$, is the $i^{th}$ homology group of the chain complex $IC_*^{\overline{p}}(X)$.

Goresky and MacPherson proved that these groups are independent of the choice of the stratification and are finitely generated [3, 4].

**Theorem 2.9** (Goresky, MacPherson [3])**.** *For any orientable compact stratified semi-algebraic $m$-dimensional pseudomanifold $X$, generalized Poincaré duality holds:*

$$IH_k^{\overline{p}}(X) \simeq IH_{m-k}^{\overline{q}}(X), \tag{2.9}$$

*where $\overline{p}$ and $\overline{q}$ are complementary perversities.*

In the non-compact case the above isomorphism holds for Borel-Moore homology:

$$IH_k^{\overline{p}}(X) \simeq IH_{m-k,BM}^{\overline{q}}(X), \tag{2.9}$$

where $IH_{*,BM}$ denotes the intersection homology with respect to Borel-Moore chains [4, 2]. A relative version is also true in the case where $X$ has boundary.

**Proposition 2.10** (Topological invariance, [3, 4])**.** *Let $X$ be a locally compact stratified pseudomanifold and $\overline{p}$ a perversity, then the intersection homology groups $IH_*^{\overline{p}}(X)$ and $IH_{*,BM}^{\overline{p}}(X)$ do not depend on the stratification of $X$.*

2.3. $\mathcal{L}^\infty$ **cohomology.** Let $M \subset \mathbb{R}^n$ be a smooth submanifold.

**Definition 2.11.** We say that a differential form $\omega$ on $M$ is $\mathcal{L}^\infty$ if there exists a constant $K$ such that for any $x \in M$:

$$|\omega(x)| \leq K.$$

We denote by $\Omega_\infty^j(M)$ the cochain complex constituted by all the $j$-forms $\omega$ such that $\omega$ and $d\omega$ are both $\mathcal{L}^\infty$. The cohomology groups of this cochain complex are called the $\mathcal{L}^\infty$**-cohomology groups of** $M$ and will be denoted by $H_\infty^*(M)$.

The third author showed that the $\mathcal{L}^\infty$ cohomology of the differential forms defined on the regular part of a pseudomanifold $X$ coincides with the intersection cohomology of $X$ in the maximal perversity ([15], Theorem 1.2.2):

**Theorem 2.12.** *Let $X$ be a compact subanalytic pseudomanifold (possibly with boundary). Then, for any $j$:*

$$H_\infty^j(Reg(X)) \simeq IH_j^{\bar{t}}(X).$$

*Furthermore, the isomorphism is induced by the natural mapping provided by integration on allowable simplices.*

2.4. **The Jelonek set.** Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial mapping. We denote by $S_F$ the set of points at which the mapping $F$ is not proper, *i.e.*,

$$S_F = \{y \in \mathbb{C}^n \text{ such that } \exists \{x_k\} \subset \mathbb{C}^n, |x_k| \to \infty, F(x_k) \to y\},$$

and call it the **asymptotic variety** or **Jelonek set** of $F$. The geometry of this set was studied by Jelonek in a series of papers [6, 7, 8]. Jelonek obtained a nice description of this set and gave an upper bound for its degree. For the details and applications of these results we refer to the works of Jelonek. In our paper, we will need the following powerful theorems.

**Theorem 2.13** ([6])**.** *If $F : \mathbb{C}^n \to \mathbb{C}^n$ is a generically finite polynomial mapping, then $S_F$ is either an $(n-1)$ pure dimensional $\mathbb{C}$-uniruled algebraic variety or the empty set.*

**Theorem 2.14** ([6])**.** *If $F : X \to Y$ is a dominant polynomial map of smooth affine varieties of the same dimension then $S_F$ is either empty or is a hypersurface.*

Here, by a $\mathbb{C}$-uniruled variety $X$ we mean that through any point of $X$ passes a rational complex curve included in $X$. In other words, $X$ is $\mathbb{C}$-uniruled if for all $x \in X$ there exists a non-constant polynomial mapping $\varphi_x : \mathbb{C} \to X$ such that $\varphi_x(0) = x$.

In the real case, the Jelonek set is an $\mathbb{R}$-uniruled semi-algebraic set but, if nonempty, its dimension can be any integer between 1 and $(n-1)$ [8].

## 3. THE VARIETY $N_F$

The variety $N_F$ was constructed by the second and third authors in [14]. Let us recall briefly this construction.

3.1. **Construction of the variety $N_F$ ([14]).** We will consider polynomial mappings $F : \mathbb{C}^n \to \mathbb{C}^n$ as real ones $F : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$. By $Sing(F)$ we mean the singular locus of $F$ that is the zero set of its Jacobian determinant and we denote by $K_0(F)$ the set of critical values of $F$, *i.e.*, the set $F(Sing(F))$.

We denote by $\rho$ the Euclidean Riemannian metric in $\mathbb{R}^{2n}$. We can pull it back in a natural way:

$$F^*\rho_x(u, v) := \rho(d_xF(u), d_xF(v)).$$

Define the Riemannian manifold $M_F := (\mathbb{R}^{2n} \setminus Sing(F); F^*\rho)$ and observe that the mapping $F$ induces a local isometry near any point of $M_F$.

**Lemma 3.1** ([14])**.** *There exists a finite covering of $M_F$ by open semi-algebraic subsets such that on every element of this covering, the mapping $F$ induces a diffeomorphism onto its image.*

**Proposition 3.2** ([14])**.** *Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial mapping. There exists a real semi-algebraic pseudomanifold $N_F \subset \mathbb{R}^\nu$, for some $\nu = 2n + p$, where $p > 0$ such that*

$$Sing(N_F) \subset (S_F \cup K_0(F)) \times \{0_{\mathbb{R}^p}\},$$

*and there exists a semi-algebraic bi-Lipschitz mapping*

$$h_F : M_F \to N_F \setminus (S_F \cup K_0(F)) \times \{0_{\mathbb{R}^p}\}$$

*where* $N_F \setminus (S_F \cup K_0(F)) \times \{0_{\mathbb{R}^p}\}$ *is equipped with the Riemannian metric induced by* $\mathbb{R}^\nu$.

The variety $N_F$ is constructed as follows: let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial mapping. Thanks to Lemma 3.1, there exists a covering $\{U_1, \ldots, U_p\}$ of $M_F = \mathbb{R}^{2n} \setminus Sing(F)$ by open semi-algebraic subsets (in $\mathbb{R}^{2n}$) such that on every element of this covering, the mapping $F$ induces a diffeomorphism onto its image. We may find some semi-algebraic closed subsets $V_i \subset U_i$ (in $M_F$) which cover $M_F$ as well. Thanks to Mostowski's Separation Lemma (see Separation Lemma in [10], page 246), for each $i$, $i = 1, \ldots, p$, there exists a Nash function $\psi_i : M_F \to \mathbb{R}$, such that $\psi_i$ is positive on $V_i$ and negative on $M_F \setminus U_i$. We define

$$h_F := (F, \psi_1, \ldots, \psi_p) \text{ and } N_F := \overline{h_F(M_F)}.$$

In order to prove $h_F$ is bi-Lipschitz, we do as follows: choose $x \in M_F$, then there exists $U_j$ such that $x \in U_j$ and the mapping $F_{|U_j} : U_j \to \mathbb{R}^{2n}$ is a diffeomorphism onto its image. Define, for $y \in F(U_j)$, the following functions:

$$\tilde{\psi}_i(y) := \psi_i \circ (F_{|U_j})^{-1}(y), \tag{3.3}$$

for $i = 1, \ldots, p$, and

$$\hat{\psi}(y) := (y, \tilde{\psi}_1(y), \ldots, \tilde{\psi}_p(y)). \tag{3.4}$$

We then have the formula

$$h_F(x) = (F(x), \tilde{\psi}_1(F(x)), \ldots, \tilde{\psi}_p(F(x))) = \hat{\psi}(F(x)). \tag{3.5}$$

As the map $F : (U_j, F^*\rho) \to F(U_j)$ is bi-Lipschitz, it is enough to show that $\hat{\psi} : F(U_j) \to \mathbb{R}^{2n+p}$ is bi-Lipschitz. This amounts to prove that $\tilde{\psi}_i$ has bounded derivatives for any $i = 1, \ldots, p$. In order to prove this, we chose the functions $\psi_i$ sufficiently small, by using Łojasiewicz inequality in the following form:

**Proposition 3.6.** [1] *Let* $A \subset \mathbb{R}^n$ *be a closed semi-algebraic set and* $f : A \to \mathbb{R}$ *a continuous semi-algebraic function. There exist* $c \in \mathbb{R}, c \geq 0$ *and* $q \in \mathbb{N}$ *such that for any* $x \in A$ *we have*

$$|f(x)| \leq c(1 + |x|^2)^q.$$

In fact, we can choose the Nash functions $\psi_i$ sufficiently small by multiplying $\psi_i$ by a huge power of $\frac{1}{1+|x|^2}$ which is a Nash function (see Proposition 2.3 in [14]).

Thanks to Łojasiewicz inequality, we also can choose the functions $\psi_i$ such that they tend to zero at infinity and near $Sing(F)$. This is the reason why the singular part of $N_F$ is contained in $(S_F \cup K_0(F)) \times \{0_{\mathbb{R}^p}\}$.

Moreover, the following diagram is commutative:

$$\begin{tikzcd} M_F \arrow[r, "h_F"] \arrow[dr, "F"'] & N_F \arrow[d, "\pi_F"] \\ & \mathbb{R}^{2n} \end{tikzcd} \tag{3.7}$$

where $\pi_F$ is the canonical projection on the first $2n$ coordinates, and $h_F$ is bijective onto its image $N_F \setminus ((S_F \cup K_0(F)) \times \{0_{\mathbb{R}^p}\})$.

Remark that the set $N_F$ is not unique, it depends on the covering of $M_F$ that we choose and on the choice of the Nash function $\psi_i$.

We see that in the complex case, even in the case $\mathbb{C}^2$, the real dimension of the variety $N_F$ is greater than 3, so it is difficult to draw the variety $N_F$ in this case. The natural question arises if the variety $N_F$ exists in the real case. The answer is yes, but we note that in this case, the variety $N_F$ is not necessarily a pseudomanifold, because in the real case, the real dimension of the Jelonek set of a polynomial mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ can be $n - 1$.

**Proposition 3.8** ([14], [11])**.** *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a polynomial mapping. There exist*

*a) a real semi-algebraic variety $N_F \subset \mathbb{R}^\nu$, for some $\nu = n + p$ where $p > 0$, such that*

$$Sing(N_F) \subset (S_F \cup K_0(F)) \times \{0_{\mathbb{R}^p}\} \subset \mathbb{R}^n \times \mathbb{R}^p,$$

*b) a semi-algebraic bi-Lipschitz mapping*

$$h_F : M_F \to N_F \setminus (S_F \cup K_0(F)) \times \{0_{\mathbb{R}^p}\}$$

*where $N_F \setminus (S_F \cup K_0(F)) \times \{0_{\mathbb{R}^p}\}$ is equipped with the Riemannian metric induced by $\mathbb{R}^\nu$.*

In order to understand better the variety $N_F$, we give here an example in the real case.

3.2. **Example.**

**Example 3.9.** [11] Let $F : \mathbb{R}^2_{(x,y)} \to \mathbb{R}^2_{(\alpha,\beta)}$ be the polynomial mapping defined by

$$F(x, y) = (x, x^2y(y + 2)).$$

Let us construct the variety $N_F$ in this case. By an easy computation, we find:

$$Sing(F) = \{(x, y) \in \mathbb{R}^2_{(x,y)} : x = 0 \text{ or } y = -1\},$$
$$K_0(F) = \{(\alpha, \beta) \in \mathbb{R}^2_{(\alpha,\beta)} : \beta = -\alpha^2\},$$
$$S_F = \{(0, \beta) \in \mathbb{R}^2_{(\alpha,\beta)} : \beta \geq 0\}.$$

We see that $\mathbb{R}^2$ is divided into four open subsets $U_i$ by $Sing(F)$ (see the Figure 1a). The mapping $F$ is a diffeomorphism on each $U_i$, for $i = 1, \dots, 4$. Observe that $U_i$ is closed in $M_F$ so that we can chose $V_i = U_i$ for $i = 1, \dots, 4$ (see section 3.1). There exist Nash functions $\psi_i : M_F \to \mathbb{R}$ such that each $\psi_i$ is positive on $U_i$ and negative on $U_j$ if $j \neq i$. Since $N_F$ is the closure of $h_F(M_F)$ where $h_F = (F, \psi_1, \dots, \psi_4)$, then $N_F$ has 4 parts $(N_F)_1, \dots, (N_F)_4$ where $(N_F)_i$ is the closure of $h_F(U_i)$ for $i = 1, \dots, 4$.

Again, an easy computation shows:

$$F(U_1) = F(U_2) = \{(\alpha, \beta) \in \mathbb{R}^2_{(\alpha,\beta)} : \alpha > 0, \beta > -\alpha^2\},$$

$$F(U_3) = F(U_4) = \{(\alpha, \beta) \in \mathbb{R}^2_{(\alpha,\beta)} : \alpha < 0, \beta > -\alpha^2\}.$$

Each $(N_F)_i$ is $F(U_i)$ embedded in $\mathbb{R}_{(\alpha,\beta)} \times \mathbb{R}^4$ but $(N_F)_i$ does not lie in the plane $\mathbb{R}_{(\alpha,\beta)}$ anymore, it is "lifted" in $\mathbb{R}_{(\alpha,\beta)} \times \mathbb{R}^4$. However, the part contained in $K_0(F) \times S_F$ still remains in the plane $\mathbb{R}_{(\alpha,\beta)}$ since the functions $\psi_i$ tend to zero at infinity and near $Sing(F)$ (see the Figure 1b).

Now we want to know how the parts $(N_F)_i$ are glued together. Using diagram (3.7), for any point $a = (\alpha, \beta) \in \mathbb{R}^2 \setminus K_0(F)$ the cardinal of $\pi_F^{-1}(a) \setminus ((K_0(F) \cup S_F) \times \{0_{\mathbb{R}^4}\})$ is equal to the cardinal of $F^{-1}(a)$ since $h_F$ is bijective. Consider now the equation

$$F(x, y) = \big(x, x^2y^2 + 2x^2y\big) = (\alpha, \beta)$$

where $\beta \neq -\alpha^2$. We have

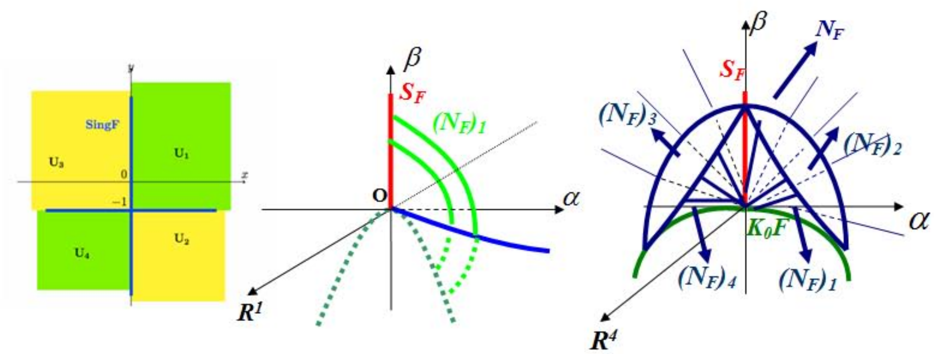
$$\alpha^2y^2 + 2\alpha^2y - \beta = 0. \tag{3.10}$$

As the reduced discriminant is $\Delta' = \alpha^4 + \alpha^2\beta = \alpha^2(\alpha^2 + \beta)$, then

1) if $\beta < -\alpha^2$, the equation (3.10) does not have any solution,

2) if $\beta > -\alpha^2$, the equation (3.10) has two solutions.

Then $(N_F)_1$ and $(N_F)_2$ are glued together along $(K_0(F) \cup S_F) \times \{0_{\mathbb{R}^4}\}$. Similarly, $(N_F)_3$ and $(N_F)_4$ are glued together along $(K_0(F) \cup S_F) \times \{0_{\mathbb{R}^4}\}$ (see the Figure 1c).




*1a* *1b* *1c*

Figure 1. The variety $N_F$.

## 3.3. Homology and intersection homology of $N_F$.

**Lemma 3.11** ([14]). *Let $F : \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial mapping. There exists a natural stratification of the variety $N_F$, by even (real) dimension strata, which is locally topologically trivial along the strata.*

In fact, the stratification of the variety $N_F$ is showed in [14] to be

$$N_F \supset (S_F \cup K_0(F)) \times \{0_{\mathbb{R}^p}\} \supset (Sing(S_F \cup K_0(F)) \cup B) \times \{0_{\mathbb{R}^p}\} \supset \emptyset,$$

where $B = S_{F|_{F^{-1}(S_F)}}$.

**Theorem 3.12** ([14])**.** *Let $F : \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial mapping with nowhere vanishing Jacobian. The following conditions are equivalent:*

*(1) $F$ is non proper,*
*(2) $H_2(N_F) \neq 0$,*
*(3) $IH_2^{\overline{p}}(N_F) \neq 0$ for any perversity $\overline{p}$,*
*(4) $IH_2^{\overline{p}}(N_F) \neq 0$ for some perversity $\overline{p}$.*

We notice that, for a given polynomial map $F : \mathbb{C}^2 \to \mathbb{C}^2$, the fact that the homology group $H_2(N_F)$ vanishes or not only depends on the behavior of $F$ at infinity.

## 4. Results

The following theorem generalizes Lemma 3.11 and shows existence of suitable stratifications of the set $S_F$ in the case of a polynomial mapping $F : \mathbb{C}^n \to \mathbb{C}^n$.

**Theorem 4.1.** *Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a generically finite polynomial mapping with nowhere vanishing Jacobian. There exists a filtration of $N_F$:*

$$N_F = V_{2n} \supset V_{2n-1} \supset V_{2n-2} \supset \cdots \supset V_1 \supset V_0 \supset V_{-1} = \emptyset$$

*such that :*

1) *for any $i < n$, $V_{2i+1} = V_{2i}$,*
2) *the corresponding stratification satisfies the Whitney $(b)$ condition.*

*Proof.* We have the following elements

+ Thanks to Sard Theorem, we have $\dim_{\mathbb{C}} Sing(S_F) \leq n-2$, *i.e.*, $\dim_{\mathbb{R}} Sing(S_F) \leq 2n-4$.

+ Let $M_{2n-2} = F^{-1}(S_F) \cap M_F$. The mapping $F$ restricted to $M_{2n-2}$ is dominant. Thanks to Jelonek's Theorem (Theorem 2.14), we have $\dim_{\mathbb{C}} S_{F_{|M_{2n-2}}} = n-2$ (since $\dim_{\mathbb{C}} M_{2n-2} = n-1$). Thus, we obtain $\dim_{\mathbb{C}} S_{F_{|M_{2n-2}}} = 2n-4$.

+ Thanks to Whitney's Theorem (Theorem 19.2, Lemma 19.3, [16]), the set $B_{2n-2}$ of points $x \in S_F$ at which the Whitney $(b)$ condition fails is contained in a complex algebraic variety of complex dimension smaller than $n-1$, so $\dim_{\mathbb{R}} B_{2n-2} \leq 2n-4$.

We will define a filtration $(\mathcal{W})$ of $\mathbb{R}^{2n}$ by algebraic varieties and compatible with $S_F$:

$$(\mathcal{W}): \qquad W_{2n} = \mathbb{R}^{2n} \supset W_{2n-1} \supset W_{2n-2} = S_F \supset \cdots \supset W_{2k+1} \supset W_{2k} \supset \cdots \supset W_1 \supset W_0 \supset \emptyset$$

by decreasing induction on $k$. Assume that $W_{2k}$ has been constructed. If $\dim_{\mathbb{R}} W_{2k} < 2k$ then we put

$$W_{2k-1} = W_{2k-2} = W_{2k}$$

otherwise we denote $M_{2k} = F^{-1}(W_{2k}) \cap M_F$ and $W'_{2k} = W_{2k} \setminus (Sing(W_{2k}) \cup S_{F_{|M_{2k}}})$. We put

$$W_{2k-1} = W_{2k-2} = Sing(W_{2k}) \cup S_{F_{|M_{2k}}} \cup A_{2k}, \tag{4.2}$$

where $A_{2k}$ is the smallest algebraic set which contains the set:

$$B_{2k} = \left\{ x \in W'_{2k} : \begin{array}{l} \text{if } x \in W_h \text{ with } h > 2k \text{ then} \\ \text{the Whitney } (b) \text{ condition fails at } x \text{ for the pair } (W'_{2k}, W_h) \end{array} \right\}.$$

Now, consider the filtration $(\mathcal{V})$ of $N_F$

$$(\mathcal{V}): \qquad N_F = V_{2n} \supset V_{2n-1} \supset V_{2n-2} \supset \cdots \supset V_{2k+1} \supset V_{2k} \supset \cdots \supset V_1 \supset V_0 \supset \emptyset$$

where $V_i = \pi_F^{-1}(W_i)$ and $\pi_F$ is the canonical projection from $N_F$ to $\mathbb{R}^{2n}$, on the first $2n$ coordinates (see diagram (3.7)).

Let $S'_{2i} = W_{2i} \setminus W_{2i-2}$. We claim that $F_{|F^{-1}(S'_{2i})}$ is proper. This is obvious if $S'_{2i}$ is empty. If $S'_{2i}$ is not empty, suppose that there exists a sequence $\{x_l\}$ in $F^{-1}(S'_{2i})$ such that $F(x_l)$ goes to a point $a$ in $S'_{2i}$. We have to show that the sequence $\{x_l\}$ does not go to infinity. Since $S'_{2i} = W_{2i} \setminus W_{2i-2}$, where $W_{2i-2} = Sing(W_{2i}) \cup S_{F_{|M_{2i-2}}} \cup A_{2i}$, we have $a \notin S_{F_{|M_{2i-2}}}$. If $x_l$ tends to infinity then $a \in S_{F_{|F^{-1}(S'_{2i})}}$, which is a contradiction.

Let $X$ be a connected component of $\pi_F^{-1}(Z)$, where $Z \subseteq W_{2i} \setminus W_{2i-2}$. We have $X \subseteq V_{2i} \setminus V_{2i-2}$. We claim that either $X \subseteq Z \times \{0_{\mathbb{R}^p}\}$ or $X \cap (S_F \times \{0_{\mathbb{R}^p}\}) = \emptyset$. Assume that there exist $x' \in X$ but $x' \notin Z \times \{0_{\mathbb{R}^p}\}$ and $x'' \in X \cap (S_F \times \{0_{\mathbb{R}^p}\})$. Then we have $x'' = (x, 0_{\mathbb{R}^p})$, where $x \in S_F$. There exists a curve $\gamma(t) = (\gamma_1(t), \gamma_2(t)) \subseteq X$ where $\gamma_1(t) \subseteq \mathbb{R}^n$ and $\gamma_2(t) \subseteq \mathbb{R}^p$, such that $\gamma(0) = x'$ and $\gamma(1) = x''$. Let us call $u = \gamma(t_0)$ the first point at which $\gamma$ meets $S_F \times \{0_{\mathbb{R}^p}\}$. Thus, we have $\gamma_2(t) \neq 0$ whenever $t < t_0$ and $h_F^{-1}(\gamma(t))$ is in $M_{2i}$, for $t < t_0$. Moreover, $F(h_F^{-1}(\gamma(t))) = \pi_F(\gamma(t))$ tends to $\pi_F(u)$ and $h_F^{-1}(\gamma(t))$ tends to infinity as $t$ tends to $t_0$. Hence, $\pi_F(u) \in S_{F|M_{2i}} \subset W_{2i-2}$, so $u$ is in $V_{2i-2}$, contradicting $u \in X \subset V_{2i} \setminus V_{2i-2}$.

Let us show that $S_{2i} := V_{2i} \setminus V_{2i-2}$ is a smooth manifold, for all $i$. Because $F_{|F^{-1}(S'_{2i})}$ is proper, the restriction of $\pi_F$ to $\pi_F^{-1}(S'_{2i}) \setminus (S_F \times \{0_{\mathbb{R}^p}\}) = h_F(F^{-1}(S'_{2i}))$ is proper. Consequently, $\pi_F$ is a covering map on $S_{2i}$. This implies that $S_{2i}$ is a smooth manifold.

Observe that in the case where $\overline{X} \cap (S_F \times \{0_{\mathbb{R}^p}\})$ is nonempty then it is included in $W_{2i-2} \times \{0_{\mathbb{R}^p}\}$, if $\dim X = 2i$, since every point of $\overline{X} \cap (S_F \times \{0_{\mathbb{R}^p}\})$ is a point of $S_{F|M_{2i}} \subseteq W_{2i-2}$. As $\pi_F$ is a covering map on $N_F \setminus (S_F \times \{0_{\mathbb{R}^p}\})$, this implies that $S'_{2i} \times \{0_{\mathbb{R}^p}\}$ is open in $\pi_F^{-1}(S'_{2i})$.

Let us prove that the filtration $(\mathcal{V})$ defines a Whitney stratification: at first, we prove that the stratification $(\mathcal{W})$ is a Whitney stratification. If the stratum $S'_{2i} = W_{2i} \setminus W_{2i-2}$ is not empty, then by (4.2), we have

$$S'_{2i} = W_{2i} \setminus W_{2i-2} \subset W_{2i} \setminus A_{2i} \subset W_{2i} \setminus B_{2i}.$$

This shows that the stratification $(\mathcal{W})$ satisfies Whitney conditions.

We denote

$$\Sigma_{\mathcal{W}} := \{X' : X' \text{ is a connected component of } W_{2i} \setminus W_{2i-2}, 0 \leq i \leq n\},$$

$$\Sigma_{\mathcal{V}} := \{X : X \text{ is a connected component of } V_{2i} \setminus V_{2i-2}, 0 \leq i \leq n\}.$$

We now prove that if $X \in \Sigma_{\mathcal{V}}$ then $\pi_F(X) \in \Sigma_{\mathcal{W}}$. If $X \subseteq S_F \times \{0_{\mathbb{R}_p}\}$ then $\pi_{F_{|X}}$ is the identity and thus $X$ belongs to $\Sigma_{\mathcal{W}}$. Otherwise, $X \subseteq N_F \setminus (S_F \times \{0_{\mathbb{R}^p}\}$. Assume that $X \subseteq V_{2i} \setminus V_{2i-2}$. This implies that $X \cap \pi_F^{-1}(W_{2i-2}) = \emptyset$. This amounts to say that $\pi_F(X) \cap W_{2i-2} = \emptyset$. Thus $\pi_F(X) \subseteq W_{2i} \setminus W_{2i-2}$. Therefore, to show that $\pi_F(X) \in \Sigma_{\mathcal{W}}$, we have to check that $\pi_F(X)$ is open and closed in $W_{2i} \setminus W_{2i-2}$. As $\pi_F$ is a local diffeomorphism at any point $x$ of $X$, the set $\pi_F(X)$ is a manifold of dimension $2i$, which is open in $S'_{2i}$. Let us show that it is closed in

$S'_{2i}$. Take a sequence $y_m \subset \pi_F(X)$ such that $y_m$ tends to $y \notin \pi_F(X)$. Let $x_m \in X$ be such that $\pi_F(x_m) = y_m$. Since $\pi_F$ is proper, $x_m$ does not tend to infinity. Taking a subsequence if necessary, we can assume that $x_m$ is convergent. Denote its limit by $x$. As $\pi_F(x) = y \notin \pi_F(X)$, then the point $x$ cannot be in $X$ and thus belongs to $V_{2i-2}$ since $X$ is closed in $V_{2i} \setminus V_{2i-2}$. This implies that $y = \pi_F(x) \in W_{2i-2}$, as required.

Let us consider a pair of strata $(X, Y)$ of the stratification $(\mathcal{V})$ such that $\overline{X} \cap Y \neq \emptyset$ and let us prove that $(X, Y)$ satisfies the Whitney $(b)$ condition. That is clear if $X, Y \subseteq S_F \times \{0_{\mathbb{R}^p}\}$. If none of them is included in $S_F \times \{0_{\mathbb{R}^p}\}$, then, as $\pi_F$ is a local diffeomorphism and Whiney $(b)$ condition is a $\mathcal{C}^1$ invariant, this is also clear. Therefore, we can assume that $X \cap (S_F \times \{0_{\mathbb{R}^p}\}) = \emptyset$ and $Y \subseteq S_F \times \{0_{\mathbb{R}^p}\}$ (if $X \subseteq S_F \times \{0_{\mathbb{R}^p}\}$, then $Y$ meets $S_F \times \{0_{\mathbb{R}^p}\}$ at the points of $\overline{X}$ and then $Y \subseteq (S_F \times \{0_{\mathbb{R}^p}\})$). Set for simplicity $Y := Y' \times \{0_{\mathbb{R}^p}\}$.

As $Y$ is open in $\pi_F^{-1}(Y')$, there exists a subanalytic open subset $U'$ of $N_F$ such that

$$\overline{U'} \cap \pi_F^{-1}(Y') = Y' \times \{0_{\mathbb{R}^p}\}.$$

Let $U'' := h_F^{-1}(\overline{U'} \cap N_F \setminus (S_F \times \{0_{\mathbb{R}^p}\}))$. We have

$$U'' \cap F^{-1}(Y') = \emptyset$$

(see diagram (3.7)). Consequently, the function distance $d(F(x); Y')$ nowhere vanishes on $U''$. As $U''$ is a closed subset of $\mathbb{R}^{2n}$, by Łojasiewicz inequality, multiplying the $\psi_i$'s by a huge power of $\frac{1}{1+|x|^2}$, we can assume that on $U''$, for every $i$

$$\psi_i(z_m) << d(F(z_m); Y') \tag{4.3}$$

for any sequence $z_m$ tending to infinity.

Now, in order to check that Whitney $(b)$ condition holds, we take $x_m \in X$ and $y_m \in Y$ tending to $y \in \overline{Y} \cap \overline{X}$. Assume that $l = \lim \overline{x_m y_m}$ and $\tau = \lim T_{x_m} X$ exist, we have to check that $l$ is included in $\tau$.

For every $m$, $x_m$ belongs to $h_F(U_j)$ for some $j$. Extracting a subsequence if necessary, we may assume that it lies in the same $h_F(U_j)$. On $U_j$, $\pi_F$ is invertible and its inverse is

$$\hat{\psi}(y) = (y, \tilde{\psi}_1(y), \dots, \tilde{\psi}_p(y)),$$

where $\tilde{\psi}_i(y) = \psi_i \circ F^{-1}_{|U_j}$ (see section 3.1, see also Proposition 2.3 in [14]).

Let $x_m = (x'_m; \tilde{\psi}(x'_m))$ and $y_m = (y'_m, 0_{\mathbb{R}^p})$, where $x'_m = \pi_F(x_m)$ and $y'_m = \pi_F(y_m)$, then $x_m - y_m = (x'_m - y'_m, \tilde{\psi}(x'_m))$. We claim that

$$\tilde{\psi}(x'_m) << |x'_m - y'_m|. \tag{4.4}$$

If $z_m = F^{-1}(x'_m)$ then $F(z_m) = x'_m$, so that by (4.3), we have

$$\tilde{\psi}_i(x'{}_m) << d(x'{}_m; Y') \leq |x'{}_m - y'{}_m|,$$

showing (4.4).

On one hand, the sets $\pi_F(X)$ and $\pi_F(Y)$ belong to $\Sigma_{\mathcal{W}}$, they satisfy the Whitney $(b)$ condition. As a matter of fact

$$\lim \frac{x'_m - y'_m}{|x'_m - y'_m|} = l' \subseteq \tau' = \lim T_{x'_m} \pi_F(X)$$

(extracting a sequence if necessary, we may assume $\frac{x'_m - y'_m}{|x'_m - y'_m|}$ is convergent). We have

$$\frac{x_m - y_m}{|x_m - y_m|} = \frac{(x'_m - y'_m, \tilde{\psi}(x_m))}{|x_m - y_m|} \to (l', 0) = l.$$

On the other hand, observe that

$$d_x \pi_F^{-1} = (Id, \partial_x \tilde{\psi}_1, \ldots, \partial_x \tilde{\psi}_p).$$

Multypling $\psi$ by a huge power of $\frac{1}{1+|x|^2}$, we can assume that the first order partial derivatives of $\tilde{\psi}$ at $x'_m$ tend to zero as $m$ goes to infinity. Then $T_{x_m} X$ tends to $\tau = \lim T_{x_m} \pi_F(X) \times \{0_{\mathbb{R}^p}\} = \tau' \times \{0_{\mathbb{R}^p}\}$. But since $l' \in \tau'$, $l = (l', 0) \in \tau = \tau' \times \{0_{\mathbb{R}^p}\}$. $\square$

We now generalize Theorem 3.12. Firstly we notice that a polynomial map $F_i : \mathbb{C}^n \to \mathbb{C}$ can be written

$$F_i = \Sigma_j F_{ij}$$

where $F_{ij}$ is the homogeneous part of degree $d_j$ in $F_i$. Let $d_k$ the highest degree in $F_i$, the leading form $\hat{F}_i$ of $F_i$ is defined as

$$\hat{F}_i := F_{ik}.$$

**Theorem 4.5.** *Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial mapping with nowhere vanishing Jacobian. If $\operatorname{rank}_{\mathbb{C}}(D\hat{F}_i)_{i=1,\ldots,n} > n-2$, where $\hat{F}_i$ is the leading form of $F_i$, then the following conditions are equivalent:*

1. *$F$ is non proper,*
2. *$H_2(N_F) \neq 0$,*
3. *$IH_2^{\overline{p}}(N_F) \neq 0$ for any (or some) perversity $\overline{p}$,*
4. *$IH_{2n-2,BM}^{\overline{p}}(N_F) \neq 0$, for any (or some) perversity $\overline{p}$.*

Before proving this theorem, we give here some necessary definitions and lemmas.

**Definition 4.6.** *A* **semi-algebraic family of sets** *(parametrized by $\mathbb{R}$)* is a semi-algebraic set $A \subset \mathbb{R}^n \times \mathbb{R}$, the last variable being considered as parameter.

**Remark 4.7.** A semi-algebraic set $A \subset \mathbb{R}^n \times \mathbb{R}$ will be considered as a family parametrized by $t \in \mathbb{R}$. We write $A_t$, for "the fiber of $A$ at $t$", *i.e.*,

$$A_t := \{x \in \mathbb{R}^n : (x, t) \in A\}.$$

**Lemma 4.8** ([14]). *Let $\beta$ be a $j$-cycle and let $A \subset \mathbb{R}^n \times \mathbb{R}$ be a compact semi-algebraic family of sets with $|\beta| \subset A_t$ for any $t$. Assume that $|\beta|$ bounds a $(j+1)$-chain in each $A_t$, $t > 0$ small enough. Then $\beta$ bounds a chain in $A_0$.*

**Definition 4.9** ([14])**.** Given a subset $X \subset \mathbb{R}^n$, we define the **"tangent cone at infinity"**, called "**contour apparent à l'infini**" in [11] by:

$$C_\infty(X) := \{\lambda \in \mathbb{S}^{n-1}(0,1) \text{ such that } \exists \varphi : (t_0, t_0+\varepsilon] \to X \text{ semi-algebraic},$$
$$\lim_{t \to t_0} \varphi(t) = \infty, \lim_{t \to t_0} \frac{\varphi(t)}{|\varphi(t)|} = \lambda\}.$$

**Lemma 4.10** ([11])**.** *Let $F = (F_1, \ldots, F_n) : \mathbb{R}^n \to \mathbb{R}^n$ be a polynomial mapping and $V$ the zero locus of $\hat{F} := (\hat{F}_1, \ldots, \hat{F}_n)$, where $\hat{F}_i$ is the leading form of $F_i$ for $i = 1, \ldots, n$. If $X$ is a subset of $\mathbb{R}^n$ such that $F(X)$ is bounded, then $C_\infty(X)$ is a subset of $\mathbb{S}^{n-1}(0,1) \cap V$.*

*Proof.* By definition, $C_\infty(X)$ is included in $\mathbb{S}^{n-1}(0,1)$. We prove now that $C_\infty(X)$ is included in $V$. In fact, given $\lambda \in C_\infty(X)$, then there exists a semi-algebraic curve $\gamma : (t_0, t_0+\varepsilon] \to X$ such that $\lim\limits_{t \to t_0} \gamma(t) = \infty$ and $\lim\limits_{t \to t_0} \frac{\gamma(t)}{|\gamma(t)|} = \lambda$. Then $\gamma(t)$ can be written as $\gamma(t) = \lambda t^m + \ldots$ and $\hat{F}_i = \hat{F}_i(\lambda) t^{md_i} + \ldots$ where $d_i$ is the homogeneous degree of $\hat{F}_i$. Since $F(X)$ is bounded, then $F_i$ cannot tend to infinity when $t$ tends to $t_0$, hence $\hat{F}_i(\lambda) = 0$ for all $i = 1, \ldots, n$. □

Let us prove now Theorem 4.5. We use the idea and technique of the second and third authors in [14].

*Proof of the Theorem 4.5.* (4) $\Leftrightarrow$ (3) : By Goresky-MacPherson Poincaré duality Theorem, we have

$$IH_2^{\overline{p}}(N_F) = IH_{2n-2,BM}^{\overline{q}}(N_F),$$

where $\overline{q}$ is the complementary perversity of $\overline{p}$. Since $IH_2^{\overline{p}}(N_F) \neq 0$ for all perversities $\overline{p}$, then $IH_{2n-2}^{\overline{q}}(N_F) \neq 0$, for all perversities $\overline{q}$.

(3) $\Rightarrow$ (1), (3) $\Rightarrow$ (2) : If $F$ is proper then the sets $S_F$ and $K_0(F)$ are empty. So $Sing(N_F)$ is empty and $N_F$ is homeomorphic to $\mathbb{R}^{2n}$. It implies that $H_2(N_F) = 0$ and $IH_2^{\overline{p}}(N_F) = 0$.

(1) $\Rightarrow$ (2), (1) $\Rightarrow$ (3) : Assume that $F$ is not proper. That means that there exists a complex Puiseux arc $\gamma : D(0,\eta) \to \mathbb{R}^{2n}$, $\gamma = uz^\alpha + \ldots$, (with $\alpha$ negative integer and $u$ is an unit vector of $\mathbb{R}^{2n}$) tending to infinity in such a way that $F(\gamma)$ converges to a generic point $x_0 \in S_F$. Let $\delta$ be an oriented triangle in $\mathbb{R}^{2n}$ whose barycenter is the origin. Then, as the mapping $h_F \circ \gamma$ (where $h_F = (F, \psi_1, \ldots, \psi_p)$) extends continuously at 0, it provides a singular 2-simplex in $N_F$ that we will denote by $c$.
Since $\operatorname{codim}_{\mathbb{R}} S_F = 2$, then

$$0 = \dim_{\mathbb{R}}\{x_0\} = \dim_{\mathbb{R}}((S_F \times \{0_{\mathbb{R}^p}\}) \cap |c|) \leq 2 - 2 + p_2,$$

because $p_2 = 0$ for any perversity $\overline{p}$. So the simplex $c$ is $(\overline{0}, 2)$-allowable for any perversity $\overline{p}$.
The support of $\partial c$ lies in $N_F \setminus S_F \times \{0_{\mathbb{R}^p}\}$. By definition of $N_F$, we have $N_F \setminus S_F \times \{0_{\mathbb{R}^p}\} \simeq \mathbb{R}^{2n}$.
Since $H_1(\mathbb{R}^{2n}) = 0$, the chain $\partial c$ bounds a singular chain $e \in C^2(N_F \setminus S_F \times \{0_{\mathbb{R}^p}\})$. So $\sigma = c - e$ is a $(\overline{p}, 2)$-allowable cycle of $N_F$.

We claim that $\sigma$ may not bound a 3-chain in $N_F$. Assume otherwise, *i.e.*, assume that there is a chain $\tau \in C_3(N_F)$, satisfying $\partial \tau = \sigma$. Let

$$A := h_F^{-1}(|\sigma| \cap (N_F \setminus (S_F \times \{0_{\mathbb{R}^p}\}))),$$

$$B := h_F^{-1}(|\tau| \cap (N_F \setminus (S_F \times \{0_{\mathbb{R}^p}\}))).$$

By definition, $C_\infty(A)$ and $C_\infty(B)$ are subsets of $\mathbb{S}^{2n-1}(0,1)$. Observe that, in a neighborhood of infinity, $A$ coincides with the support of the Puiseux arc $\gamma$. The set $C_\infty(A)$ is equal to $\mathbb{S}^1.a$ (denoting the orbit of $a \in \mathbb{C}^n$ under the action of $\mathbb{S}^1$ on $\mathbb{C}^n$, $(e^{i\eta}, z) \mapsto e^{i\eta}z$). Let $V$ be the zero locus of the leading forms $\hat{F} := (\hat{F_1}, \ldots, \hat{F_n})$. Since $F(A)$ and $F(B)$ are bounded, by Lemma 4.10, $C_\infty(A)$ and $C_\infty(B)$ are subsets of $V \cap \mathbb{S}^{2n-1}(0,1)$.

For $R$ large enough, the sphere $\mathbb{S}^{2n-1}(0,R)$ with center 0 and radius $R$ in $\mathbb{R}^{2n}$ is transverse to $A$ and $B$ (at regular points). Let

$$\sigma_R := \mathbb{S}^{2n-1}(0,R) \cap A, \qquad \tau_R := \mathbb{S}^{2n-1}(0,R) \cap B.$$

After a triangulation, the intersection $\sigma_R$ is a chain bounding the chain $\tau_R$.

Consider a semi-algebraic strong deformation retraction $\rho : W \times [0;1] \to \mathbb{S}^1.a$, where $W$ is a neighborhood of $\mathbb{S}^1.a$ in $\mathbb{S}^{2n-1}(0,1)$ onto $\mathbb{S}^1.a$.

Considering $R$ as a parameter, we have the following semi-algebraic families of chains:

1) $\tilde{\sigma_R} := \frac{\sigma_R}{R}$, for $R$ large enough, then $\tilde{\sigma_R}$ is contained in $W$,
2) $\sigma'_R = \rho_1(\tilde{\sigma_R})$, where $\rho_1(x) := \rho(x,1), \qquad x \in W$,
3) $\theta_R = \rho(\tilde{\sigma_R})$, we have $\partial\theta_R = \sigma'_R - \tilde{\sigma_R}$,
4) $\theta'_R = \tau_R + \theta_R$, we have $\partial\theta'_R = \sigma'_R$.

As, near infinity, $\sigma_R$ coincides with the intersection of the support of the arc $\gamma$ with $\mathbb{S}^{2n-1}(0,R)$, for $R$ large enough the class of $\sigma'_R$ in $\mathbb{S}^1.a$ is nonzero.

Let $r = 1/R$, consider $r$ as a parameter, and let $\{\tilde{\sigma_r}\}$, $\{\sigma'_r\}$, $\{\theta_r\}$ as well as $\{\theta'_r\}$ the corresponding semi-algebraic families of chains.

Denote by $E_r \subset \mathbb{R}^{2n} \times \mathbb{R}$ the closure of $|\theta_r|$, and set $E_0 := (\mathbb{R}^{2n} \times \{0\}) \cap E$. Since the strong deformation retraction $\rho$ is the identity on $C_\infty(A) \times [0,1]$, we see that

$$E_0 \subset \rho(C_\infty(A) \times [0,1]) = \mathbb{S}^1.a \subset V \cap \mathbb{S}^{2n-1}(0,1).$$

Denote $E'_r \subset \mathbb{R}^{2n} \times \mathbb{R}$ the closure of $|\theta'_r|$, and set $E'_0 := (\mathbb{R}^{2n} \times \{0\}) \cap E'$. Since $A$ bounds $B$, so $C_\infty(A)$ is contained in $C_\infty(B)$. We have

$$E'_0 \subset E_0 \cup C_\infty(B) \subset V \cap \mathbb{S}^{2n-1}(0,1).$$

The class of $\sigma'_r$ in $\mathbb{S}^1.a$ is, up to a product with a nonzero constant, equal to the generator of $\mathbb{S}^1.a$. Therefore, since $\sigma'_r$ bounds the chain $\theta'_r$, the cycle $\mathbb{S}^1.a$ must bound a chain in $|\theta'_r|$ as well. By Lemma 4.8, this implies that $\mathbb{S}^1.a$ bounds a chain in $E'_0$ which is included in $V \cap \mathbb{S}^{2n-1}(0,1)$.

The set $V$ is a projective variety which is an union of cones in $\mathbb{R}^{2n}$. Since

$$\operatorname{rank}_{\mathbb{C}}(D\hat{F_1})_{i=1,\ldots,n} > n-2,$$

it follows that $\operatorname{corank}_{\mathbb{C}}(D\hat{F_1})_{i=1,\ldots,n} = \dim_{\mathbb{C}} V \leq 1$, so $\dim_{\mathbb{R}} V \leq 2$ and $\dim_{\mathbb{R}} V \cap \mathbb{S}^{2n-1}(0,1) \leq 1$. The cycle $\mathbb{S}^1.a$ thus bounds a chain in $E'_0 \subseteq V \cap \mathbb{S}^{2n-1}(0,1)$, which is a finite union of circles. A contradiction. $\square$

We have the following corollary

**Corollary 4.11.** *Let $F = (F_1, \dots, F_n) : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial mapping with nowhere vanishing Jacobian and such that $\operatorname{rank}_{\mathbb{C}}(D\hat{F}_i)_{i=1,\dots,n} > n-2$, where $\hat{F}_i$ is the leading form of $F_i$. The following conditions are equivalent:*

1. *$F$ is nonproper,*
2. *$H^2_\infty(\operatorname{Reg}(N^R_F)) \neq 0$,*
3. *$H^{n-2}_\infty(\operatorname{Reg}(N^R_F)) \neq 0$,*

*where $N^R_F := N_F \cap \bar{B}(0,R)$, which $R$ is large enough.*

The proof is similar to the one in [14].

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(Nguyen Thi Bich Thuy) I.M.L., Aix-Marseille University, Case 907, Luminy 13288-Marseille Cedex 9, France

*E-mail address*: `ntbthuy@ctu.edu.vn`

(Anna Valette) Instytut Matematyki Uniwersytetu Jagiellońskiego, ul. S Łojasiewicza, Kraków, Poland

*E-mail address*: `anna.valette@im.uj.edu.pl`

(Guillaume Valette) Instytut Matematyczny PAN, ul. Św. Tomasza 30, 31-027 Kraków, Poland

*E-mail address*: `gvalette@impan.pl`

**Journal *of* Singularities**
**Volume 7 (2013), 205-219**

received 3 January 2013
in revised form 6 May 2013
DOI: 10.5427/jsing.2013.7k

# THE UNIVERSAL ABELIAN COVER OF A GRAPH MANIFOLD

HELGE MØLLER PEDERSEN

Abstract. Complex surfaces singularities with rational homology sphere links play an important role in singularity theory. They include all rational and splice quotient singularities, and in particular in the latter case the universal abelian cover of the link is a key element of the theory. All such links of singularities are graph manifolds, and to a rational homology sphere graph manifold one can associate a weighted tree invariant called splice diagram. It is known that the splice diagram determines the universal abelian cover of the manifold. In this paper we give an explicit method for constructing the universal abelian cover from the splice diagram, which works for most of the graph manifolds in particular for all links of singularities.

## 1. Introduction

Splice quotient singularities are an important class of normal complex surface singularities with rational homology sphere links ($\mathbb{Q}$HS) recently discovered by Neumann and Wahl (see [NW05b] and [NW05a]). They include all rational and minimally elliptic singularities with $\mathbb{Q}$HS links by work of Okuma [Oku04], and all weighted homogeneous singularities with $\mathbb{Q}$HS links ([Neu83a]). Splice quotient singularities also play an important role in recent works of Némethi and Okuma ([NO08, NO09]). Their analytic structures are defined by the corresponding analytic structures of their universal abelian covers which in turn are given by complete intersection equations called *splice diagram equations*. Although these equations are fairly simple, the topology of the universal abelian cover is in general rather complicated. The aim of this paper is to give a general way to describe it.

Links of normal complex surface singularities belong to a specific class of 3-manifolds called graph manifolds, which are defined as having only Seifert fibered pieces in their JSJ-decompositions, or alternatively having no hyperbolic pieces in their geometric decompositions. If one restricts to $\mathbb{Q}$HS's, then one has a non complete invariant of graph manifolds called splice diagrams. Splice diagrams were original introduced by Eisenbud and Neumann in [EN85] and by Siebermann in [Sie80] for integer homology sphere graph manifolds, and were then later generalized by Neumann and Wahl to $\mathbb{Q}$HS's in [NW02]. In [NW05a], Neumann and Wahl define the splice diagram equations when the splice diagram $\Gamma$ of $M$ satisfies, what they call the *semigroup condition*. The splice diagram equations define an isolated complete intersection. If $M$ also satisfies the *congruence condition*, they show that there exists a splice quotient singularity whose link is $M$, and that the link of the isolated complete intersection is the universal abelian cover of $M$.

Based on the result for links of singularities in [NW05a] Neumann and Wahl conjectured that the splice diagram determines the universal abelian cover for a $\mathbb{Q}$HS graph manifold, even when the graph manifold is not a singularity link. In [Ped10] the following theorem proved this:

**Theorem 1.1** ([Ped10]; 6.3). *Let $M_1$ and $M_2$ be two $\mathbb{Q}HS$ graph manifolds having the same splice diagram. Let $\widetilde{M}_i \to M_i$ be the universal abelian cover. Then $\widetilde{M}_1$ and $\widetilde{M}_2$ are homeomorphic.*

2000 *Mathematics Subject Classification.* 57M10, 57M27.

*Key words and phrases.* rational homology sphere, abelian cover.

Graph manifolds are also the 3-manifolds which are boundaries of plumbed 4-manifolds. A very common method to describe a graph manifold $M$ is to give a plumbing diagram of a 4-manifold $X$, such that $M = \partial X$. Neumann gave a complete calculus for changing $X$ but keeping $M$ fixed in [Neu81]. In section 3 we are going to construct a plumbing diagram of the universal abelian cover.

The proof of Theorem 1.1 consists of inductively constructing the universal abelian cover from the splice diagram, and the purpose of this article is to extract from this proof an algorithm for constructing the topology of the universal abelian cover. The explicit construction given will only work under the assumption that the splice diagram has no edge weight of 0. This assumption is always satisfied if the manifold is a singularity link.

In the proof of Theorem 1.1 one had to extend the notion of splice diagram to a class of orbifolds called *graph orbifolds*, and in [Pedb] the congruence condition is extended to graph orbifolds. Hence if $X$ is a singularity defined by the splice diagram equations associated to the splice diagram $\Gamma$, one can use this algorithm to construct a dual resolution diagram, provided that there is a manifold or orbifold satisfying the congruence condition and having $\Gamma$ as its splice diagram. This is for example always true if $\Gamma$ only has two nodes (see [Pedb]). Neumann and Wahl conjecture that in fact the splice diagram equations always define the universal abelian cover.

In section 2 we recall the definition of splice diagrams from [Ped10], and their relation with plumbing diagrams, and we state the results needed for the algorithm. In section 3 we describe the algorithm giving the plumbing diagram of the universal abelian cover by performing it on an example, and we give further examples.

**Acknowledgements:** The auther was supported by the Hungarian Academy of Sciences' *Lendület* LDT program.

## 2. Splice Diagrams

A *splice diagram* is a weighted tree with no vertices of valence two. By the valence of a vertex we mean the number of adjacent edges. We call vertices of valence greater than two *nodes*. At a node one assigns a sign, and on edges adjacent to nodes one assigns a non negative integer weight.

Let $M$ be a $\mathbb{Q}$HS graph manifold. Let $M = \bigcup_v M_v$ be the JSJ-decomposition of $M$, that is the unique minimal decomposition of $M$ into Seifert fibered pieces $M_v$ with $\partial M_v$ a union of tori. We associate a splice diagram $\Gamma(M)$ to $M$ by the following procedure:

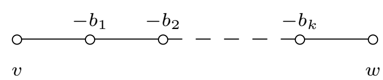
- Take a vertex $v$ for each $M_v$.
- Connect two vertices $v$ and $w$ by an edge if $M_v \bigcap M_w \neq \emptyset$.
- Add a *leaf*, i.e., a valence one vertex connected by an edge, to a vertex $v$ for each singular fiber of the Seifert fibration of $M_v$.
- To each vertex $v$ assign the sign of the linking number of two nonsingular fibers of $M_v$. See Definition 2.1 in [Ped10] for the precise definition of these linking numbers.
- Let $v$ be a node and $e$ an edge adjacent to $v$. Then the edge weight $d_{ve}$ is determined in the following way. Cut $M$ along the torus $T$ corresponding to $e$ (either a torus of the boundary of $M_v$ or the boundary of a tubular neighborhood of a singular fiber) into the pieces $M'_v$ and $M'_{ve}$, where $M_v \subset M'_v$. Then glue a solid torus into the boundary of $M'_{ve}$ by identifying a meridian with the image of a fiber of $M_v$, and call this new closed graph

manifold $M_{ve}$. Then

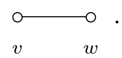
$$d_{ve} := \begin{cases} |H_1(M_{ve})| & \text{if } H_1(M_{ve}) \text{ is finite} \\ 0 & \text{if } H_1(M_{ve}) \text{ is infinite.} \end{cases}$$

A common way to represent graph manifolds is by plumbing diagrams, and we will next describe how to get the splice diagram from a plumbing diagram $\Delta$ of $M$.

To construct the graph structure of $\Gamma(M)$ from $\Delta$ one just suppresses all vertices of valence two, i.e., replacing any configuration like



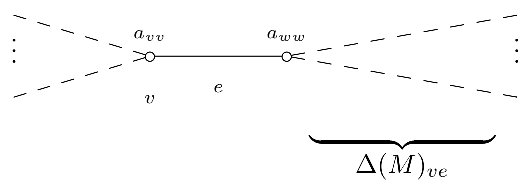

with an edge




Let $A(\Delta)$ be the intersection matrix of the 4 manifold defined by $\Delta$. The edge weights and signs are computed by the following two results.

**Lemma 2.1** ([Ped10] 2.1)**.** *Let $v$ be a node in $\Gamma(M)$ and $e$ be an edge at that node. We get the weight $d_{ve}$ by $d_{ve} = |\det(-A(\Delta(M)_{ve}))|$, where $\Delta(M)_{ve}$ is the connected component of $\Delta(M)$ after we remove $e$, which does not contain $v$.*

$\Delta(M) =$




**Lemma 2.2** ([Ped10] 2.3)**.** *Let $v$ be a node in $\Gamma(M)$. Then the sign $\varepsilon$ at $v$ is $\varepsilon = -\operatorname{sign}(a_{vv})$, where $a_{vv}$ is the entry of $A(M)^{-1}$ corresponding to the node $v$.*

In the algorithm the *rational Euler number* of a Seifert fibered piece of $M$ will play an important role. If $M$ is a closed Seifert fibered manifold, then the rational Euler number $e_M$ is defined by

$$e_M := \sum_{i=0}^{n} \frac{q_i}{p_i},$$

where $(p_0, q_0), \dots, (p_n, q_n)$ are the unnormalized Seifert invariants (see [NR78]). Notice that we use the opposite choice of orientation when we define the invariants, this is the reason for the sign difference in our formula for $e_M$ compared to the one they use. If we consider $M$ as a plumbed manifold, then the plumbing diagram is star shaped with $n$ strings connected to the central vertex. As explained in [NR78] one can change the Seifert invariants such that $p_0 = 1$ and $p_i > q_i > 0$ for $i = 1, \dots n$. Then $q_0$ is the weight at the central vertex, and $p_i/q_i$ for $i = 1, \dots n$ is the continued fraction

$$[a_{i1}, a_{i2}, \dots, a_{ik_i}] = a_{i1} - \cfrac{1}{a_{i2} - \frac{1}{a_{i3} - \dots}},$$

where $-a_{i1}, -a_{i2}, \dots, -a_{ik_i}$ are the weights along the i'th string of the plumbing diagram leading from the central vertex. In this case the splice diagram of $M$ is also star shaped, it has $n$ leaves with weights $p_1, \dots, p_n$, and the sign at the node is $-\operatorname{sign}(e_M)$.

We need the rational Euler number $e_{M_v}$ of a Seifert fibered pieces $M_v$ of the JSJ-decomposition of $M$. Since $M_v$ is not closed, we need additional information to define it. We consider a simple closed curve in each boundary component of $M_v$. Each curve is the image of a fiber of the Seifert fibered piece on the other side of the corresponding torus. One glues a solid torus in each of the boundary components of $M_v$, by identifying the simple closed curve with a meridian, and takes $e_{M_v}$ to be the rational Euler number of this closed manifold. If $M$ is given by a plumbing diagram $\Delta$, then the $M_v$'s correspond to the vertices $v$'s with valence $\geq 3$ (or vertices with non zero genus). One gets $e_{M_v}$ as the rational Euler number of the starshaped piece containing $v$, after one removes from $\Delta$ all the vertices corresponding to $M_w$'s with $w \neq v$.

The splice diagram itself does not determine neither the rational Euler numbers nor $|H_1(M)|$. But $|H_1(M)|$ and the splice diagram do determine the rational Euler number of any of the Seifert fibered pieces of $M$, for this result we need the following definition. The edge determinant $D(e)$ associated to an edge $e$ between two nodes $v$ and $w$ is

$$D(e) := r_v r_w - \varepsilon_v \varepsilon_w (\prod_i n_{vi})(\prod_j n_{wj}),$$

where $r_v$ and $r_w$ are the edge weights on $e$, where $\varepsilon_v$ and $\varepsilon_w$ are the signs on the nodes and where the $n_{vi}$'s and $n_{wj}$'s are the weights adjacent to the nodes not on $e$.

**Proposition 2.3** ([Ped10] 3.4)**.** *Let $v$ be a node in a splice diagram decorated as in Figure 1 below, with $r_i \neq 0$ for $i \neq 1$, and let $e_v$ be the rational Euler number of $M_v$. Then*

$$e_v = -d\left(\frac{\varepsilon s_1}{N D_1 \prod_{j=2}^{k} r_k} + \sum_{i=2}^{k} \frac{\varepsilon_i M_i}{r_i D_i}\right), \tag{1}$$

*where $d = |H_1(M)|$, $N = \prod_{j=1}^{k} n_j$, $M_i = \prod_{j=1}^{l_i} m_{ij}$, and $D_i$ is the edge determinant associated to the edge between $v$ and $v_i$.*

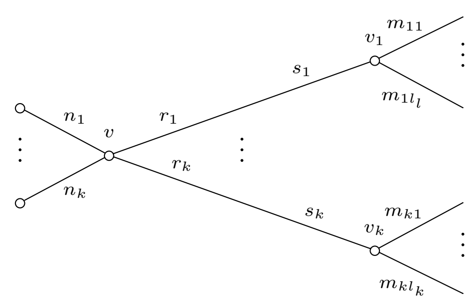



Figure 1 .

Note that this does give a formula for $e_v/|H_1(M)|$ from $\Gamma$, which we will need later.

In the algorithm to construct the universal abelian cover of $M$ from $\Gamma(M)$, a number associated to each end of an edge in $\Gamma(M)$ is going to be very important. It is the *ideal generator*, which is constructed in the following way. Let $v$ and $w$ be two vertices of $\Gamma(M)$, then we define the *linking number* $l_{vw}$ of $v$ and $w$ as the product of all edge weights adjacent to but not on the shortest path from $v$ to $w$. We define $l'_{vw}$ in the same way, except that we omit weights adjacent to $v$ and $w$. If $e$ is an edge adjacent to $v$, we let $\Gamma_{ve}$ be the connected component of $\Gamma(M) - e$ not containing $v$, and define the following ideal in $\mathbb{Z}$

$$I_{ve} := \langle l'_{vw} |\ w \text{ a leaf in } \Gamma_{ve} \rangle.$$

Then we define the ideal generator $\overline{d}_{ve}$ associated to $v$ and $e$ to be the positive generator of $I_{ve}$.

**Definition 2.4.** A splice diagram $\Gamma$ satisfies the *ideal condition* if the ideal generator $\overline{d}_{ve}$ divides the edge weight $d_{ve}$ for all nodes $v$ and adjacent edges $e$.

**Proposition 2.5.** *Let $M$ be a $\mathbb{Q}HS$ graph manifold. Then $\Gamma(M)$ satisfies the ideal condition.*

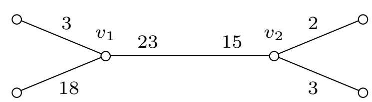
This proposition follows from the following topological description of the ideal generator in Appendix 1 of [NW05a].

**Theorem 2.6.** *The ideal generator $\overline{d}_{ve}$ equals $|H_1(M/M'_v)|$.*

Remember we defined $M'_v$, when we constructed $\Gamma(M)$ at the beginning of the present section.

## 3. Construction of the Universal Abelian Cover: An Example

In this section we explain how the proof of Theorem 1.1 [Ped10] can be used to construct the universal abelian cover $\widetilde{M}$ of a graph manifold $M$ from the splice diagram $\Gamma(M)$. We specify $\widetilde{M}$ by constructing a plumbing diagram $\Delta$ for $\widetilde{M}$. To illustrate the construction we use the following example

$\Gamma =$ 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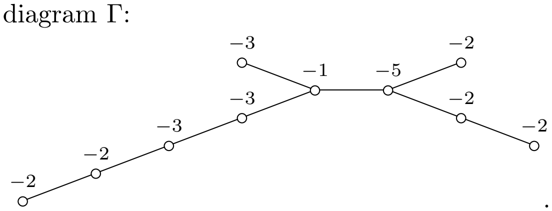
There are four different manifolds which have $\Gamma$ as their splice diagram, and also several non manifold graph orbifolds. By Theorem 4.1 in [Ped10] $\Gamma$ is the splice diagram of a singularity link, and [Peda] gives that $\widetilde{M}$ is a rational homology sphere. The example is also interesting, since none of the manifolds having splice diagram $\Gamma$ satisfy the congruence condition of Neumann and Wahl (see [NW05a]). But there are non manifold orbifolds with splice diagram $\Gamma$ which satisfy the orbifold congruence condition (see [Pedb]). Below are plumbing diagrams for the four manifolds having splice diagram $\Gamma$:

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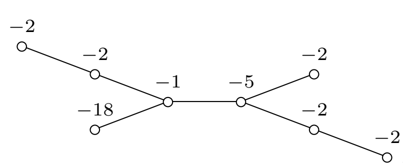


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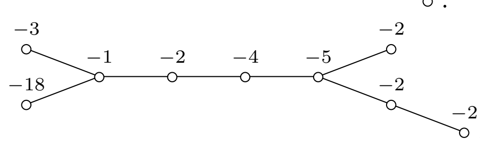


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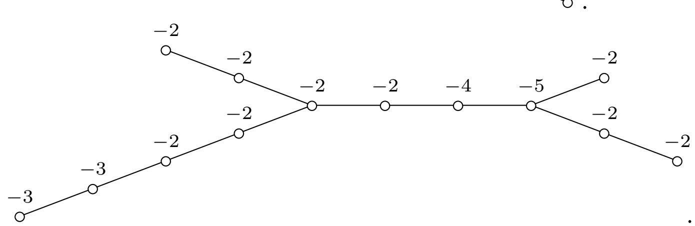


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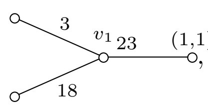
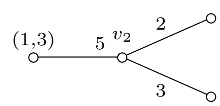
The construction of the universal abelian cover is done in two steps. First we construct a one-node splice diagram for each node in $\Gamma$, each of these one-node splice diagrams is then used to define a Seifert fibered manifold. We call these Seifert fibered manifolds *the building blocks*. In the second step we take a number of copies of the building blocks, and use the information given by $\Gamma$ to glue them together to create the universal abelian cover.

3.1. **Constructing the building blocks.** The inductive procedure in the construction of the universal abelian cover consists of taking an edge $e$ between two nodes of $\Gamma$, and making a new non connected splice diagram $\Gamma_e$, where $e$ has been replaced by two leaves. So starting with the edge called $e_1$ and going through this process of cutting the edges until we have cut the last edge between two nodes $e_{N-1}$, we get that $\Gamma_{e_{N-1}}$ is a collection of one-node splice diagrams $\Gamma_{e_{N-1}} = \{\mathscr{G}_i\}_{i=1}^N$. For each of these one-node splice diagrams $\mathscr{G}_i$ one then takes a number of copies of a specific manifold $M_i$, and uses the information from the $\Gamma_{e_j}$'s to glue the pieces together. So the first step is to determine these manifolds $\{M_i\}_{i=1}^N$, which are the building blocks of the universal abelian cover.

First let us describe the $\Gamma_{e_j}$'s. Each time we cut an edge $e$ between the nodes $w_1$ and $w_2$ in $\Gamma$, we divide every edge weight $d_{ve'}$ such that $w_1$ or $w_2$ is in $\Gamma_{ve'}$, by the ideal generator $\overline{d}_{w_i e}$ of the edge weight $d_{w_i e}$, where $v$ is not in $\Gamma_{w_i e}$. In our example we only have two edge weights, which have to be divided when we cut along the central edge $e$ namely $d_{v_1 e} = 23$ and $d_{v_2 e} = 15$, and $\overline{d}_{v_1 e} = 1$ and $\overline{d}_{v_2 e} = 3$. So the two one-node splice diagrams $\mathscr{G}_1$ and $\mathscr{G}_2$ are

$\mathscr{G}_1 =$ 

, $\mathscr{G}_2 =$ 

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The pair added to the new leaves, which will be used to describe the gluings, is defined as follows: the first number specifies the order the sequence of cuttings this is, and the second number is the ideal generator associated to the weight before cutting.

Next we want to find the building block $M_i$ associated to each of the $\mathscr{G}_i$'s. To do this we have to separate the $\mathscr{G}_i$'s into two types. The first type consists of the $\mathscr{G}_i$'s that do not have an edge weight of 0, and the second type consists of the $\mathscr{G}_i$'s that have an edge weight of 0. At most one weight adjacent to a node can be 0, since if there were two edge weights of 0 adjacent to a node the edge determinant of any edge with the edge weight 0 would be 0. Then by using The Edge Determinant Equation (Corollary 3.3 of [Ped10]) the Seifert fibration can be extended over the torus corresponding to the edge, hence we would not have cut along this torus in the JSJ-decomposition of $M$.

In the first case we use the following theorem

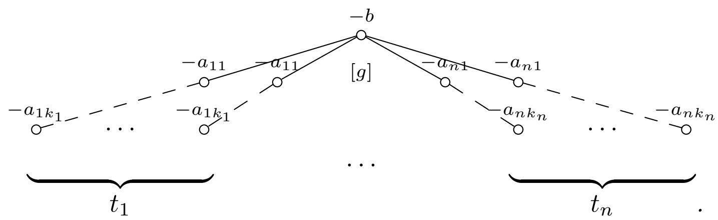
**Theorem 3.1.** *Let $M$ be a $\mathbb{Q}HS$ orbifold $S^1$-fibration over a orbifold surface with Seifert invariants $(\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n)$. Then the universal abelian cover of $M$ is the link of the Brieskorn complete intersection $\Sigma(\alpha_1, \dots, \alpha_n)$.*

The way one constructs the manifolds after cutting an edge may result in graph orbifolds instead of just graph manifolds as explained in the proof of 6.3 in [Ped10]. Hence we need this theorem for orbifold $S^1$-fibrations. Neumann proves this theorem for Seifert fibered manifolds in [Neu83a] and [Neu83b], but the proof given in [Neu83b] also works in the general case of an orbifold $S^1$-fibration. These theorems assume that the rational Euler number $e_M$ is positive, but if $e_M < 0$ one just composes with an orientation reversing map. Notice that $\alpha_1, \dots, \alpha_n$ are exactly the edge weights of $\Gamma(M)$. The value of the sign $\varepsilon$ at the node does not matter, since reversing the orientation of a Seifert fibered manifold only changes the $\beta_i$'s not the $\alpha_i$'s, and hence only changes the splice diagrams by replacing $\varepsilon$ with $-\varepsilon$.

So in our example $M_1$ is the link of $\Sigma(3,18,23)$, and $M_2$ is the link of $\Sigma(2,3,5)$.

Next we use the description of the Seifert invariants of $\Sigma(\alpha_1,\dots,\alpha_n)$ given by Neumann and Raymond in [NR78] to get plumbing diagrams for the $M_i$'s.

**Theorem 3.2.** *Let $M$ be the link of the Brieskorn complete intersection $\Sigma(\alpha_1,\dots,\alpha_n)$. A plumbing diagram for $M$ is given by*




*The values of $g$ and of the $t_i$'s are given by*

$$t_i = \frac{\prod_{j\neq i}(\alpha_j)}{\operatorname{lcm}_{j\neq i}(\alpha_j)} \tag{2}$$

$$g = \tfrac{1}{2}\big(2 + \frac{(n-2)\prod_i \alpha_i}{\operatorname{lcm}_i(\alpha_i)} - \sum_{i=1}^{n} t_i\big). \tag{3}$$

*One calculates numbers $p_1,\dots,p_n$ as*

$$p_i = \frac{\operatorname{lcm}_j(\alpha_j)}{\operatorname{lcm}_{j\neq i}(\alpha_j)}, \tag{4}$$

*and finds numbers $q_1,\dots,q_n$ as the smallest no negative solutions to the equations*

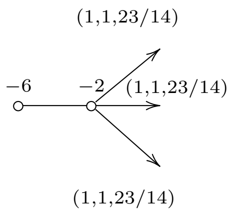
$$\frac{\operatorname{lcm}_j(\alpha_j)}{\alpha_i} q_i \equiv -1(\mod p_i). \tag{5}$$

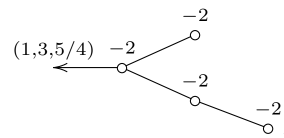
*The $a_{ij}$'s are given by the continued fraction $p_i/q_i = [a_{i1},\dots,a_{ik_i}]$. If $p_i = 1$ then the string of valence two vertices is empty. Finally $b$ is given by*

$$b = \frac{\prod_i \alpha_i + \operatorname{lcm}_k(\alpha_k)\sum_i q_i \prod_{j\neq i}\alpha_j}{(\operatorname{lcm}_k \alpha_k)^2}. \tag{6}$$

Before we use this theorem to make a plumbing diagram $\Delta_i$ for $M_i$, notice that we have to remove some solid tori from $M_i$ to make the gluing, so we need to record this data in $\Delta_i$. Some leaves in $\mathscr{G}_i$ have a pair of integers attached. These leaves correspond to the tori in $M$ we cut along when we created $\mathscr{G}_i$. Since $M_i$ is the universal abelian cover of any graph orbifold having splice diagram $\mathscr{G}_i$, several fibers sit above the singular fiber corresponding to these leaves. We have to remove a neighborhood of each of these fibers. So if $\alpha_j$ is an edge weight in $\mathscr{G}_i$ to a leaf with a pair attached, the $t_j$ fibers above the leaf correspond to all the strings with the weights $-a_{j1},\dots,-a_{jn_j}$. So in the plumbing diagram for $M_i$ we replace these strings by arrows, and add a triple which consists of the pair attached to the leaf and $p_j/q_j = [a_{j1},\dots,a_{jk_j}]$ to each of the arrows. If the fibers sitting above are non singular, i.e., the set $\{a_{ji}\}$ is empty, we still add $t_j$ arrows and triples, and in this case the third number is $1/0$.

Using this on our example we get the following plumbing diagrams

$\Delta_1 =$ 

$\Delta_2 =$ 


Notice that the weight of the node is only well define for a closed manifold, so when we remove the solid torus corresponding to an arrow we lose that information. The weight of the nodes are then gotten trough the gluing process in the next section.

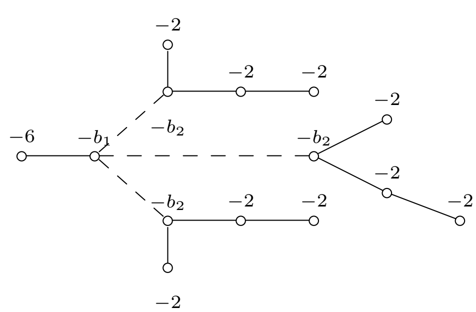
The second case, i.e., when there is an edge weight of 0, is not as easy. The proof of Theorem 6.3 in [Ped10] gives an explicit construction as a gluing of 3-spheres along $S^2$ boundaries in this case. But it might not be a Seifert fibered manifold, and I have at the present no simple way to find a plumbing diagram for the building blocks in this case. Hence the explicit algorithm does not work in this case.

3.2. **Gluing the building blocks.** The only thing that remains to construct the universal abelian cover is to glue together the building blocks $M_i$. This will be done by using the plumbing diagrams $\Delta_i$ to create a plumbing diagram $\Delta$ for $\widetilde{M}$.

We start by taking two of the $\Delta_i$'s and create a plumbing diagram $G_1$. Then we take another of the $\Delta_i$'s and glue this to $G_1$ to create $G_2$. We continue this process until all the $\Delta_i$'s have been used, and then $\Delta = G_{N-1}$ where $G_{N-1}$ is the last created plumbing diagram.

Now the order we glue the $\Delta_i$'s together in is important. This is why we added a triple to the arrows. We start by taking $\Delta_i$ which has at least one arrow having the triple $(N-1, d_i, r_i)$, where $N-1$ is the highest value for the first number in any triple. We next take $\Delta_j$ such that at least one arrow has the triple $(N-1, d_j, r_j)$. By the method we constructed the $\Delta_i$'s, there are exactly two graphs $\Delta_i$ and $\Delta_j$ satisfying respectively these conditions. Then we take $d_i$ copies of $\Delta_i$ and $d_j$ copies of $\Delta_j$. We create an intermediate $\widetilde{G}_1$ by removing the arrows with triple $(N-1, d_i, r_i)$ (respectively $(N-1, d_j, r_j)$) on the $\Delta_i$'s (respectively $\Delta_j$'s), and by replacing these with dashed lines between copies of $\Delta_i$ and $\Delta_j$, such that a copy of $\Delta_i$ is only connected to a copy of $\Delta_j$ once. We also replace the weights at the nodes in the $\Delta_i$ piece by an unknown variable $b_i$ and in the $\Delta_j$ piece by an unknown variable $b_j$. This will create a connected weighted graph $\widetilde{G}_1$, with no arrows which have first number in the triple equal to $N-1$.

Let us see how this is done in our example. We only have two $\Delta_i$'s, so we start by gluing $\Delta_1$ to $\Delta_2$. The triples are $(1, 1, 23/14)$ and $(1, 3, 5/4)$. So we start by taking one copy of $\Delta_1$ and three copies of $\Delta_2$, replacing each of the arrows in the copy of $\Delta_1$ with a dashed line to one of the copies of $\Delta_2$ replacing its arrow, and replace the weights at the nodes. We get

$\widetilde{G}_1 =$ 

The next step to create $G_1$ is to replace the dashed lines by a string of valence two vertices and find the weights at the nodes. We have to compute the number of vertices along the string and their Euler numbers. First by symmetry all the strings will be the same, so we only have to calculate one of them. Likewise the weights at the nodes are the same at the corresponding ends of the identified strings.

Now $G_1 \bigcup (\bigcup_{l \neq i,j} \Delta_l)$ is a plumbing diagram for the non-connected manifold which is the universal abelian cover of any non-connected manifold with splice diagram $\Gamma_{e_{N-2}}$. Hence it is $\Gamma_{e_{N-2}}$ we need to use, when we make the calculations in the following.

Choose nodes $v_i$ and $v_j$ of $\widetilde{G}_1$, which are attached to each other by a dashed line such that $v_k$ comes from a $\Delta_k$ piece. First we find the fiber intersection number $p$, which is also the numerator of the two continued fractions associated to the string. Now $p = f_i \cdot f_j$ where $f_k$ is a fibre in the boundary of $M_k$. These fibres are gotten as connected components of the preimage of fibres $\tilde{f}_k$ of a graph orbifold $\widetilde{M}$ with splice diagram $\Gamma(\widetilde{M}) = \Gamma_{e_{N-2}}$. This implies that $\pi^{-1}(\tilde{f}_i) \cdot \pi^{-1}(\tilde{f}_j) = d(\tilde{f}_i \cdot \tilde{f}_l)$ where $d = |H_1^{orb}(\widetilde{M})|$. The intersection number $\tilde{f}_i \cdot \tilde{f}_j = |D|/d$ by The Edge Determinant Equation (Corollary 3.3 of [Ped10]), where $D$ is the edge determinant of the corresponding edge in $\Gamma_{e_{N-2}}$. Hence $n_i n_j p = |D|$ where $n_k$ is the number of connected pieces of $\pi^{-1}(f_k)$. We have that $n_k = \deg(\pi|_{M_k})/\deg(\pi|_{f_k})$. Now $\deg(\pi|_{M_k}) = d/d_k$ where $d_k$ is given by the triple attached to the arrow in $\Delta_k$, and $\deg(\pi|_{f_k})$ is calculated in the end of the proof of Theorem 6.3 in [Ped10] to be $d/\lambda_k$. Here $\lambda_k = \prod m_j / \operatorname{lcm}(m_1/\overline{d}_1, \dots, m_l/\overline{d}_l)$, where the $m_j$'s are the edge weights adjacent to the node corresponding to $v_k$ in $\Gamma_{e_{N-2}}$, and the $\overline{d}_j$'s are the ideal generators associated to the edges. Putting this together we get the following formula for calculating $p$:

$$p = \frac{d_i d_j}{\lambda_i \lambda_j} |D|$$

In our example $\Gamma_{e_{N-2}} = \Gamma$, so $|D| = 21$ and $\lambda_1 = \lambda_2 = 3$ and we get that $p = 7$.

To find the complete string and the $b_k$'s we use that there are two different ways to calculate the rational Euler number of the Seifert fibered piece corresponding to a node in $G_1$. One using $G_1$ and one given by the splice diagram by a formula derived at the end of the proof of Theorem 6.3 in [Ped10].

From $G_1$ the rational Euler number $e_{v_k}$ is given by $b_k + \sum_e q_e/p_e$, where the sum is taken over all edges adjacent to $v_k$ (including the dashed lines), and $(p_e, q_e)$ is the Seifert pair associated to the string starting with the edge $e$. Now there are four types of different edges attached to $v_i$, and we need to see how to get $(p_e, q_e)$ from each type of the edge. We will first explain how to get $(p_e, q_e)$ for an edge $e$ if $e$ is not a dashed line. Then use this to give an equation relating $e_{v_k}$ to $b_k$ and the Seifert pair associated to the dashed lines, notice that all the dashed lines have the same Seifert pair $(p, q_k)$. We will then calculate $e_{v_k}$ in another way, and use this to get the $q_k$'s and the $b_k$'s.

If $e$ is on a string that ends at a valence one vertex then we get $(p_e, q_e)$ from the continued fraction associated to the string, i.e., $p_e/q_e = [a_{e1}, \dots, a_{ek_e}]$.

If $e$ is on a string that leads to a node (when one makes $G_1$ these do not exist, but they can be there when we are going to make $G_2$). We again get the Seifert pair from the continued fraction, this time from the string between $v_k$ and the other node.

If $e$ is an arrow, we get $(p_e, q_e)$ from the triple $(n_e, d_e, r_e)$ attached to the arrow as $p_e/q_e = r_e$.

We can now write the equation relating $e_{v_k}$, $b_k$ and $(p, q_k)$.

$$e_{v_k} = d'_k \frac{q_k}{p} - b_k + \sum_e \frac{q_e}{p_e}, \tag{7}$$

where the sum is taken over all edges at $v_k$ except the dashed lines, and $d'_k$ is the number of dashed lines at $v_k$. Notice that if $v_k = v_i$ is a node sitting in a $\Delta_i$ piece, then $d'_k = d_j$.

Returning to our example, if we use the leftmost node as $v_1$, the equation becomes

$$e_{v_1} = 3\frac{q_1}{7} - b_1 + \frac{1}{6}.$$

For one of the rightmost nodes the equation becomes

$$e_{v_2} = \frac{q_1}{7} - b_2 + \frac{1}{2} + \frac{2}{3} = \frac{q_2}{7} - b_1 + \frac{7}{6}.$$

From the end of the proof of Theorem 6.3 in [Ped10] one gets that if $v_k$ sits in a $\Delta_k$ piece,

$$e_{v_k} = \frac{\lambda_k^2}{\overline{D}}\tilde{e}_{v_k}/d. \tag{8}$$

Remember we defined $\lambda_k$ and $d$ when we calculated $p$ in the beginning of this section, and $\overline{D} = \prod_l \overline{d}_l$ where the $\overline{d}_l$'s are the ideal generators of all the edges adjacent to $v_k$ in $\Gamma_{e_{N-2}}$. Notice that there is a mistake in the formula in [Ped10], there one only divides by $d_k$ and not $\overline{D}$. It does not change the result of that article, but it is important when one wants to actually calculate $e_{v_k}$ as we do. Now neither $\tilde{e}_{v_k}$ nor $d$ are determined by $\Gamma_{e_{N-2}}$, but proposition 2.3 gives a formula for $\tilde{e}_{v_k}/d$ only using $\Gamma_{e_{N-2}}$ (the proposition also works for graph orbifolds).

In our example we find that $\lambda_1 = \lambda_2 = 3$, $\tilde{e}_{v_1}/d = -5/378$ and $\tilde{e}_{v_2}/d = -23/126$, so $e_{v_1} = -5/42$ and $e_{v_2} = -23/42$.

Now one finds an equation relating $b_k$ and $q_k$ by combining the equations (7) and (8). Since $b_k$ is an integer this equation gives us an congruence equation $\mod p$ involving $q_k$ as the only unknown. This equation involving $q_k$ might not determine $q_k(\mod p)$, since it is possible that $q_k$ is multiplied by a divisor of $p$. But the equation involving $q_i$ and the equation involving $q_j$ together with the equation $q_i q_j \equiv -1(\mod p)$ enable us to find $q_i$ and $q_j$ $(\mod p)$, and since we know $0 \leq q_i, q_j < p$ we can determine $q_i$ and $q_j$.

In our example the equations relating $b_1$ and $q_1$ becomes

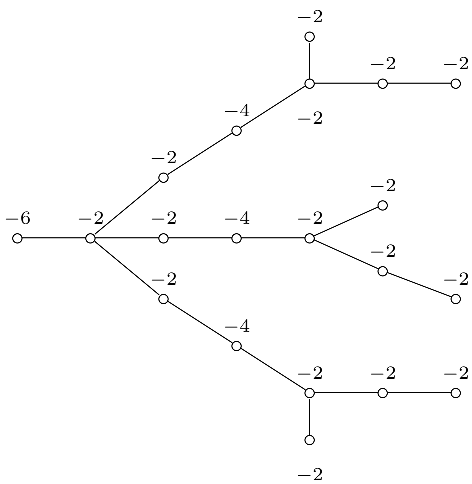
$$b_1 = 3\frac{q_1}{7} + \frac{1}{6} + \frac{5}{42} = 3\frac{q_1}{7} + \frac{2}{7},$$

and the equations relating $b_2$ and $q_2$ is

$$b_2 = \frac{q_2}{7} + \frac{7}{6} + \frac{23}{42} = \frac{q_2}{7} + \frac{12}{7}.$$

We get that $q_1 = 4$ and $q_2 = 2$. Remember that $p/q_1$ is the continued fraction associated to the string replacing the dashed lines when seen from $v_1$. We find $b_k$ by putting $q_k$ back into the equation we just used. In our example this gives that $b_1 = 2$ and $b_2 = 2$.

Replacing all the dashed lines with the strings corresponding to the continued fractions and adding the $b_i$'s one gets $G_1$ from $\widetilde{G}_1$. In our example we get the following plumbing diagram:

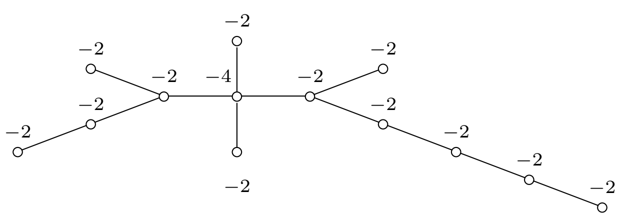
$G_1 =$ 


.

If $\Gamma_{e_{N-2}} \neq \Gamma$, then one adds $G_1$ to the collection of $\Delta_i$'s not used, and one repeats the process by taking the two plumbing diagrams of this collection having arrows whose triples start with $N-2$. One continues this process until all the $\Delta_i$'s have been used, and the final $G_{N-1}$ is then a plumbing diagram for the universal abelian cover $\widetilde{M}$ of $M$.

We will finish by performing the algorithm on a couple of other examples. We will leave the details of the calculation to the readers.

**Example 3.3.** Let $M$ be the manifold defined by the following plumbing diagram:

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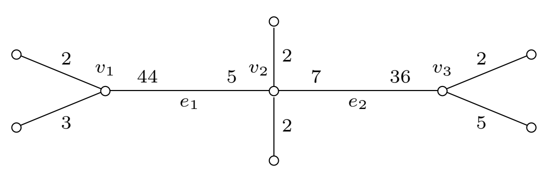
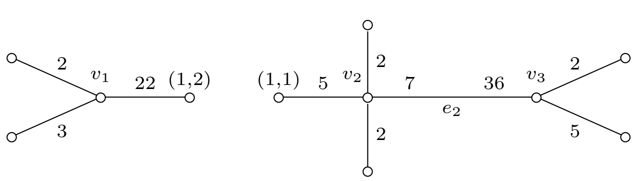
Its splice diagram is:

$\Gamma =$ 


.

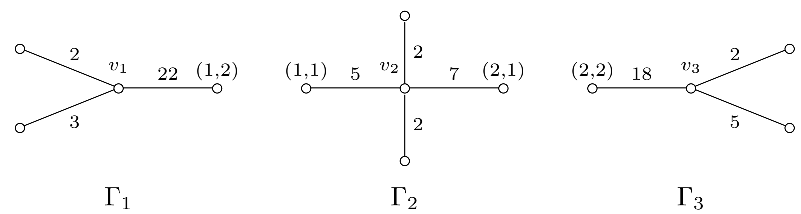
If we first cut along the edge called $e_1$, we get:

$\Gamma_{e_1} =$ 

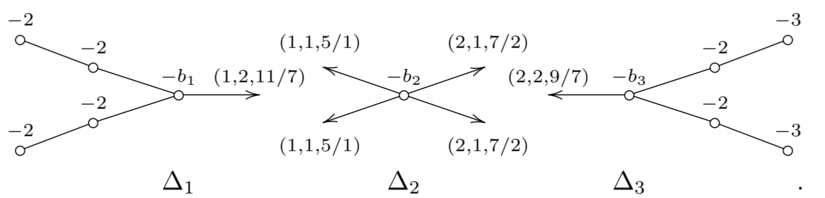

,

and cutting along $e_2$ gives us:

$\Gamma_{e_2} =$

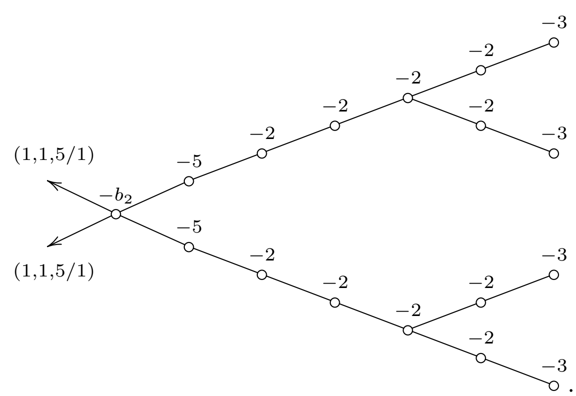



Next one determines the 3 building blocks and gets the following plumbing diagrams:



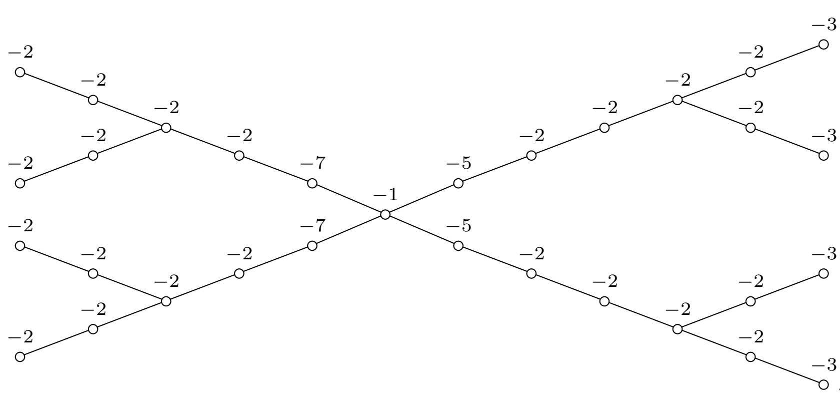

One first glues one copy of $\Delta_2$ to two copies of $\Delta_3$, and gets after calculating the strings and weights at nodes:

$G_1 =$

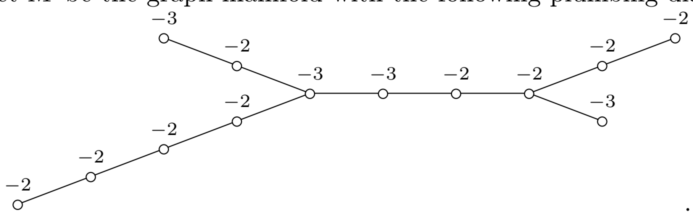



Then gluing two copies of $\Delta_1$ to $G_1$ and calculating the strings and weights at nodes gives the following plumbing diagram for the universal abelian cover:

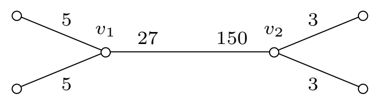
$G =$



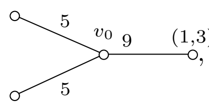
**Example 3.4.** Let $M$ be the graph manifold with the following plumbing diagram:

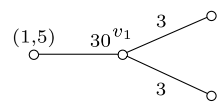



Its splice diagram is:

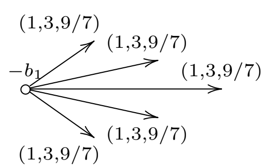
$\Gamma =$ 

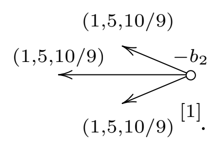

Cutting the edge gives us the one-node splice diagrams:

$\Gamma_1 =$ 

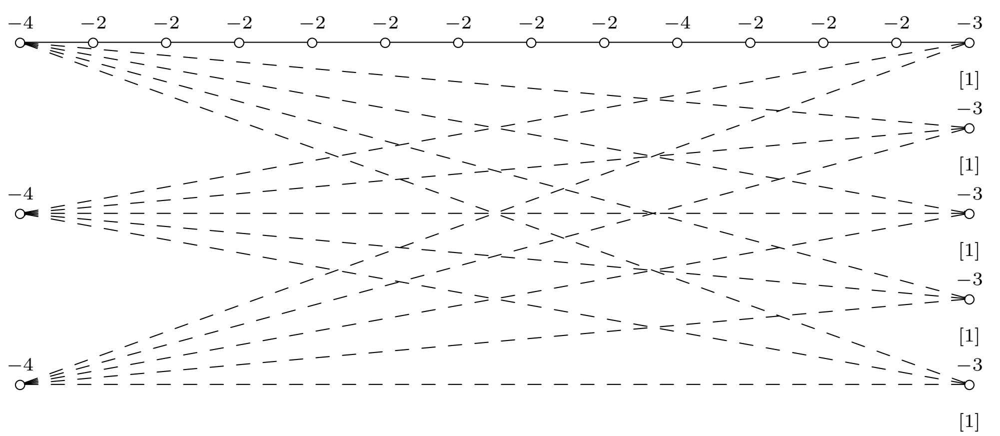
$\Gamma_2 =$ 


and the building blocks become

$\Delta_1 =$ 

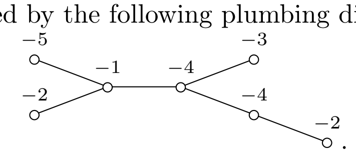
$\Delta_2 =$ 


So to create the plumbing diagram $G$ of the universal abelian cover, we glue 3 copies of $\Delta_1$ to 5 copies of $\Delta_2$, we calculate the string and the weights at nodes, and get

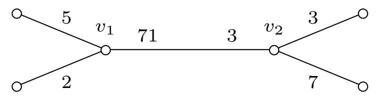



where all the dashed lines represent strings identical to the string at the top. Notice that the graph is not a planar graph. So any intersections between the strings represented by the dashed lines do not represent intersections in $G$, just crossings arising from a planar projection of $G$, which is what we see here.

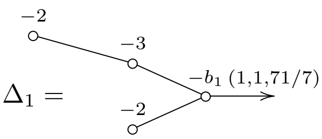
**Example 3.5.** Let $M$ be defined by the following plumbing diagram

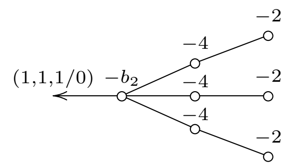


Its splice diagram is:

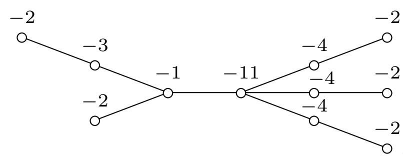
$\Gamma =$ 


Since both ideal generators are 1, the one-node splice diagrams $\Gamma_1$ and $\Gamma_2$ have the same weights as $\Gamma$, and the pairs added to the new leaves are $(1, 1)$ for both. The building blocks become:

$\Delta_1 =$ 


$\Delta_2 =$ 


If we use Theorem 3.2 to find the values at the node of the closed Seifert fibered manifolds we got the building blocks from, we will get that at the node in $\Delta_1$ the Euler number is $-1$ and at the node in $\Delta_2$ the Euler number is $-2$. Gluing a copy of $\Delta_1$ to a copy of $\Delta_2$ and finding the remaining Euler numbers gives us the following plumbing diagram for the universal abelian cover of $M$:




Notice that the Euler number of the rightmost node in the universal abelian cover is $-11$, which is very different of the $-2$ that was the Euler number of the building block.

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Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, 13-15 Reáltanoda u. 1053 Budapest, Hungary

*E-mail address*: helge@renyi.hu

**Journal *of* Singularities**
**Volume 7 (2013), 220-237**

received 13 June 2013
DOI: 10.5427/jsing.2013.7l

# MILNOR FIBERS OF REAL LINE ARRANGEMENTS

MASAHIKO YOSHINAGA

ABSTRACT. We study Milnor fibers of complexified real line arrangements. We give a new algorithm computing monodromy eigenspaces of the first cohomology. The algorithm is based on the description of minimal CW-complexes homotopic to the complements, and uses the real figure, that is, the adjacency relations of chambers. It enables us to generalize a vanishing result of Libgober, give new upper-bounds and characterize the $A_3$-arrangement in terms of non-triviality of Milnor monodromy.

## 1. INTRODUCTION

The Milnor fiber is a central object in the study of the topology of complex hypersurface singularities. In particular, the monodromy action on its cohomology groups has been intensively studied. Monodromy eigenspaces contain subtle geometric information. For example, for projective plane curves, the Betti numbers of Milnor fiber of the cone detect Zariski pairs [1]. In other words, Betti numbers of Milnor fiber of the cone of a plane curve are not in general determined by local and combinatorial data of singularities.

In the theory of hyperplane arrangements, one of the central problems is to what extent topological invariants of the complements are determined combinatorially. For example, the cohomology ring is combinatorially determined (Orlik and Solomon [16]), while the fundamental group is not (Rybnikov [1, 11]). Between these two cases, local system cohomology groups and monodromy eigenspaces of Milnor fibers recently received a considerable amount of attention.

There are several ways to compute monodromy eigenspaces of the Milnor fiber, especially for line arrangements. One is the topological method developed by Cohen and Suciu [4]. They first give a presentation of the fundamental group of the complement. Then, using Fox calculus, they compute the monodromy eigenspaces. Another approach is the algebraic method, which computes the multiplicities of monodromy eigenvalues as the superabundance of singular points. This approach has recently been well developed, especially for line arrangements having only double and triple points [14].

The purpose of this paper is to develop a topological method of computing Milnor monodromy for complexified real arrangements following Cohen and Suciu. The new ingredient is a recent study of minimal cell structures for the complements of complexified real arrangements [19, 22]. By using the description of twisted minimal chain complexes, we obtain an algorithm which computes monodromy eigenspaces directly from real figures without passing through the presentations of $\pi_1$.

The paper is organized as follows. In §2 we recall a few results which are used in this paper. §3 is the main section of the paper. First, in §3.1, we introduce discrete geometric notions, the so-called *k-resonant band* and the *standing wave* on this band. These notions are used in §3.2 for

**Acknowledgment**: Part of this work was done while the author was visiting Universidad de Zaragoza. The author gratefully acknowledges Professor E. Artal Bartolo and Professor J. I. Cogolludo-Agustín for their support, hospitality and encouragement. The author also thanks Michele Torielli for the comments to the preliminary version of this paper. This work is supported by a JSPS Grant-in-Aid for Young Scientists (B).

the computation of eigenspaces. Several consequences of our algorithm are discussed in §3.3, §3.4 and §3.5. Among other things, we prove that if the arrangement contains more than 6 lines and the cohomological monodromy action (of degree one) is non-trivial, then each line has at least three multiple points (see Corollary 3.24 for a precise statement). Such arrangements have been studied in discrete geometry as "configurations", and several examples are provided in [8, 9]. In §4, we apply our algorithm to arrangements appearing in papers by Grünbaum [8, 9]. We also present several examples and conjectures.

## 2. Preliminaries

2.1. **Milnor fiber of arrangements.** Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an affine line arrangement in $\mathbb{R}^2$ with the defining equation $Q_{\mathcal{A}}(x, y) = \prod_{i=1}^{n} \alpha_i$, where $\alpha_i$ is a defining linear equation for $H_i$. In this paper, we assume that not all lines are parallel (or equivalently, $\mathcal{A}$ has at least one intersection). The coning $c\mathcal{A}$ of $\mathcal{A}$ is an arrangement of $n+1$ planes in $\mathbb{R}^3$ defined by the equation $Q_{c\mathcal{A}}(x, y, z) = z^{n+1}Q(\frac{x}{z}, \frac{y}{z})$. The line $\{z = 0\} \in c\mathcal{A}$ is called the line at infinity and is denoted by $H_\infty$. The space $\mathsf{M}(\mathcal{A}) = \mathbb{C}^2 \setminus \{Q_{\mathcal{A}} = 0\} = \mathbb{P}^2_{\mathbb{C}} \setminus \{Q_{c\mathcal{A}} = 0\}$ is called the complexified complement. In this article, $\mathcal{A}$ always denotes a line arrangement in $\mathbb{R}^2$ and $c\mathcal{A}$ denotes a line arrangement in $\mathbb{RP}^2$. We call $p \in \mathbb{RP}^2$ a *multiple point* if the multiplicity of $c\mathcal{A}$ at $p$ (that is, the number of lines passing through $p$) is greater than or equal to 3.

**Definition 2.1.** $F_{\mathcal{A}} = \{(x, y, z) \in \mathbb{C}^3 \mid Q_{c\mathcal{A}}(x, y, z) = 1)\}$ is called the Milnor fiber of $\mathcal{A}$. The automorphism $\rho : F_{\mathcal{A}} \longrightarrow F_{\mathcal{A}},\ (x, y, z) \longmapsto (\zeta x, \zeta y, \zeta z)$, with $\zeta = \exp(2\pi i/(n + 1))$, is called the monodromy action.

The automorphism $\rho$ has order $n + 1$. It generates the cyclic group $\langle\rho\rangle \simeq \mathbb{Z}/(n + 1)\mathbb{Z}$. The monodromy $\rho$ induces a linear map $\rho^* : H^1(F_{\mathcal{A}}, \mathbb{C}) \longrightarrow H^1(F_{\mathcal{A}}, \mathbb{C})$. Since $(\rho^*)^{n+1}$ is the identity, we have the eigenspace decomposition $H^1(F_{\mathcal{A}}, \mathbb{C}) = \bigoplus_{\lambda^{n+1}=1} H^1(F_{\mathcal{A}}, \mathbb{C})_\lambda$, where $H^1(F_{\mathcal{A}}, \mathbb{C})_\lambda$ is the the set of $\lambda$-eigenvectors with eigenvalue $\lambda \in \mathbb{C}^*$. When $\lambda = 1$, $H^1(F_{\mathcal{A}})_1 = H^1(F_{\mathcal{A}})^{\rho^*}$ is the subspace of elements fixed by $\rho^*$, which is isomorphic to $H^1(F_{\mathcal{A}}/\langle\rho\rangle)$. It is easily seen that the quotient by the monodromy action is $F_{\mathcal{A}}/\langle\rho\rangle \simeq \mathsf{M}(\mathcal{A})$. Therefore, the 1-eigenspace of the first cohomology is combinatorially determined, $H^1(F_{\mathcal{A}})_1 \simeq H^1(\mathsf{M}(\mathcal{A})) \simeq \mathbb{C}^n$. In general, let $\mathcal{L}_\lambda$ be a complex rank one local system associated with a representation

$$\pi_1(\mathsf{M}(\mathcal{A})) \longrightarrow \mathbb{C}^*,\ \gamma_H \longmapsto \lambda,$$

where $\gamma_H$ is a meridian loop of the line $H$. Then it is known that

$$H^1(F_{\mathcal{A}})_\lambda \simeq H^1(\mathsf{M}(\mathcal{A}), \mathcal{L}_\lambda). \tag{1}$$

(See [4] for details.)

2.2. **Multinets and Milnor monodromy.** In this section, we recall a relation between the combinatorial structures known as multinets and the eigenvalues of Milnor monodromy. We note that a $k$-multinet gives a lower bound on the eigenspace.

**Definition 2.2.** A *$k$-multinet* on $c\mathcal{A}$ is a pair $(\mathcal{N}, \mathcal{X})$, where $\mathcal{N}$ is a partition of $c\mathcal{A}$ into $k \geq 3$ classes $\mathcal{A}_1, \dots, \mathcal{A}_k$ and $\mathcal{X}$ is a set of multiple points such that

- (i) $|\mathcal{A}_1| = \cdots = |\mathcal{A}_k|$;
- (ii) $H \in \mathcal{A}_i$ and $H' \in \mathcal{A}_j$ $(i \neq j)$ imply that $H \cap H' \in \mathcal{X}$;
- (iii) for all $p \in \mathcal{X}$, $|\{H \in \mathcal{A}_i \mid H \ni p\}|$ is constant and independent of $i$;
- (iv) for any $H, H' \in \mathcal{A}_i$ $(i = 1, \dots, k)$, there is a sequence $H = H_0, H_1, \dots, H_r = H'$ in $\mathcal{A}_i$ such that $H_{j-1} \cap H_j \notin \mathcal{X}$ for $1 \leq j \leq r$.

The following is a consequence of [7, Theorem 3.11] and [6, Theorem 3.1 (i)]

**Theorem 2.3.** *Suppose there exists a $k$-multinet on $c\mathcal{A}$ for some $k \geq 3$ and set $\lambda = e^{2\pi i/k}$. Then*

$$\dim H^1(F_{\mathcal{A}})_\lambda \geq k - 2.$$

2.3. **Twisted minimal cochain complexes.** In this section, we recall the construction of the twisted minimal cochain complex from [19, 20, 21], which will be used for the computation of the right hand side of (1).

A connected component of $\mathbb{R}^2 \setminus \bigcup_{H\in\mathcal{A}} H$ is called a chamber. The set of all chambers is denoted by $\mathsf{ch}(\mathcal{A})$. A chamber $C \in \mathsf{ch}(\mathcal{A})$ is called bounded (resp. unbounded) if the area is finite (resp. infinite). For an unbounded chamber $U \in \mathsf{ch}(\mathcal{A})$, the opposite unbounded chamber is denoted by $U^\vee$ (see [21, Definition 2.1] for the definition; see also Figure 1 below).

Let $\mathcal{F}$ be a generic flag in $\mathbb{R}^2$

$$\mathcal{F} : \emptyset = \mathcal{F}^{-1} \subset \mathcal{F}^0 \subset \mathcal{F}^1 \subset \mathcal{F}^2 = \mathbb{R}^2,$$

where $\mathcal{F}^k$ is a generic $k$-dimensional affine subspace.

**Definition 2.4.** For $k = 0, 1, 2$, define the subset $\mathsf{ch}^k_{\mathcal{F}}(\mathcal{A}) \subset \mathsf{ch}(\mathcal{A})$ by

$$\mathsf{ch}^k_{\mathcal{F}}(\mathcal{A}) := \{C \in \mathsf{ch}(\mathcal{A}) \mid C \cap \mathcal{F}^k \neq \emptyset, C \cap \mathcal{F}^{k-1} = \emptyset\}.$$

The set of chambers decomposes into a disjoint union as

$$\mathsf{ch}(\mathcal{A}) = \mathsf{ch}^0_{\mathcal{F}}(\mathcal{A}) \sqcup \mathsf{ch}^1_{\mathcal{F}}(\mathcal{A}) \sqcup \mathsf{ch}^2_{\mathcal{F}}(\mathcal{A}).$$

The cardinality of $\mathsf{ch}^k_{\mathcal{F}}(\mathcal{A})$ is equal to $b_k(\mathsf{M}(\mathcal{A}))$ for $k = 0, 1, 2$.

We further assume that the generic flag $\mathcal{F}$ satisfies the following conditions:

- $\mathcal{F}^1$ does not separate intersections of $\mathcal{A}$,
- $\mathcal{F}^0$ does not separate $n$-points $\mathcal{A} \cap \mathcal{F}^1$.

Then we can choose coordinates $x_1, x_2$ so that $\mathcal{F}^0$ is the origin $(0, 0)$, $\mathcal{F}^1$ is given by $x_2 = 0$, all intersections of $\mathcal{A}$ are contained in the upper-half plane $\{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 > 0\}$ and $\mathcal{A} \cap \mathcal{F}^1$ is contained in the half-line $\{(x_1, 0) \mid x_1 > 0\}$.

We set $H_i \cap \mathcal{F}^1$ to have coordinates $(a_i, 0)$. By changing the numbering of lines and the signs of the defining equation $\alpha_i$ of $H_i \in \mathcal{A}$ we may assume that

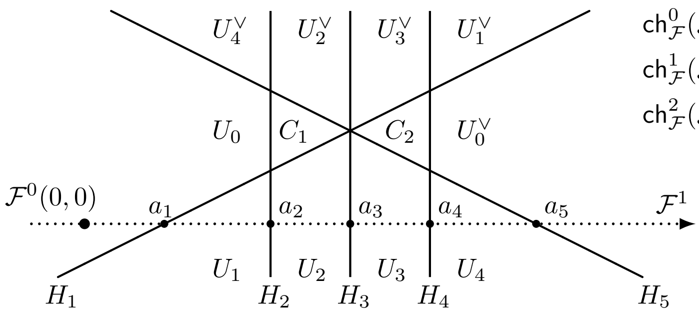
- $0 < a_1 < a_2 < \cdots < a_n$,
- the origin $\mathcal{F}^0$ is contained in the negative half-plane $H_i^- = \{\alpha_i < 0\}$.

We set $\mathsf{ch}^{\mathcal{F}}_0(\mathcal{A}) = \{U_0\}$ and $\mathsf{ch}^{\mathcal{F}}_1(\mathcal{A}) = \{U_1, \dots, U_{n-1}, U_0^\vee\}$ so that $U_p \cap \mathcal{F}^1$ is equal to the interval $(a_p, a_{p+1})$ for $p = 1, \dots, n-1$. It is easily seen that the chambers $U_0, U_1, \dots, U_{n-1}$ and $U_0^\vee$ have the following expression:

$$\begin{aligned}
U_0 &= \bigcap_{i=1}^{n} \{\alpha_i < 0\}, \\
U_p &= \bigcap_{i=1}^{p} \{\alpha_i > 0\} \cap \bigcap_{i=p+1}^{n} \{\alpha_i < 0\}, \ (p = 1, \dots, n-1), \\
U_0^\vee &= \bigcap_{i=1}^{n} \{\alpha_i > 0\}.
\end{aligned} \tag{2}$$

The notations introduced to this point are illustrated in Figure 1.

Let $\mathcal{L}$ be a complex rank-one local system on $\mathsf{M}(\mathcal{A})$. The local system $\mathcal{L}$ is determined by non-zero complex numbers (monodromy around $H_i$) $q_i \in \mathbb{C}^*$, $i = 1, \dots, n$. Fix a square root $q_i^{1/2} \in \mathbb{C}^*$ for each $i$.




$$\mathsf{ch}^0_{\mathcal{F}}(\mathcal{A}) = \{U_0\}$$
$$\mathsf{ch}^1_{\mathcal{F}}(\mathcal{A}) = \{U_0^\vee, U_1, \dots, U_4\}$$
$$\mathsf{ch}^2_{\mathcal{F}}(\mathcal{A}) = \{U_1^\vee, \dots, U_4^\vee, C_1, C_2\}$$

FIGURE 1. Numbering of lines and chambers.

**Definition 2.5.** (1) For $C, C' \in \mathsf{ch}(\mathcal{A})$, let us denote by $\mathrm{Sep}(C, C')$ the set of lines $H_i \in \mathcal{A}$ which separate $C$ and $C'$.
(2) Define the complex number $\Delta(C, C') \in \mathbb{C}$ by

$$\Delta(C, C') := \prod_{H_i \in \mathrm{Sep}(C,C')} q_i^{1/2} - \prod_{H_i \in \mathrm{Sep}(C,C')} q_i^{-1/2}.$$

Now we construct the cochain complex $(\mathbb{C}[\mathsf{ch}^\bullet_{\mathcal{F}}(\mathcal{A})], d_{\mathcal{L}})$.

(i) The map $d_{\mathcal{L}} : \mathbb{C}[\mathsf{ch}^0_{\mathcal{F}}(\mathcal{A})] \longrightarrow \mathbb{C}[\mathsf{ch}^1_{\mathcal{F}}(\mathcal{A})]$ is defined by

$$d_{\mathcal{L}}([U_0]) = \Delta(U_0, U_0^\vee)[U_0^\vee] + \sum_{p=1}^{n-1} \Delta(U_0, U_p)[U_p].$$

(ii) $d_{\mathcal{L}} : \mathbb{C}[\mathsf{ch}^1_{\mathcal{F}}(\mathcal{A})] \longrightarrow \mathbb{C}[\mathsf{ch}^2_{\mathcal{F}}(\mathcal{A})]$ is defined by

$$d_{\mathcal{L}}([U_p]) = -\sum_{\substack{C \in \mathsf{ch}^2_{\mathcal{F}}(\mathcal{A}) \\ \alpha_p(C) > 0 \\ \alpha_{p+1}(C) < 0}} \Delta(U_p, C)[C] + \sum_{\substack{C \in \mathsf{ch}^2_{\mathcal{F}}(\mathcal{A}) \\ \alpha_p(C) < 0 \\ \alpha_{p+1}(C) > 0}} \Delta(U_p, C)[C], \ (\text{for } p = 1, \dots, n-1),$$

$$d_{\mathcal{L}}([U_0^\vee]) = -\sum_{\alpha_n(C) > 0} \Delta(U_0^\vee, C)[C].$$

**Example 2.6.** Let $\mathcal{A} = \{H_1, \dots, H_5\}$, and let the flag $\mathcal{F}$ be as in Figure 1. Then

$$d_{\mathcal{L}}([U_0]) = ([U_1], [U_2], [U_3], [U_4], [U_0^\vee]) \begin{pmatrix} q_1^{1/2} - q_1^{-1/2} \\ q_{12}^{1/2} - q_{12}^{-1/2} \\ q_{123}^{1/2} - q_{123}^{-1/2} \\ q_{1234}^{1/2} - q_{1234}^{-1/2} \\ q_{12345}^{1/2} - q_{12345}^{-1/2} \end{pmatrix},$$

$$d_{\mathcal{L}}([U_1], [U_2], [U_3], [U_4], [U_0^\vee]) = ([U_1^\vee], [U_2^\vee], [U_3^\vee], [U_4^\vee], [C_1], [C_2])$$

$$\times \begin{pmatrix} q_{12345}^{1/2} - q_{12345}^{-1/2} & 0 & 0 & 0 & -(q_1^{1/2} - q_1^{-1/2}) \\ q_{125}^{1/2} - q_{125}^{-1/2} & -(q_{15}^{1/2} - q_{15}^{-1/2}) & 0 & q_{1345}^{1/2} - q_{1345}^{-1/2} & -(q_{134}^{1/2} - q_{134}^{-1/2}) \\ q_{1235}^{1/2} - q_{1235}^{-1/2} & 0 & -(q_{15}^{1/2} - q_{15}^{-1/2}) & q_{145}^{1/2} - q_{145}^{-1/2} & -(q_{14}^{1/2} - q_{14}^{-1/2}) \\ 0 & 0 & 0 & q_{12345}^{1/2} - q_{12345}^{-1/2} & -(q_{1234}^{1/2} - q_{1234}^{-1/2}) \\ q_{12}^{1/2} - q_{12}^{-1/2} & -(q_1^{1/2} - q_1^{-1/2}) & 0 & 0 & 0 \\ 0 & 0 & -(q_5^{1/2} - q_5^{-1/2}) & q_{45}^{1/2} - q_{45}^{-1/2} & -(q_4^{1/2} - q_4^{-1/2}) \end{pmatrix}.$$

**Theorem 2.7.** *Under the above notation,* $(\mathbb{C}[\mathsf{ch}^{\bullet}_{\mathcal{F}}(\mathcal{A})], d_{\mathcal{L}})$ *is a cochain complex and*

$$H^k(\mathbb{C}[\mathsf{ch}^{\bullet}_{\mathcal{F}}(\mathcal{A})], d_{\mathcal{L}}) \simeq H^k(M(\mathcal{A}), \mathcal{L}).$$

See [19, 20, 21] for details.

## 3. Resonant band algorithm

Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an arrangement of affine lines in $\mathbb{R}^2$, and let $\mathcal{F}$ be a generic flag as in §2.3.

Fix an integer $k > 1$ with $k|(n+1)$, and set $\lambda = e^{2\pi i/k}$. In this section, we will give an algorithm for computing the $\lambda$-eigenspace $H^1(F_{\mathcal{A}})_\lambda$ of the first cohomology of a Milnor fiber.

### 3.1. **Resonant bands and standing waves.**

**Definition 3.1.** A *band* $B$ is a region bounded by a pair of consecutive parallel lines $H_i$ and $H_{i+1}$.

Each band $B$ includes two unbounded chambers $U_1(B), U_2(B) \in \mathsf{ch}(\mathcal{A})$. By definition, $U_1(B)$ and $U_2(B)$ are opposite each other, $U_1(B)^\vee = U_2(B)$ and $U_2(B)^\vee = U_1(B)$.

Define the adjacency distance $d(C, C')$ between two chambers $C$ and $C'$ to be the number of lines $H \in \mathcal{A}$ that separate $C$ and $C'$, that is,

$$d(C, C') = |\operatorname{Sep}(C, C')|.$$

The distance $d(U_1(B), U_2(B))$ is called the length of the band $B$.

**Remark 3.2.** Let $\overline{B}$ be the closure of $B$ in the real projective plane $\mathbb{RP}^2$. $\overline{B}$ intersects $H_\infty$ in one point, $\overline{B} \cap H_\infty$. Each line $H \in \mathcal{A} \cup \{H_\infty\}$ either passes $\overline{B} \cap H_\infty$ or separates $U_1(B)$ and $U_2(B)$. Therefore the length of $B$ is equal to $n + 1 - \operatorname{mult}(\overline{B} \cap H_\infty)$.

**Definition 3.3.** A band $B$ is called *$k$-resonant* if the length of $B$ is divisible by $k$. We denote the set of all $k$-resonant bands by $\mathrm{RB}_k(\mathcal{A})$.

To a $k$-resonant band $B \in \mathrm{RB}_k(\mathcal{A})$, we can associate a *standing wave* $\nabla(B) \in \mathbb{C}[\mathsf{ch}(\mathcal{A})]$ on the band $B$ as follows:

$$\begin{aligned}
\nabla(B) &= \sum_{\substack{C \in \mathsf{ch}(\mathcal{A}),\\ C \subset B}} \left(e^{\frac{\pi i d(U_1(B), C)}{k}} - e^{-\frac{\pi i d(U_1(B), C)}{k}}\right) \cdot [C] \\
&= \sum_{\substack{C \in \mathsf{ch}(\mathcal{A}),\\ C \subset B}} \left(\lambda^{\frac{d(U_1(B), C)}{2}} - \lambda^{-\frac{d(U_1(B), C)}{2}}\right) \cdot [C] \qquad (3) \\
&= 2i \cdot \sum_{\substack{C \in \mathsf{ch}(\mathcal{A}),\\ C \subset B}} \sin\left(\frac{\pi d(U_1(B), C)}{k}\right) \cdot [C].
\end{aligned}$$

**Remark 3.4.** Since the length $d(U_1(B), U_2(B))$ of the band $B$ is divisible by $k$, the coefficients of $[U_1(B)]$ and $[U_2(B)]$ in the linear combination in (3) are zero. Hence the chambers in the summations in (3) run only over bounded chambers contained in $B$. We also note that exchanging of $U_1(B)$ and $U_2(B)$ affects at most the sign of $\nabla(B)$.

**Remark 3.5.** To indicate the choice of $U_1(B)$ and $U_2(B)$, we always put the name $B$ of the band in the unbounded chamber $U_1(B)$ (see Figure 2).

3.2. **Eigenspaces via resonant bands.** The map $B \longmapsto \nabla(B)$ can be naturally extended to the linear map

$$\nabla : \mathbb{C}[\mathrm{RB}_k(\mathcal{A})] \longrightarrow \mathbb{C}[\mathsf{ch}(\mathcal{A})]. \tag{4}$$

**Theorem 3.6.** *The kernel of $\nabla$ is isomorphic to the $\lambda$-eigenspace of the Milnor fiber monodromy, that is,*

$$\mathrm{Ker}\,(\nabla : \mathbb{C}[\mathrm{RB}_k(\mathcal{A})] \longrightarrow \mathbb{C}[\mathsf{ch}(\mathcal{A})]) \simeq H^1(F_{\mathcal{A}})_\lambda.$$

*In particular, $\dim H^1(F_{\mathcal{A}})_\lambda$ is equal to the number of linear relations among the standing waves $\nabla(B)$, $B \in \mathrm{RB}_k(\mathcal{A})$.*

*Proof.* Let $\mathcal{L}_\lambda$ be the rank-one local system on $\mathsf{M}(\mathcal{A})$ defined by $q_1 = \cdots = q_n = \lambda \in \mathbb{C}^*$ (see §2.1 and §2.3). In this case, $\Delta(C, C')$ depends only on the adjacency distance $d(C, C')$, or more precisely,

$$\Delta(C, C') = \lambda^{\frac{d(C,C')}{2}} - \lambda^{-\frac{d(C,C')}{2}}.$$

Now, we consider the first cohomology group $H^1(\mathbb{C}[\mathsf{ch}^\bullet_{\mathcal{F}}(\mathcal{A})], d_{\mathcal{L}})$ of the twisted minimal cochain complex. The image $d_{\mathcal{L}} : \mathbb{C}[\mathsf{ch}^0_{\mathcal{F}}(\mathcal{A})] \longrightarrow \mathbb{C}[\mathsf{ch}^1_{\mathcal{F}}(\mathcal{A})]$ is generated by

$$d_{\mathcal{L}}([U_0]) = \sum_{p=1}^{n-1} (\lambda^{\frac{p}{2}} - \lambda^{-\frac{p}{2}})[U_p] + \left(\lambda^{\frac{n}{2}} - \lambda^{-\frac{n}{2}}\right)[U_0^\vee].$$

Since $\lambda = e^{2\pi i/k}$ with $k > 1$ and $k|(n+1)$, we have $\lambda^{\frac{n}{2}} - \lambda^{-\frac{n}{2}} = \lambda^{-\frac{n}{2}}(\lambda^n - 1) \neq 0$. Thus the coefficient of $[U_0^\vee]$ in $d_{\mathcal{L}}([U_0])$ is non-zero. Define the subspace $V$ of $\mathbb{C}[\mathsf{ch}^1_{\mathcal{F}}(\mathcal{A})]$ by

$$\begin{aligned} V &= \bigoplus_{p=1}^{n-1} \mathbb{C} \cdot [U_p] \\ &(\simeq \mathrm{Coker}\left(d_{\mathcal{L}} : \mathbb{C}[\mathsf{ch}^0_{\mathcal{F}}(\mathcal{A})] \longrightarrow \mathbb{C}[\mathsf{ch}^1_{\mathcal{F}}(\mathcal{A})]\right)). \end{aligned} \tag{5}$$

Then $H^1(\mathbb{C}[\mathsf{ch}^\bullet_{\mathcal{F}}(\mathcal{A})], d_{\mathcal{L}})$ is isomorphic to $\mathrm{Ker}\left(d_{\mathcal{L}}|_V : V \longrightarrow \mathbb{C}[\mathsf{ch}^2_{\mathcal{F}}(\mathcal{A})]\right)$. It is sufficient to show that $\mathrm{Ker}(d_{\mathcal{L}}|_V) \simeq \mathrm{Ker}\,\nabla$, which will be done in several steps. Suppose that

$$\varphi = \sum_{p=1}^{n-1} c_p \cdot [U_p] \in \mathrm{Ker}(d_{\mathcal{L}}|_V).$$

(i) If $H_i$ and $H_{i+1}$ are not parallel, then $c_i = 0$.

Note that if $j \neq i$, then the chamber $[U_i^\vee]$ does not appear in $d_{\mathcal{L}}([U_j])$. Thus the coefficient of $[U_i^\vee]$ in

$$d_{\mathcal{L}}(\varphi) = \sum_{p=1}^{n-1} c_p \cdot d_{\mathcal{L}}([U_p])$$

is $c_i \cdot \Delta(U_i, U_i^\vee) = c_i(\lambda^{\frac{n}{2}} - \lambda^{-\frac{n}{2}})$. This equals zero if and only if $c_i = 0$.

Now we may assume that $\varphi = \sum_p c_p \cdot [U_p] \in \mathrm{Ker}(d_{\mathcal{L}})$ is a linear combination of $[U_p]$s such that $H_p$ and $H_{p+1}$ are parallel. Suppose that $H_i$ and $H_{i+1}$ are parallel and denote by $B_i$ the band determined by these lines.

(ii) If $B_i$ is not $k$-resonant, then $c_i = 0$.

In this case, $\Delta(U_i, U_i^\vee) = \lambda^{\frac{d(U_i,U_i^\vee)}{2}} - \lambda^{-\frac{d(U_i,U_i^\vee)}{2}}$. By the assumption that $d(U_i, U_i^\vee)$ is not divisible by $k$, we have $\Delta(U_i, U_i^\vee) \neq 0$. Since $\varphi$ is a linear combination of $[U_p]$s with parallel boundaries $H_p$ and $H_{p+1}$, the term $[U_i^\vee]$ appears only in $d_{\mathcal{L}}([U_i])$, which is equal to $c_i \cdot \Delta(U_i, U_i^\vee)[U_i^\vee]$. Therefore $c_i = 0$.

Finally we may assume that $\varphi$ is a linear combination of $[U_p]$s such that the boundaries $H_p$ and $H_{p+1}$ are parallel and the length of the corresponding band $B_p$ is divisible by $k$. In this case, it is straightforward to check that the maps $d_{\mathcal{L}}$ and $\nabla$ are identical. This completes the proof. □

**Example 3.7.** ($A_3$-arrangement, $\mathcal{A}(6,1)$ or $\mathcal{B}_6$) The three arrangements in Figure 2 are projectively equivalent, and are respectively called $A_3$-arrangement, $\mathcal{A}(6,1)$ or $\mathcal{B}_6$. (See §4 for the latter two notations.) We use the left figure to compute $\dim H^1(F_{\mathcal{A}})_\lambda$. (The symbol $\infty$ indicates that the line at infinity is an element of $\mathcal{A}$.) Since $|c\mathcal{A}| = n+1 = 6$, $k \in \{2,3,6\}$ and we have $\mathrm{RB}_2(\mathcal{A}) = \mathrm{RB}_6(\mathcal{A}) = \emptyset$, $\mathrm{RB}_3(\mathcal{A}) = \{B_1, B_2\}$. By definition, we have

$$\nabla(B_1) = \sqrt{-3}\cdot[C_1] + \sqrt{-3}\cdot[C_2]$$
$$\nabla(B_2) = \sqrt{-3}\cdot[C_1] + \sqrt{-3}\cdot[C_2].$$

Hence we have a linear relation $\nabla(B_1 - B_2) = 0$ and $\dim H^1(F_{\mathcal{A}})_\lambda = 1$ for $\lambda = e^{2\pi i/3}$. (Hence the $A_3$-arrangement is pure-tone; see Definition 4.1.)

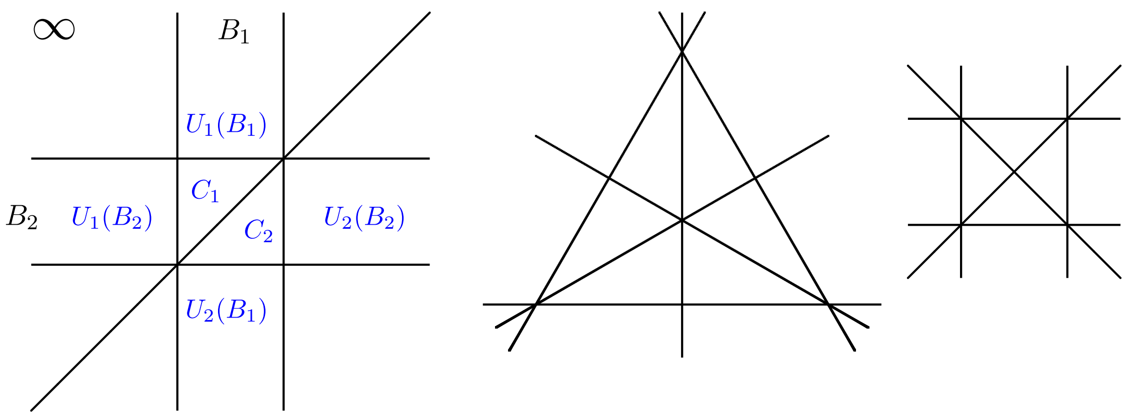



FIGURE 2. The $A_3$-arrangement ($= \mathcal{A}(6,1) = \mathcal{B}_6$)

**Example 3.8.** ($\mathcal{A}(12,2)$ from [8]) Let $\mathcal{A}$ be the line arrangement in Figure 3 (together with the line at infinity). Then $|c\mathcal{A}| = n+1 = 12$. There are seven bands, $B_1, \dots, B_7$. Among them, $B_5, B_6$ and $B_7$ have length 7 which is coprime with 12 so we can ignore them. We have $\mathrm{RB}_3(\mathcal{A}) = \{B_1, B_4\}$ and $\mathrm{RB}_2(\mathcal{A}) = \mathrm{RB}_4(\mathcal{A}) = \{B_2, B_3\}$. First consider the case $k = 3$. Then

$$\nabla(B_1) = \sqrt{-3}\cdot[C_1] + \dots,$$
$$\nabla(B_4) = \sqrt{-3}\cdot[C_6] + \dots.$$

Since the chamber $C_6$ is not contained in the band $B_1$, it does not appear in the linear combination for $\nabla(B_1)$. Hence $\nabla(B_1)$ and $\nabla(B_4)$ are linearly independent. We conclude that $H^1(F_{\mathcal{A}})_\lambda = 0$ for $\lambda = e^{2\pi i/3}$. The cases $k = 2$ and $k = 4$ are similar. More precisely, since $B_2, B_3 \in \mathrm{RB}_2(\mathcal{A}) = \mathrm{RB}_4(\mathcal{A})$ are parallel and they do not overlap, $\nabla(B_2)$ and $\nabla(B_3)$ are linearly independent. Consequently we have $H^1(F_{\mathcal{A}})_{\neq 1} = 0$ and so the cohomology does not have non-trivial eigenvalues.

The argument used in Example 3.8 is generalized in the next section. See §4 for further examples.

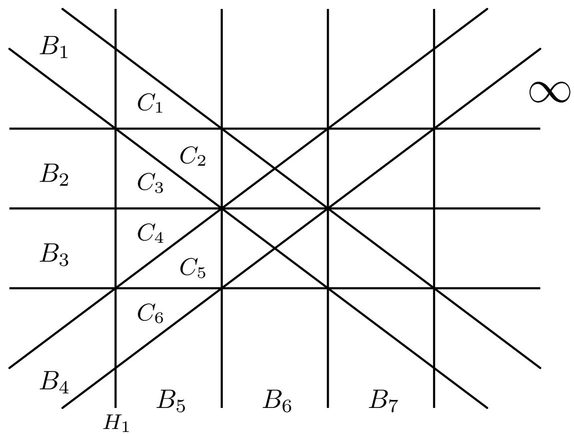



FIGURE 3. $\mathcal{A}(12, 2)$

**Remark 3.9.** The cohomology of the Milnor fiber $H^1(F_{\mathcal{A}})$ depends only on the projective arrangement $\mathcal{A} \cup \{H_\infty\}$. The change of the line at infinity $H_\infty$ sometimes makes the structure of resonant bands $\mathrm{RB}_k$ simpler. This fact will be used in Corollary 3.16.

3.3. **Vanishing.** Fix $k$ and $\lambda$ as above. We describe some corollaries to Theorem 3.6.

**Corollary 3.10.** *If* $\mathrm{RB}_k(\mathcal{A}) = \emptyset$, *then* $H^1(F_{\mathcal{A}})_\lambda = 0$.

*Proof.* Since $\mathbb{C}[\mathrm{RB}_k(\mathcal{A})] = 0$, obviously $\mathrm{Ker}(\nabla : \mathbb{C}[\mathrm{RB}_k(\mathcal{A})] \to \mathbb{C}[\mathsf{ch}(\mathcal{A})]) = 0$. By Theorem 3.6, $H^1(F_{\mathcal{A}})_\lambda = 0$. □

Using the interpretations in Remark 3.2, we have the following.

**Proposition 3.11.** A band $B$ is $k$-resonant if and only if $\mathrm{mult}(\overline{B} \cap H_\infty)$ is divisible by $k$.

**Corollary 3.12.** *Suppose that there are no points on* $H_\infty$ *where the multiplicity of*

$$c\mathcal{A} = \mathcal{A} \cup \{H_\infty\}$$

*is divisible by* $k$. *Then* $H^1(F_{\mathcal{A}})_\lambda = 0$.

*Proof.* By Proposition 3.11, the assumption is equivalent to $\mathrm{RB}_k(\mathcal{A}) = \emptyset$. We then use Corollary 3.10. □

**Remark 3.13.** Corollary 3.12 is proved by Libgober [13, Corollary 3.5] for more general complex arrangement cases.

For the real case, we obtain a stronger result as follows.

**Theorem 3.14.** *Suppose that all* $k$*-resonant bands are parallel to each other. Then* $H^1(F_{\mathcal{A}})_\lambda = 0$.

*Proof.* By the assumption, $\mathrm{RB}_k(\mathcal{A}) = \{B_1, \dots, B_m\}$ consists of parallel bands. Now, the supports of $\nabla(B_1), \dots, \nabla(B_m)$, that is, the set of chambers appearing in each standing wave, are mutually disjoint. They are obviously linearly independent. (Recall that, in this paper, we assume that the arrangement $\mathcal{A}$ has at least one intersection.) Hence $H^1(F_{\mathcal{A}})_\lambda = 0$. (See Example 3.8.) □

**Corollary 3.15.** *Suppose that there is at most one point* $p \in H_\infty$ *such that the multiplicity of* $c\mathcal{A} = \mathcal{A} \cup \{H_\infty\}$ *at* $p$ *is divisible by* $k$. *Then* $H^1(F_{\mathcal{A}})_\lambda = 0$.

*Proof.* Again by Proposition 3.11, the assumption is equivalent to $\mathrm{RB}_k(\mathcal{A})$ consisting of parallel bands. We then use Theorem 3.14. $\square$

Let us denote by $H^1(F_{\mathcal{A}})_{\neq 1} = \bigoplus_{\lambda \neq 1} H^1(F_{\mathcal{A}})_\lambda$ the direct sum of non-trivial eigenspaces. The following is immediate from Corollary 3.15 and Remark 3.9.

**Corollary 3.16.** (1) *Suppose that $H^1(F_{\mathcal{A}})_\lambda \neq 0$. Then each line $H \in c\mathcal{A} = \mathcal{A} \cup \{H_\infty\}$ has at least two multiple points, such that the multiplicity is divisible by $k$.*

(2) *Suppose that $H^1(F_{\mathcal{A}})_{\neq 1} \neq 0$. Then each line $H \in c\mathcal{A} = \mathcal{A} \cup \{H_\infty\}$ has at least two multiple points.*

*Proof.* Let $H$ be such a line. Choose an affine open set in such a way that $H$ is a line at infinity. $\square$

**Remark 3.17.** We do not know whether Corollary 3.16 holds for complex arrangements. We will prove a stronger result in §3.5.

3.4. **Upper-bound.** Recall that two lines $H, H'$ in the real projective plane $\mathbb{RP}^2$ divide the space into two regions.

**Definition 3.18.** Let $c\mathcal{A}$ be a line arrangement in the real projective plane $\mathbb{RP}^2$. Then the pair of lines $H_i, H_j \in c\mathcal{A}$ is said to be a *sharp pair* if all intersection points of $c\mathcal{A} \setminus \{H_i, H_j\}$ are contained in one of two regions or lie on $H_i \cup H_j$. (In other words, there are no intersection points in one of the two regions determined by $H_i$ and $H_j$.)

**Example 3.19.** A fiber-typer arrangement has a sharp pair of lines.

**Example 3.20.** In the Pappus arrangement (Figure 7), the line at infinity and the leftmost vertical line form a sharp pair. So do the two boundary lines of the band $B_1$. Furthermore, all line arrangements appearing in this paper contain sharp pairs of lines. (There also exist arrangements which have no sharp pairs.)

**Theorem 3.21.** *Assume that the arrangement $c\mathcal{A}$ contains a sharp pair of lines. Then:*

(i) $\dim H^1(F)_\lambda \leq 1$ *for* $\lambda \neq 1$.

(ii) *Suppose that the pair $H_1, H_2 \in c\mathcal{A}$ is sharp. Let $p = H_1 \cap H_2$ be the intersection. If the multiplicity of $c\mathcal{A}$ at $p$ is not divisible by $k$, then $H^1(F)_\lambda = 0$ for $\lambda = e^{2\pi i/k}$.*

*Proof.* By the $PGL_3(\mathbb{C})$ action, we may assume that the line at infinity $H_\infty$ and $H_1 = \{x = 0\}$ form a sharp pair and that there are no intersections in the region $\{(x, y) \in \mathbb{R}^2 \mid x < 0\}$ (see Figure 3). The intersection is $p = H_\infty \cap H_1 = \{(0 : 1 : 0)\}$. Let $B$ be a horizontal (that is, non-vertical) band, that does not passing through the point $p$. Denote by $C_B$ the leftmost bounded chamber in $B$ (e.g., in Figure 3, $C_{B_1} = C_1, C_{B_2} = C_3, C_{B_3} = C_4$ and $C_{B_4} = C_6$).

First, consider the case where the multiplicity of $c\mathcal{A}$ at $p$ is not divisible by $k$. Then all $k$-resonant bands are horizontal. Let $B \in \mathrm{RB}_k(\mathcal{A})$. Then

$$\nabla(B) = 2i \sin\left(\frac{\pi}{k}\right) \cdot [C_B] + \cdots , \tag{6}$$

and so $[C_B]$ has a non-zero coefficient. Since $C_B$ is contained in the unique $k$-resonant band $B$, $[C_B]$ does not appear in the linear combinations of other $k$-resonant bands. Hence,

$$\nabla(B), B \in \mathrm{RB}_k(\mathcal{A})$$

are linearly independent. Thus (ii) is proved.

Now we assume that the multiplicity of $\mathcal{A}$ at $p$ is divisible by $k$. In this case, there are vertical $k$-resonant bands. Denote by $B_{left}$ the leftmost vertical band (in Figure 3, $B_{left} = B_5$). Suppose that

$$c_{left} \cdot B_{left} + \cdots \in \operatorname{Ker}(\nabla).$$

Let $B \in \mathrm{RB}_k(\mathcal{A})$ be a horizontal $k$-resonant band. Then, since $C_B$ is contained in only $B$ and $B_{left}$, the coefficient $c_{left}$ of $B_{left}$ determines the coefficient of $B$. The coefficients of other vertical $k$-resonant bands are also determined by those of the horizontal bands. Hence $\operatorname{Ker}(\nabla)$ is at most one-dimensional. $\square$

**Example 3.22.** Let $\mathcal{A}$ be as in Figure 4, with $|c\mathcal{A}| = 12$. Let $k = 3$. Then

$$\mathrm{RB}_3 = \{B_1^1, B_2^1, B_3^1, B_4^1, B_1^2, B_2^2, B_3^2, B_4^2\}$$

contains eight bands. Suppose that $\sum_{i=1}^{2}\sum_{j=1}^{4} c_{ij}[B_j^i] \in \operatorname{Ker}(\nabla)$. By computing

$$\sum_{i=1}^{2}\sum_{j=1}^{4} c_{ij}\nabla(B_j^i),$$

as in the figure, we conclude that all the coefficients are $c_{ij} = 0$. Hence $H^1(F_{\mathcal{A}})_\lambda = 0$ for $\lambda = e^{2\pi i/3}$. Note that the multiple points on the diagonal line are triple points. If we put the diagonal line at infinity, then $\mathrm{RB}_2 = \mathrm{RB}_4 = \mathrm{RB}_6 = \emptyset$. Therefore,

$$H^1(F_{\mathcal{A}})_{-1} = H^1(F_{\mathcal{A}})_i = H^1(F_{\mathcal{A}})_{e^{2\pi i/6}} = 0$$

by Corollary 3.16.

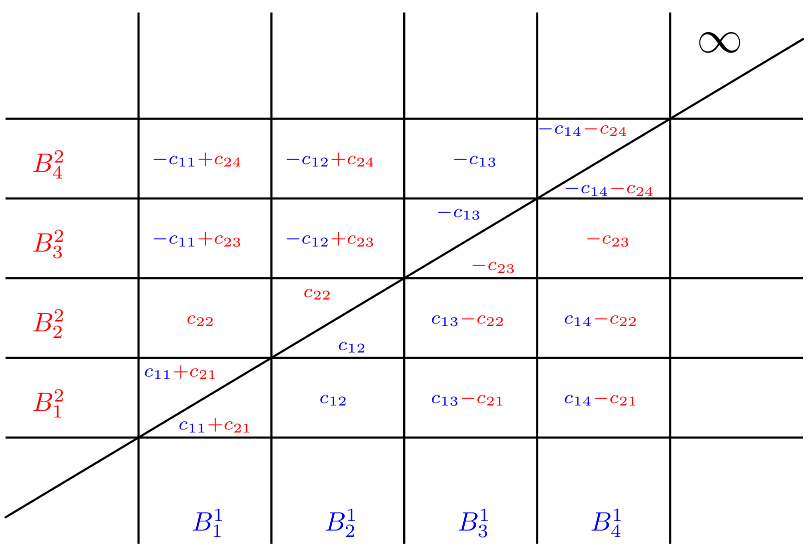



FIGURE 4. Example 3.22

3.5. **A characterization of the $A_3$-arrangement.** Now we give a characterization of the $A_3$-arrangement in terms of non-trivial Milnor monodromy.

**Theorem 3.23.** *Assume that $H^1(F_{\mathcal{A}})_\lambda \neq 0$ with $\lambda = e^{2\pi i/k} \neq 1$, and that the set of $k$-resonant bands $\mathrm{RB}_k(\mathcal{A})$ consists of at most two directions (this condition is equivalent to $H_\infty$ containing at most two multiple points which have multiplicities divisible by $k$). Then $c\mathcal{A}$ is equivalent to the $A_3$-arrangement.*

*Proof.* If $\mathrm{RB}_k(\mathcal{A})$ consists of one direction, then by Theorem 3.14, $H^1(F_{\mathcal{A}})_\lambda = 0$. Thus we may assume that $\mathrm{RB}_k(\mathcal{A})$ consists of two directions. After a suitable change of coordinates, we assume the following (see Figure 5):

- $\mathrm{RB}_k(\mathcal{A}) = \{B_1^1, B_2^1, \dots, B_p^1, B_1^2, B_2^2, \dots, B_q^2\}$.
- $B_1^1, B_2^1, \dots, B_p^1$ are parallel to the vertical line $x = 0$ and may be expressed as
$$B_i^1 = \{(x, y) \in \mathbb{R}^2 \mid a_i < x < a_{i+1}\}$$
with $a_1 < \cdots < a_{p+1}$. The lines $H_i^1 = \{x = a_i\}$, $i = 1, \dots, p+1$, which are vertical lines, are boundaries of these bands.
- $B_1^2, B_2^2, \dots, B_q^2$ are parallel to the horizontal line $y = 0$ and may be expressed as
$$B_i^2 = \{(x, y) \in \mathbb{R}^2 \mid b_i < y < b_{i+1}\}$$
with $b_1 < \cdots < b_{q+1}$. The lines $H_i^2 = \{y = b_i\}$, $i = 1, \dots, q+1$, which are horizontal lines, are boundaries of these bands.
- Let $\sum_{i=1}^{p} c_{1i} \cdot B_i^2 + \sum_{i=1}^{q} c_{2i} \cdot B_i^2 \in \mathrm{Ker}(\nabla)$ be a non-trivial relation among $k$-resonant bands.

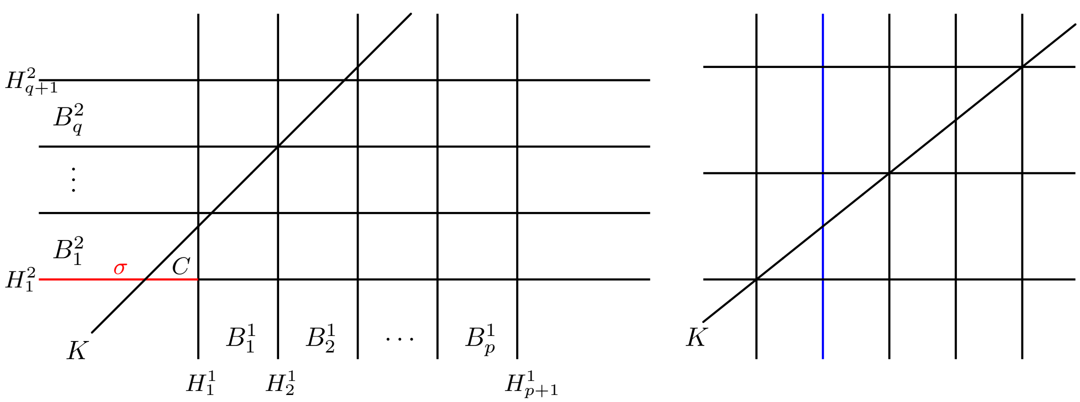



FIGURE 5. Proof of Theorem 3.23

The multiplicity of $c\mathcal{A}$ at $(0:1:0) = \overline{B_i^1} \cap H_\infty$ is $p+2$, which must be divisible by $k$ by Proposition 3.11. Hence $p$ can be expressed as $p = ks - 2$ ($s \in \mathbb{Z}_{>0}$). Similarly, set $q = kt - 2$ ($t \in \mathbb{Z}_{>0}$). So far, together with $H_\infty$, we have $(s+t)k-1$ lines. The remaining $n+1-(s+t)k+1$ can be expressed as $ku+1$, which in particular cannot be zero. We prove that (1) $u = 0$, (2) $k = 3$, (3) $p = q$, (4) $p = q = 1$, and conclude that $c\mathcal{A}$ is the $A_3$-arrangement.

(1) We first prove that $u = 0$. Consider the open segment
$$\sigma = \{(x:y:1) \in \mathbb{R}^2 \mid y = b_1, -\infty < x < a_1\} \subset H_1^2$$

which is bounded by the two points $(1:0:0)$ and $(a_1:b_1:1)$. (See Figure 5.) Let us prove that there are no intersections on $\sigma$. Suppose that the line $K \in \mathcal{A}$ intersects $\sigma$. The leftmost chamber $C$ in $B_1^2$ is not contained in the other $k$-resonant bands and satisfies $d(U_1(B_1^2), C) = 1$. Since

$$\nabla(B_1^1) = 2i\sin\left(\frac{\pi}{k}\right)\cdot[C] + \cdots,$$

and the coefficient of $[C]$ is non-zero, we have $c_{21} = 0$. This implies that

$$c_{11} = c_{12} = \cdots = c_{1p} = 0.$$

Then we have $c_{22} = \cdots = c_{2q} = 0$. This contradicts the hypothesis that $H^1(F_{\mathcal{A}})_\lambda \neq 0$. This contradiction proves that there are no intersections on the segment $\sigma$. Similarly there are no intersections on the seven other similar segments, that is, the boundaries of the four regions

$$\{(x:y:1) \mid x < a_1, y < b_1\},\ \{(x:y:1) \mid x < a_1, y > b_q\},$$
$$\{(x:y:1) \mid x > a_p, y < b_1\},\ \{(x:y:1) \mid x > a_p, y > b_q\}.$$

Thus $K \in \mathcal{A}$ must be one of the two diagonals

$$K_1 = \text{the line connecting } (a_1:b_1:1) \text{ and } (a_p:b_q:1),$$
$$K_2 = \text{the line connecting } (a_p:b_1:1) \text{ and } (a_1:b_q:1).$$

Hence $ku+1 \leq 2$, and we have $u=0$.

(2) Now we prove $k=3$. Using the above notation, we may assume that

$$\mathcal{A} = \{H_1^1, \dots, H_{ks-1}^1, H_1^2, \dots, H_{kt-1}^2, K\},$$

where $K$ is the diagonal line connecting $(a_1:b_1:1)$ and $(a_p:b_q:1)$. Then the point $(a_1:b_1:1)$ has multiplicity 3. The line $H_1^1$ has exactly two multiple points, $(a_1:b_1:1)$ and $(0:1:0)$. By Corollary 3.16, $k$ is a common divisor of 3 and the multiplicity of $(0:1:0)$. Since $k \neq 1$, we have $k=3$.

(3) If $p \neq q$, then there exists a (either vertical or horizontal) line which intersects the diagonal line $K$ normally (that is, with multiplicity 2, the right-hand side of Figure 5). Then the line has only one multiple point on $H_\infty$ (either $(0:1:0)$ or $(1:0:0)$). This contradicts Corollary 3.16. Hence $p=q$.

(4) If $p=q>1$, then we can prove that $H^1(F_{\mathcal{A}})_{e^{2\pi i/3}} = 0$ by an argument similar to Example 3.22. Hence $p=q=1$. This obviously implies that $c\mathcal{A}$ is isomorphic to the $A_3$-arrangement. □

**Corollary 3.24.** *Assume that $\mathcal{A}$ is a real arrangement as above, and assume that*

$$|c\mathcal{A}| = n+1 \geq 7.$$

*If $H^1(F_{\mathcal{A}})_\lambda \neq 0$, then each line $H \in c\mathcal{A}$ passes through at least three multiple points which have multiplicities divisible by $k$.*

**Remark 3.25.** We do not know whether Theorem 3.23 and Corollary 3.24 hold for complex arrangements.

## 4. Examples and Conjectures

By the previous result (Corollary 3.24), the Milnor fiber cohomology has non-trivial eigenspaces only when each line has at least three multiple points. Classes of line arrangements known as "simplicial arrangements" and "configurations" provide such examples. In this section, we present examples of non-trivial eigenspaces $H^1(F_{\mathcal{A}})_{\neq 1} \neq 0$.

4.1. **Observation.** As far as the author knows, all examples of real arrangements with

$$H^1(F_{\mathcal{A}})_{\neq 1} \neq 0$$

have the following "pure-tone" property, that is, only the third root of 1 appears with multiplicity one.

**Definition 4.1.** $\mathcal{A}$ is said to be *pure-tone* if $H^1(F_{\mathcal{A}})_\lambda = 0$ for $\lambda^3 \neq 1$ and $\dim H^1(F_{\mathcal{A}})_\lambda = 1$ for $\lambda = e^{\pm 2\pi i/3}$.

Furthermore, it is observed that all known examples with $H^1(F_{\mathcal{A}})_{\neq 1} \neq 0$ satisfy

- $c\mathcal{A}$ has a sharp pair of lines,
- $c\mathcal{A}$ has a $k$-multinet structure with $k = 3$.

These two properties imply by Theorem 3.21 and Theorem 2.3 that $\mathcal{A}$ is pure-tone.

4.2. **Simplicial arrangements.** Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a line arrangement in $\mathbb{R}^2$. Then the projective arrangement $c\mathcal{A} = \mathcal{A} \cup \{H_\infty\}$ in the real projective plane $\mathbb{RP}^2$ is called *simplicial* if each chamber is a triangle. Grünbaum [8] presents a catalogue of known simplicial arrangements with up to 37 lines (see [5] for additional information).

**Notation.** The symbol $\infty$ in a figure indicates that the $(n+1)$-st line is $H_\infty$. The notation $\mathcal{A}(n, k)$ comes from [8], which is the $k$-th simplicial arrangement of $n$-lines.

Example 3.7 can be generalized in two ways.

**Definition 4.2.** For a positive integer $n \in \mathbb{Z}_{>0}$, $\mathcal{A}(2n, 1)$ is described as follows. Starting with a regular convex $n$-gon in the Euclidean plane, $\mathcal{A}(2n, 1)$ is obtained by taking $n$ lines determined by the sides of the $n$-gon together with the $n$-lines of symmetry of that $n$-gon. $\mathcal{A}(2n, 1)$ is a simplicial arrangement of $2n$-lines.

Obviously, the $A_3$-arrangement is equivalent to $\mathcal{A}(6, 1)$.

**Example 4.3.** Let $c\mathcal{A} = \mathcal{A}(12, 1)$ (Figure 6). Then $\mathrm{RB}_3(\mathcal{A}) = \{B_1, \dots, B_7\}$.

$$\begin{array}{rlllllllllll}
\nabla(B_2) = & \sqrt{-3}(C_1 & & +C_3 & & -C_5 & -C_6 & & +C_8 & & +C_{10}) \\
\nabla(B_3) = & \sqrt{-3}(C_1 & & +C_3 & & & & -C_7 & & -C_9 & ) \\
\nabla(B_6) = & \sqrt{-3}( & C_2 & & +C_4 & & & & -C_8 & & -C_{10}) \\
\nabla(B_7) = & \sqrt{-3}( & C_2 & & +C_4 & -C_5 & -C_6 & +C_7 & & +C_9 & )
\end{array}$$

Hence we have a linear relation

$$\nabla(B_2) - \nabla(B_3) + \nabla(B_6) - \nabla(B_7) = 0,$$

and so we have that $\mathcal{A}(12, 1)$ is pure-tone.

More generally, using Theorem 2.3 and Theorem 3.21, we can prove that $\mathcal{A}(6m, 1)$ is pure-tone. All other examples except for $\mathcal{A}(6m, 1)$ in the catalogue [8] (and [5]) satisfy $H^1(F_{\mathcal{A}})_{\neq 1} = 0$. It seems natural to pose the following.

**Conjecture 4.4.** *Assume that $c\mathcal{A}$ is a simplicial arrangement. Then the following are equivalent.*

(a) $c\mathcal{A} = \mathcal{A}(6m, 1)$ *for some* $m > 0$.
(b) $H^1(F_{\mathcal{A}})_{\neq 1} \neq 0$.
(c) $\mathcal{A}$ *is pure-tone.*
(d) $c\mathcal{A}$ *has a $k$-multinet structure for some* $k \geq 3$.
(e) $c\mathcal{A}$ *has a 3-multinet structure.*

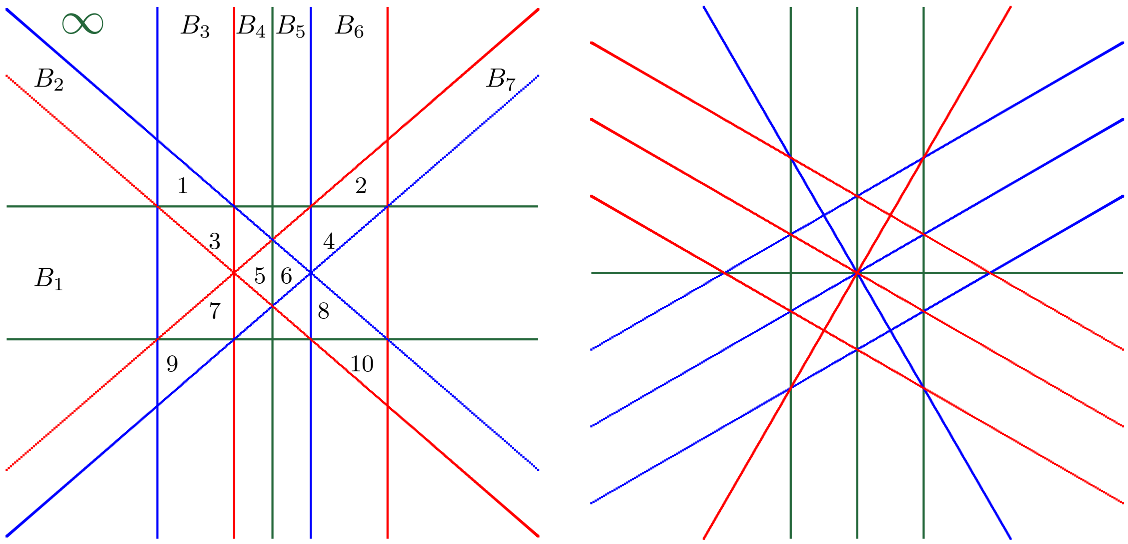



FIGURE 6. $\mathcal{A}(12, 1)$

### 4.3. **Zoo of non-trivial eigenspaces.**

**Example 4.5.** Let $c\mathcal{A}$ be the Pappus arrangement (Figure 7), so that $|c\mathcal{A}| = n + 1 = 9$. Let $k = 3$. Then $\mathrm{RB}_3(\mathcal{A}) = \{B_1, B_2, B_3\}$. By the expressions

$$\begin{array}{llllllllll}
\nabla(B_1) = & \sqrt{-3}(C_1 & & +C_3 & & & -C_9 & & -C_{11}) \\
\nabla(B_2) = & \sqrt{-3}(C_1 & +C_2 & +C_3 & +C_4 & -C_8 & -C_9 & -C_{10} & -C_{11}) \\
\nabla(B_3) = & \sqrt{-3}( & C_2 & & +C_4 & -C_8 & & -C_{10} & )
\end{array}$$

there is a unique relation $\nabla(B_1) - \nabla(B_2) + \nabla(B_3) = 0$. Hence the Pappus arrangement is pure-tone.

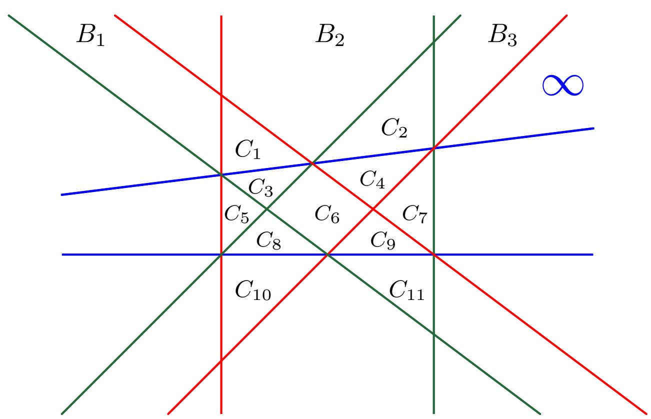



FIGURE 7. Pappus arrangement (Example 4.5)

**Example 4.6.** (Taken from [9, page 244].) Let $\mathcal{A}$ be as in the right-hand side of Figure 8. Then $|c\mathcal{A}| = n + 1 = 12$ and $\mathrm{RB}_3(\mathcal{A}) = \{B_1, \ldots, B_5\}$. There is a unique linear relation

$$\nabla(B_1) - \nabla(B_2) + \nabla(B_3) - \nabla(B_4) = 0$$

($B_5$ does not appear). Hence $\mathcal{A}$ is pure-tone.

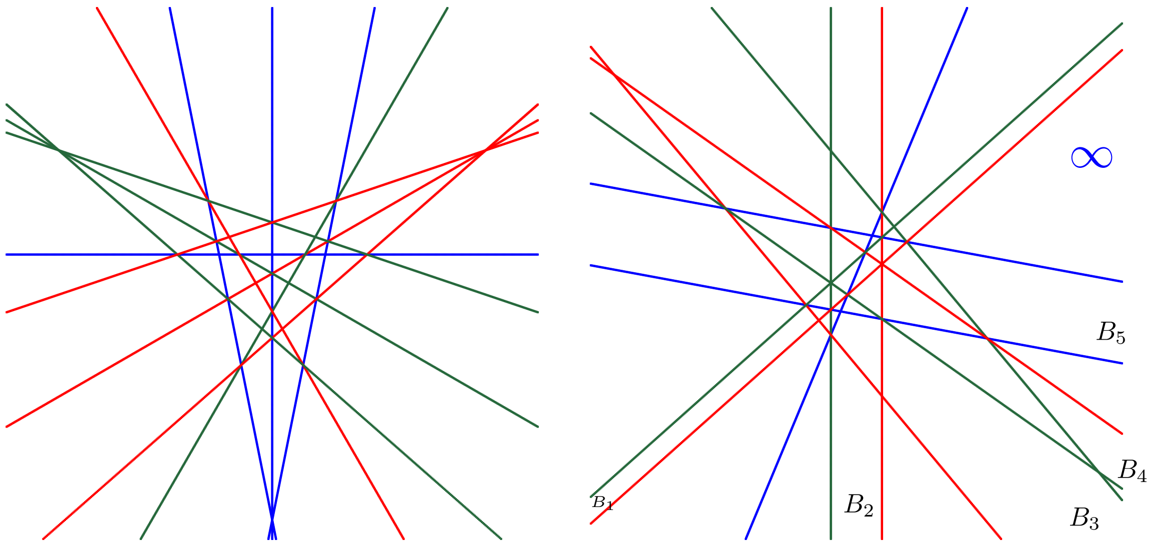



Figure 8. Example 4.6

**Example 4.7.** (Taken from [9, page 244].) Let $\mathcal{A}$ be as in the right-hand side of Figure 9. Then $|c\mathcal{A}| = n + 1 = 12$ and $\mathrm{RB}_3(\mathcal{A}) = \{B_1, \ldots, B_4\}$. There is a unique linear relation

$$\nabla(B_1) - \nabla(B_2) + \nabla(B_3) - \nabla(B_4) = 0.$$

Hence $\mathcal{A}$ is pure-tone.

**Example 4.8.** (Taken from [9, page 44].) Let $\mathcal{A}$ be as in the right-hand side of Figure 10. Then $|c\mathcal{A}| = n + 1 = 15$ and $\mathrm{RB}_3(\mathcal{A}) = \{B_1, \ldots, B_7\}$. There is a unique linear relation

$$\nabla(B_1) - \nabla(B_3) + \nabla(B_4) - \nabla(B_6) + \nabla(B_7) = 0$$

($B_2$ and $B_5$ do not appear). Hence $\mathcal{A}$ is pure-tone.

**Definition 4.9.** For a positive integer $m \in \mathbb{Z}_{>0}$, $\mathcal{B}_{3m}$ is described as follows. Starting with a regular convex $2m$-gon in the Euclidean plane, $\mathcal{B}_{3m}$ is obtained by taking $2m$ lines determined by the sides of the $2m$-gon together with $m$-diagonal lines connecting opposite vertices. (Note that $\mathcal{B}_6$ is equivalent to the $A_3$-arrangement, see Figure 2.)

**Example 4.10.** Using Theorem 2.3 and Theorem 3.21, we can prove that the $\mathcal{B}_{3m}$-arrangement is pure-tone.

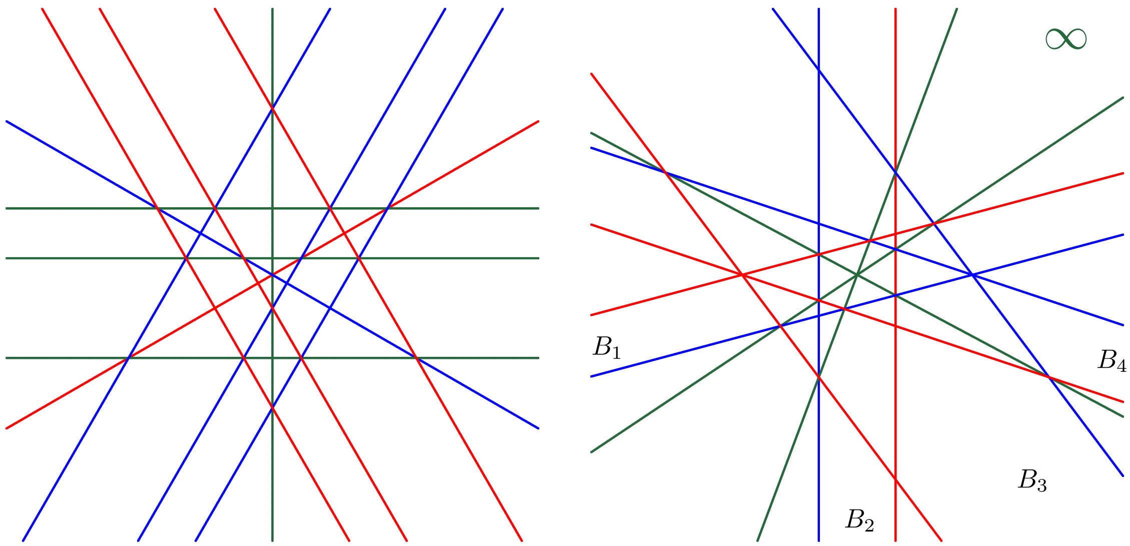



FIGURE 9. Example 4.7

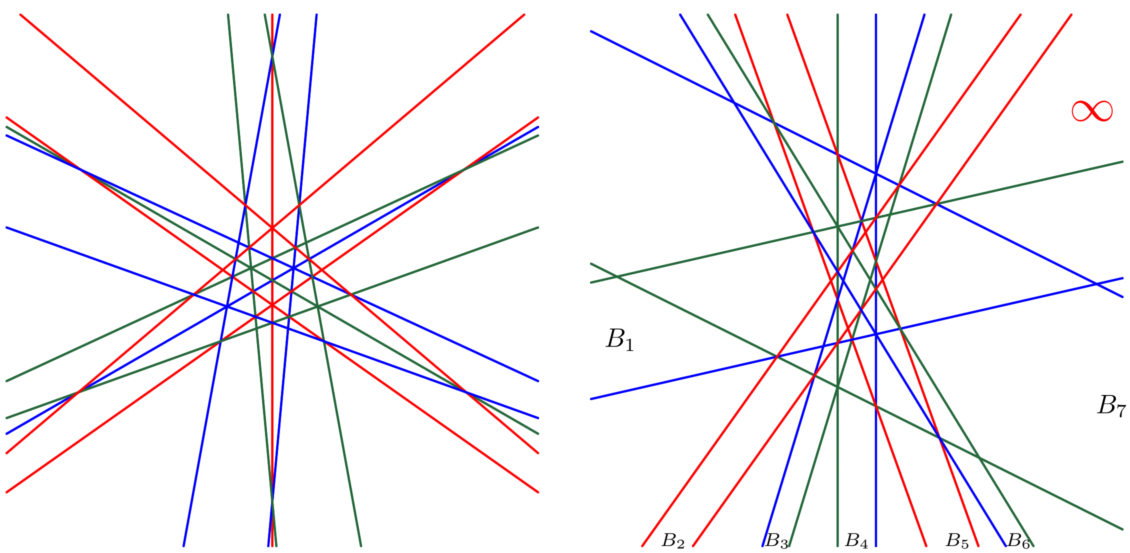



FIGURE 10. Example 4.8

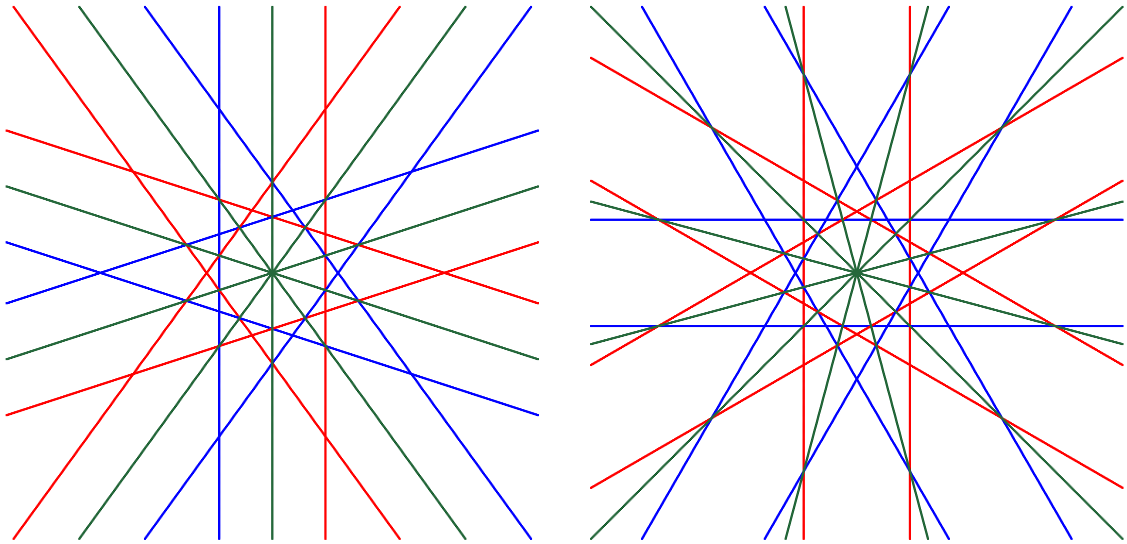


FIGURE 11. $\mathcal{B}_{15}$ and $\mathcal{B}_{18}$

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Department of Mathematics, Hokkaido University, North 10, West 8, Kita-ku, Sapporo 060-0810, JAPAN E-mail: yoshinaga@math.sci.hokudai.ac.jp

**Journal *of* Singularities**
**Volume 7 (2013), 238-252**

received 24 April 2013
in revised form 7 August 2013
DOI: 10.5427/jsing.2013.7m

# ARC SPACES OF $cA$-TYPE SINGULARITIES

JENNIFER M. JOHNSON AND JÁNOS KOLLÁR

Let $X$ be a complex variety or an analytic space and $x \in X$ a point. A formal arc through $x$ is a morphism $\phi : \operatorname{Spec} \mathbb{C}[[t]] \to X$ such that $\phi(0) = x$. The set of formal arcs through $x$ – denoted by $\widehat{\operatorname{Arc}}(x \in X)$ – is naturally a (non-noetherian) scheme.

A preprint of Nash, written in 1968 but only published later as [Nas95], describes an injection – called the *Nash map* – from the irreducible components of $\widehat{\operatorname{Arc}}(x \in X)$ to the set of so called *essential divisors.* These are the divisors whose center on every resolution $\pi : X' \to X$ is an irreducible component of $\pi^{-1}(x)$. The *Nash problem* asks if this map is also surjective or not. Surjectivity fails in dimensions $\geq 3$ [IK03, dF12] but holds in dimension 2 [FdBP12b].

In all dimensions, the most delicate cases are singularities whose resolutions contain many rational curves. For example, although it is easy to describe all arcs and their deformations on Du Val singularities of type $A$, the type $E$ cases have been notoriously hard to treat [PS12, Per13].

The first aim of this note is to determine the irreducible components of the arc spaces of $cA$-type singularities in all dimensions. In Section 1 we prove the following using quite elementary arguments.

**Theorem 1.** *Let $f(z_1, \dots, z_n)$ be a holomorphic function whose multiplicity at the origin is $m \geq 2$. Let $X := \big(xy = f(z_1, \dots, z_n)\big) \subset \mathbb{C}^{n+2}$ denote the corresponding cA-type singularity. Assume that $n \geq 1$.*

1. *$\widehat{\operatorname{Arc}}(0 \in X)$ has $(m-1)$ irreducible components $\widehat{\operatorname{Arc}}_i(0 \in X)$ for $0 < i < m$.*
2. *There are dense, open subsets $\widehat{\operatorname{Arc}}_i^{\circ}(0 \in X) \subset \widehat{\operatorname{Arc}}_i(0 \in X)$ such that*

$$\big(\psi_1(t), \psi_2(t), \phi_1(t), \dots, \phi_n(t)\big) \in \widehat{\operatorname{Arc}}_i^{\circ}(0 \in X)$$

*iff* $\operatorname{mult} \psi_1(t) = i$, $\operatorname{mult} \psi_2(t) = m - i$ *and* $\operatorname{mult} f\big(\phi_1(t), \dots, \phi_n(t)\big) = m$.

We found it much harder to compute the set of essential divisors and we have results only if $\operatorname{mult}_0 f = 2$. If $\dim X = 3$ then, after a coordinate change, we can write the equation as $(xy = z^2 - u^m)$. Already [Nas95] proved that these singularities have at most 2 essential divisors: an easy one obtained by blowing-up the origin and a difficult one obtained by blowing-up the origin twice. In Section 2 we use ideas of [dF12] to determine the cases when the second divisor is essential. The following is obtained by combining Theorem 1 and Proposition 9.

**Example 2.** For the singularities $X_m := (xy = z^2 - u^m) \subset \mathbb{C}^4$ the Nash map is not surjective for odd $m \geq 5$ but surjective for even $m$ and for $m = 3$.

Thus the simplest counter example to the Nash conjecture is the singularity

$$(x^2 + y^2 + z^2 + t^5 = 0) \subset \mathbb{C}^4.$$

In higher dimensions our answers are less complete. We describe the situation for the divisors obtained by the first and second blow-ups as above, but we do not control other exceptional divisors. Using Theorem 1 and Proposition 22 we get the following partial generalization of Example 2.

**Example 3.** Let $g(u_1, \ldots, u_r)$ be an analytic function near the origin. Set $m = \operatorname{mult}_0 g$ and let $g_m$ denote the degree $m$ homogeneous part of $g$. If $m \geq 4$ and the Nash map is surjective for the singularity

$$X_g := \bigl(xy = z^2 - g(u_1, \ldots, u_r)\bigr) \subset \mathbb{C}^{r+3}$$

then $g_m(u_1, \ldots, u_r)$ is a perfect square.

Since we do not determine all essential divisors, the cases when $g_m(u_1, \ldots, u_r)$ is a perfect square remain undecided.

On the one hand, this can be interpreted to mean that the Nash conjecture hopelessly fails in dimensions $\geq 3$. On the other hand, the proof leads to a reformulation of the Nash problem and to an approach that might be feasible, at least in dimension 3; see Section 5.

In Section 4 we observe that the deformations constructed in Section 1 also lead to an enumeration of the irreducible components of the space of short arcs – introduced in [KN13] – for $cA$-type singularities.

**Question 4** (Arcs on $cDV$ singularities)**.** It is easy to see that Theorem 1 is equivalent to saying that the image of every general arc on $X$ is contained in an $A$-type surface section of $X$.

It is natural to ask if this holds for all $cDV$ singularities. That is, let $(0 \in X) \subset \mathbb{C}^n$ be a hypersurface singularity such that $X \cap L^3$ is a Du Val singularity for every general 3-dimensional linear space (or smooth 3–fold) $0 \in L^3 \subset \mathbb{C}^n$.

Let $\phi$ be a general arc on $X$. Is it true that there is a 3–fold $L^3 \subset \mathbb{C}^n$ containing the image of $\phi$ such that $X \cap L^3$ is a Du Val singularity?

**Acknowledgments.** We thank V. Alexeev, T. de Fernex, R. Lazarsfeld, C. Plénat and M. Spivakovsky for corrections and helpful discussions. Partial financial support to JK was provided by the NSF under grant number DMS-07-58275 and by the Simons Foundation. Part of the paper was written while the authors visited Stanford University.

## 1. Arcs on $cA$-type singularities

**Definition 5** ($cA$-type singularities)**.** In some coordinates write a hypersurface singularity as

$$X := \bigl(f(x_1, \ldots, x_{n+1}) = 0\bigr) \subset \mathbb{C}^{n+1}.$$

Assume that $X$ is singular at the origin and let $f_2$ denote the quadratic part of $f$. If $\operatorname{mult}_0 f = 2$ then $(f_2 = 0)$ is the tangent cone of $X$ at the origin. We say that $X$ has *$cA$-type* if $\operatorname{rank} f_2 \geq 2$ and *$cA_1$-type* if $\operatorname{rank} f_2 \geq 3$. By the Morse lemma, if $\operatorname{rank} f_2 = r$ then we can choose local analytic or formal coordinates $y_i$ such that

$$f = y_1^2 + \cdots + y_r^2 + g(y_{r+1}, \ldots, y_{n+1}) \quad \text{where} \quad \operatorname{mult}_0 g \geq 3.$$

In the sequel we also use other forms of the quadratic part if that is more convenient.

Note that by adding 2 squares in new variables we get a map from hypersurface singularities in dimension $n-2$ (modulo isomorphism) to $cA$-type hypersurface singularities in dimension $n$ (modulo isomorphism). This map is one-to-one and onto; see [AGZV85, Sec.11.1]. Thus $cA$-type singularities are quite complicated in large dimensions.

We rename the coordinates and write a $cA$-type singularity as

$$X := \bigl(xy = f(z_1, \ldots, z_n)\bigr) \subset \mathbb{C}^{n+2}.$$

Thus an arc through the origin is written as

$$t \mapsto \bigl(\psi_1(t), \psi_2(t), \phi_1(t), \ldots, \phi_n(t)\bigr),$$

where $\psi_i, \phi_j$ are power series such that $\operatorname{mult} \psi_i, \operatorname{mult} \phi_j \geq 1$ for $i = 1, 2$ and $j = 1, \ldots, n$. We set $\vec{\phi}(t) = \big(\phi_1(t), \ldots, \phi_n(t)\big)$.

A deformation of $\vec{\phi}(t)$ is given by power series $\big(\Phi_1(t,s), \ldots, \Phi_n(t,s)\big)$. Then we compute

$$f\big(\Phi_1(t,s), \ldots, \Phi_n(t,s)\big) \in \mathbb{C}[[t,s]]$$

and try to factor it as

$$\Psi_1(t,s)\Psi_2(t,s) = f\big(\Phi_1(t,s), \ldots, \Phi_n(t,s)\big)$$

where $\Psi_i(t,0) = \psi_i(t)$. Usually $f\big(\Phi_1(t,s), \ldots, \Phi_n(t,s)\big)$ is irreducible, but Newton's method of rotating rulers (Lemma 7 below) says that

$$f\big(\Phi_1(t,s^r), \ldots, \Phi_n(t,s^r)\big)$$

factors for some $r \geq 1$.

**6** (Proof of Theorem 1)**.** If $\vec{\phi}(0) = \mathbf{0}$ then $\operatorname{mult} f\big(\vec{\phi}(t)\big) \geq m$. Thus, for every $0 < i < m$ we can choose any $\psi_1(t)$ such that $\operatorname{mult} \psi_1(t) = i$ and then set $\psi_2(t) = \psi_1(t)^{-1} f\big(\vec{\phi}(t)\big)$. This shows that the families $\widehat{\operatorname{Arc}}_i^{\circ}(0 \in X)$ are nonempty and open in $\widehat{\operatorname{Arc}}(0 \in X)$.

In order to show that their union is dense, after a linear change of coordinates we may assume that $z_1^m$ appears in $f$ with nonzero constant coefficient.

Set $D := \operatorname{mult}_t f\big(\phi_1(t), \ldots, \phi_n(t)\big)$. Assume first that $D < \infty$ and consider

$$F(t,s) := f\big(\phi_1(t) + st, \phi_2(t), \ldots, \phi_n(t)\big) = \sum_i \frac{\partial^i f}{\partial z_1^i}\big(\vec{\phi}\big) \cdot \frac{(st)^i}{i!}.$$

We know that $t^m$ divides $F(s,t)$ (since $\operatorname{mult}_0 f = m$) and $(st)^m$ appears in $F$ with nonzero coefficient (since $z_1^m$ appears in $f$ with nonzero coefficient). Thus $t^m$ is the largest $t$-power that divides $F(s,t)$.

Furthermore, $t^D$ is the smallest $t$-power that appears in $F$ with nonzero constant coefficient. Thus, by Lemma 7 below, there is an $r \geq 1$ such that

$$F(t,s^r) = u(t,s) \prod_{i=1}^{D} \big(t - \sigma_i(s)\big),$$

where $u(0,0) \neq 0$ and $\sigma_i(0) = 0$. Furthermore, exactly $m$ of the $\sigma_i$ are identically zero.

For $j = 1, 2$ write $\psi_j(t) = t^{a_j} v_j(t)$ where $v_j(0) \neq 0$. Note that $a_1 + a_2 = D$ and

$$u(t,0) = v_1(t) v_2(t).$$

Divide $\{1, \ldots, D\}$ into two disjoint subsets $A_1, A_2$ such that $|A_j| = a_j$ and they both contain at least 1 index $i$ such that $\sigma_i(t) \equiv 0$. Finally set

$$\Psi_1(t,s) = v_1(t) \cdot \prod_{i \in A_1} \big(t - \sigma_i(s)\big) \quad \text{and} \quad \Psi_2(t,s) = \frac{u(t,s)}{v_1(t)} \cdot \prod_{i \in A_2} \big(t - \sigma_i(s)\big).$$

Then

$$\big(\Psi_1(t,s), \Psi_2(t,s), \phi_1(t) + st, \phi_2(t), \ldots, \phi_n(t)\big)$$

is a deformation of $\big(\psi_1(t), \psi_2(t), \phi_1(t), \ldots, \phi_n(t)\big)$ whose general member is in the $r$th irreducible component as in Theorem 1.2 iff exactly $r$ of the $\{\sigma_i : i \in A_1\}$ are identically zero.

(This also shows that arcs with $\operatorname{mult} \psi_1(t) \geq m-1$ and $\operatorname{mult} \psi_2(t) \geq m-1$ constitute the intersection of all of the $\widehat{\operatorname{Arc}}_i(0 \in X)$.)

If $D = \infty$, that is, when $f\big(\phi_1(t), \ldots, \phi_n(t)\big)$ is identically zero, we need to perform some similar preliminary deformations first.

First, if both $\psi_1(t), \psi_2(t)$ are identically zero then we can take

$$\big(st, 0, \phi_1(t), \phi_2(t), \dots, \phi_n(t)\big).$$

Hence, up-to interchanging $x$ and $y$, we may assume that $d := \operatorname{mult} \psi_1(t) < \infty$. Again assuming that $z_1^m$ appears in $f$ with nonzero coefficient, we see that

$$F(t,s) := f\big(\phi_1(t) + st^{d+1}, \phi_2(t), \dots, \phi_n(t)\big)$$

is not identically zero and divisible by $t^{d+1}$. Thus $F(t,s)/\psi_1(t)$ is holomorphic and divisible by $t$. Therefore

$$\Big(\psi_1(t), \frac{F(t,s)}{\psi_1(t)}, \phi_1(t) + st^{d+1}, \phi_2(t), \dots, \phi_n(t)\Big)$$

is a deformation of $\big(\psi_1(t), 0, \phi_1(t), \phi_2(t), \dots, \phi_n(t)\big)$ such that

$$\operatorname{mult}_t f\big(\phi_1(t) + st^{d+1}, \phi_2(t), \dots, \phi_n(t)\big) < \infty$$

for $0 < |s| \ll 1$. □

We used Newton's lemma on Puiseux series solutions in the following form.

**Lemma 7.** *Let $g(x,y) \in \mathbb{C}[[x,y]]$ be a power series. Assume that $m := \operatorname{mult}_0 g(x,0) < \infty$. Then there is an $r \geq 1$ such that one can write $g(x, z^r)$ as*

$$g(x, z^r) = u(x,z) \prod_{i=1}^{m} \big(x - \sigma_i(z)\big)$$

*where $u(0,0) \neq 0$ and $\sigma_i(0) = 0$ for every $i$. The representation is unique, up-to permuting the $\sigma_i(z)$.*

*Furthermore, if $g(x,y)$ is holomorphic on the bidisc $\overline{\mathbb{D}}_x \times \mathbb{D}_y$ then $u(x,z)$ and the $\sigma_i(z)$ are holomorphic on the smaller bidisc $\overline{\mathbb{D}}_x \times \mathbb{D}_z(\epsilon)$ for some $0 < \epsilon \leq 1$.* □

## 2. Essential divisors on $cA_1$-type 3-fold singularities

In dimension 3, the only $cA_1$-type singularities are $X_m := (xy = z^2 - t^m)$ for $m \geq 2$. Already [Nas95, p.37] proved that they have at most 2 essential divisors. We use the method of [dF12, 4.1] to determine the precise count.

**Definition 8.** Let $X$ be a normal variety or analytic space and $E$ a divisor over $X$. That is, there is a birational or bimeromorphic morphisms $p : X' \to X$ such that $E \subset X'$ is an exceptional divisor. The closure of $p(E) \subset X$ is called the *center* of $E$ on $X$; it is denoted by $\operatorname{center}_X E$. If $\operatorname{center}_X E = \{x\}$, we say that $E$ is a divisor over $(x \in X)$.

We say that $E$ is an *essential divisor* over $X$ if for every resolution of singularities $\pi : Y \to X$, $\operatorname{center}_Y E$ is an irreducible component of $\pi^{-1}\big(\operatorname{center}_X E\big)$. (Note that $\pi^{-1} \circ p : X' \dashrightarrow Y$ is regular on a dense subset of $E$, hence $\operatorname{center}_Y E$ is defined.)

If $X$ is an analytic space, then $Y$ is allowed to be any analytic resolution. If $X$ is algebraic, one gets slightly different notions depending on whether one allows $Y$ to be a quasi-projective variety, an algebraic space or an analytic space; see [dF12]. We believe that for the Nash problem it is natural to allow analytic resolutions.

**Proposition 9.** *Set $X_m := (xy = z^2 - t^m) \subset \mathbb{C}^4$.*

(1) *If $m \geq 5$ is odd, there are 2 essential divisors.*
(2) *If $m \geq 2$ is even or $m = 3$, there is 1 essential divisor.*

Even in dimension 3, it seems surprisingly difficult to determine the set of essential divisors. A basic invariant is given by the discrepancy.

**Definition 10.** Let $X$ be a normal variety or analytic space. Assume for simplicity that the canonical class $K_X$ is Cartier. (This holds for all hypersurface singularities.) Let $\pi : Y \to X$ be a resolution of singularities and write

$$K_Y \sim \pi^* K_X + \sum_i a(E_i, X) E_i,$$

where the $E_i$ are the $\pi$-exceptional divisors. The integer $a(E_i, X)$ is called the *discrepancy* of $E_i$. (See [KM98, Sec.2.3] for basic references and more general definitions.)

For example, let $X$ be smooth and $Z \subset X$ a smooth subvariety of codimension $r$. Let $\pi_Z : B_Z X \to X$ denote the blow-up and $E_Z \subset B_Z X$ the exceptional divisor. Then $a(E_Z, X) = r - 1$ and easy induction shows that $a(F, X) \geq r$ for every other divisor whose center on $X$ is $Z$.

We say that $X$ is *canonical* (resp. *terminal*) of $a(E_i, X) \geq 0$ (resp. $a(E_i, X) > 0$) for every resolution and every exceptional divisor.

For instance, normal $cA$-type singularities are canonical and a $cA$-type singularity is terminal iff its singular set has codimension $\geq 3$; see [Rei83] for a proof that applies to all $cDV$ singularities or [Kol13, 1.42] for a simpler argument in the $cA$ case.

**11** (Resolving $X_m$)**.** Blow up the origin to get $\pi_1 : X_{m,1} := B_0 X_m \to X_m$. The exceptional divisor is the singular quadric $E_1 \cong (xy - z^2 = 0) \subset \mathbb{P}^3(x, y, z, t)$.

If $m \in \{2, 3\}$ then $B_0 X$ is smooth, hence the only essential divisor is $E_1$.

For $m \geq 4$ the resulting $B_0 X_m$ has one singular point, visible in the chart

$$(x_1, y_1, z_1, t) := \bigl(x/t, y/t, z/t, t\bigr)$$

where the local equation is $x_1 y_1 = z_1^2 - t^{m-2}$. We can thus blow up the origin again and continue. After $r := \lfloor \frac{m}{2} \rfloor$ steps we have a resolution

$$\Pi_r : X_{m,r} \to X_{m,r-1} \to \cdots \to X_{m,1} \to X_m.$$

We get $r$ exceptional divisors $E_r, \dots, E_1$. For $1 \leq c \leq r$ the divisor $E_c$ first appears on $X_{m,c}$. At the unique singular point one can write the local equation as

$$X_{m,c} = \bigl(x_c y_c = z_c^2 - t^{m-2c}\bigr) \quad \text{and} \quad E_c = (t = 0).$$

where $(x_c, y_c, z_c, t) := \bigl(x/t^c, y/t^c, z/t^c, t\bigr)$.

We thus need to decide which of the divisors $E_1, \dots, E_{\lfloor \frac{m}{2} \rfloor}$ are essential. It is easy to see that $E_1$ is essential and a direct computation (Lemma 15 below) shows that $E_3, \dots, E_{\lfloor \frac{m}{2} \rfloor}$ are not. (This is actually not needed in order to establish Example 2.) The hardest is to decide what happens with $E_2$.

**Lemma 12.** *Notation as above. Then*

(1) $a(E_c, X_m) = c$ *for every* $c$.
(2) $E_1$ *is the only exceptional divisor whose center is the origin and whose discrepancy is* 1.
(3) $E_1$ *appears on every resolution of* $X_m$ *whose exceptional set is a divisor.*
(4) *Let* $p : Y \dashrightarrow X_m$ *be any (not necessarily proper) bimeromorphic map from a smooth analytic space* $Y$ *such that* $\operatorname{center}_Y E_1 \subset Y$ *is not empty. Then* $\operatorname{center}_Y E_1$ *is an irreducible component of the exceptional set* $\operatorname{Ex}(p)$.

Proof. The first claim follows from the formula

$$\Pi_r^*\Bigl(\tfrac{dx \wedge dy \wedge dt}{z}\Bigr) = t^{-c} \cdot \tfrac{dx_c \wedge dy_c \wedge dt}{z_c}.$$

Let $F$ be any other exceptional divisor whose center is the origin. Then $\operatorname{center}_{X_r} F$ lies on one of the $E_c$, thus $a(F, X) > a(E_c, X) \geq 1$. (This also proves that $X_m$ is terminal.)

To see (3) set $W_1 := \text{center}_Y E_1 \subset Y$. Let $F_i \subset Y$ be the exceptional divisors and note that, as in [KM98, 2.29],

$$a(E_1, X_m) \geq \big(\text{codim}_Y W_1 - 1\big) + \textstyle\sum_i \text{mult}_{W_1} F_i \cdot a(F_i, X_m). \tag{12.5}$$

Note that $a(E_1, X_m) = 1$ and $a(F_i, X_m) \geq 1$ for every $i$. If $W_1$ is not an irreducible component of $\text{Ex}(p)$ then $W_1 \subset F_i$ form some $i$ and then both terms on the right hand side of (12.5) are positive, a contradiction. □

**13** (Small resolutions and factoriality of $X_m$)**.** If $m = 2a$ is even, then $X_m$ has a small resolution obtained by blowing up either $(x = z - t^a = 0)$ or $(x = z + t^a = 0)$. The resulting blow-ups $Y_{2a}^{\pm} \subset \mathbb{C}^4_{xyzt} \times \mathbb{P}^1_{uv}$ are defined by the equations

$$Y_{2a}^{\pm} := \text{rank}\begin{pmatrix} x & z \pm t^a & u \\ z \mp t^a & y & v \end{pmatrix} \leq 1 \tag{13.1}$$

We show that $X_m$ does not have small resolutions if $m$ is odd. More generally, let

$$X_f := \big(xy = f(z,t)\big) \subset \mathbb{C}^4$$

be an isolated $cA$-type singularity. Write $f = \prod_j f_j$ as a product of irreducibles. The $f_j$ are distinct since the singularity is isolated. Set $D_j := (x = f_j = 0)$. By [Kol91, 2.2.7] the local divisor class group is

$$\text{Div}\big(0 \in X_f\big) = \big(\textstyle\sum_j \mathbb{Z}[D_j]\big)\big/\textstyle\sum_j [D_j]. \tag{13.2}$$

In particular, $X_f$ is factorial iff $f$ is irreducible.

This formula works both algebraically and analytically. If we are interested in the affine variety $X_f$, then we consider factorizations of $f$ in the polynomial ring. If we are interested in the complex analytic germ $X_f$, then we consider factorizations of $f$ in the ring of germs of analytic functions. Thus, for example,

$$(xy = z^2 - t^2 - t^3) \subset \mathbb{C}^4$$

is algebraically factorial, since $z^2 - t^2 - t^3$ is an irreducible polynomial, but it is not analytically factorial, since

$$z^2 - t^2 - t^3 = \big(z - t\sqrt{1+t}\big)\big(z + t\sqrt{1+t}\big).$$

Thus if $m$ is odd then $X_m$ is factorial (both algebraically and analytically) and it does not have small resolutions; see Lemma 17 below for stronger results.

**Lemma 14.** *If $m$ is even then there is a divisorial resolution whose sole exceptional divisor is birational to $E_1$. Thus the only essential divisor is $E_1$.*

Proof. The $m = 2$ case was noted in Paragraph 11, hence we may assume that $m = 2a \geq 4$.

There are 2 ways to obtain such resolutions. First, we can blow up the exceptional curve in either of the $Y_{2a}^{\pm}$ as in (13.1).

Alternatively, we first blow up the origin to get $B_0 X_m$ which has one singular point with local equation $x_1 y_1 = z_1^2 - t_1^{2a-2}$ and then blow up

$$D^+ := (x_1 = z_1 + t_1^{a-1} = 0)$$

or

$$D^- := (x_1 = z_1 - t_1^{a-1} = 0).$$

□

**Lemma 15.** [Nas95, p.37] *The divisors $E_3, \ldots, E_r$ are not essential.*

Proof. If $m$ is even, this follows from Lemma 14, but for the proof below the parity of $m$ does not matter.

If $2b \geq a \geq 0$ and $m \geq a$ then $(u, v, w, t) \mapsto (ut, vt^{a+1}, wt^{b+1}, t) = (x, y, z, t)$ defines a birational map

$$g(a, b, m) : Z_{abm} := \left(uv = w^2 t^{2b-a} - t^{m-2-a}\right) \to X_m.$$

Note that $\mathrm{Ex}\big(g(a, b, m)\big) = (t = 0)$ is mapped to the origin and $Z_{abm}$ is smooth along the $v$-axis, save at the origin.

If $1 \leq c \leq m/2$ then $(x_c, y_c, z_c, t) \mapsto (x_c t^c, y_c t^c, z_c t^c, t) = (x, y, z, t)$ defines a birational map

$$h(c, m) : X_{m,c} := \left(x_c y_c = z_c^2 - t^{m-2c}\right) \to X_m.$$

By composing we get a birational map $g(a, b, m)^{-1} \circ h(c, m) : Y_c \dashrightarrow Z_{abm}$ given by

$$(x_c, y_c, z_c, t) \mapsto (x_c t^{c-1}, y_c t^{c-a-1}, z_c t^{c-b-1}, t) = (u, v, w, t)$$

which is a morphism if $c \geq a + 1, b + 1$. If $c = a + 1$ and $c > b + 1$ then we have

$$(x_c, y_c, z_c, t) \mapsto (x_c t^{c-1}, y_c, z_c t^{c-b-1}, t) = (u, v, w, t)$$

which maps $E_c$ to the $v$-axis.

If $c \geq 3$ then by setting $a = c - 1, b = c - 2$ we get a birational morphism

$$p(c, m) := g(c, c{-}1, m)^{-1} \circ h(c, m)$$

given by

$$(x_c, y_c, z_c, t) \mapsto (x_c t^c, y_c, z_c t, t) = (u, v, w, t).$$

Note that

$$p(c, m) : Y_c = \left(x_c y_c = z_c^2 - t^{m-2c}\right) \to \left(uv = w^2 t^{c-2} - t^{m-c}\right) = Z_{c,c-1,m}$$

maps $E_c$ onto the $v$-axis. Thus $E_c$ is not essential for $c \geq 3$. □

**Lemma 16.** *If $m \geq 5$ is odd then $E_2$ is essential.*

Proof. We follow the arguments in [dF12, 4.1]. Let $p : Y \to X_m$ be any resolution and set $Z := \mathrm{center}_Y\, E_2 \subset Y$. Since $X_m$ is factorial (here we use that $m$ is odd), $\mathrm{Ex}(p)$ has pure dimension 2 by Lemma 17.2.

Assume to the contrary that $Z$ is not a divisor. Using that $a(E_2, X_m) = 2$, (12.5) implies that $Z$ is a curve, there is a unique exceptional divisor $F \subset Y$ that contains $Z$, $F$ is smooth at general points of $Z$ and $a(F, X_m) = 1$.

If $p(F)$ is a curve then $Z$ is an irreducible component of $p^{-1}(0)$. The remaining case is when $p(F) = 0$, thus $F = E_1$ by Lemma 12.2.

Since $t$ vanishes along $E_2$ with multiplicity 1, it also vanishes along $Z$ with multiplicity 1. Since $p^*x, p^*y, p^*z, p^*t$ all vanish along $E_1$, the rational functions $p^*(x/t), p^*(y/t), p^*(z/t)$ are regular generically along $Z$. Thus $p_1 := \pi_1^{-1} \circ p : Y \dashrightarrow X_{m,1}$ is a morphism generically along $Z$. Note that our $E_2$ is what we would call $E_1$ if we started with $X_{m,1}$. Applying Lemma 12.4 to $p_1 : Y \dashrightarrow X_{m,1}$ we see that $Z$ is an irreducible component of $\mathrm{Ex}(p_1)$. Since $m$ is odd, $X_{m,1}$ is analytically factorial by Paragraph 13, hence $Z$ is a divisor by Lemma 17.2 below. This is a contradiction. □

**Lemma 17.** *Let $X, Y$ be normal varieties or analytic spaces and $g : Y \to X$ a birational or bimeromorphic morphism. Then the exceptional set $\mathrm{Ex}(g)$ has pure codimension 1 in $Y$ in the following cases.*

(1) *$Y$ is an algebraic variety and $X$ is $\mathbb{Q}$-factorial.*
(2) *$\dim Y = 3$ and $X$ is analytically locally $\mathbb{Q}$-factorial.*

Proof. The algebraic case is well known; see for instance the method of [Sha74, Sec.II.4.4].

If $\dim Y = 3$ and $\mathrm{Ex}(g)$ does not have pure codimension 1 then it has a 1-dimensional irreducible component $C \subset Y$. After replacing $X$ by a suitable neighborhood of $g(C) \in X$ we may assume that there is a divisor $D_Y \subset Y$ such that $\mathrm{Ex}(g) \cap D_Y$ is a single point of $C$ and $g|_{D_Y}$ is proper. Thus $D_X := g(D_Y)$ is a divisor on $X$. If $mD_X$ is Cartier then so is $g^*(mD_X)$ hence its support has pure codimension 1 in $Y$. On the other hand, $\mathrm{Supp}\bigl(g^*(mD_X)\bigr) = \mathrm{Ex}(g) \cup D_Y$ does not have pure codimension 1. (Note that there are many possible choices for $D_Y$; the resulting $D_X$ determine an algebraic equivalence class of divisors.) $\square$

Somewhat surprisingly, the analog of Lemma 17.2 fails in dimension 4.

**Example 18.** Let $W \subset \mathbb{P}^4$ be a smooth quintic 3–fold and $C \subset W$ a line whose normal bundle is $\mathcal{O}(-1) + \mathcal{O}(-1)$. Let $X \subset \mathbb{C}^5$ denote the cone over $W$ with vertex 0; it is analytically locally factorial by [Gro68, XI.3.14].

The exceptional divisor of the blow-up $B_0X \to X$ can be identified with $W$; let $C \subset B_0X$ be our line. Its normal bundle is $\mathcal{O}(-1) + \mathcal{O}(-1) + \mathcal{O}(-1)$.

Blow up the line $C$ to obtain $B_CB_0X \to B_0X$. Its exceptional divisor is $E \cong \mathbb{P}^1 \times \mathbb{P}^2$. One can contract $E$ in the other direction to obtain $g : Y \to X$.

By construction, $\mathrm{Ex}(g)$ is the union of $\mathbb{P}^2$ and of a 3-fold obtained from $W$ by flopping the line $C$. The two components intersect along a line.

This completes our analysis of 3–dimensional $cA_1$-type singularities. Our study of the higher dimensional cases relies on a deeper understanding of the proof of Lemma 17.2 for

$$X_c := \bigl(xy = z^2 - ct^m\bigr),$$

where $c \neq 0$.

The reader may wish to jump to Section 3 and return to this point once formula (22.3) shows why the question answered in Proposition 19 is of interest.

Let $g_c : Y_c \to X_c$ be a proper birational or bimeromorphic morphism and $E_c \subset \mathrm{Ex}(g_c)$ a 1-dimensional irreducible component.

The proof of Lemma 17.2 associates to $E_c$ an algebraic equivalence class of non-Cartier divisors on $X_c$. Thus $m$ has to be even by Paragraph 13.

If $m = 2a$ is even then the divisor class group is $\mathrm{Div}(X_c) \cong \mathbb{Z}$. The two possible generators correspond to $(x = z - \sqrt{c}t^a = 0)$ and $(x = z + \sqrt{c}t^a = 0)$. Starting with $E_c$ we constructed a divisor $D_c \subset X_c$ which is a nontrivial element of $\mathrm{Div}(X_c)$. Thus $[D_c]$ is a positive multiple of either $(x = z - \sqrt{c}t^a = 0)$ or $(x = z + \sqrt{c}t^a = 0)$. Hence, to $E_c \subset Y_c$ we can associate a choice of $\sqrt{c}$.

This may not be very interesting for a fixed value of $c$ (since many other choices are involved) but it turns out to be quite useful when $c$ varies.

**Proposition 19.** *Let $g(u_1, \dots, u_r, v)$ be a holomorphic function for $u_i \in \mathbb{C}$ and $|v| < \epsilon$ such that $g(u_1, \dots, u_r, 0)$ is not identically zero. For $m \geq 4$ set*

$$X := \bigl(xy = z^2 - v^m g(u_1, \dots, u_r, v)\bigr) \subset \mathbb{C}^{r+4}.$$

*Let $\pi : Y \to X$ be a birational or bimeromorphic morphism. Assume that there is an irreducible component $Z \subset \mathrm{Ex}(\pi)$ that dominates $(x = y = z = v = 0) \subset X$, has codimension $\geq 2$ in $Y$ and such that $\pi|_Z : Z \to (x = y = z = v = 0)$ has connected fibers.*

*Then $m$ is even and $g(u_1, \dots, u_r, 0)$ is a perfect square.*

Proof. For general $\mathbf{c} = (c_1, \dots, c_r) \in \mathbb{C}^r$ the repeated hyperplane section

$$X(\mathbf{c}) := \bigl(xy = z^2 - v^m g(\mathbf{c}, v)\bigr) \subset \mathbb{C}^4$$

has an isolated singularity at the origin and we get a proper birational or bimeromorphic morphism $\pi(\mathbf{c}) : Y(\mathbf{c}) \to X(\mathbf{c})$ where $Y(\mathbf{c}) \subset Y$ is the preimage of $X(\mathbf{c})$.

Furthermore, $Z(\mathbf{c}) := Z \cap Y(\mathbf{c})$ is an irreducible component of $\operatorname{Ex}\bigl(\pi(\mathbf{c})\bigr)$ and has codimension $\geq 2$ in $Y(\mathbf{c})$.

Thus, as we noted above, $m = 2a$ is even and our construction gives a function

$$(c_1, \dots, c_r) \mapsto \text{a choice of } \sqrt{g(c_1, \dots, c_r, 0)}.$$

It is clear that this function is continuous on a Zariski open set $U \subset \mathbb{C}^r$. Therefore $g(u_1, \dots, u_r, 0)$ is a perfect square. □

**Remark 20.** Conversely, assume that $m$ is even and $g(u_1, \dots, u_r, 0) = h^2(u_1, \dots, u_r)$ is a square. Write the equation of $X$ as

$$xy = z^2 - v^m\bigl(h^2(u_1, \dots, u_r) + vR(u_1, \dots, u_r, v)\bigr)$$

Over the open set $X^0 \subset X$ where $h \neq 0$, change coordinates to $w := h^{-2}v$. (Equivalently, blow up $(v = h = 0)$ twice.) Then

$$D := \bigl(x = z - w^{m/2}h^{m+1}\sqrt{1 + wR(u_1, \dots, u_r, h^2 w)}\bigr)$$

is a globally well defined analytic divisor. Blowing it up gives a bimeromorphic morphism $X_D \to X$ whose exceptional set over $X^0$ has codimension 2.

It seems that even if $X$ is algebraic, usually $X_D$ is not an algebraic variety.

## 3. Essential divisors on $cA_1$-type singularities

In higher dimensions $cA_1$-type singularities are more complicated and their resolutions are much harder to understand. There is no simple complete answer as in dimension 3.

In the previous Section, the key part was to understand the exceptional divisors that correspond to the first 2 blow-ups. These are the 2 divisors that we understand in higher dimensions as well.

**21** (Defining $E_1$ and $E_2$)**.** In order to fix notation, write the equation as

$$X := \bigl(xy = z^2 - g(u_1, \dots, u_r)\bigr) \subset \mathbb{C}^{r+3}. \tag{21.1}$$

Set $m := \operatorname{mult}_0 g$ and let $g_s(u_1, \dots, u_r)$ denote the homogeneous degree $s$ part of $g$. In a typical local chart the 1st blow-up $\sigma_1 : X_1 := B_0 X \to X$ is given by

$$x_1 y_1 = z_1^2 - \bigl(u_r'\bigr)^{-2} g(u_1' u_r', \dots, u_{r-1}' u_r', u_r') \tag{21.2}$$

where $x = x_1 u_r', y = y_1 u_r', z = z_1 u_r'$, $u_1 = u_1' u_r', \dots, u_{r-1} = u_{r-1}' u_r'$ and $u_r = u_r'$. The exceptional divisor is the rank 3 quadric

$$E_1 := \bigl(x_1 y_1 - z_1^2 = 0\bigr) \subset \mathbb{P}^{r+2}. \tag{21.3}$$

Note also that

$$\begin{aligned} &\bigl(u_r'\bigr)^{-2} g(u_1' u_r', \dots, u_{r-1}' u_r', u_r') = \\ &\quad = \bigl(u_r'\bigr)^{m-2}\Bigl(g_m(u_1', \dots, u_{r-1}', 1) + u_r' g_{m+1}(u_1', \dots, u_{r-1}', 1) + \cdots\Bigr). \end{aligned} \tag{21.4}$$

From this we see that, for $m \geq 4$, the blow-up $X_1$ is singular along the closure of the linear space

$$L := (x_1 = y_1 = z_1 = u_r' = 0), \tag{21.5}$$

$X_1$ has terminal singularities and a general 3-fold section has equation

$$x_1 y_1 = z_1^2 - \bigl(u_r'\bigr)^{m-2}\Bigl(g_m(c_1, \dots, c_{r-1}, 1) + u_r' g_{m+1}(c_1, \dots, c_{r-1}, 1) + \cdots\Bigr).$$

Blowing up the closure of $L$ we obtain $X_2$ with exceptional divisor $E_2$. As in Lemma 12 we compute that

(6) $a(E_1, X) = r$,
(7) $a(E_2, X) = r + 1$,
(8) $a(F, X) \geq r + 1$ for every other exceptional divisor whose center on $X$ is the origin and
(9) the pull-backs of the $u_i$ vanish along $E_1, E_2$ with multiplicity 1.

The key computation is the following.

**Proposition 22.** *Notation as above and assume that $m \geq 4$.*

(1) *$E_1$ is an essential divisor.*
(2) *$E_2$ is an essential divisor iff $g_m(u_1, \ldots, u_r)$ is not a perfect square.*

Proof. By (21.6) and (21.8), $E_1$ has the smallest discrepancy among all divisors over $X$ whose center on $X$ is the origin. Thus $E_1$ is essential by Proposition 24.

If $E_2$ is non-essential then there is a resolution $\pi : Y \to X$ and an irreducible component $W \subset \operatorname{Supp} \pi^{-1}(0)$ such that $Z := \operatorname{center}_Y E_2 \subsetneq W$. By (21.9), the $\pi^* u_i$ vanish at a general point of $Z$ with multiplicity 1. Since the $\pi^* u_i$ vanish along $W$, this implies that $\operatorname{Supp} \pi^{-1}(0)$ is smooth at a general point of $Z$. In particular, $W$ is the only irreducible component of $\operatorname{Supp} \pi^{-1}(0)$ that contains $Z$ and $W$ is smooth at general points of $Z$. Therefore the blow-up $B_W Y$ is smooth over the generic point of $Z$. So, if we replace $Y$ by a suitable desingularization of $B_W Y$, we get a situation as before where, in addition, $W$ is a divisor.

The $\pi^* u_i$ are local equations of $W$ at general points of $Z$ and $\pi^* x, \pi^* y, \pi^* z$ all vanish along $W$. Thus the rational functions

$$\pi^*(x/u_r), \pi^*(y/u_r), \pi^*(z/u_r), \pi^*(u_1/u_r), \ldots, \pi^*(u_{r-1}/u_r),$$

are all regular at general points of $Z$. Hence the birational map $\sigma_1^{-1} \circ \pi : Y \to B_0 X = X_1$ is a morphism at general points of $Z$. Furthermore, $\sigma_1^{-1} \circ \pi$ maps $W$ birationally to $E_1 \subset X_1$ and it is not a local isomorphism along $Z$ since $Y$ is smooth but $X_1$ is singular along the center $L$ of $E_2$. Thus $Z$ is an irreducible component of $\operatorname{Ex}(\sigma_1^{-1} \circ \pi)$. Since $E_2 \to L$ has connected fibers, all the assumptions of Proposition 19 are satisfied by the equation of the blow-up

$$x_1 y_1 = z_1^2 - (u_r')^{m-2} \Big( g_m(u_1', \ldots, u_{r-1}', 1) + u_r' g_{m+1}(u_1', \ldots, u_{r-1}', 1) + \cdots \Big). \tag{22.3}$$

Thus $m$ is even and $g_m(u_1', \ldots, u_{r-1}', 1)$ is a perfect square. Since it is a dehomogenization of $g_m(u_1, \ldots, u_{r-1}, u_r)$, the latter is also a perfect square.

The converse follows from Remark 20. □

**Definition 23.** For $(x \in X)$ let min-discrep$(x \in X)$ be the infimum of the discrepancies $a(E, X)$ where $E$ runs through all divisors over $X$ such that $\operatorname{center}_X E = \{x\}$. (It is easy to see that either min-discrep$(x \in X) \geq -1$ and the infimum is a minimum or min-discrep$(x \in X) = -\infty$; cf. [KM98, 2.31]. We do not need these facts.)

**Proposition 24.** *Let $(x \in X)$ be a canonical singularity and $E$ a divisor over $X$ such that* $\operatorname{center}_X E = \{x\}$ *and $a(E, X) < 1 +$* min-discrep$(x \in X)$. *Then $E$ is essential.*

Proof. Let $F$ be any non-essential divisor over $X$ whose center on $X$ is the origin. Thus there is a resolution $\pi : Y \to X$ and an irreducible component $W \subset \operatorname{Supp} \pi^{-1}(x)$ such that $Z := \operatorname{center}_Y F \subsetneq W$. Let $E_W$ be the divisor obtained by blowing up $W \subset Y$. As we noted in Definition 10,

$$a(E_W, Y) = \operatorname{codim}_Y W - 1 \quad \text{and} \quad a(F, Y) \geq \operatorname{codim}_Y Z - 1 \geq \operatorname{codim}_Y W. \tag{24.1}$$

Write $K_Y = \pi^* K_X + D_Y$ where $D_Y$ is effective since $X$ is canonical and note that

$$a(E_W, X) = a(E_W, Y) + \operatorname{mult}_W D_Y \quad \text{and} \quad a(F, X) \geq a(F, Y) + \operatorname{mult}_Z D_Y. \tag{24.2}$$

Since $\operatorname{mult}_Z D_Y \geq \operatorname{mult}_W D_Y$, we conclude that

$$a(F, X) \geq 1 + a(E_W, X) \geq 1 + \text{min-discrep}(x \in X). \tag{24.3}$$

Thus any divisor $E$ with $a(E, X) < 1 + \text{min-discrep}(x \in X)$ is essential. $\square$

## 4. Short arcs

Let $\mathbb{D} \subset \mathbb{C}$ denote the open unit disk and $\overline{\mathbb{D}} \subset \mathbb{C}$ its closure. The open (resp. closed) disc of radius $\epsilon$ is denoted by $\mathbb{D}(\epsilon)$ (resp. $\overline{\mathbb{D}}(\epsilon)$). If several variables are involved, we use a subscript to indicate the name of the coordinate.

**25** (Short arcs)**.** [KN13] Let $X$ be an analytic space and $p \in X$ a point. A *short arc* on $(p \in X)$ is a holomorphic map $\phi(t) : \overline{\mathbb{D}}_t \to X$ such that $\operatorname{Supp} \phi^{-1}(p) = \{0\}$.

The space of all short arcs is denoted by $\operatorname{ShArc}(p \in X)$. It has a natural topology and most likely also a complex structure that, at least for isolated singularities, locally can be written as the product of a finite dimensional complex space and of a complex Banach space; see [KN13, Sec.11] for details.

A *deformation* of short arcs is a holomorphic map $\Phi(t, s) : \overline{\mathbb{D}}_t \times \mathbb{D}_s \to X$ such that

$$\Phi(t, s_0) : \overline{\mathbb{D}}_t \to X$$

is a short arc for every $s_0 \in \mathbb{D}_s$. Equivalently, if $\operatorname{Supp} \Phi^{-1}(p) = \{0\} \times \mathbb{D}_s$.

In general the space of short arcs has more connected components than the space of formal arcs. As a simple example, consider arcs on $(xy = z^m) \subset \mathbb{C}^3$. For $0 < i < m$ the deformations

$$(t, s) \mapsto \big(t^i (t + s)^{m-i}, t^{m-i}(t + s)^i, t(t + s)\big) \tag{25.1}$$

show that the arc $(t^m, t^m, t^2)$ is in the closure of the families $\widehat{\operatorname{Arc}}{}^{\circ}_i(0 \in X)$, provided we work in the space of formal arcs. However, (25.1) is *not* a deformation of short arcs and $(t^m, t^m, t^2)$ is a typical member of a new connected component of $\operatorname{ShArc}\big(0 \in (xy = z^m)\big)$.

By contrast, adding one more variable kills this component. For example, starting with the arc $(t^m, t^m, t^2, 0)$ on $(xy = z^m) \subset \mathbb{C}^4$, we have deformations of short arcs

$$(t, s) \mapsto \big(t^i (t + s)^{m-i}, t^{m-i}(t + s)^i, t(t + s), ts\big). \tag{25.2}$$

This example turns out to be typical and it is quite easy to modify the deformations in the proof of Theorem 1 to yield the following.

**Theorem 26.** *Let $X = (xy = f(z_1, \dots, z_n) \subset \mathbb{C}^{n+2}$ be a cA-type singularity. Assume that $n \geq 2$ and $m := \operatorname{mult}_0 f \geq 2$.*

*Then $\operatorname{ShArc}(0 \in X)$ has $(m-1)$ irreducible components as in Theorem 1.2.*

It is not always clear if a deformation $\Phi(t, s)$ is short or not. There is, however, one case when this is easy, at least over a smaller disc $\mathbb{D}_s(\epsilon) \subset \mathbb{D}_s$.

**Lemma 27.** *Let $\Phi(t, s) = \big(\Phi_1(t, s), \dots, \Phi_r(t, s)\big)$ be a deformation of arcs on $X \subset \mathbb{C}^r$. Assume that $\Phi(t, 0)$ is short and $\Phi_i(t, s)$ is independent of $s$ and not identically zero for some $i$. Then $\Phi(t, s_0) : \overline{\mathbb{D}}_t \to X$ is short for $|s_0| \ll 1$.*

Proof. By assumption $\Phi(*, s_0)^{-1}(p) \subset \Phi_i(*, s_0)^{-1}(p) = \Phi_i(*, 0)^{-1}(p)$ for every $s_0 \in \mathbb{D}_s$, thus there is a finite subset $Z = \Phi_i(*, 0)^{-1}(p) \subset \overline{\mathbb{D}}_t$ such that

$$\Phi^{-1}(0) \subset Z \times \mathbb{D}_s \quad \text{and} \quad \Phi^{-1}(0) \cap (s = 0) = \{(0, 0)\}.$$

Since $\Phi^{-1}(0)$ is closed, this implies that

$$\Phi^{-1}(0) \cap \left(\overline{\mathbb{D}}_t \times \mathbb{D}_s(\epsilon)\right) \subset \{0\} \times \mathbb{D}_s(\epsilon) \quad \text{for } 0 < \epsilon \ll 1. \qquad \square$$

**28** (Proof of Theorem 26)**.** At the very beginning of the proof of Theorem 1, after a linear change of coordinates we may assume that $z_1^m$ appears in $f$ with nonzero coefficient and $\phi_2$ is not identically zero. Then the construction gives a deformation of short arcs by Lemma 27.

The deformations at the end of the proof were written to yield short arcs. $\square$

## 5. A revised version of the Nash problem

As we saw, the Nash map is not surjective in dimensions $\geq 3$. In this section we develop a revised version of the notion of essential divisors. This leads to a smaller target for the Nash map, so surjectivity should become more likely. Our proposed variant of the Nash problem at least accounts for all known counter examples.

We start with a reformulation of the original definition of essential divisors.

**29.** Let $Y$ be a complex variety and $Z \subset Y$ a closed subset. Let $\widehat{\mathrm{Arc}}(Z \subset Y)$ denote the scheme of formal arcs $\phi : \operatorname{Spec} \mathbb{C}[[t]] \to Y$ such that $\phi(0) \in Z$.

An easy but key observation is the following.

29.1. If $Y$ is smooth, then the irreducible components of $\widehat{\mathrm{Arc}}(Z \subset Y)$ are in a natural one–to–one correspondence with the irreducible components of $Z$.

We say that a divisor $E$ over $Y$ is *essential* for $Z \subset Y$ if $E$ is obtained by blowing up one of the irreducible components of $Z$. (For each irreducible component $Z_i \subset Z$, the blow-up $B_Z Y$ contains a unique divisor that dominates $Z_i$.)

The definition of essential divisors can now be reformulated as follows.

29.2. Let $(x \in X)$ be a singularity. A divisor $E$ is *essential* for $(x \in X)$ if $E$ is essential for $\left(\operatorname{Supp} \pi^{-1}(x) \subset Y\right)$ for every resolution $\pi : Y \to X$.

In order to refine the Nash problem, we need to understand singular spaces for which the analog of (29.1) still holds.

**Definition 30** (Sideways deformations)**.** Let $X$ be a variety (or an analytic space) and

$$\phi : \operatorname{Spec} \mathbb{C}[[t]] \to X$$

a formal arc such that $\phi(0) \in \operatorname{Sing} X$. A *sideways deformation* of $\phi$ is a morphism

$$\Phi : \operatorname{Spec} \mathbb{C}[[t, s]] \to X$$

such that

$$\Phi^* I_{\operatorname{Sing} X} \supset (t, s)^m \quad \text{for some } m \geq 1,$$

where $I_{\operatorname{Sing} X} \subset \mathcal{O}_X$ is the ideal sheaf defining $\operatorname{Sing} X$.

If $\Phi$ comes from a convergent arc $\Phi^{\mathrm{an}} : \mathbb{D}_t \times \mathbb{D}_s \to X$ then this is equivalent to assuming that for every $0 \neq |s_0| \ll 1$ the nearby arc $\Phi^{\mathrm{an}}(t, s_0)$ maps $\mathbb{D}_t(\epsilon)$ to $X \setminus \operatorname{Sing} X$ for some $0 < \epsilon \leq 1$.

We say that $(x \in X)$ is *arc-wise Nash-trivial* if every general arc in $\widehat{\mathrm{Arc}}(x \in X)$ has a sideways deformation. (By [FdBP12a], this implies that every arc in $\widehat{\mathrm{Arc}}(x \in X)$ has a sideways deformation.)

**Comment 31.** If $(x \in X)$ is an isolated singularity with a small resolution $\pi : X' \to X$ then every arc has a sideways deformation. We can lift the arc to $X'$ and there move it away from the $\pi$-exceptional set. This is not very interesting and the notion of essential divisors captures this phenomenon.

To exclude these cases, we are mainly interested in arc-wise Nash-trivial singularities that do not have small modifications. If arc-wise Nash-trivial singularities are log terminal then assuming analytic $\mathbb{Q}$-factoriality captures this restriction, but in general one needs to be careful of the difference between analytic $\mathbb{Q}$-factoriality and having no small modifications.

Also, in the few examples of which we know, general arcs of every irreducible component of $\widehat{\operatorname{Arc}}(x \in X)$ have sideways deformations. If there are singularities where sideways deformations exist only for some of the irreducible components, the following outline needs to be suitably modified.

The main observation is that, for the purposes of the Nash problem, $\mathbb{Q}$-factorial arc-wise Nash-trivial singularities should be considered as good as smooth points. The first evidence is the following straightforward analog of (29.1).

**Lemma 32.** *Let $Y$ be a complex space with isolated, arc-wise Nash-trivial singularities. Let $Z \subset Y$ a closed subset that is the support of an effective Cartier divisor. Then the irreducible components of $\widehat{\operatorname{Arc}}(Z \subset Y)$ are in a natural one–to–one correspondence with the irreducible components of $Z$.* □

If $Z$ has lower dimensional irreducible components, the situation seems more complicated, but, at least in dimension 3, the following seems to be the right generalization of (29.1).

**Conjecture 33.** *Let $Y$ be a 3–dimensional complex space with isolated, $\mathbb{Q}$-factorial, arc-wise Nash-trivial singularities. Let $Z \subset Y$ be a closed subset. Then the irreducible components of $\widehat{\operatorname{Arc}}(Z \subset Y)$ are in a natural one–to–one correspondence with the union of the following two sets.*

(1) *Irreducible components of $Z$.*
(2) *Irreducible components of $\widehat{\operatorname{Arc}}(p \in Y)$, where $p \in Y$ is any singular point such that $p \in Z$ and $\dim_p Z \leq 1$.*

**Definition 34.** With the above assumptions, a divisor over $Y$ is *essential* for $Z \subset Y$ if it corresponds to one of the irreducible components of $\widehat{\operatorname{Arc}}(Z \subset Y)$, as enumerated in Conjecture 33.1–2.

**Definition 35.** Let $(x \in X)$ be a 3–dimensional, normal singularity. A divisor $E$ over $X$ is called *very essential* for $(x \in X)$ if $E$ is essential for $\big(\operatorname{Supp} \pi^{-1}(x) \subset Y\big)$ for every proper bimeromorphic morphism $\pi : Y \to X$ where $Y$ has only isolated, $\mathbb{Q}$-factorial, arc-wise Nash-trivial singularities. (As in Definition 8, it is better to allow $Y$ to be an analytic space.)

It is easy to see that the Nash map is an injection from the irreducible components of $\widehat{\operatorname{Arc}}(x \in X)$ into the set of very essential divisors. One can hope that there are no other obstructions.

**Problem 36** (Revised Nash problem)**.** Is the Nash map surjective onto the set of very essential divisors for normal 3-fold singularities?

As a first step, one should consider the following.

**Problem 37.** In dimension 3, classify all $\mathbb{Q}$-factorial, arc-wise Nash-trivial singularities.

Hopefully they are all terminal and a complete enumeration is possible. The papers [Hay05a, Hay05b] contain several results about partial resolutions of terminal singularities.

We treat two easy cases next. A positive solution of Question 4 would imply that all isolated, 3-dimensional cDV singularities are arc-wise Nash-trivial.

**Theorem 38.** *Let* $(0 \in X)$ *be a cA-type singularity such that* $\dim \operatorname{Sing} X \le \dim X - 3$. *Then all arcs in* $\widehat{\operatorname{Arc}}_i^\circ(0 \in X)$ *(as in Theorem 1.2) have sideways deformations.*

Proof. We use the notation of the proof of Theorem 1.

Since $\operatorname{mult} f\big(\phi_1(t), \dots, \phi_n(t)\big) = m$, we see that $\operatorname{mult} \phi_j(t) = 1$ for at least one index $j$. We may assume that $j = 1$ and $\phi_1(t) = t$. Thus, after the coordinate change $z_i \mapsto z_i - \phi_i(z_1)$ for $i = 2, \dots, n$ and an additional general linear coordinate change among the $z_2, \dots, z_n$ we may assume that

(1) $\phi_1(t) = t$,
(2) $\phi_j(t) \equiv 0$ for $j > 1$,
(3) $\big(xy = g(z_1, z_2)\big) \subset \mathbb{C}^4$ has an isolated singularity at the origin and $g(z_1, z_2)$ is divisible neither by $z_1$ nor by $z_2$ where $g(z_1, z_2) = f(z_1, z_2, 0, \dots, 0)$.

By Lemma 7 there is an $r \ge 1$ such that

$$g(t, s^r) = u(t, s) \prod_{i=1}^{m} \big(t - \sigma_i(s)\big).$$

Since $g(z_1, z_2)$ is not divisible by $z_1$, none of the $\sigma_i$ are identically zero. Since $g(t, s)$ has an isolated critical point at the origin and is not divisible by $s$, $g(t, s^r)$ also has an isolated critical point at the origin. Thus all the $\sigma_i(s)$ are distinct.

As before, for $j = 1, 2$ write $\psi_j(t) = t^{a_j} v_j(t)$ where $v_j(0) \ne 0$. Note that $a_1 + a_2 = m$ and $u(t, 0) = v_1(t) v_2(t)$.

Divide $\{1, \dots, m\}$ into two disjoint subsets $A_1, A_2$ such that $|A_j| = a_j$. Finally set

$$\Psi_1(t, s) = v_1(t) \cdot \prod_{i \in A_1} \big(t - \sigma_i(s)\big) \quad \text{and} \quad \Psi_2(t, s) = \frac{u(t, s)}{v_1(t)} \cdot \prod_{i \in A_2} \big(t - \sigma_i(s)\big).$$

Then

$$\big(\Psi_1(t, s), \Psi_2(t, s), t, s^r, 0, \dots, 0\big)$$

is a sideways deformation of $\big(\psi_1(t), \psi_2(t), t, 0, \dots, 0\big)$. $\square$

The opposite happens for quotient singularities.

**Proposition 39.** *Let* $(0 \in X) := \mathbb{C}^n / G$ *be an isolated quotient singularity. Then arcs with a sideways deformation are nowhere dense in* $\widehat{\operatorname{Arc}}(0 \in X)$.

Proof. Let $\Phi : \operatorname{Spec} \mathbb{C}[[t, s]] \to X$ be a sideways deformation of an arc $\phi(t) = \Phi(t, 0)$. By the purity of branch loci, $\Phi$ lifts to an arc $\tilde{\Phi} : \operatorname{Spec} \mathbb{C}[[t, s]] \to \mathbb{C}^n$. In particular, $\phi : \operatorname{Spec} \mathbb{C}[[t]] \to X$ lifts to $\tilde{\phi} : \operatorname{Spec} \mathbb{C}[[t]] \to \mathbb{C}^n$.

By [KN13], such arcs constitute a connected component of $\operatorname{ShArc}(0 \in X)$. We claim, however, that these arcs do not cover a whole irreducible component of $\widehat{\operatorname{Arc}}(0 \in X)$.

It is enough to show the latter on some intermediate cover of $X$. The simplest is to use $(0 \in Y) := \mathbb{C}^n / C$ where $C \subset G$ is any cyclic subgroup.

Set $r := |C|$, fix a generator $g \in C$ and diagonalize its action as

$$(x_1, \dots, x_n) \mapsto \big(\epsilon^{a_1} x_1, \dots, \epsilon^{a_n} x_n\big),$$

where $\epsilon$ is a primitive $r$th root of unity. Thus $Y$ is the toric variety corresponding to the free abelian group $N = \mathbb{Z}^n + \mathbb{Z}\big(a_1/r, \dots, a_n/r\big)$ and the $\Delta = \big(\mathbb{Q}_{\geq 0}\big)^n$. The Nash conjecture is true for toric singularities and by [IK03, Sec.3] the essential divisors are all toric and correspond to interior vectors of $N \cap \Delta$ that can not be written as the sum of an interior vector of $N \cap \Delta$ and of a nonzero vector of $N \cap \Delta$. In our case, all such vectors are of the form $\big(\overline{ca_1}/r, \dots, \overline{ca_n}/r\big)$ for $c = 1, \dots, r-1$ where $\overline{ca_i}$ denotes remainder mod $r$.

Arcs that lift to $\mathbb{C}^n$ correspond to the vector $(1, \dots, 1)$, which is not minimal. In fact $(1, \dots, 1) = \big(\overline{a_1}/r, \dots, \overline{a_n}/r\big) + \big(\overline{(r-1)a_1}/r, \dots, \overline{(r-1)a_n}/r\big)$. □

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Princeton University, Princeton NJ 08544-1000

jmjohnso@math.princeton.edu

kollar@math.princeton.edu

**Journal *of* Singularities**
**Volume 7 (2013), 253-274**

received 24 September 2011
in revised form 9 April 2013
DOI: 10.5427/jsing.2013.7n

# ADJOINT DIVISORS AND FREE DIVISORS

DAVID MOND AND MATHIAS SCHULZE

Abstract. We describe two situations where adding the adjoint divisor to a divisor $D$ with smooth normalization yields a free divisor. Both also involve stability or versality. In the first, $D$ is the image of a corank 1 stable map-germ $(\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$, and is not free. In the second, $D$ is the discriminant of a versal deformation of a weighted homogeneous function with isolated critical point (subject to certain numerical conditions on the weights). Here $D$ itself is already free.

We also prove an elementary result, inspired by these first two, from which we obtain a plethora of new examples of free divisors. The presented results seem to scratch the surface of a more general phenomenon that is still to be revealed.

## 1. Introduction

Let $M$ be an $n$-dimensional complex analytic manifold and $D$ a hypersurface in $M$. The $\mathscr{O}_M$-module $\mathrm{Der}(-\log D)$ of *logarithmic vector fields* along $D$ consists of all vector fields on $M$ tangent to $D$ at all smooth points of $D$. If this module is locally free, $D$ is called a *free divisor*. This terminology was introduced by Kyoji Saito in [Sai80b]. As freeness is evidently a local condition, so we may pass to germs of analytic spaces $D \subset X := (\mathbb{C}^n, 0)$, and pick coordinates $x_1, \ldots, x_n$ on $X$ and a defining equation $h \in \mathscr{O}_X = \mathbb{C}\{x_1, \ldots, x_n\}$ for $D$.

The module $\mathrm{Der}(-\log D)$ is an infinite-dimensional Lie sub-algebra of $\mathrm{Der}_X$, to be more precise, a Lie algebroid. Thus free divisors bring together commutative algebra, Lie theory and the theory of $\mathscr{D}$-modules, see [CMNM05]. Freeness has been used by Jim Damon and the first author to give an algebraic method for computing the vanishing homology of sections of discriminants and other free divisors, see [DM91] and [Dam96]. More recently the idea of adding a divisor to another in order to make the union free has been used by Damon and Brian Pike as a means of extending this technique to deal with sections of non-free divisors, see [DP11a] and [DP11b].

Saito formulated the following elementary freeness test, called *Saito's criterion* (see [Sai80b, Thm. 1.8.(ii)]): If the determinant of the so-called *Saito matrix* $(\delta_i(x_j))$ generates the defining ideal $\langle h \rangle$ for some $\delta_1, \ldots, \delta_n \in \mathrm{Der}(-\log D)$, then $D$ is free and $\delta_1, \ldots, \delta_n$ is a basis of $\mathrm{Der}(-\log D)$. While any smooth hypersurface is free, singular free divisors are in fact highly singular: Let $\mathrm{Sing}\, D$ be the singular locus of $D$ with structure defined by the Jacobian ideal of $D$. By the theorem of Aleksandrov–Terao (see [Ale88, §1 Thm.] or [Ter80, Prop. 2.4]), freeness of $D$ is equivalent to $\mathrm{Sing}\, D$ being a Cohen–Macaulay space of (pure) codimension 1 in $D$.

The simplest example of a free divisor, whose importance in algebraic geometry is well-known, is the normal crossing divisor $D$ defined by $h := x_1 \cdots x_n$; here, due to Saito's criterion,

1991 *Mathematics Subject Classification.* 32S25, 17B66, 32Q26, 32S30, 14M17.

*Key words and phrases.* free divisor, adjoint divisor, stable map, versal deformation, isolated singularity, prehomogeneous vector space.

The research leading to these results has received funding from the People Programme (Marie Curie Actions) of the European Union's Seventh Framework Programme (FP7/2007-2013) under REA grant agreement n° PCIG12-GA-2012-334355.

$\mathrm{Der}(-\log D)$ is freely generated by the vector fields $x_1\frac{\partial}{\partial x_1}, \ldots, x_n\frac{\partial}{\partial x_n}$. In general free divisors are rather uncommmon: given $n$ vector fields $\delta_1, \ldots, \delta_n \in \mathrm{Der}_X$, let $h$ be the determinant of their Saito matrix, and suppose that $h$ is reduced. Then $h$ defines a free divisor if and only if the $\mathscr{O}_{\mathbb{C}^n}$-submodule of $\mathrm{Der}_M$ generated by the $\delta_j$ is a Lie algebra, see [Sai80b, Lem. 1.9]. Thus to generate examples, special techniques are called for. Non-trivial examples of free divisors first appeared as discriminants and bifurcation sets in the base of versal deformations of isolated hypersurface singularities, see [Sai80a], [Ter83], [Loo84], [Bru85], [vS95], [Dam98], [BEGvB09]. Here freeness follows essentially from the fact that $\mathrm{Der}(-\log D)$ is the kernel of the *Kodaira–Spencer map* from the module of vector fields on the base to the relative $T^1$ of the deformation.

In this paper, we construct new examples of free divisors by a quite different procedure. Recall that we denote by $X$ the germ $(\mathbb{C}^n, 0)$. Let now $f\colon X \to (\mathbb{C}^{n+1}, 0) =: T$ be a finite and generically 1-to-1 holomorphic map germ. In particular, $X$ is a normalization of the reduced image $D$ of $f$. Denote by $\mathscr{F}_i$ the $i$th Fitting ideal of $\mathscr{O}_X$ considered as $\mathscr{O}_T$-module. Mond and Pellikaan [MP89, Props. 3.1, 3.4, 3.5] showed that $D$ is defined by $\mathscr{F}_0$, $\mathscr{F}_1$ is perfect ideal of height 2 restricting to the *conductor ideal* $\mathscr{C}_D := \mathrm{Ann}_{\mathscr{O}_D}(\mathscr{O}_X/\mathscr{O}_D)$ which in turn is a principal ideal of $\mathscr{O}_X$. In particular, the reduced singular locus $\Sigma$ of $D$ is the closure of the set of double points of $f$ and $\mathscr{C}_D$ is radical provided $D$ is normal crossing in codimension 1. We call any member of $\mathscr{F}_1$ whose pull-back under $f$ generates $\mathscr{C}_D$ an *adjoint equation*, and its zero locus $A$ an *adjoint divisor*. Set-theoretically, this implies that $\Sigma = A \cap D$. Ragni Piene [Pie79, §3] showed that an adjoint equation is given by the quotient

$$\frac{\partial h/\partial x_j}{\partial(f_1, \ldots, \widehat{f_j}, \ldots, f_{n+1})/\partial(x_1, \ldots, x_n)}.$$

For example, if $f(x_1, x_2) = (x_1, x_2^2, x_1x_2)$ is the parameterisation of the Whitney umbrella, with image $D = \{t_3^2 - t_1^2 t_2 = 0\}$, then this recipe gives $t_1$ as adjoint equation. Figure 1 below shows $D + A$ in this example.

We show

**Theorem 1.1.** *Let $D$ be the image of a stable map germ $(\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ of corank 1, and let $A$ be an adjoint divisor for $D$. Then $D + A$ is a free divisor.*

However, we show by an example that $D + A$ is not free when $D$ is the image of a stable germs of corank $\geq 2$ (see Example 2.4.(3)). We recall the standard normal forms for stable map germs of corank 1 in §2.

In §3, we prove the following analogous result for the discriminants of certain weighted homogeneous isolated function singularities. Recall that the discriminant has smooth normalization, so that the preceding definitions apply.

**Theorem 1.2.** *Let $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a weighted homogeneous polynomial of degree $d$ with isolated critical point and Milnor number $\mu$. Let $d_1 \geq d_2 \geq \cdots \geq d_\mu$ denote the degrees of the members of a weighted homogeneous $\mathbb{C}$-basis of the Jacobian algebra $\mathscr{O}_{\mathbb{C}^n,0}/\left\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right\rangle$. Assume that $d - d_1 + 2d_i \neq 0 \neq d - d_i$ for $i = 1, \ldots, \mu$.*

*Let $D$ be the discriminant in the base space of an $\mathscr{R}_e$-versal deformation of $f$ and let $A$ be an adjoint divisor for $D$. Then $D + A$ is a free divisor.*

Theorem 1.2 evidently applies to the simple singularities, since for these $d_1 < d$. It also applies in many other cases. For example, it is easily checked that the hypotheses on the weights hold for plane curve singularities of the form $x^p + y^q$ with $p$ and $q$ coprime. We do not know whether the conclusion continues to holds without the hypotheses on the weights.

The adjoint $A$ of the discriminant is closely related to the bifurcation set; this is discussed at the end of Section 3.

We remark that an adjoint can be defined verbatim in case $X$ is merely Gorenstein rather than smooth. In [GMS12, §6] the techniques used here are applied to construct new free divisors from certain Gorenstein varieties lying canonically over the discriminants of Coxeter groups.

In Theorem 1.2, $D$ is already a free divisor as remarked above. In contrast, in Theorem 1.1, $D$ itself is not free: the argument with the Kodaira Spencer map referred to above shows that $\mathrm{Der}(-\log D)$ has depth $n$ rather than $n+1$. So by adding $A$ we are making a non-free divisor free. Something similar was already done by Jim Damon, for the same divisor $D$, in [Dam98, Ex. 8.4]; Damon showed that after the addition of a certain divisor $E$ (not an adjoint divisor for $D$) with two irreducible components, $D+E$ is the discriminant of a $\mathscr{K}_V$-versal deformation of a non-linear section of another free divisor $V$. Freeness of $D+E$ followed by his general theorem on $\mathscr{K}_V$-versal discriminants. It seems that our divisor $D+A$ does not arise as a discriminant using Damon's procedure.

A crucial step in the proofs of both of Theorems 1.1 and 1.2 is the following fact (see Propositions 2.8 and 3.4).

**Proposition 1.3.** *In the situations of Theorems 1.1 and 1.2, $\mathscr{F}_1$ is cyclic as module over the Lie algebra* $\mathrm{Der}(-\log D)$*, and generated by an adjoint equation $a \in \mathscr{O}_{\mathbb{C}^{n+1},0}$ for $D$.*

Indeed, Theorem 1.2 follows almost trivially from this (see Proposition 3.9 below). We cannot see how to deduce Theorem 1.1 in an equally transparent way. Unfortunately our proof of Proposition 1.3 is combinatorial and not very revealing, something we hope to remedy in future work.

From Proposition 1.3 it follows that the adjoint is unique up to isomorphism preserving $D$ (see Corollaries 2.9 and 3.8).

In the process of proving Theorem 1.2, we note an easy argument which shows that the preimage of the adjoint divisor in the normalization of the discriminant is itself a free divisor. This is our Theorem 3.2.

Motivated by Theorems 1.1 and 1.2, we describe in §4 a general procedure which constructs, from a triple consisting of a free divisor $D$ with $k$ irreducible components and a free divisor in $(\mathbb{C}^k, 0)$ containing the coordinate hyperplanes, a new free divisor containing $D$. By this means we are able to construct a surprisingly large number of new examples of free divisors.

**Notation.** We shall denote by $\mathscr{F}^R_\ell(M)$ the $\ell$th *Fitting ideal* of the $R$-module $M$. For any presentation

$$R^m \xrightarrow{A} R^k \longrightarrow M \longrightarrow 0$$

it is generated by all $(k-\ell)$-minors of $A$ and defines the locus where $M$ requires more than $\ell$ generators.

For any analytic space germ $X$, we denote by $\Theta_X := \mathrm{Hom}_{\mathscr{O}_X}(\Omega^1_X, \mathscr{O}_X)$ the $\mathscr{O}_X$-module of vector fields on $X$. The $\mathscr{O}_X$-module of vector fields along an analytic map germ $f\colon X \to Y$ is defined by $\Theta(f) := f^*\Theta_Y$. For $X$ and $Y$ smooth, we shall use the standard operators $tf : \Theta_X \to \Theta(f)$ and $\omega f : \Theta_Y \to \Theta(f)$ of singularity theory defined by $tf(\xi) := Tf \circ \xi$ and $\omega f(\eta) = \eta \circ f$. For background in singularity theory we suggest the survey paper [Wal81] of C.T.C. Wall.

Throughout the paper, all the hypersurfaces we consider will be assumed reduced, without further mention.

**Acknowledgments.** We are grateful to Eleonore Faber for pointing out a gap in an earlier version of Theorem 4.1, and to the referee for a number of valuable suggestions.

## 2. Images of stable maps

Let $f\colon X := (\mathbb{C}^n,0) \to (\mathbb{C}^{n+1},0) =: T$ be a finite and generically one-to-one map-germ with image $D$. Note that $X = \tilde{D}$ is a normalization of $D$. By [MP89, Prop. 2.5], the $\mathscr{O}_T$-module $\mathscr{O}_X$ has a free resolution of the form

$$0 \longrightarrow \mathscr{O}_T^k \xrightarrow{\lambda} \mathscr{O}_T^k \xrightarrow{\alpha} \mathscr{O}_X \longrightarrow 0, \tag{2.1}$$

in which the matrix $\lambda$ can be chosen symmetric (we shall recall the proof below). For $1 \le i,j \le k$, we denote by $m^i_j$ the minor obtained from $\lambda$ by deleting the $i$th row and the $j$th column. The map $\alpha$ sends the $i$th basis vector $e_i$ to $g_i \in \mathscr{O}_X$, where $g = g_1,\dots,g_k$ generates $\mathscr{O}_X$ over $\mathscr{O}_T$. It will be convenient to assume, after reordering the $g_i$, that $g_k = 1$. This leads to a free presentation

$$\mathscr{O}_T^k \xrightarrow{\lambda^{\hat{k}}} \mathscr{O}_T^{k-1} \xrightarrow{\alpha} \mathscr{O}_X/\mathscr{O}_D \longrightarrow 0,$$

where $\lambda^{\hat{k}}$ is obtained from $\lambda$ by deleting the $k$th row (corresponding to the generator $g_k = 1$ of $\mathscr{O}_X$). By a theorem of Buchsbaum and Eisenbud [BE77], this shows that

$$\mathscr{F}_1' := \mathscr{F}_0^{\mathscr{O}_T}(\mathscr{O}_X/\mathscr{O}_D) = \operatorname{Ann}_{\mathscr{O}_T}(\mathscr{O}_X/\mathscr{O}_D) = \mathscr{C}_D\mathscr{O}_T. \tag{2.2}$$

As $\mathscr{F}_\ell^{\mathscr{O}_T}(\mathscr{O}_X)$ defines the locus where $\mathscr{O}_X$ requires more than $\ell$ $\mathscr{O}_T$-generators, $\det(\lambda)$ is an equation for $D$. By the hypothesis that $f$ is generically one-to-one, it is a reduced equation for $D$ (see [MP89, Prop. 3.1]).

By Cramer's rule one finds that in $\mathscr{O}_X$, $g_i m^j_s = \pm g_j m^i_s$ for $1 \le i,j,s \le k$ (see [MP89, Lem. 3.3]); invoking the symmetry of $\lambda$, this gives

$$g_i m^k_j = \pm g_j m^k_i, \quad 1 \le i,j \le k. \tag{2.3}$$

Combining this with the structure equations $g_ig_j = \sum_{\ell=1}^k \alpha^\ell_{i,j} g_\ell$ for $\mathscr{O}_X$ as $\mathscr{O}_T$-algebra, one shows that all of the $m^i_j$ lie in $\operatorname{Ann}_{\mathscr{O}_T}(\mathscr{O}_X/\mathscr{O}_D)$; then from (2.2), one deduces that (see [MP89, Thm. 3.4])

**Lemma 2.1.** *$\mathscr{F}_1 = \mathscr{F}_1' = \langle m^\mu_j \mid j = 1,\dots,\mu\rangle$ is a determinantal ideal.*

Since $\mathscr{F}_1$ therefore gives $\operatorname{Sing} D$ a Cohen-Macaulay structure, this structure is reduced if generically reduced. If $f$ is stable, then at most points of $\operatorname{Sing} D$, $D$ consists of two smooth irreducible components meeting transversely, from which generic reducedness follows.

As $g_k = 1$, it follows from (2.3) that $m^k_i = \pm g_i m^k_k$ and hence

**Lemma 2.2.** *$\mathscr{F}_1\mathscr{O}_X = \langle m^k_k\rangle_{\mathscr{O}_X}$ is a principal ideal.*

From this, we deduce

**Lemma 2.3.** *Any adjoint divisor $A$ for $D$ is of the form*

$$A = V\big(m^k_k + \sum_{j=1}^{k-1} c_j m^k_j\big), \quad c_j \in \mathscr{O}_T.$$

Mather [Mat69] showed how to construct normal forms for stable map germs: one begins with a germ $f\colon(\mathbb{C}^k,0)\to(\mathbb{C}^\ell,0)$ whose components lie in $\mathfrak{m}^2_{\mathbb{C}^k,0}$, and unfolds it by adding to $f$ terms of the form $u_ig_i$, where the $u_i$ are unfolding parameters and the $g_i$ form a basis for $\mathfrak{m}_{\mathbb{C}^k,0}\Theta(f)/(tf(\Theta_{\mathbb{C}^k,0}) + \mathfrak{m}_{\mathbb{C}^\ell,0}\Theta(f))$. Applying this construction to $f(x) = (x^k,0)$, one obtains the stable corank-1 map germ $f_k : (\mathbb{C}^{2k-2},0) \to (\mathbb{C}^{2k-1},0)$ given by

$$f_k(u,v,x) := (u,v,x^k + u_1x^{k-2} + \dots + u_{k-2}x, v_1x^{k-1} + \dots + v_{k-1}x) = (u,v,w), \tag{2.4}$$

where we abbreviate $u := u_1,\dots,u_{k-2}$, $v := v_1,\dots,v_{k-1}$, $w := w_1,w_2$.

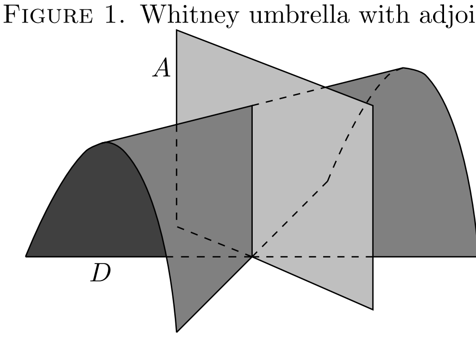
*Example* 2.4.

(1) When $k = 2$ in (2.4), $f_2(v, x) = (v, x^2, vx)$ parameterizes the Whitney umbrella in $(\mathbb{C}^3, 0)$, whose equation is $w_2^2 - v^2 w_1 = 0$. With respect to the basis $g = x, 1$ of $\mathscr{O}_S$ over $\mathscr{O}_T$, one has the symmetric presentation matrix

$$\lambda = \begin{pmatrix} v & -w_2 \\ -w_2 & vw_1 \end{pmatrix},$$

and this gives $a = v$ as equation of an adjoint $A$ (see Figure 1).

FIGURE 1. Whitney umbrella with adjoint




One calculates that $\mathrm{Der}(-\log D)$ and $\mathrm{Der}(-\log(D + A))$ are generated, respectively, by the vector fields whose coefficients are displayed as the columns of the matrices

$$\begin{pmatrix} v & v & 0 & w_2 \\ 2w_1 & 0 & 2w_2 & 0 \\ 0 & w_2 & v^2 & vw_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} v & v & 0 \\ 2w_1 & 0 & 2w_2 \\ 2w_2 & w_2 & w^2 \end{pmatrix}.$$

Note that here the basis of the free module $\mathrm{Der}(-\log(D + A))$ is included in a (minimal) list of generators of $\mathrm{Der}(-\log D)$. As we will see, this is always the case for the germs $f_k$ described above.

(2) Let $D_0$ be the image of the stable corank-2 map germ

$$(u_1, u_2, u_3, u_4, x, y) \mapsto (u_1, u_2, u_3, u_4, x^2 + u_1 y, xy + u_2 x + u_3 y, y^2 + u_4 x).$$

One calculates that $D_0 + A$ is *not* free.

(3) Every stable map germ of corank $k > 2$ is adjacent to the germ considered in the preceding example. That is, there are points on the image $D$ where $D$ is isomorphic to the product of the divisor $D_0$ we have just considered in (2) and a smooth factor. It follows that that $D + A$ also is not free.

(4) If $D$ is the image of an unstable corank 1 germ then in general $D + A$ is not free.

(5) A multi-germ of immersions $(\mathbb{C}^n, \{p_1, \ldots, p_k\}) \to (\mathbb{C}^{n+1}, 0)$ is stable if and only if it is a normal crossing. There are strata of such normal crossing points, of different multiplicities, on the image $D$ of a stable map germ such as (2.4). It is easy to show that adding an adjoint divisor to $D$ gives a free divisor at such normal crossing points. The normal crossing divisor $D = V(y_1 \cdots y_k) \subset (\mathbb{C}^k, 0)$ is normalized by separating its irreducible components. The equation $a := \sum_{j=1}^k y_1 \cdots \hat{y}_j \cdots y_k$ restricts to $y_1 \cdots \hat{y}_j \cdots y_k$ on the component $V(y_j)$, and so generates the conductor there, and thus $A := V(a)$ is an adjoint for $D$. The Euler vector field $\chi = \sum_{j=1}^k y_j \partial_{y_j}$, together with the $n - 1$ vector fields $\delta_j := y_j^2 \partial_{y_j} - y_{j+1}^2 \partial_{y_{j+1}}$, $j = 1, \ldots, k - 1$, all lie in $\mathrm{Der}(-\log(D + A))$. An application of Saito's criterion yields freeness of $D + A$ and shows that $\chi, \delta_1, \ldots, \delta_{k-1}$ form a basis for $\mathrm{Der}(-\log(D + A))$.

Our proof of Theorem 1.1 is based on Saito's criterion. By Mather's construction, we are concerned with the map $f := f_k$ of (2.4) where now $n = 2k-2$. Using an explicit list of generators of $\mathrm{Der}(-\log D)$ constructed by Houston and Littlestone in [HL09], and testing them on the equation $m_k^k$ of $A$, we find a collection of vector fields $\xi_1, \dots, \xi_{2k-1}$ in $\mathrm{Der}(-\log D)$ which are in $\mathrm{Der}(-\log A)$ "to first order", in the sense that for $j = 1, \dots, 2k-1$, we have $\xi_j \cdot m_k^k \in \langle m_k^k \rangle + \mathfrak{m}_T \mathscr{F}_1$. Note that $\mathscr{F}_1$ is intrinsic to $D$, and therefore invariant under any infinitesimal automorphism of $D$, so that necessarily $\xi_j \cdot m_k^k \in \mathscr{F}_1$. In the process of testing, we show that the map $\mathrm{Der}(-\log D) \to \mathscr{F}_1$ sending $\xi$ to $\xi \cdot m_k^k$ is surjective. Using this, we can then adjust the $\xi_j$, without altering their linear part, so that now $\xi_j \cdot m_k^k \in \langle m_k^k \rangle$ for $j = 1, \dots, 2k-1$. As a consequence, the determinant of their Saito matrix must be divisible by the equation of $D + A$. This determinant contains a distinguished monomial also present in the equation of $D + A$, so the quotient of the determinant by the equation of $D + A$ is a unit, the determinant is a reduced equation for $D + A$, and $D + A$ is a free divisor, by Saito's criterion.

To begin this process, we need more detailed information about the matrix $\lambda$ of (2.1). We use a trick from [MP89, §2]: embed $X$ as $X \times \{0\}$ into $X \times (\mathbb{C}, 0) := S$, and let the additional variable in $S$ be denoted by $t$. Extend $f\colon X \to T$ to a map $F\colon S \to T$ by adding $t$ to the last component of $f$. Applying this procedure to the map $f = f_k$ of (2.4), gives

$$F(u, v, x, t) := (u, v, x^k + u_1 x^{k-2} + \cdots + u_{k-2} x, v_1 x^{k-1} + \cdots + v_{k-1} x + t) = (u, v, w). \tag{2.5}$$

It is clear that $\mathscr{O}_S / \mathfrak{m}_T \mathscr{O}_S$ is generated over $\mathbb{C}$ by the classes of $1, x, \dots, x^{k-1}$, from which it follows by [Gun74, Cor. 2, p. 137] that $\mathscr{O}_S$ is a free $\mathscr{O}_T$-module on the basis

$$g_i := x^{k-i}, \quad i = 1, \dots, k. \tag{2.6}$$

Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{O}_S \arrow[r, "t"] & \mathscr{O}_S \arrow[r] & \mathscr{O}_X \arrow[r] & 0 \\
0 \arrow[r] & \mathscr{O}_T^k \arrow[r, "{[t]_g^g}"'] \arrow[u, "\varphi_g"] & \mathscr{O}_T^k \arrow[r] \arrow[u, "\varphi_g"] & \mathscr{O}_X \arrow[r] \arrow[u, equal] & 0
\end{tikzcd} \tag{2.7}$$

in which $\varphi_g$ is the $\mathscr{O}_T$-isomorphism sending $(c_1, \dots, c_k) \in \mathscr{O}_T^k$ to $\sum_{j=1}^k c_j g_j \in \mathscr{O}_S$, and where now $[t]_g^g$ denotes the matrix of multiplication by $t$ with respect to the basis $g$ of $\mathscr{O}_S$ as $\mathscr{O}_T$-module. The lower row is thus a presentation of $\mathscr{O}_X$ as $\mathscr{O}_T$-module. This can be improved by a change of basis on the source of $[t]_g^g$, as follows.

Since $\mathscr{O}_S$ is Gorenstein, $\mathrm{Hom}_{\mathscr{O}_T}(\mathscr{O}_S, \mathscr{O}_T) \cong \mathscr{O}_S$ as $\mathscr{O}_S$-module. Let $\Phi$ be an $\mathscr{O}_S$-generator of $\mathrm{Hom}_{\mathscr{O}_T}(\mathscr{O}_S, \mathscr{O}_T)$. It induces a symmetric perfect pairing

$$\langle \cdot, \cdot \rangle : \mathscr{O}_S \times \mathscr{O}_S \to \mathscr{O}_T, \quad \langle a, b \rangle = \Phi(ab)$$

with respect to which multiplication by $t$ is self-adjoint. We refer to this pairing as the Gorenstein pairing. Now choose a basis $\check{g} = \check{g}_1, \dots, \check{g}_k$ for $\mathscr{O}_S$ over $\mathscr{O}_T$ dual to $g$ with respect to $\langle \cdot, \cdot \rangle$; that is, such that $\langle g_i, \check{g}_j \rangle = \delta_{i,j}$. Then the $(i,j)$th entry of $[t]_g^{\check{g}}$ equals $\langle t\check{g}_j, \check{g}_i \rangle$, and so redefining

$$\lambda = (\lambda_j^i) = (\lambda_1, \dots, \lambda_k) := [t]_g^{\check{g}} \tag{2.8}$$

yields a symmetric presentation matrix.

**Lemma 2.5.** *With an appropriate choice of generator $\Phi$ of $\operatorname{Hom}_{\mathscr{O}_T}(\mathscr{O}_S, \mathscr{O}_T)$, we have*

$$\lambda \equiv \begin{pmatrix} -v_1 & -v_2 & -v_3 & \cdots & -v_{k-1} & w_2 \\ -v_2 & -v_3 & & \iddots & w_2 & 0 \\ -v_3 & & & \iddots & \iddots & \vdots \\ \vdots & \iddots & \iddots & & & \vdots \\ -v_{k-1} & w_2 & \iddots & & & \vdots \\ w_2 & 0 & \cdots & & \cdots & 0 \end{pmatrix} \mod \langle u, w_1 \rangle. \tag{2.9}$$

*Proof.* Let $b \in \mathscr{O}_S$ map to the socle of the 0-dimensional Gorenstein ring $\mathscr{O}_S/\mathfrak{m}_T\mathscr{O}_S$. This means that

$$\mathfrak{m}_S b \subset \mathfrak{m}_T \mathscr{O}_S.$$

Thus, for any $a \in \mathfrak{m}_S$ and $\Psi \in \operatorname{Hom}_{\mathscr{O}_T}(\mathscr{O}_S, \mathscr{O}_T)$, we have

$$(a\Psi)(b) = \Psi(ab) = \Psi(\sum_i c_i b_i) = \sum_i c_i \Psi(b_i) \in \mathfrak{m}_T$$

where $c_i \in \mathfrak{m}_T$ and $b_i \in \mathscr{O}_S$. In other words, any $\Phi \in \mathfrak{m}_S \operatorname{Hom}_{\mathscr{O}_T}(\mathscr{O}_S, \mathscr{O}_T)$ maps $b$ to $\mathfrak{m}_T$. Conversely, for any $\Phi \in \operatorname{Hom}_{\mathscr{O}_T}(\mathscr{O}_S, \mathscr{O}_T)$ with $\Phi(b) \in \mathscr{O}_T^*$, its class $\bar{\Phi}$ in

$$\operatorname{Hom}_{\mathscr{O}_T}(\mathscr{O}_S, \mathscr{O}_T)/\mathfrak{m}_S \operatorname{Hom}_{\mathscr{O}_T}(\mathscr{O}_S, \mathscr{O}_T)$$

is non-zero. As $\operatorname{Hom}_{\mathscr{O}_T}(\mathscr{O}_S, \mathscr{O}_T) \cong \mathscr{O}_S$, this latter space is 1-dimensional over $\mathbb{C}$ which shows that $\bar{\Phi}$ is a generator. By Nakayama's lemma, $\Phi$ is then an $\mathscr{O}_S$-basis of $\operatorname{Hom}_{\mathscr{O}_T}(\mathscr{O}_S, \mathscr{O}_T)$. In particular, we may take as $\Phi(h)$, for $h \in \mathscr{O}_S$, the coefficient of $b = x^{k-1}$ in the representation of $h$ in the basis (2.6); then $\Phi(b) = 1$.

In the following, we implicitly compute modulo $\langle u, w_1 \rangle$. Using the relation

$$w_1 = x^k + \sum_{i=1}^{k-2} u_i x^{k-1-i} \tag{2.10}$$

from (2.5) we compute

$$\check{g}_1 = 1,\ \check{g}_j = \left( w_1 - \sum_{i=j-1}^{k-2} u_i x^{k-1-i} \right) \Big/ x^{k+1-j} = x^{j-1} + \sum_{i=1}^{j-2} u_i x^{j-i-2},\ j = 2, \ldots, k.$$

Note that

$$\check{g}_2 = x\check{g}_1, \quad \check{g}_j = x\check{g}_{j-1} + u_{j-2}\check{g}_1,\ j = 3, \ldots, k. \tag{2.11}$$

Now let us calculate the columns $\lambda_1, \ldots, \lambda_k$ of the matrix (2.8). Using $\check{g}_1 = 1$ and the relation $t + \sum_{i=1}^{k-1} v_i g_i = w_2 = w_2 g_k$ from (2.5), we first compute

$$\lambda_1 = [t\check{g}_1]_g = (\langle t, \check{g}_j \rangle)_j = \left( \left\langle w_2 g_k - \sum_{i=1}^{k-1} v_i g_i,\ \check{g}_j \right\rangle \right)_j = (-v_1, \ldots, -v_{k-1}, w_2)^t.$$

By (2.11), each of the remaining columns $\lambda_j$ is obtained by multiplying by $x$ the vector represented by its predecessor, and, for $j \geq 3$, adding $u_{j-2}\lambda_1$. Thus,

$$\lambda_2 = [x]_g^g \lambda_1, \quad \lambda_j = [x]_g^g \lambda_{j-1} + u_{j-2}\lambda_1,\ j = 3, \ldots, k. \tag{2.12}$$

Using (2.10) again, observe that

$$[x]_g^g = \begin{pmatrix} w_1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ -u_1 & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & 1 \\ -u_{k-2} & 0 & \cdots & \cdots & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & 1 \\ 0 & 0 & \cdots & \cdots & 0 \end{pmatrix} \quad \mathrm{mod}\ \langle u, w_1 \rangle. \tag{2.13}$$

The result follows. □

**Corollary 2.6.** *The reduced equation $h$ of the image $D$ of the map $f_k$ of (2.4) contains the monomial $w_2^k$ with coefficient $\pm 1$. The minor $m_k^k$ contains the monomial $w_2^{k-2}v_1$ with coefficient $\pm 1$.*

*Proof.* The determinant of the matrix $\lambda$ of (2.9) is a reduced equation for the image of $f$ (see [MP89, Prop. 3.1]). Both statements then follow from Lemma 2.5. □

*Example* 2.7. For the stable map-germs

$$f_3(u_1, v_1, v_2, x) = (u_1, v_1, v_2, x^3 + u_1 x, v_1 x^2 + v_2 x)$$

and

$$f_4(u_1, u_2, v_1, v_2, v_3, x) = (u_1, u_2, v_1, v_2, v_3, x^4 + u_1 x^2 + u_2 x, v_1 x^3 + v_2 x^2 + v_3 x)$$

of (2.4), the matrix $\lambda$ is equal to

$$\begin{pmatrix} -v_1 & -v_2 & w_2 \\ -v_2 & w_2 + u_1 v_1 & -v_1 w_1 \\ w_2 & -v_1 w_1 & v_2 w_1 - u_1 w_2 \end{pmatrix}$$

and to

$$\begin{pmatrix} -v_1 & -v_2 & -v_3 & w_2 \\ -v_2 & u_1 v_1 - v_3 & w_2 + u_2 v_1 & -v_1 w_1 \\ -v_3 & w_2 + u_2 v_1 & u_2 v_2 - u_1 v_3 - v_1 w_2 & -v_2 w_1 + u_1 w_2 \\ w_2 & -v_1 w_2 & -v_2 w_1 + u_1 w_2 & -v_3 w_1 + u_2 w_2 \end{pmatrix}$$

respectively.

*Proof of Theorem 1.1.* In [HL09, Thms. 3.1-3.3], Houston and Littlestone give an explicit list of generators for $\mathrm{Der}(-\log D)$. Their proof that these generators lie in $\mathrm{Der}(-\log D)$ simply exhibits, for each member $\xi$ of the list, a lift $\eta \in \Theta_X$ in the sense that $tf(\eta) = \omega f(\xi)$. The Houston-Littlestone list consists of the Euler field $\xi_e$ and three families $\xi_j^i$, $1 \leq i \leq 3$, $1 \leq j \leq k-1$. Denote by $\bar{\xi}_j^i$ the linear part of $\xi_j^i$. After dividing by 1, $k$, $k$, $k$ and $k^2$ respectively, these linear parts are

$$\bar{\xi}_e = \sum_{i=1}^{k-2}(i+1)u_i\partial_{u_i} + \sum_{i=1}^{k-1} iv_i\partial_{v_i} + kw_1\partial_{w_1} + kw_2\partial_{w_2},$$
$$\bar{\xi}^1_j = -w_2\partial_{v_j} + \sum_{i<j} v_{i-j+k}\partial_{v_i}, \quad 1 \le j \le k-1,$$
$$\bar{\chi} = -\bar{\xi}^2_1 = -\sum_{i=1}^{k-2}(i+1)u_i\partial_{u_i} + \sum_{i=1}^{k-1}(k-i)v_i\partial_{v_i} + kw_1\partial_{w_1},$$
$$\bar{\xi}^2_j = \sum_{i<k-j}(i+j)u_{i+j-1}\partial_{u_i} - \sum_{i<k-j+1}(k-i-j+1)v_{i+j-1}\partial_{v_i}, \quad 1 < j \le k-1,$$
$$\bar{\xi}^3_j = -(1-\delta_{j,1})w_2\partial_{u_{k-j}} + \sum_{i<k-j} v_{i+j}\partial_{u_i} + \delta_{j,1}w_2\partial_{w_1}, \quad 1 \le j \le k-1.$$

Also let

$$\bar{\sigma} = (\bar{\xi}_e + \bar{\chi})/k = \sum_{i=1}^{k-1} v_i\partial_{v_i} + w_2\partial_{w_2}.$$

We now test the above vector fields for tangency to $A = V(m^k_k)$. Vector fields in $\mathrm{Der}(-\log D)$ preserve $\mathscr{F}_1$, so for each $\xi \in \mathrm{Der}(-\log D)$ there exist $c_j \in \mathscr{O}_T$, unique modulo $\mathfrak{m}_T$, such that

$$\xi(m^k_k) = \sum_j c_j m^k_j.$$

We determine their value modulo $\mathfrak{m}_T$ with the help of distinguished monomials. Let $\iota$ be the sign of the order-reversing permutation of $1, \dots, k-1$. Then, by Lemma 2.5, for $1 < j \le k$, the monomial $w_2^{k-2}v_{k-j+1}$ appears in the polynomial expansion of $m^k_j$ with coefficient $(-1)^{j-1}\iota$ but does not appear in the polynomial expansion of $m^k_\ell$ for $\ell \ne j$. Similarly, the monomial $w_2^{k-1}$ has coefficient $\iota$ in the polynomial expansion of $m^k_1$ but does not appear in that of $m^k_j$ for $j \ge 2$.

Let $\lambda'_1, \dots, \lambda'_k$ denote the columns of the matrix $\lambda$ of (2.9) with its last row deleted. For any $\delta \in \Theta_T$, we have

$$\delta(m^k_k) = \sum_{r=1}^{k-1} \det(\lambda'_1, \dots, \delta(\lambda'_r), \dots, \lambda'_{k-1}). \tag{2.14}$$

For $\delta = \xi^2_j$, the only distinguished monomial to appear in any of the summands in (2.14) is $v_j w_2^{k-2}$, which appears in the summand for $r = 1$, with coefficient $(k-j)(-1)^k\iota$. Thus

$$\xi^2_j(m^k_k) = (-1)^j(k-j)m^k_{k-j+1} \mod \mathfrak{m}_T\mathscr{F}_1, \quad \text{for } 1 \le j \le k-1. \tag{2.15}$$

Similarly we find

$$\xi^1_j(\lambda'_r) = \begin{cases} \lambda'_{k-j+r} & \text{if } r \le j, \\ 0 & \text{otherwise.} \end{cases}$$

Using (2.14) with $\delta = \xi^1_j$, it follows that

$$\xi^1_j(m^k_i) = (-1)^{k-j+1}m^k_{i+j-k} \mod \mathfrak{m}_T\mathscr{F}_1, \quad \text{for } 1 \le i \le k,\ 1 \le j \le k-1. \tag{2.16}$$

Note that (2.15), and (2.16) with $i = k$, imply

**Proposition 2.8.** *The map*

$$dm^k_k : \mathrm{Der}(-\log D) \to \mathscr{F}_1 \tag{2.17}$$

*sending $\xi \in \mathrm{Der}(-\log D)$ to $\xi(m^k_k)$ is an $\mathscr{O}_T$-linear surjection.* □

Combining (2.15) and (2.16) with $i = k$, we construct vector fields

$$\eta_j = (j-1)\xi^1_j - \xi^2_{k-j+1}, \quad 2 \le j \le k-1$$

with linear part

$$\bar{\eta}_j \equiv (1-j)w_2\partial_{v_j} + \sum_{i<j}(2j-i-1)v_{k+i-j}\partial_{v_i} \mod \langle \partial_u, \partial_{w_1}\rangle,$$

which lie in $\mathrm{Der}(-\log(D+A))$ to first order, since $\eta_j(m^k_k) \in \mathfrak{m}_T\mathscr{F}_1$.

Both $\bar{\chi}$ and $\bar{\sigma}$ are semi-simple, and so by consideration of the distinguished monomials, $\chi$ and $\sigma$ must therefore lie in $\mathrm{Der}(-\log A)$ to first order. The vector fields $\xi^3_j$ lie in $\mathrm{Der}(-\log A)$ to first order, since it is clear by consideration of the distinguished monomials that $\xi^3_j(m^k_k) \in \mathfrak{m}_T\mathscr{F}_1$.

Thus we have $2k-1$ vector fields $\eta_2, \dots, \eta_{k-1}, \chi, \sigma, \xi^3_1, \dots, \xi^3_{k-1}$ in $\mathrm{Der}(-\log D)$ which are also in $\mathrm{Der}(-\log A)$ to first order. By Proposition 2.8, we can modify these by the addition of suitable linear combinations, with coefficients in $\mathfrak{m}_T$, of the Houston–Littlestone generators of $\mathrm{Der}(-\log D)$, so that they are indeed in $\mathrm{Der}(-\log A)$ and therefore in $\mathrm{Der}(-\log(D+A))$. The determinant of the Saito matrix of the modified vector fields $\tilde{\eta}_2, \dots, \tilde{\eta}_{k-1}, \tilde{\chi}, \tilde{\sigma}, \tilde{\xi}^3_1, \dots, \tilde{\xi}^3_{k-1}$ must be a multiple $\alpha h m^k_k$ of the equation of $D + A$. We now show that $\alpha$ is a unit, from which it follows, by Saito's criterion, that $D + A$ is a free divisor.

The modification of the vector fields does not affect the lowest order terms in the determinant of their Saito matrix, and these are the same as the lowest order terms in the determinant of the Saito matrix of their linear parts. With the rows representing the coefficients of $\partial_{u_1}, \dots, \partial_{u_{k-2}}, \partial_{w_1}, \partial_{v_1}, \dots, \partial_{v_{k-1}}, \partial_{w_2}$ in this order, this matrix is of the form

$$\begin{pmatrix} * & \bar{B}_1 \\ \bar{B}_2 & 0 \end{pmatrix},$$

with

$$\bar{B}_1 = \begin{pmatrix} v_2 & v_3 & v_4 & \cdots & v_{k-1} & -w_2 \\ v_3 & v_4 & & \iddots & -w_2 & 0 \\ v_4 & & & \iddots & \iddots & \vdots \\ \vdots & \iddots & \iddots & & & \\ v_{k-1} & -w_2 & \iddots & & & \vdots \\ w_2 & 0 & \cdots & & \cdots & 0 \end{pmatrix}$$

and

$$\bar{B}_2 = \begin{pmatrix} 2v_{k-1} & 4v_{k-2} & 6v_{k-3} & \cdots & \cdots & (2k-4)v_2 & (k-1)v_1 & v_1 \\ -w_2 & 3v_{k-1} & 5v_{k-2} & \cdots & \cdots & (2k-5)v_3 & (k-2)v_2 & v_2 \\ 0 & -2w_2 & 4v_{k-1} & \ddots & & \vdots & \vdots & \vdots \\ \vdots & 0 & -3w_2 & \ddots & \ddots & \vdots & \vdots & \vdots \\ & \vdots & 0 & \ddots & \ddots & kv_{k-2} & 3v_{k-3} & v_{k-3} \\ \vdots & \vdots & \vdots & \ddots & \ddots & (k-1)v_{k-1} & 2v_{k-2} & v_{k-2} \\ 0 & 0 & 0 & \cdots & 0 & -(k-2)w_2 & v_{k-1} & v_{k-1} \\ 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & w_2 \end{pmatrix}$$

In its determinant we find the monomial $w_2^{2k-2}v_1$ with coefficient $\pm(k-1)!$. By Corollary 2.6, this monomial is present in the equation of $D + A$. This proves that $\alpha$ is a unit and completes the proof that $D + V(m^k_k)$ is a free divisor.

The following consequence of Proposition 2.8 is needed to prove that Theorem 1.1 holds for *any* adjoint divisor $A$ of $D$, and not just for $A = V(m_k^k)$.

**Corollary 2.9.** *The adjoint divisor $A$ is unique up to isomorphism preserving $D$.*

*Proof.* Let $A_0 := V(m_k^k)$. By Lemma 2.3, any adjoint divisor $A_1$ must have an equation of the form $m_1 := m_k^k + \sum_{i=1}^{k-1} c_i m_i^k$. Consider the family of divisors

$$A := V(m) \subset T \times (\mathbb{C}, 1), \quad m := m_k^k + s \sum_{j=1}^{k-1} c_j m_j^k, \tag{2.18}$$

where $s$ is a coordinate on $(\mathbb{C}, 1)$. We claim that there exists a vector field

$$\Xi \in \mathrm{Der}_{T\times(\mathbb{C},1)/(\mathbb{C},1)}(-\log(D \times (\mathbb{C}, 1)))$$

such that

$$\Xi(m) = \partial_s(m). \tag{2.19}$$

Then the vector field $\partial_s - \Xi$ is tangent to $D \times (\mathbb{C}, 1)$, and its integral flow trivializes the family (2.18). Passing to representatives of germs and scaling the $c_j$, the latter holds true in an open neighborhood of any point of the interval $\{0\} \times [0, 1] \subset \mathbb{C}^{n+1} \times \mathbb{C}$. A finite number of such neighborhoods cover this interval and it follows that $A_0$ and $A_1$ are isomorphic by an isomorphism preserving $D$.

To construct $\Xi$ we first show, using Proposition 2.8, that

$$dm_1 : \mathrm{Der}(-\log D) \to \mathscr{F}_1 \tag{2.20}$$

is surjective. By (2.17), there are vector fields $\delta_1, \ldots, \delta_k \in \mathrm{Der}(-\log D)$, homogeneous with respect to the grading determined by the vector field $\chi$, such that $\delta_j(m_k^k) = m_j^k$ for $j = 1, \ldots, k$. Pick $\alpha_{i,j}^\ell \in \mathscr{O}_T$ such that

$$\delta_i(m_j^k) = \sum_{\ell=1}^{k} \alpha_{i,j}^\ell m_\ell^k, \quad 1 \le i, j \le k,$$

and set $L_j := (\alpha_{i,j}^\ell)_{1\le \ell, i \le k}$. The constant parts $L_j(0)$ are uniquely defined and $L_k$ is the identity matrix by choice of the $\delta_j$. For $j < k$, with respect to the grading determined by $\chi$, we have

$$\deg(m_k^k) < \deg(m_{k-1}^k) \le \cdots \le \deg(m_1^k)[1].$$

This gives

$$\deg(\delta_i(m_j^k)) > \deg(\delta_i(m_k^k)) = \deg(m_i^k) \ge \deg(m_{i+1}^k) \ge \cdots \ge \deg(m_k^k),$$

and hence

$$\delta_i(m_j^k) \in \langle m_1^k, \ldots, m_{i-1}^k \rangle + \mathfrak{m}_T \mathscr{F}_1.$$

The constant matrices $L_j(0)$, $j < k$, are therefore strictly upper triangular, and the constant part $L_k + \sum_{j=1}^{k-1} c_j(0) L_j(0)$ of the matrix of $dm_1$ is invertible. Thus,

$$dm_1(\mathrm{Der}(-\log D)) + \mathfrak{m}_T \mathscr{F}_1 = \mathscr{F}_1,$$

and (2.20) follows by Nakayama's Lemma. As $m \equiv m_1 \mod \mathfrak{m}_{T\times(\mathbb{C},1)}$, (2.20) gives

$$dm(\mathrm{Der}_{T\times(\mathbb{C},1)/(\mathbb{C},1)}(-\log(D \times (\mathbb{C}, 1))) + \mathfrak{m}_{T\times(\mathbb{C},1)} \mathscr{O}_{T\times(\mathbb{C},1)} \mathscr{F}_1 \supset \mathscr{O}_{T\times(\mathbb{C},1)} \mathscr{F}_1$$

and Nakayama's Lemma yields

$$dm(\mathrm{Der}_{T\times(\mathbb{C},1)/(\mathbb{C},1)}(-\log(D \times (\mathbb{C}, 1))) \supset \mathscr{O}_{T\times(\mathbb{C},1)} \mathscr{F}_1. \tag{2.21}$$

[1]All the inequalities here are in fact strict, but we want to use the argument again later in a context where we assume only what is written here (see Corollary 3.8).

Then any preimage $\Xi$ of $\partial_s(m) \in \mathscr{F}_1$ under $dm$ solves (2.19). $\square$

The proof of Theorem 1.1 is now complete. $\square$

## 3. Discriminants of hypersurface singularities

Let $f\colon X := (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) =: T$ be weighted homogeneous of degree $d$ (with respect to positive weights) and have an isolated critical point at 0. Let $\chi_0$ be an Euler vector field for $f$, with $\chi_0(f) = d \cdot f$. Denote by $J_f := \langle \partial_{x_1}(f), \dots, \partial_{x_n}(f)\rangle$ the Jacobian ideal of $f$. Pick a weighted homogeneous $g = g_1, \dots, g_\mu \in \mathscr{O}_X$ with decreasing degrees $d_i := \deg(g_i)$ inducing a $\mathbb{C}$-basis of the Jacobian algebra

$$M_f := \mathscr{O}_X / J_f.$$

We may take $g_\mu := 1$ and $g_1 := H_0$ to be the Hessian determinant $H_0$ of $f$, which generates the socle of $M_f$. Then

$$F(x, u) := f(x) + g_1(x)u_1 + \cdots + g_\mu(x)u_\mu \tag{3.1}$$

defines an $\mathscr{R}_e$-versal unfolding

$$F \times \pi_S \colon Y := X \times S \to T \times S$$

of $f$, with base space $S := (\mathbb{C}^\mu, 0)$, where

$$\pi = \pi_S \colon Y = X \times S \to S$$

is the natural projection. Setting $\deg(u_i) = w_i := d - d_i$ makes $F$ weighted homogeneous of degree $\deg(F) = d = \deg(f)$. We denote by $\chi$ the Euler vector field $\chi_0 + \delta_1$ where $\delta_1 = \sum_{i=1}^{\mu} w_i u_i \partial_{u_i}$.

Let $\Sigma \subset Y$ be the relative critical locus of $F$, defined by the relative Jacobian ideal

$$J_F^{\mathrm{rel}} = \big\langle \partial_{x_1}(F), \dots, \partial_{x_\mu}(F)\big\rangle,$$

and set $\Sigma^0 = \Sigma \cap V(F)$. Then $\mathscr{O}_\Sigma$ is a finite free $\mathscr{O}_S$-module with basis $g$. As $\Sigma$ is smooth and hence Gorenstein, $\mathrm{Hom}_{\mathscr{O}_S}(\mathscr{O}_\Sigma, \mathscr{O}_S) \cong \mathscr{O}_\Sigma$ as $\mathscr{O}_\Sigma$-modules, and a basis element $\Phi$ defines a symmetric perfect pairing

$$\langle\cdot,\cdot\rangle\colon \mathscr{O}_\Sigma \otimes_{\mathscr{O}_S} \mathscr{O}_\Sigma \to \mathscr{O}_S, \quad \langle g, h\rangle := \Phi(gh),$$

which we refer to as the Gorenstein pairing. As in §2 (see the proof of Lemma 2.5), a generator $\Phi$ may be defined by projection to the socle of the special fiber: We let $\Phi(h)$ be the coefficient of the Hessian $g_1$ in the expression of $h$ in the basis $g$. By

$$\langle\cdot,\cdot\rangle_0\colon M_f \otimes_{\mathbb{C}} M_f \to \mathbb{C}$$

we denote the induced Gorenstein pairing on $\mathscr{O}_\Sigma/\mathfrak{m}_S\mathscr{O}_\Sigma = \mathscr{O}_X/J_f = M_f$.

Let $\check{g} = \check{g}_1, \dots, \check{g}_\mu$ denote the dual basis of $g$ with respect to the Gorenstein pairing, and denote by $\check{d}_i$ the degree of $\check{g}_i$. We have $d_i + \check{d}_i = d_1$, so $\check{d}_i = d_{\mu+i-1}$ (recall that we have ordered the $g_i$ by descending degree).

The discriminant $D = \pi_S(\Sigma^0) \subset S$ was shown by Kyoji Saito (see [Sai80a]) to be a free divisor. The following argument proves this, and shows also that it is possible to choose a basis for $\mathrm{Der}(-\log D)$ whose Saito matrix is symmetric.

**Theorem 3.1.** *There is a free resolution of $\mathscr{O}_{\Sigma^0}$ as $\mathscr{O}_S$-module*

$$0 \longrightarrow \mathscr{O}_S^\mu \xrightarrow{\Lambda} \Theta_S \xrightarrow{dF} \mathscr{O}_{\Sigma^0} \longrightarrow 0$$

*in which $\Lambda$ is symmetric, and is the Saito matrix of a basis of* $\mathrm{Der}(-\log D)$.

*Proof.* As in (2.7), there is a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{O}_\Sigma \arrow[rr, "F"] & & \mathscr{O}_\Sigma \arrow[r] & \mathscr{O}_{\Sigma^0} \arrow[r] & 0 \\
0 \arrow[r] & \mathscr{O}_S^\mu \arrow[u, "\varphi_{\check{g}}", "\cong"'] \arrow[rr, "\Lambda"] \arrow[dr, "\cong"] & & \Theta_S \arrow[u, "\varphi_g", "\cong"'] \arrow[ur, "dF"'] & & \\
& & \mathrm{Der}(-\log D) \arrow[ur, hook] & & &
\end{tikzcd}$$

where $\Lambda = (\lambda^i_j)_{1\leq i,j\leq \mu}$ is the matrix of multiplication by $F$ with respect to bases $\check{g}$ in the source and $g$ in the target. As in (2.8), symmetry of $\Lambda$ follows from self-adjointness of multiplication by $F$ with respect to the Gorenstein pairing. Because of the form of $F$, the map $\varphi_g : \Theta_S \to \mathscr{O}_\Sigma$ sending $\eta = \sum_j \alpha_j \partial_{u_j}$ to $\sum_j \alpha_j g_j$ coincides with evaluation of $dF$ on a(ny) lift $\tilde{\eta} \in \Theta_Y$ of $\eta$; different lifts of the same vector field differ by a sum $\sum_j \alpha_j \partial_{x_j} \in \Theta_{Y/S}$, and the evaluation of $dF$ on such a sum vanishes on $\Sigma$. The kernel of the composite $\Theta_S \to \mathscr{O}_{\Sigma_0}$ consists of vector fields on $S$ which lift to vector fields on $Y$ which are tangent to $V(F)$, since $dF(\tilde{\eta})$ is divisible by $F$ if and only if $\tilde{\eta} \in \mathrm{Der}(-\log V(F))$. It is well known (see e.g. [Loo84, Lem. 6.14]) that the set of vector fields on $S$ which lift to vector fields tangent to $V(F)$ is equal to $\mathrm{Der}(-\log D)$. $\square$

Denote by $m^i_j$ the $(\mu-1)$-minor of $\Lambda$ obtained by deleting the $i$th row and the $j$th column. Then Lemmas 2.1, 2.2 and 2.3 of §2 remain valid in this new context. That is,

$$\mathscr{F}_1 := \mathscr{F}_1^{\mathscr{O}_S}(\mathscr{O}_{\Sigma^0}) = \left\langle m^\mu_j \mid j = 1, \ldots, \mu \right\rangle_{\mathscr{O}_S}, \quad \mathscr{F}_1 \mathscr{O}_{\Sigma^0} = \left\langle m^\mu_\mu \right\rangle_{\mathscr{O}_{\Sigma^0}}, \tag{3.2}$$

and the adjoints of $D$ are divisors of the form

$$A = V\bigl(m^\mu_\mu + \sum_{j=1}^{\mu-1} c_j m^\mu_j\bigr).$$

Although it is not part of the main thrust of our paper, the following result seems to be new, and is easily proved. It assumes that $D$ is the discriminant of an $\mathscr{R}_e$-versal deformation, but does not require any assumption of weighted homogeneity. We denote by $H$ the relative Hessian determinant of the deformation (3.1).

**Theorem 3.2.** *Let $A$ be any adjoint divisor for $D$. Then*

$$\tilde{A} := \pi^{-1}(A) \cap \Sigma^0$$

*is a free divisor in $\Sigma^0$ containing $V(H) \cap \Sigma^0$, with reduced defining equation $(m^\mu_\mu \circ \pi)/H$.*

*Proof.* By Corollary 3.8 below, we may assume that $A = V(m^\mu_\mu)$, and hence $\tilde{A} = V(m^\mu_\mu \circ \pi) \cap \Sigma_0$. First, it is necessary to show that $H^2$ divides $m^\mu_\mu \circ \pi$ and that $m^\mu_\mu / H$ is reduced. Since $\Sigma^0$ is smooth, it is enough to check this at generic points of $V(H)$. This reduces to checking that it holds at an $A_2$-point. The miniversal deformation of an $A_2$-singularity is given by $G(x, v_1, v_2) = x^3 + v_1 x + v_2$. In this case, $m^\mu_\mu$ is, up to multiplication by a unit, simply the coefficient of $\partial_{v_1}$ in the Euler vector field, namely $v_1$, and $v_1 = -3x^2$ on $\Sigma^0$. The Hessian $H$ is equal to $6x$, so $H^2$ does divide $m^\mu_\mu \circ \pi$, and moreover the quotient $(m^\mu_\mu \circ \pi)/H$ is reduced.

As $\Sigma^0$ is the normalization of $D$ (see [Loo84, Thm. 4.7]), vector fields tangent to $D$ lift to vector fields tangent to $\Sigma^0$ (see [Sei66]). Let $\tilde{\delta}_1, \ldots, \tilde{\delta}_\mu$ be the lifts to $\Sigma^0$ of the symmetric basis $\delta_1, \ldots, \delta_\mu$ of $\mathrm{Der}(-\log D)$ constructed in Theorem 3.1. We will show that $\tilde{\delta}_1, \ldots, \tilde{\delta}_{\mu-1}$ form a basis for $\mathrm{Der}(-\log \tilde{A})$.

To see this, pick coordinates for $\Sigma^0$, and denote by $(\tilde{\delta}_1, \dots, \tilde{\delta}_\mu)$ the matrix whose $j$th column consists of the coefficients of the vector field $\tilde{\delta}_j$ with respect to these coordinates. Similarly, denote by $(\delta_1, \dots, \delta_\mu)$ the matrix whose $j$th column consists of the coefficients of $\delta_j$ with respect to the coordinates $u_1, \dots, u_\mu$. We abbreviate $\pi_{\Sigma^0} := \pi|_{\Sigma^0}$ and $\pi_\Sigma := \pi|_\Sigma$. There is a matrix equality

$$[T\pi_{\Sigma^0}] \cdot (\tilde{\delta}_1, \dots, \tilde{\delta}_\mu) = (\delta_1, \dots, \delta_\mu) \circ \pi, \tag{3.3}$$

where $[T\pi_{\Sigma^0}]$ is the Jacobian matrix of $\pi$ with respect to the chosen coordinates. Let $\pi^\mu$ be obtained from $\pi$ by omitting the $\mu$th component, and let $(\delta_1, \dots, \delta_\mu)^\mu_\mu$ denote the submatrix of $(\delta_1, \dots, \delta_\mu)$ obtained by omitting its $\mu$th row and column. Then (3.3) gives

$$[T\pi^\mu_{\Sigma^0}] \cdot (\tilde{\delta}_1, \dots, \tilde{\delta}_{\mu-1}) = (\delta_1, \dots, \delta_\mu)^\mu_\mu \circ \pi,$$

so that

$$\det[T\pi^\mu_{\Sigma^0}] \det(\tilde{\delta}_1, \dots, \tilde{\delta}_{\mu-1}) = m^\mu_\mu \circ \pi. \tag{3.4}$$

We will now compute $\det[T\pi^\mu_{\Sigma^0}]$ in terms of $F$. Because $g_\mu = 1$, $u_\mu$ does not appear in the equations of $\Sigma$, so $\partial_{u_\mu} \in T_{(x,u)}\Sigma$ for all $(x, u) \in \Sigma$. It follows that at any point of $\Sigma^0$, $T_{(x,u)}\Sigma$ has a basis consisting of a basis of $T_{(x,u)}\Sigma^0$ followed by the vector $\partial_{u_\mu}$. With respect to such a basis, the matrix of $[T\pi_\Sigma]$ takes the form

$$[T\pi_\Sigma] = \begin{bmatrix} T\pi^\mu_{\Sigma^0} & * \\ 0 & 1 \end{bmatrix}$$

from which it follows that

$$\det[T\pi^\mu_{\Sigma^0}] = \det[T\pi_\Sigma]. \tag{3.5}$$

In order to express the latter in terms of $F$, we compare two representations of the zero-dimensional Gorenstein ring

$$\mathscr{O}_\Sigma/\langle u_1, \dots, u_\mu\rangle = \mathscr{O}_\Sigma/\pi^*\mathfrak{m}_S = \mathscr{O}_Y/\langle \partial_{x_1}(F), \dots, \partial_{x_n}(F), u_1, \dots, u_\mu\rangle \tag{3.6}$$

as a quotient of a regular $\mathbb{C}$-algebra. In both cases, by [SS75, (4.7) Bsp.], the socle is generated by the Jacobian determinant of the generators of the respective defining ideal. The first representation then shows that the socle is generated by $\det[T\pi_\Sigma]$, the second one, that it is also generated by the (relative) Hessian $H = \det(\partial^2 F/\partial x^2)$ of $F$. Hence, up to multiplication by a unit, we obtain

$$\det[T\pi_\Sigma] = H \tag{3.7}$$

in $\mathscr{O}_\Sigma/\langle u_1, \dots, u_\mu\rangle$. It is easy to see that the non-immersive locus of $\pi_\Sigma$ is the vanishing set of $H$. This, together with (3.7), shows that $\det[T\pi_{\Sigma^0}] = H$. Now combining (3.4), (3.5) and (3.7),

$$\det(\tilde{\delta}_1, \dots, \tilde{\delta}_{\mu-1}) = (m^\mu_\mu \circ \pi)/H,$$

and so is a reduced defining equation for $\tilde{A} = V(m^\mu_\mu \circ \pi)$. Finally, each $\tilde{\delta}_j$ is tangent to $\tilde{A} = (\pi_{\Sigma^0})^{-1}(A)$ at its smooth points, since $\delta_j$ is necessarily tangent to the non-normal locus $A \cap D$ of $D$. The theorem now follows by Saito's criterion. □

*Remark* 3.3. Computation with examples appears to show that closure

$$C_v := \overline{\tilde{A} \setminus V(H)}$$

is also a free divisor.

We now go on to show first that the divisor $D+V(m^\mu_\mu)$ is free and then (see Corollary 3.8) that all adjoints are isomorphic. Just as in §2, our proof makes use of the representation of $\mathrm{Der}(-\log D)$ on $\mathscr{F}_1$, and relies on the surjectivity of $dm^\mu_\mu\colon \mathrm{Der}(-\log D)\to\mathscr{F}_1$.

**Proposition 3.4.** *Assume that $d-d_1+2d_i\neq 0\neq d-d_i$ for $i=1,\dots,\mu$. Then*

$$dm^\mu_\mu(\mathrm{Der}(-\log D))=\mathscr{F}_1.$$

Inclusion of the left hand side in the right is a consequence of the $\mathrm{Der}(-\log D)$-invariance of $\mathscr{F}_1$. To show equality, it is enough to show that it holds modulo $\mathfrak{m}_S\mathscr{F}_1$. This will cover most of the remainder of this section.

Denote by $\bar\Lambda=(\bar\lambda^i_j)_{1\le i,j\le\mu}$ the linear part of $\Lambda$, and let $\bar\delta_i=\sum_j\bar\lambda^i_j u_i\partial_{u_i}$ be the linear part of $\delta_i$.

**Theorem 3.5.** *The entries of $\Lambda$ are given by $\lambda^i_j=\sum_{k=1}^\mu\langle\check g_i\check g_j,g_k\rangle w_k u_k$. In particular,*

$$\bar\lambda^i_j=\sum_{k=1}^\mu\langle\check g_i\check g_j,g_k\rangle_0 w_k u_k.$$

*Proof.* Since $\chi_0(F)\in J^{\mathrm{rel}}_F$, we have

$$F=\chi(F)\equiv\delta_1(F)=\sum_k w_k u_k g_k \mod J^{\mathrm{rel}}_F,$$

and hence

$$\lambda^i_j=\langle\check g_i,F\check g_j\rangle=\langle\check g_i\check g_j,F\rangle=\left\langle\check g_i\check g_j,\sum_k w_k u_k g_k\right\rangle=\sum_k\langle\check g_i\check g_j,g_k\rangle w_k u_k.$$

□

We call a homogeneous basis $g$ of $M_f$ self-dual if

$$\check g_i=g_{\mu+1-i}. \tag{3.8}$$

**Lemma 3.6.** *$M_f$ admits self-dual bases.*

*Proof.* Denote by $W_j\subset M_f$ the subspace of degree-$d_j$ elements. The space $W_1$ is 1-dimensional generated by the Hessian of $f$. Therefore $W_j$ and $W_k$ are orthogonal unless $d_j+d_k=d_1$, in which case $\langle\cdot,\cdot\rangle_0$ induces a non-degenerate pairing $W_j\otimes_{\mathbb{C}}W_k\to\mathbb{C}$. If $j\neq k$, one can choose the basis of $W_j$ to be the reverse dual basis of a basis of $W_k$. Otherwise, $W_j=W_k$ and (since quadratic forms are diagonalizable) there is a basis of $W_j$ for which the matrix of $\langle\cdot,\cdot\rangle_0$ is diagonal. Self-duality on $W_j$ is then achieved by a coordinate change with matrix

$$\begin{pmatrix}1&0&\cdots&\cdots&0&1\\0&\ddots&&&\iddots&0\\\vdots&&1&1&&\vdots\\\vdots&&i&-i&&\vdots\\0&\iddots&&&\ddots&0\\i&0&\cdots&\cdots&0&-i\end{pmatrix}\quad\text{or}\quad\begin{pmatrix}1&0&\cdots&&\cdots&0&1\\0&\ddots&&&&\iddots&0\\\vdots&&1&&1&&\vdots\\&&&1&&&\\\vdots&&i&&-i&&\vdots\\0&\iddots&&&&\ddots&0\\i&0&\cdots&&\cdots&0&-i\end{pmatrix}$$

where $i=\sqrt{-1}$, for $\dim_{\mathbb{C}}(W_j)$ even or odd, respectively. A self-dual basis of $M_f$ is then obtained by joining the bases of the $W_j$ constructed above. □

Let $\bar{m}^i_j$ be the $(\mu-1)$-jet of $m^i_j$, that is, the corresponding minor of $\bar{\Lambda}$.

**Lemma 3.7.** *Suppose $g$ is an $\mathscr{O}_S$-basis for $\mathscr{O}_\Sigma$ whose restriction to $M_f$ is self-dual, and that $d \neq d_i \neq 0$. Then the following equalities hold true:*

*(a)* $\mathfrak{m}_S\mathscr{F}_1 = \mathscr{F}_1 \cap \mathfrak{m}_S^\mu \mathscr{O}_S$ *and $\mathscr{F}_1$ is minimally generated by $m^\mu_1, \dots, m^\mu_\mu$. In particular, $\bar{m}^\mu_i \equiv m^\mu_i$* $\mathrm{mod}\ \mathfrak{m}_S\mathscr{F}_1$ *for $i = 1, \dots, \mu$.*

*(b)* $\bar{\delta}_i(\bar{m}^\mu_\mu) = \pm(d - d_1 + 2d_i)\bar{m}^\mu_{\mu+1-i}$ *for $i = 2, \dots, \mu$.*

*(c)* $\delta_1(m^\mu_\mu) \equiv m^\mu_\mu \mod \mathbb{C}^*$.

*Proof.* As $w_i = d - d_i \neq 0$ by hypothesis, we may introduce new variables $v_i = w_i u_i$ for $i = 1, \dots, \mu$. Under the self-duality hypothesis, Theorem 3.5 implies that the matrix $\bar{\Lambda}$ has the form

$$\bar{\Lambda} = \begin{pmatrix} v_1 & v_2 & \cdots & \cdots & v_{\mu-1} & v_\mu \\ v_2 & \star & \cdots & \star & v_\mu & 0 \\ \vdots & \vdots & \iddots & \iddots & \iddots & \vdots \\ \vdots & \star & \iddots & \iddots & & \vdots \\ v_{\mu-1} & v_\mu & \iddots & & & \vdots \\ v_\mu & 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix} \tag{3.9}$$

where $\star$ entries do not involve the variable $v_\mu$. For the first row and column, this is clear. For the remaining entries, we note that, by Theorem 3.5, $v_\mu$ appears in $\bar{\lambda}^i_j$ if and only if

$$0 \neq \langle \check{g}_i \check{g}_j, g_\mu \rangle_0 = \langle \check{g}_i, \check{g}_j \rangle_0.$$

By the self-duality assumption (3.8), this is equivalent to $i + j = \mu + 1$, in which case $\langle \check{g}_i \check{g}_j, g_\mu \rangle = 1$ and $\bar{\lambda}^i_{\mu-i+1} = v_\mu$.

As in the proof of Theorem 1.1, it is convenient to use $\iota$ to denote the sign of the order-reversing permutation of $1, \dots, \mu - 1$. From (3.9) it follows that $\bar{m}^\mu_{\mu+1-i}$ involves a distinguished monomial $v_i v_\mu^{\mu-2}$, with coefficient $(-1)^{\mu-i-1}\iota$ for $i = 1, \dots, \mu - 1$ and $\iota$ for $i = \mu$; this monomial does not appear in any other minor $\bar{m}^\mu_{\mu+1-j}$ for $i \neq j$. This implies (a).

In order to prove (b), assume, for simplicity of notation, that $\Lambda$ and $\delta$ are linear, and fix $i \in \{2, \dots, \mu - 1\}$; the case $i = \mu$ is similar. We know that $\delta_i(m^\mu_\mu)$ is a linear combination of $m^\mu_1, \dots, m^\mu_\mu$. We will show that

$$\delta_i(m^\mu_\mu) = (-1)^{i-1}(w_1 - 2w_{\mu-i+1})m^\mu_{\mu-i+1} \tag{3.10}$$

by computing that the coefficient $c_{i,j}$ of the distinguished monomial $v_j v_\mu^{\mu-2}$ in $\delta_i(m^\mu_\mu)$ satisfies

$$c_{i,j} = (-1)^{\mu-2}\iota(w_1 - 2w_{\mu-i+1})\delta_{i,j}. \tag{3.11}$$

The self-duality assumption (3.8) implies that $d_1 - d_\ell = \check{d}_\ell = d_{\mu-\ell+1}$. Using $w_\ell = d - d_\ell$, this gives

$$w_1 - 2w_{\mu-i+1} = d - d_1 + 2d_{\mu-i+1} - 2d = d - d_1 + 2d_1 - 2d_i - 2d = -(d - d_1 + 2d_i).$$

So (b) will follow from (3.10).

By linearity of $\delta_i$, the only monomials in the expansion of $m^\mu_\mu$ that could conceivably contribute to a non-zero $c_{i,j}$ are of the following three forms:

$$v_\mu^{\mu-1}, \quad v_j v_\mu^{\mu-2}, \quad v_j v_k v_\mu^{\mu-3}. \tag{3.12}$$

The first monomial does not figure in the expansion of $m^\mu_\mu$. Monomials of the other two types do appear. The second type of monomial in (3.12) must satisfy $j = 1$ and arises as the product

$$(-1)^{\mu-2}\iota v_1 v_\mu^{\mu-2} = (-1)^{\mu-2}\iota \lambda^1_1 \lambda^2_{\mu-1} \lambda^3_{\mu-2} \cdots \lambda^{\mu-1}_2. \tag{3.13}$$

Monomials of the third type in (3.12) must satisfy $k = \mu - j + 1$. Each such monomial arises in the expansion of $m^\mu_\mu$ in two ways, which coincide when $j = \mu - j + 1$:

$$(-1)^{\mu-3}\iota v_j v_k v_\mu^{\mu-3} = (-1)^{\mu-3}\iota\lambda_j^1\lambda_{\mu-1}^2\cdots\lambda_{j+1}^{\mu-j}\lambda_1^{\mu-j+1}\lambda_{j-1}^{\mu-j+2}\cdots\lambda_2^{\mu-1} \tag{3.14}$$

$$(-1)^{\mu-3}\iota v_j v_k v_\mu^{\mu-3} = (-1)^{\mu-3}\iota\lambda_{\mu-j+1}^1\lambda_{\mu-1}^2\cdots\lambda_{\mu-j+2}^{j-1}\lambda_1^{j}\lambda_{\mu-j}^{j+1}\cdots\lambda_2^{\mu-1}. \tag{3.15}$$

In terms of the coordinates $v_1, \ldots, v_\mu$, $\delta_i$ contains monomials

$$w_i u_i \partial_{u_1} = w_1 v_i \partial_{v_1}, \tag{3.16}$$

$$w_\mu u_\mu \partial_{u_{\mu-i+1}} = w_{\mu-i+1} v_\mu \partial_{v_{\mu-i+1}}. \tag{3.17}$$

Now (3.16) applied to (3.13) contributes $w_1(-1)^{\mu-2}\iota$ to $c_{i,i}$, (3.17) applied to one or two copies of (3.14) for $i = j$ contributes $2(-1)^{\mu-3}\iota w_{\mu-i+1}$ to $c_{i,i}$ in both cases. There are no contributions to the coefficient of any other distinguished monomial.

We have proved (3.10), from which (b) follows; (c) is clear, since $\delta_1$ is the Euler vector field. $\square$

By Nakayama's lemma, Proposition 3.4 now follows immediately from (3.2) and Lemma 3.7.

The next result, closely analogous to Corollary 2.9, follows from Proposition 3.4 by the same argument by which Corollary 2.9 is deduced from Proposition 2.8.

**Corollary 3.8.** *Assume the hypothesis of Proposition 3.4. Then any two adjoint divisors of $D$ are isomorphic by an isomorphism preserving $D$.* $\square$

**Proposition 3.9.** *Let $D = V(h)$ and $A = V(m)$ be divisors in $S$, and suppose that $D$ is a free divisor. Let $\mathscr{F}$ be the $\mathscr{O}_S$-ideal $dm(\mathrm{Der}(-\log D))$, and suppose that $m \in \mathscr{F}$. Then the following two statements are equivalent:*

*(1)* $\operatorname{depth}_{\mathscr{O}_S}\mathscr{F} = \mu - 1$.
*(2)* *$D + A$ is a free divisor.*

*Proof.* Apply the depth lemma (see [BH93, Prop. 1.2.9]) to the two short exact sequences:

$$0 \longrightarrow \langle m\rangle \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}/\langle m\rangle \longrightarrow 0$$

$$0 \longrightarrow \mathrm{Der}(-\log(D+A)) \longrightarrow \mathrm{Der}(-\log D) \xrightarrow{dm} \mathscr{F}/\langle m\rangle \longrightarrow 0$$ $\square$

*Proof of Theorem 1.2.* By Corollary 3.8, we may assume that $A = V(m^\mu_\mu)$. Recall that

$$dm^\mu_\mu(\mathrm{Der}(-\log D)) = \mathscr{F}_1$$

by Proposition 3.4 and hence $\operatorname{depth}_{\mathscr{O}_S}\mathscr{F}_1 = \mu - 1$ by the Hilbert–Burch Theorem (see [BH93, Thm. 1.4.17]). So Proposition 3.9 with $m = m^\mu_\mu$ and $\mathscr{F} = \mathscr{F}_1$ yields the claim. $\square$

*Remark* 3.10. We remark that the very simple deduction of Theorem 1.2 from Propositions 3.4 and 3.9 does not have a straightforward analogue by which Theorem 1.1 can be deduced from Propositions 2.8 and 3.9. Firstly, the image $D$ of a stable map-germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ is not free: $\mathrm{Der}(-\log D)$ has depth $n$ and not $n+1$. Secondly, there can be no way of adapting the argument to deal with this difference without some other input, since when $f$ is the stable germ of corank 2 of Example 2, the corresponding map $\mathrm{Der}(-\log D) \to \mathscr{F}_1$ is surjective, and $\mathscr{F}_1$ has depth $n$, but even so $\mathrm{Der}(-\log(D+A))$ is not free.

We conclude this section with a description of the relation between the adjoint divisor of $D$ and the bifurcation set of the deformation. For $u \in S$, we set $X_u := \pi_S^{-1}(u)$ and define $f_u\colon X_u \to T$

by $f_u(x) := F(x,u)$. We consider $S' := (\mathbb{C}^{\mu-1},0)$ with coordinates $u' = u_1,\dots,u_{\mu-1}$, and we denote by

$$\rho\colon S \to S' \quad u \mapsto u', \tag{3.18}$$

the natural projection forgetting the last coordinate. Recall that the *bifurcation set* is the set $B \subset S'$ of parameter values $u'$ such that $f_{u'} := f_{(u',0)}$ has fewer than $\mu$ distinct critical values. The coefficient $u_\mu$ of $g_\mu = 1$ is set to 0 since it has no bearing on the number of critical values. The bifurcation set consists of two parts: the level bifurcation set $B_v$ consisting of parameter values $u'$ for which $f_{u'}$ has distinct critical points with the same critical value, and the local bifurcation set $B_\ell$ where $f_{u'}$ has a degenerate critical point. H. Terao, in [Ter83], and J.W. Bruce in [Bru85] proved that $B$ is a free divisor and gave algorithms for constructing a basis for $\mathrm{Der}(-\log B)$. The free divisor $B$ is of course singular in codimension 1. The topological double points (points at which $B$ is reducible) are of four generic types:

- Type 1: $f_{u'}$ has two distinct degenerate critical points, $x_1$ and $x_2$.
- Type 2: $f_{u'}$ has two distinct pairs of critical points, $x_1$, $x_2$ and $x_3, x_4$, such that

$$f_{u'}(x_1) = f_{u'}(x_2) \text{ and } f_{u'}(x_3) = f_{u'}(x_4).$$

- Type 3: $f_{u'}$ has a pair of critical points $x_1$ and $x_2$ with the same critical value, and also a degenerate critical point $x_3$.
- Type 4: $f_{u'}$ has three critical points $x_1, x_2$ and $x_3$ with the same critical value.

In the neighborhood of a double point of type 1, 2 or 3, $B$ is a normal crossing of two smooth sheets. In the neighborhood of a double point of type 4, $B$ is isomorphic to a product

$$B_0 \times (\mathbb{C}^{\mu-2},0),$$

where $B_0 = V(uv(u-v)) \subset (\mathbb{C}^2,0)$.

**Proposition 3.11.** *For any adjoint divisor $A$ for $D$,* (3.18) *induces a surjection*

$$\rho\colon D \cap A \twoheadrightarrow B. \tag{3.19}$$

*Proof.* A point $u \in S$ lies in $D \cap A$ if the sum of the lengths of the Jacobian algebras of $f_u$ at points $x \in f_u^{-1}(0)$ is greater than 1. The sum may be greater than 1 because for some $x$ the dimension of the Jacobian algebra is greater than 1 – in which case $f_u$ has a degenerate critical point at $x$ – or because $f_u$ has two or more critical points with critical value 0. In either case, it is clear that $\rho(u) \in B$. If $u' \in B$, then $f_{u'}$ has either a degenerate critical point or a repeated critical value (or both). In both, cases let $v$ be the corresponding critical value. Then $(u', -v) \in D \cap A$, proving surjectivity. ☐

*Remark* 3.12. The projection (3.19) is a partial normalization, in the sense that generically, topological double points of $u' \in B$ of types 1, 2 and 3 are separated. Indeed, in each such case $f_{u'}$ has two critical points with different critical values, and hence with different preimages under $\rho$. However, a general point $u'$ of type 4 has only one preimage, $(u', -f_{(u',0)}(x_i))$, in $D \cap A$. Generically, at such a point $D$ is a normal crossing of three smooth divisors, and $D \cap A$ is the union of their pairwise intersections. Thus $B_0$ is improved to a curve isomorphic to the union of three coordinate axes in 3-space.

Finally, our free divisors $D + A$ and $\tilde{A}$ of Theorems 1.2 and 3.2, and the conjecturally free divisor $C_v$ of Remark 3.3, fit into the following commutative diagram, in which $A$ is any adjoint divisor for $D$, and the simple and double underlinings indicate conjecturally free and free divisors.

$$\begin{tikzcd}
V(H)\cap\Sigma^0 \arrow[dd, two heads, "\rho\circ\pi|"'] \arrow[rr, hook, bend left] & \underline{C_v} \arrow[r, hook] \arrow[dd, two heads, "\rho\circ\pi|"'] & \underline{\underline{\tilde{A}}} \arrow[r, hook] \arrow[d, "\pi"] & \Sigma^0 \arrow[d, "\pi"] & \\
 & & D\cap A \arrow[r, hook] \arrow[d, two heads, "\rho"'] & \underline{\underline{D}} \arrow[r, hook] & \underline{\underline{\underline{D+A}}} \\
B_\ell \arrow[rr, hook, bend right] & B_v \arrow[r, hook] & \underline{\underline{B}} & &
\end{tikzcd}$$

## 4. Pull-back of free divisors

In this section, we describe a procedure for constructing new free divisors from old by a pull-back construction. It is motivated by Example 2.4.(5).

**Theorem 4.1.** *Suppose that $D = \bigcup_{i=1}^k D_i \subset (\mathbb{C}^n, 0) =: X$ is a germ of a free divisor. Let $f\colon X \to Y := (\mathbb{C}^k, 0)$ be the map whose $i$th component $f_i \in \mathscr{O}_X$, for $i = 1, \dots, k$, is a reduced equation for $D_i$. Suppose that, for $j = 1, \dots, k$, there exist vector fields $\varepsilon_j \in \Theta_X$ such that*

$$df_i(\varepsilon_j) = \delta_{i,j} \cdot f_i. \tag{4.1}$$

*Let $N := V(y_1 \cdots y_k) \subset Y$ be the normal crossing divisor, so that $D = f^{-1}(N)$. Let $E \subset Y$ be a divisor such that $N + E$ is free. Then provided that no component of $f^{-1}(E)$ lies in $D$, $f^{-1}(N + E) = D + f^{-1}(E)$ is a free divisor.*

*Proof.* The vector fields $\varepsilon_1, \dots, \varepsilon_k$ can be incorporated into a basis $\varepsilon_1, \dots, \varepsilon_n$ for $\operatorname{Der}(-\log D)$ such that (4.1) holds for $j = 1, \dots, n$ and hence

$$tf(\varepsilon_j) = \sum_{i=1}^k df_i(\varepsilon_j)\partial_{y_i} = \begin{cases} \omega f(y_j \partial_{y_j}), & \text{if } j \le k, \\ 0, & \text{otherwise.} \end{cases} \tag{4.2}$$

Any Saito matrix of $N + E$ can be written in the form $S_{N+E} = S_N \cdot A$, where

$$S_N = \operatorname{diag}(y_1, \dots, y_k)$$

is the standard Saito matrix of $N$ and $A = (a_{i,j}) \in \mathscr{O}_Y^{k \times k}$. Then, by Saito's criterion, $h := \det A$ and $g := y_1 \cdots y_n h$ are reduced equations for $E$ and $N + E$ respectively. For $j = 1, \dots k$, consider the vector fields $v_j := \sum_{i=1}^k a_{i,j} y_i \partial_{y_i} \in \operatorname{Der}(-\log(N + E))$ whose coefficients are the columns of $S_{N+E}$ and let

$$\tilde{v}_j := \sum_{i=1}^k (a_{i,j} \circ f)\varepsilon_i \in \Theta_X.$$

By (4.2), we have $tf(\tilde{v}_j) = \omega f(v_j)$; by construction of the $v_j$, it follows that

$$d(g \circ f)(\tilde{v}_j) = (dg(v_j)) \circ f \in (g \circ f)\mathscr{O}_X,$$

so that $\tilde{v}_j \in \operatorname{Der}(-\log f^{-1}(N + E))$, and moreover $\varepsilon_j \in \operatorname{Der}(-\log f^{-1}(N + E))$ for $j > k$.

Let $S_D$ be the Saito matrix of $D$, whose columns are the coefficients of the vector fields $\varepsilon_1, \dots, \varepsilon_n$. The matrix of coefficients of the vector fields $\tilde{v}_1, \dots, \tilde{v}_k, \varepsilon_{k+1}, \dots, \varepsilon_n$ is equal to

$$S_D \cdot \begin{pmatrix} A \circ f & 0 \\ 0 & I_{n-k} \end{pmatrix}$$

and thus its determinant $\det(S_D) \cdot (h \circ f)$ defines $D + f^{-1}(E) = f^{-1}(N + E)$. By Saito's criterion, this shows that the latter is a free divisor, provided

(1) $h \circ f$ is reduced and
(2) $h \circ f$ has no irreducible factor in common with $f_1 \cdots f_k$.

Now $h \circ f$ is reduced where $f$ is a submersion. So provided no component of $f^{-1}(E)$ is contained in the critical set $\Sigma_f$ of $f$, $h \circ f$ is reduced. In fact $\Sigma_f = D_{\mathrm{Sing}}$. To see this, consider the "logarithmic Jacobian matrix" $(\varepsilon_i(f_j))_{1 \le i \le n, 1 \le j \le k}$ of $f$. The determinant of the first $k$ columns is equal to $f_1 \cdots f_k$; thus $f_1 \cdots f_k$ is in the Jacobian ideal of $f$, so $\Sigma_f \subset D$. Thus $D_{\mathrm{Sing}} = D \cap \Sigma_f = \Sigma_f$. This shows that condition (2) above implies condition (1). $\square$

*Example* 4.2. Let $D := V(x_1 \cdots x_n) \subset (\mathbb{C}^n, 0)$ be the normal crossing divisor, define

$$f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^2, 0)$$

by $f(x_1, \ldots, x_n) = (x_1 \cdots x_k, x_{k+1} \cdots x_n)$, and let $g \in \mathscr{O}_{\mathbb{C}^2,0}$ be any germ not divisible by either of the coordinates. Take $N := V(y_1 y_2)$ and $E := V(g(y_1, y_2))$. By Theorem 4.1 it follows that $V(x_1 \cdots x_n g(x_1 \cdots x_k, x_{k+1} \cdots x_n))$ is a free divisor. The condition that no component of $f^{-1}(E)$ should lie in $D$ is guaranteed by the requirement that neither $y_1$ nor $y_2$ should divide $g(y_1, y_2)$.

The existence hypothesis (4.1) in the theorem is not fulfilled for every reducible free divisor. In the graded case, the vector fields $\varepsilon_j$ must have degree zero. If $D$ is the discriminant of a versal deformation of a weighted homogeneous isolated hypersurface singularity meeting the hypotheses of Theorem 1.2, and $A$ is an adjoint divisor, then $D + A$ is free but the only vector fields of weight zero in $\mathrm{Der}(-\log D)$ are multiples of the Euler field. Thus, hypothesis (4.1) cannot hold.

However there is an interesting class, namely linear free divisors, for which this requirement always holds. We recall from [GMNRS09] that a free divisor $D$ in the $n$-dimensional vector space $V$ is said to be linear if $\mathrm{Der}(-\log D)$ has a basis consisting of vector fields of weight 0. The linear span $\mathrm{Der}(-\log D)_0$ of the basic fields is an $n$-dimensional Lie algebra. It is naturally identified with the Lie algebra of the algebraic subgroup $\iota\colon G_D \hookrightarrow \mathrm{GL}(V)$ consisting of the identity component of the set of automorphisms preserving $D$. It follows that $(V, G_D, \iota)$ is a prehomogeneous vector space (see [SK77]) with discriminant $D$.

Let $D \subset V$ be a linear free divisor and $D = \bigcup_{i=1}^k D_i$ a decomposition into irreducible components. The corresponding defining equations $f_1, \ldots, f_k$ are polynomial relative invariants of $(V, G_D, \iota)$ with associated characters $\chi_1, \ldots, \chi_k$; that is, for $g \in G_D$ and $x \in V$, $f_j(gx) = \chi_j(g) f_j(x)$. These characters are multiplicatively independent, by [SK77, §4, proof of Prop. 5]. Let $\mathfrak{g}_D$ denote the Lie algebra of $G_D$. By differentiating the character map $\chi = (\chi_1, \ldots, \chi_k)\colon G \to (\mathbb{C}^*)^k$ we obtain an epimorphism of Lie algebras $d\chi\colon \mathfrak{g}_D \to \mathbb{C}^k$. This yields a decomposition

$$\mathfrak{g}_D = \ker d\chi \oplus \bigoplus_{i=1}^k \mathbb{C}\varepsilon_i, \quad d\chi_i(\varepsilon_j) = \delta_{i,j}.$$

For $\delta \in \mathfrak{g}_D$, the equality $f_i(gx) = \chi_i(g) f_i(x)$ differentiates to $\delta(f_i) = d\chi_i(\delta) \cdot f_i$, which implies (4.1). We have proved

**Proposition 4.3.** *Any germ of a linear free divisor $D$ satisfies the existence hypothesis* (4.1) *of Theorem 4.1.* $\square$

*Example* 4.4. Let $\sigma_i(y)$ be the $i$th symmetric function of $y = y_1, \ldots, y_k$ and set $N := V(\sigma_k(y))$ and $E := V(\sigma_{k-1}(y))$. As seen in Example 2.4.(5), the divisor $N + E$ is free. So by Theorem 4.1 and Proposition 4.3, for any germ $D$ of a linear free divisor with distinct irreducible components $D_i = V(f_i)$, the divisor germ $V(\sigma_k(f_1, \ldots, f_k)\sigma_{k-1}(f_1, \ldots, f_k))$ is also free. No component of $V(\sigma_{k-1}(f_1, \ldots, f_k))$ can lie in $D$, since were this the case, some $f_i$ would divide $\sigma_{k-1}(f_1, \ldots, f_k)$ and therefore would divide $f_1 \cdots \widehat{f_i} \cdots f_k$.

If each of the $D_i$ is normal, then in fact $f^{-1}(E)$ is an adjoint divisor of the normalization of $D$. As the singular locus of any free divisor has pure codimension 1, the singular locus of $D$ is equal to its non-normal locus. The ring of functions on the normalization $\bar{D} = \coprod_{i=1}^k D_i$ has presentation matrix $\operatorname{diag}(f_1, \ldots, f_k)$. Thus $V(\sigma_{k-1} \circ f)$ is an adjoint divisor of $D$.

There is another class of divisors that fits naturally into the setup of Theorem 4.1, namely that of hyperplane arrangements.

**Proposition 4.5.** *Given the hypothesis (4.1), any germ $N+E$ of a free hyperplane arrangement automatically satisfies the hypothesis on $f^{-1}(E)$ in Theorem 4.1.*

*Proof.* By assumption, $N = V(y_1, \ldots, y_k)$ and $E = \bigcup_{i=k+1}^m H_i$ where $H_i = V(\ell_i)$ for some linear equation $\ell_i(y) = \sum_j \alpha_{i,j} y_j$ for $i = k+1, \ldots, m$. We need to show that no component of any of the $\ell_i \circ f$ is divisible by any $f_j$. Suppose to the contrary that $\ell_i \circ f = g \cdot f_t$. For $s \neq t$, $\varepsilon_s$ applied to this equation gives $\alpha_{i,s} f_s = \varepsilon_s(g) \cdot f_t$. Since $f_t$ cannot divide $f_s$ as $D$ is reduced, it follows that $\alpha_{i,s} = 0$ for any $s \neq t$, and thus $\ell_i = \alpha_{i,t} y_t$. This is absurd since $A$ is supposed reduced. $\square$

Combining Propositions 4.3 and 4.5 proves

**Corollary 4.6.** *Let $A = \bigcup_{i=1}^m H_i \subset (\mathbb{C}^k, 0)$ be the germ of a free hyperplane arrangement containing the normal crossing divisor $\{y_1 \cdots y_k = 0\}$, and let $D \subset (\mathbb{C}^n, 0)$ be the germ of a linear free divisor whose irreducible components have equations $f_1, \ldots, f_k$. If $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^k, 0)$ is defined by $f(x) = (f_1(x), \ldots, f_k(x))$, then $f^{-1}(A)$ is a free divisor.*

Note that the corollary applies to any essential free arrangement, once suitable coordinates are chosen.

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D. Mond, Mathematics Institute, University of Warwick, Coventry CV47AL, England
*E-mail address*: D.M.Q.Mond@warwick.ac.uk

M. Schulze, Department of Mathematics, University of Kaiserslautern, 67663 Kaiserslautern, Germany
*E-mail address*: mschulze@mathematik.uni-kl.de

**Journal *of* Singularities**
**Volume 7 (2013), 275-291**

received 17 February 2012
in revised form 24 November 2013
DOI: 10.5427/jsing.2013.7o

# GENERICITY OF CAUSTICS ON A CORNER

TAKAHARU TSUKADA

*Dedicated to Professor Masahiko Suzuki on his sixtieth birthday*

Abstract. We introduce the notions of *caustic-equivalence* and *weak caustic-equivalence relations* of reticular Lagrangian maps in order to give a generic classification of "shapes" of caustics on a corner. We give the figures of all generic caustics on a corner in a smooth manifold of dimension 2 and 3 under these equivalence relations.

## 1. Introduction

Lagrangian singularities can be found in many problems of differential geometry, calculus of variations and mathematical physics. One of the most successful applications is the study of singularities of caustics. For example, the particles incident along geodesics from a smooth hypersurface $V^{n-1}$ in a Riemannian manifold $M^n$ to conormal directions define a Lagrangian submanifold at a point in the cotangent bundle. The caustic generated by the hypersurface

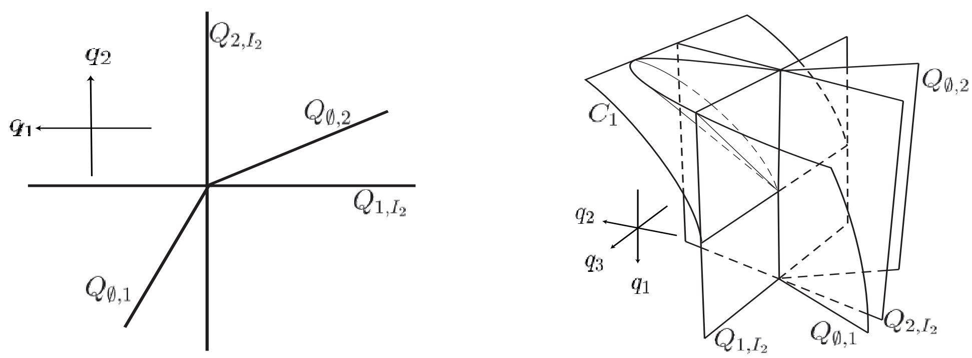


Figure 1. Example of caustics on a corner

is regarded as the caustic of the Lagrangian map defined by the restriction of the cotangent bundle projection to the Lagrangian submanifold. In [5] we investigated the theory of *reticular Lagrangian maps*, which is a generalized notion of Lagrangian maps and can be described as stable caustics generated by a hypersurface germ with a boundary($r = 1$), a corner($r = 2$), or an $r$-corner in a smooth manifold. In [6] we gave classification lists of *generic* caustics in the case $r = 0, 1$ respectively. In the case $r = 2$, that is the initial hypersurface has a corner, the method used in [6] does not work well by the modality of generating families since the transversality theorem can not work in this case.

In the case $r = 2$ we consider the following situation: Let $V$ be a 2-dimensional hypersurface germ in a 3-dimensional manifold $M$. We suppose that $V$ is the light source with a corner and there exist local coordinates $(x_1, x_2, x_3)$ such that $V = \{(x_1, x_2, 0) \in (\mathbb{R}^3, 0) | x_1 \geq 0, x_2 \geq 0\}$. We

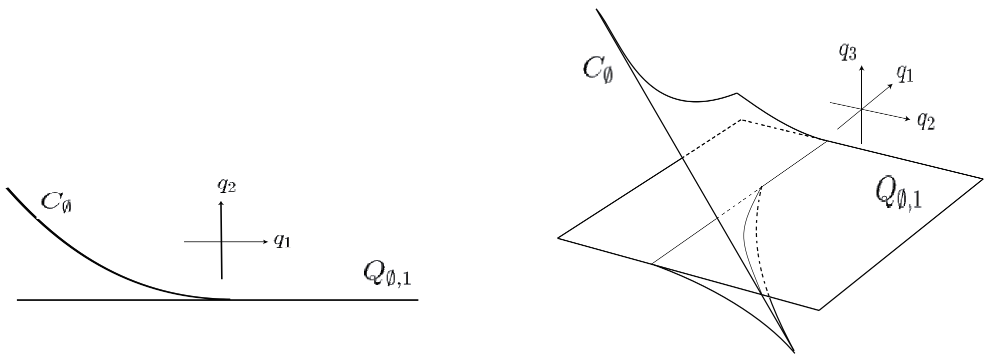


FIGURE 2. Example of caustics on a boundary

consider that the light rays are incident from each of $V$, $V_1 = V \cap \{x_1 = 0\}$, $V_2 = V \cap \{x_2 = 0\}$, and $V_{I_2} = V \cap \{x_1 = x_2 = 0\}$ to the conormal directions, where $I_2 = \{1, 2\}$. We denote them by $L^0_\emptyset, L^0_1, L^0_2, L^0_{I_2}$ respectively. Then they are reduced to the following normal forms by a suitable symplectic diffeomorphism on $(T^*\mathbb{R}^3, 0)$:

$$L^0_\sigma = \{(q,p) \in (T^*\mathbb{R}^3, 0) | q_\sigma = p_{I_2 \setminus \sigma} = q_3 = 0, q_{I_2\setminus\sigma} \geq 0\},$$

for $\sigma = \emptyset, 1, 2, I_2$. Then all $L^0_\sigma$ are transposed around a point in $T^*M$ by a symplectic diffeomorphism $S$ on $T^*M$ along geodesics. By taking a Lagrangian equivalence around this point, we may assume that $S$ is given by $S : (T^*\mathbb{R}^3, 0) \to (T^*\mathbb{R}^3, 0)$. Let

$$\mathbb{L} = \{(q,p) \in T^*\mathbb{R}^3 | q_1p_1 = q_2p_2 = q_3 = 0, q_1 \geq 0, q_2 \geq 0\}$$

be a representative as a germ of the union of all $L^0_\sigma$.

We define $i = S|_{\mathbb{L}}$. Let $\pi : (T^*\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ be the canonical Lagrangian projection. We consider the following map

$$(\mathbb{L}, 0) \xrightarrow{i} (T^*\mathbb{R}^3, 0) \xrightarrow{\pi} (\mathbb{R}^3, 0).$$

We define the caustic of $\pi \circ i$ to be the union of *the caustics* $C_\sigma$ of the Lagrangian maps $\pi \circ i|_{L^0_\sigma}$ for all $\sigma \subset I_2$ and *the quasi-caustic* $Q_{\sigma,\tau} = \pi \circ i(L^0_\sigma \cap L^0_\tau)$ for all $\sigma, \tau \subset I_r (\sigma \neq \tau)$.

In the case $r = 2$, the caustic of $\pi \circ i$ is

$$C_\emptyset \cup C_1 \cup C_2 \cup C_{\{1,2\}} \cup Q_{\emptyset,1} \cup Q_{\emptyset,2} \cup Q_{1,\{1,2\}} \cup Q_{2,\{1,2\}}.$$

Then the quasi-caustic $Q_{\sigma,\tau}$ expresses the intersection of light rays incident from $V_\sigma$ and $V_\tau$.

Our purpose is the investigation of generic caustics under perturbations of $S$. All functions and maps are smooth, unless stated otherwise.

We here give a review of the theory of reticular Lagrangian maps which is developed in [5].

**Reticular Lagrangian maps**: Let $I_r = \{1, \ldots, r\}$,

$$\mathbb{L} = \{(q,p) \in T^*\mathbb{R}^n |\ q_1p_1 = \cdots = q_rp_r = q_{r+1} = \cdots = q_n = 0, q_{I_r} \geq 0\}$$

be a representative of the union of

$$L^0_\sigma = \{(q,p) \in (T^*\mathbb{R}^n, 0) | q_\sigma = p_{I_r\setminus\sigma} = q_{r+1} = \cdots = q_n = 0, q_{I_r\setminus\sigma} \geq 0\}$$

for all $\sigma \subset I_r$. We write $q_\sigma = 0$ for the condition $q_i = 0$ for all $i \in \sigma$ and write $q_\sigma \geq 0$ for the condition $q_i \geq 0$ for all $i \in \sigma$ and write other notations analogously. The set $L^0_\sigma$ is the normalization of the particles incident from the $\sigma$-edge $V_\sigma := V \cap \{q_\sigma = 0\}$ of the light source hypersurface $V = \{(q_1, \ldots, q_n) \in (\mathbb{R}^n, 0) | q_1 \geq 0, \ldots, q_r \geq 0, q_n = 0\}$ with an $r$-corner for some local coordinate system $q$ of $M$ to conormal directions. Let $\pi : (T^*\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be the canonical Lagrangian projection.

A map germ $\pi \circ i : (\mathbb{L}, 0) \to (T^*\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ is called *a reticular Lagrangian map* if there exists a symplectic diffeomorphism germ $S$ on $(T^*\mathbb{R}^n, 0)$ such that $i = S|_{\mathbb{L}}$.

We note that the particles incident from all $V_\sigma{}'s$ to the conormal directions are mapped along geodesics. This map is extended to *the reticular Lagrangian immersion* $i$ which is the generalized notion of Lagrangian immersion.

**Equivalence relation**: We call a symplectic diffeomorphism germ $\phi$ on $(T^*\mathbb{R}^n, 0)$ *a reticular diffeomorphism* if $\phi(L^0_\sigma) = L^0_\sigma$ for $\sigma \subset I_r$. We say that reticular Lagrangian maps

$$\pi \circ i_1, \pi \circ i_2 : (\mathbb{L}, 0) \to (T^*\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$$

are *Lagrangian equivalent* if there exist a reticular diffeomorphism $\phi$ and a Lagrangian equivalence $\Theta$ (a symplectic diffeomorphism which preserves the fiber of $\pi$) such that the following diagram is commutative:

$$\begin{array}{ccccc} (\mathbb{L}, 0) & \xrightarrow{i_1} & (T^*\mathbb{R}^n, 0) & \xrightarrow{\pi} & (\mathbb{R}^n, 0) \\ \phi|_{\mathbb{L}} \downarrow & & \Theta \downarrow & & g \downarrow \\ (\mathbb{L}, 0) & \xrightarrow{i_2} & (T^*\mathbb{R}^n, 0) & \xrightarrow{\pi} & (\mathbb{R}^n, 0), \end{array}$$

where $g$ is the diffeomorphism of the base space of $\pi$ induced by $\Theta$.

It may be thought that a reticular diffeomorphism does not have to be a symplectic diffeomorphism. But a reticular diffeomorphism consists of compositions of two symplectic diffeomorphisms and a Lagrangian equivalence, it follows that a reticular diffeomorphism is automatically a symplectic diffeomorphism. We also remark that if two reticular Lagrangian maps $\pi \circ i_1$ and $\pi \circ i_2$ are Lagrangian equivalent, then the Lagrangian maps $\pi \circ i_1|_{L^0_\sigma}$ and $\pi \circ i_2|_{L^0_\sigma}$ are Lagrangian equivalent for each $\sigma \subset I_r$.

Let $U, V$ be open sets in $\mathbb{R}^{m_1}, \mathbb{R}^{m_2}$ respectively. We denote by $C^\infty(U, V)$ the set which consists of smooth maps from $U$ to $V$. We define

$$N_f(l, \varepsilon, K) = \{\ g \in C^\infty(U, V) \mid |D^\alpha(g - f)_x| < \varepsilon\ \forall x \in K, |\alpha| < l\ \}$$

for each $f \in C^\infty(U, V), l \in \mathbb{N}, \varepsilon > 0$ and compact set $K$ in $U$. Then the family of sets $N_f(l, \varepsilon, K)$ forms a basis for the $C^\infty$-topology on $C^\infty(U, V)$.

Let $U$ be an open set in $T^*\mathbb{R}^n$, and $S(U, T^*\mathbb{R}^n)$ be the set which consists of symplectic embeddings from $U$ to $T^*\mathbb{R}^n$. We equip $S(U, T^*\mathbb{R}^n)$ the induced topology from $C^\infty$-topology of $C^\infty(U, T^*\mathbb{R}^n)$. We define $S(T^*\mathbb{R}^n, 0)$ to be the set of symplectic diffeomorphism germs on $(T^*\mathbb{R}^n, 0)$.

We say that a reticular Lagrangian map $\pi \circ i : (\mathbb{L}, 0) \to (T^*\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ is *stable* if the following holds: For any extension $S \in S(T^*\mathbb{R}^n, 0)$ of $i$ and any representative $\tilde{S} \in S(U, T^*\mathbb{R}^n)$ of $S$, there exists a neighborhood $N_{\tilde{S}}$ of $\tilde{S}$ such that for any $\tilde{T} \in N_{\tilde{S}}$ the reticular Lagrangian maps $\pi \circ (\tilde{T}|_{\mathbb{L}}$ at $x_0)$ and $\pi \circ i$ are Lagrangian equivalent for some $x_0 = (0, \cdots, 0, P^0_{r+1}, \cdots, P^0_n) \in U$, where the map $(\tilde{T}|_{\mathbb{L}}$ at $x_0)$ is given by $x (\in \mathbb{L}) \mapsto \tilde{T}(x + x_0) - \tilde{T}(x_0)$.

**Generating family**: Let $\mathbb{H}^r = \{(x_1, \ldots, x_r) \in \mathbb{R}^r | x_1 \geq 0, \ldots, x_r \geq 0\}$ be an $r$-corner. We denote by $\mathcal{E}(r; k)$ the set of all germs at 0 in $\mathbb{H}^r \times \mathbb{R}^k$ of smooth maps $\mathbb{H}^r \times \mathbb{R}^k \to \mathbb{R}$ and set

$\mathfrak{M}(r;k) = \{f \in \mathcal{E}(r;k) | f(0) = 0\}$. We write $\mathcal{E}(k)$ for $\mathcal{E}(0;k)$ and $\mathfrak{M}(k)$ for $\mathfrak{M}(0;k)$. Then $\mathcal{E}(r;k)$ is an $\mathbb{R}$-algebra in the usual way and $\mathfrak{M}(r;k)$ is its unique maximal ideal.

We denote by $J^l(r+k,p)$ the set of $l$-jets at 0 of germs in $\mathcal{E}(r;k,p)$. There are natural projections:

$$\pi_l : \mathcal{E}(r;k,p) \longrightarrow J^l(r+k,p), \pi^{l_1}_{l_2} : J^{l_1}(r+k,p) \longrightarrow J^{l_2}(r+k,p) \ (l_1 > l_2).$$

We write $j^l f(0)$ for $\pi_l(f)$ for each $f \in \mathcal{E}(r;k,p)$.

A function germ $F(x,y,q) \in \mathfrak{M}(r;k+n)^2$ is called *$S$-non-degenerate* if

$$x_1, \cdots, x_r, \frac{\partial F}{\partial x_1}, \cdots, \frac{\partial F}{\partial x_r}, \frac{\partial F}{\partial y_1}, \cdots, \frac{\partial F}{\partial y_k}$$

are independent on $(\mathbb{H}^k \times \mathbb{R}^{k+n}, 0)$, that is

$$\mathrm{rank} \begin{pmatrix} \dfrac{\partial^2 F}{\partial x \partial y} & \dfrac{\partial^2 F}{\partial x \partial q} \\ \dfrac{\partial^2 F}{\partial y \partial y} & \dfrac{\partial^2 F}{\partial y \partial q} \end{pmatrix}_0 = r + k.$$

We say that an $S$-non-degenerate function germ $F(x,y,q) \in \mathfrak{M}(r;k+n)^2$ is *a generating family* of a reticular Lagrangian map $\pi \circ i$ if $F|_{x_\sigma = 0}$ is a generating family of the Lagrangian map $\pi \circ i|_{L^0_\sigma}$, that is

$$i(L^0_\sigma) = \{(q, \frac{\partial F}{\partial q}(x,y,q)) \in (T^*\mathbb{R}^n, 0) | x_\sigma = \frac{\partial F}{\partial x_{I_r \setminus \sigma}} = \frac{\partial F}{\partial y} = 0, x_{I_r \setminus \sigma} \geq 0\}$$

for $\sigma \subset I_r$.

Generating families of reticular Lagrangian maps with caustics of Figure 1,2 are given as follows:
Figure 1(left): $F(x_1,x_2,q_1,q_2) = x_1^2 - x_1 x_2 + x_2^2 + q_1 x_1 + q_2 x_2$,
Figure 1(right): $F(x_1,x_2,q_1,q_2,q_3) = x_1^2 - x_1 x_2 - x_2^3 + q_1 x_1 + q_2 x_2 + q_3 x_2^2$,
Figure 2(left): $F(x,q_1,q_2) = x^3 + q_1 x^2 + q_2 x$,
Figure 2(right): $F(x,q_1,q_2,q_3) = x^4 + q_1 x^3 + q_2 x^2 + q_1 x$.
We showed that the Lagrangian maps of figure 2 are stable in [5].

We calculate the caustic of the first example: The canonical relation

$$P_i := \{(x, i(x)) \in (\mathbb{L} \times T^*\mathbb{R}, (0,0))\}$$

of $i$ is given by the generating family $H(Q_1,Q_2,q_1,q_2) = Q_1^2 - Q_1 Q_2 + Q_2^2 + Q_1 q_1 + Q_2 q_2$ such that

$$\begin{aligned} P_i &= \{(Q_1, Q_2, -\frac{\partial H}{\partial Q_1}, -\frac{\partial H}{\partial Q_2}, q_1, q_2, \frac{\partial H}{\partial q_1}, \frac{\partial H}{\partial q_2})\} \\ &= \{Q_1, Q_2, -2Q_1 + Q_2 - q_1, Q_1 - 2Q_2 - q_2, q_1, q_2, Q_1, Q_2)\}. \end{aligned}$$

Therefore we have that

$$\begin{aligned}
i(L^0_\emptyset) = L_\emptyset &= \{(q_1, q_2, \frac{\partial F}{\partial q_1}, \frac{\partial F}{\partial q_2}) \in (T^*\mathbb{R}^2, 0) | \frac{\partial F}{\partial x_1} = \frac{\partial F}{\partial x_2} = 0\} \\
&= \{(-2x_1 + x_2, x_1 - 2x_2, x_1, x_2)\}, \\
i(L^0_1) = L_1 &= \{(q, \frac{\partial F}{\partial q}) | x_1 = \frac{\partial F}{\partial x_2} = 0\} = \{(q_1, -2x_2, 0, x_2)\}, \\
i(L^0_2) = L_2 &= \{(q, \frac{\partial F}{\partial q}) | x_2 = \frac{\partial F}{\partial x_1} = 0\} = \{(-2x_1, q_2, x_1, 0)\}, \\
i(L^0_{\{1,2\}}) = L_{\{1,2\}} &= \{(q, \frac{\partial F}{\partial q}) | x_1 = x_2 = 0\} = \{(q_1, q_2, 0, 0)\}.
\end{aligned}$$

Therefore $C_\emptyset = C_1 = C_2 = C_{\{1,2\}} = \emptyset$.

$$Q_{\emptyset,1} = \{(x_2, -2x_2) \in (\mathbb{R}^2, 0) | x_2 \geq 0\}, Q_{\emptyset,2} = \{(-2x_1, x_1) | x_1 \geq 0\},$$

$$Q_{1,\{1,2\}} = \{(q_1, 0)\}, Q_{2,\{1,2\}} = \{(0, q_2)\}.$$

**Stability of unfoldings**: We recall the theory of unfolding which is developed in [5, p.583 Section 4]. Let $(x, y) = (x_1, \dots, x_r, y_1, \dots, y_k)$ be a fixed coordinate system of $(\mathbb{H}^r \times \mathbb{R}^k, 0)$. We denote by $\mathcal{B}(r; k)$ the group of diffeomorphism germs on $(\mathbb{H}^r \times \mathbb{R}^k, 0)$ of the form:

$$\Phi(x, y) = (x_1\phi^1_1(x, y), \dots, x_r\phi^r_1(x, y), \phi^1_2(x, y), \dots, \phi^k_2(x, y)).$$

We also denote by $\mathcal{B}_n(r; k+n)$ the group of diffeomorphism germs on $(\mathbb{H}^r \times \mathbb{R}^{k+n}, 0)$ of the form:

$$\begin{aligned}
\Phi(x, y, q) = (x_1\phi^1_1(x, y, q), \dots, x_r\phi^r_1(x, y, q), \phi^1_2(x, y, q), \dots, \phi^k_2(x, y, q), \\
\phi^1_3(q), \dots, \phi^n_3(q)).
\end{aligned}$$

We write $\Phi(x, y, q) = (x\phi_1(x, y, q), \phi_2(x, y, q), \phi_3(q))$, $x\frac{\partial f}{\partial x} = (x_1\frac{\partial f}{\partial x_1}, \cdots, x_r\frac{\partial f}{\partial x_r})$ and write other notations analogously.

We say that $f, g \in \mathcal{E}(r; k)$ are *reticular $\mathcal{R}$-equivalent* if there exists $\phi \in \mathcal{B}(r; k)$ such that $g = f \circ \phi$.

We say that $F, G \in \mathcal{E}(r; k+n)$ are *reticular $\mathcal{P}$-$\mathcal{R}^+$-equivalent* if there exist $\Phi \in \mathcal{B}_n(r; k+n)$ and $a \in \mathfrak{M}(n)$ such that $G(x, y, q) = F \circ \Phi(x, y, q) + a(q)$ for $(x, y, q) \in (\mathbb{H}^r \times \mathbb{R}^{k+n}, 0)$. We say that $F \in \mathcal{E}(r; k_1+n)$ and $G \in \mathcal{E}(r; k_2+n)$ are *stably* reticular $\mathcal{P}$-$\mathcal{R}^+$-equivalent if $F$ and $G$ are reticular $\mathcal{P}$-$\mathcal{R}^+$-equivalent after the addition of nondegenerate quadratic forms of $y$.

We say that a function germ $f \in \mathfrak{M}(r; k)$ is *$\mathcal{R}$-simple* if the following holds: For a sufficiently large integer $l$, there exists a neighborhood $N$ of $j^lf(0)$ in $J^l(r+k, 1)$ such that $N$ intersects with a finite number of $\mathcal{R}$-orbits. By [1] we have that:

**Theorem 1.1.** (see also [2, p.279]) *An $\mathcal{R}$-simple function germ in $\mathfrak{M}(1; k)^2$ is stably $\mathcal{R}$-equivalent to one of the following function germs:*

$$B^\pm_l : \pm x^l \ (l \geq 2), \quad C^\pm_l : xy \pm y^l \ (l \geq 3), \quad F^\pm_4 : \pm x^2 + y^3.$$

Let $U$ be an open set in $\mathbb{R}^n$. We equip $C^\infty(U, \mathbb{R}^n)$ with the $C^\infty$-topology. Let $f \in \mathfrak{M}(r; k)$ and $F \in \mathfrak{M}(r; k+n)$ be an unfolding of $f$. We say that $F$ is *reticular $\mathcal{P}$-$\mathcal{R}^+$-stable* if the following condition holds: For any neighborhood $U$ of 0 in $\mathbb{R}^{r+k+n}$ and any representative $\tilde{F} \in C^\infty(U, \mathbb{R})$ of $F$, there exists a neighborhood $N_{\tilde{F}}$ of $\tilde{F}$ such that, for any element $\tilde{G} \in N_{\tilde{F}}$, the germ $\tilde{G}|_{\mathbb{H}^r \times \mathbb{R}^{k+n}}$ at $(0, y_0, q_0)$ is reticular $\mathcal{P}$-$\mathcal{R}^+$-equivalent to $F$ for some $(0, y_0, q_0) \in U$.

We say that $F$ is *reticular $\mathcal{P}$-$\mathcal{R}^+$-infinitesimal versal* if

$$\mathcal{E}(r;k) = \langle x\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle_{\mathcal{E}(r;k)} + \langle 1, \frac{\partial F}{\partial q}|_{q=0}\rangle_{\mathbb{R}}.$$

In [5] we define other stabilities of unfoldings: versatility, infinitesimal stability, homotopical stability.

As a consequence of Damon's theory of good action, we have the following theorem:

**Theorem 1.2.** (see [5, Theorem 4.5]) *Let $F \in \mathfrak{M}(r;k+n)$ be an unfolding of $f \in \mathfrak{M}(r;k)$. Then the following are all equivalent.*
(1) *$F$ is reticular $\mathcal{P}$-$\mathcal{R}^+$-stable.*
(2) *$F$ is reticular $\mathcal{P}$-$\mathcal{R}^+$-versal.*
(3) *$F$ is reticular $\mathcal{P}$-$\mathcal{R}^+$-infinitesimally versal.*
(4) *$F$ is reticular $\mathcal{P}$-$\mathcal{R}^+$-infinitesimally stable.*
(5) *$F$ is reticular $\mathcal{P}$-$\mathcal{R}^+$-homotopically stable.*

**The relation between reticular Lagrangian maps and their generating families** are given in the following theorems:

**Theorem 1.3.** (see [5, Theorem 3.2]) (1) *For any reticular Lagrangian map $\pi \circ i$, there exists a function germ $F \in \mathfrak{M}(r;k+n)^2$ which is a generating family of $\pi \circ i$.*
(2) *For any $S$-non-degenerate function germ $F \in \mathfrak{M}(r;k+n)^2$, there exists a reticular Lagrangian map of which $F$ is a generating family.*
(3) *Two reticular Lagrangian maps are Lagrangian equivalent if and only if their generating families are stably reticular $\mathcal{P}$-$\mathcal{R}^+$-equivalent.*

We remark that Nguyen Huu Duc, Nguyen Tien Dai and F. Pham proved the same theorem in the complex analytic category (cf., [3]). But their method does not work well for the $C^\infty$-category because $F_t = (1-t)F + tF''$ on p.14 may be degenerate at some point in $[0,1]$. We solved this problem in [5, p.577-582]. Our method is available for all of $C^\infty$, real analytic, and complex analytic categories.

**Theorem 1.4.** (see [5, Theorem 5.5] or [6, Theorem 3.1]) *Let $\pi\circ i : (\mathbb{L}, 0) \to (T^*\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a reticular Lagrangian map with the generating family $F(x,y,q) \in \mathfrak{M}(r;k+n)^2$. Then $\pi \circ i$ is stable if and only if $F$ is a reticular $\mathcal{P}$-$\mathcal{R}^+$-stable unfolding of $F|_{q=0}$.*

We investigated the genericity of caustics on an $r$-corner and gave the generic classification for the cases $r = 0$ and 1 in [6]. We also showed that the method in [6] does not work well for the case $r = 2$ by the modality of generating families. In this paper we introduce two equivalence relations of reticular Lagrangian maps which are weaker than Lagrangian equivalence in order to give a generic classification of caustics on a corner.

## 2. Caustic- and Weak Caustic-equivalences

We introduce equivalence relations on reticular Lagrangian maps and their generating families.

Let $\pi \circ i_j$ be reticular Lagrangian maps for $j = 1, 2$. We say that they are *caustic-equivalent* if there exists a diffeomorphism germ $g$ on $(\mathbb{R}^n, 0)$ such that

$$g(C^1_\sigma) = C^2_\sigma, \quad g(Q^1_{\sigma,\tau}) = Q^2_{\sigma,\tau} \quad \text{for all } \sigma, \tau \subset I_r\ (\sigma \neq \tau). \tag{1}$$

When all $C^i_\sigma$ and $Q^i_{\sigma,\tau}$ are smooth, we may define weak caustic-equivalence. We say that reticular Lagrangian maps $\pi \circ i_1$ and $\pi \circ i_2$ are *weakly caustic-equivalent* if all $C^i_\sigma$ and $Q^i_{\sigma,\tau}$ are smooth and there exists a homeomorphism germ $g$ on $(\mathbb{R}^n, 0)$ such that $g$ is smooth on all $C^1_\sigma$, $Q^1_{\sigma,\tau}$, and satisfies (1).

We shall define the stabilities of reticular Lagrangian maps under the above equivalence relations and define the corresponding equivalence relations and stabilities of their generating families.

The purpose of this paper is to show the following theorem:

**Theorem 2.1.** *Let $n = 2, 3$, $U$ a neighborhood of $0$ in $T^*\mathbb{R}^n$, $S(T^*\mathbb{R}^n, 0)$ be the set of symplectic diffeomorphism germs on $(T^*\mathbb{R}^n, 0)$, and $S(U, T^*\mathbb{R}^n)$ be the space of symplectic embeddings from $U$ to $T^*\mathbb{R}^n$ with the $C^\infty$-topology. Then there exists a residual set $O \subset S(U, T^*\mathbb{R}^n)$ such that for any $\tilde{S} \in O$ and $x \in U$, the reticular Lagrangian map $\pi \circ \tilde{S}_x|_{\mathbb{L}}$ is weakly caustic-stable or caustic-stable, where $\tilde{S}_x \in S(T^*\mathbb{R}^n, 0)$ is defined by the map $x_0 \mapsto \tilde{S}(x_0 + x) - \tilde{S}(x)$.*

A reticular Lagrangian map $\pi \circ \tilde{S}_x|_{\mathbb{L}}$ for any $\tilde{S} \in O$ and $x \in U$ is weakly caustic-equivalent to one which has a weak reticular $\mathcal{P}$-$\mathcal{C}$-stable generating family $B^{\pm,+,1}_{2,2}$, $B^{\pm,+,2}_{2,2}$, $B^{\pm,-}_{2,2}$, or is caustic equivalent to one which has a reticular $\mathcal{P}$-$\mathcal{C}$-stable generating family $B^{\pm,0}_{2,2}$, $B^{\pm,\pm}_{2,2,3}$, $B^{\pm,\pm}_{2,3}$, $B^{\pm,\pm}_{3,2}$, $C^{\pm,\pm}_{3,2}$:

$B^{\pm,+,1}_{2,2}$: $F(x_1, x_2, q_1, q_2) = x_1^2 \pm x_1x_2 + \frac{1}{5}x_2^2 + q_1x_1 + q_2x_2$,
$B^{\pm,+,2}_{2,2}$: $F(x_1, x_2, q_1, q_2) = x_1^2 \pm x_1x_2 + x_2^2 + q_1x_1 + q_2x_2$,
$B^{\pm,-}_{2,2}$: $F(x_1, x_2, q_1, q_2) = x_1^2 \pm x_1x_2 - x_2^2 + q_1x_1 + q_2x_2$,
$B^{\pm,0}_{2,2}$: $F(x_1, x_2, q_1, q_2, q_3) = x_1^2 \pm x_2^2 + q_1x_1 + q_2x_2 + q_3x_1x_2$,
$B^{\pm,\pm}_{2,2,3}$: $F(x_1, x_2, q_1, q_2, q_3) = (x_1 \pm x_2)^2 \pm x_2^3 + q_1x_1 + q_2x_2 + q_3x_2^2$,
$B^{\pm,\pm}_{2,3}$: $F(x_1, x_2, q_1, q_2, q_3) = x_1^2 \pm x_1x_2 \pm x_2^3 + q_1x_1 + q_2x_2 + q_3x_2^2$,
$B^{\pm,\pm}_{3,2}$: $F(x_1, x_2, q_1, q_2, q_3) = x_1^3 \pm x_1x_2 \pm x_2^2 + q_1x_1 + q_2x_2 + q_3x_1^2$,
$C^{\pm,\pm}_{3,2}$: $F(y, x_1, x_2, q_1, q_2, q_3) = \pm y^3 + x_1y \pm x_2y + x_2^2 + q_1y + q_2x_1 + q_3x_2$.

In order to describe the caustic-equivalence of reticular Lagrangian maps by their generating families, we introduce the following equivalence relation of function germs. We say that function germs $f, g \in \mathcal{E}(r; k)$ are *reticular $\mathcal{C}$-equivalent* if there exist $\phi \in \mathcal{B}(r; k)$ and a non-zero number $a \in \mathbb{R}$ such that $g = a \cdot f \circ \phi$. We construct the theory of unfoldings with respect to the corresponding equivalence relation. Then the relation of unfoldings is given as follows: Two function germs $F(x, y, q), G(x, y, q) \in \mathcal{E}(r; k + n)$ are *reticular $\mathcal{P}$-$\mathcal{C}$-equivalent* if there exist $\Phi \in \mathcal{B}_n(r; k + n)$ and a unit $a \in \mathcal{E}(n)$ and $b \in \mathfrak{M}(n)$ such that $G = a \cdot F \circ \Phi + b$. We define the *stable* reticular ($\mathcal{P}$-)$\mathcal{C}$-equivalence in the usual way. We remark that a reticular $\mathcal{P}$-$\mathcal{C}$-equivalence class includes the reticular $\mathcal{P}$-$\mathcal{R}^+$-equivalence classes.

We review the results of the theory of function germs under this equivalence relation. Let $F(x, y, q) \in \mathfrak{M}(r; k + n)$ be an unfolding of $f(x, y) \in \mathfrak{M}(r; k)$.

We say that $F$ is *reticular $\mathcal{P}$-$\mathcal{C}$-stable* if the following condition holds: For any neighborhood $U$ of $0$ in $\mathbb{R}^{r+k+n}$ and any representative $\tilde{F} \in C^\infty(U, \mathbb{R})$ of $F$, there exists a neighborhood $N_{\tilde{F}}$ of $\tilde{F}$ in the $C^\infty$-topology such that for any element $\tilde{G} \in N_{\tilde{F}}$ the germ $\tilde{G}|_{\mathbb{H}^r \times \mathbb{R}^{k+n}}$ at $(0, y_0, q_0)$ is reticular $\mathcal{P}$-$\mathcal{C}$-equivalent to $F$ for some $(0, y_0, q_0) \in U$.

We say that $F$ is *reticular $\mathcal{P}$-$\mathcal{C}$-versal* if all unfoldings of $f$ are reticular $\mathcal{P}$-$\mathcal{C}$-$f$-induced from $F$. That is, for any unfolding $G \in \mathfrak{M}(r; k+n')$ of $f$, there exist a map germ

$$\Phi : (\mathbb{H}^r \times \mathbb{R}^{k+n'}, 0) \to (\mathbb{H}^r \times \mathbb{R}^{k+n}, 0)$$

and a unit $a \in \mathcal{E}(n')$ and $b \in \mathfrak{M}(n')$ satisfying the following conditions:
(1) $\Phi(x, y, 0) = (x, y, 0)$ for all $(x, y) \in (\mathbb{H}^r \times \mathbb{R}^k, 0)$ and $a(0) = 1,\ b(0) = 0$,
(2) $\Phi$ can be written in the form: $\Phi(x, y, q) = (x\phi_1(x, y, q), \phi_2(x, y, q), \phi_3(q))$,
(3) $G(x, y, q) = a(q) \cdot F \circ \Phi(x, y, q) + b(q)$ for all $(x, y, q) \in (\mathbb{H}^r \times \mathbb{R}^{k+n'}, 0)$.

We say that $F$ is *reticular $\mathcal{P}$-$\mathcal{C}$-infinitesimally versal* if

$$\mathcal{E}(r; k) = \langle x\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle_{\mathcal{E}(r;k)} + \langle 1, f, \frac{\partial F}{\partial q}|_{q=0}\rangle_{\mathbb{R}}. \tag{2}$$

We say that $F$ is *reticular $\mathcal{P}$-$\mathcal{C}$-infinitesimally stable* if

$$\mathcal{E}(r; k+n) = \langle x\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}\rangle_{\mathcal{E}(r;k+n)} + \langle 1, F, \frac{\partial F}{\partial q}\rangle_{\mathcal{E}(n)}.$$

We say that $F$ is *reticular $\mathcal{P}$-$\mathcal{C}$-homotopically stable* if for any smooth path-germ

$$(\mathbb{R}, 0) \to \mathcal{E}(r; k+n), t \mapsto F_t$$

with $F_0 = F$, there exists a smooth path-germ

$$(\mathbb{R}, 0) \to \mathcal{B}_n(r; k+n) \times \mathcal{E}(n) \times \mathcal{E}(n), t \mapsto (\Phi_t, a_t, b_t)$$

with $(\Phi_0, a_0, b_0) = (id, 1, 0)$ such that each $(\Phi_t, a_t, b_t)$ is a reticular $\mathcal{P}$-$\mathcal{C}$-isomorphism from $F$ to $F_t$, that is $F_t = a_t \cdot F \circ \Phi_t + b_t$ for $t$ around 0.

The following theorem is used to prove that the stabilities of reticular Lagrangian maps and their generating families are equivalent.

**Theorem 2.2.** (cf., [5, Theorem 4.5]) *Let $F \in \mathfrak{M}(r; k+n)$ be an unfolding of $f \in \mathfrak{M}(r; k)$. Then the following are all equivalent.*
(1) *$F$ is reticular $\mathcal{P}$-$\mathcal{C}$-stable.*
(2) *$F$ is reticular $\mathcal{P}$-$\mathcal{C}$-versal.*
(3) *$F$ is reticular $\mathcal{P}$-$\mathcal{C}$-infinitesimally versal.*
(4) *$F$ is reticular $\mathcal{P}$-$\mathcal{C}$-infinitesimally stable.*
(5) *$F$ is reticular $\mathcal{P}$-$\mathcal{C}$-homotopically stable.*

For a non-quasihomogeneous function germ $f(x, y) \in \mathfrak{M}(r; k)$, if $1, f, a_1, \ldots, a_n \in \mathcal{E}(r; k)$ is a representative of a basis of the vector space

$$\mathcal{E}(r; k)/\langle x\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle_{\mathcal{E}(r;k)},$$

then the function germ $f + a_1q_1 + \cdots + a_nq_n \in \mathfrak{M}(r; k+n)$ is a reticular $\mathcal{P}$-$\mathcal{C}$-stable unfolding of $f$ by (2). We call $n$ the reticular $\mathcal{C}$-codimension of $f$. We remark that the dimension of the vector space is $(n+2)$, but the reticular $\mathcal{C}$-codimension is $n$.

If $f$ is quasihomogeneous then $f$ is included in $\langle x\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle_{\mathcal{E}(r;k)}$. This means that the reticular $\mathcal{C}$-codimension of a quasihomogeneous function germ is equal to its reticular $\mathcal{R}^+$-codimension. Therefore if $1, a_1, \ldots, a_n \in \mathcal{E}(r; k)$ is a representative of a basis of the vector space, the function germ $f + a_1q_1 + \cdots + a_nq_n \in \mathfrak{M}(r; k+n)$ is a reticular $\mathcal{P}$-$\mathcal{C}$-stable unfolding of $f$. In this case the dimension of the vector space is $(n+1)$, but the reticular $\mathcal{C}$-codimension is $n$.

We define the *simplicity* of function germs under the reticular $\mathcal{C}$-equivalence in the usual way (cf., the definition before Theorem 1.1).

**Theorem 2.3.** (cf., Theorem 1.1) *A reticular $\mathcal{C}$-simple function germ in $\mathfrak{M}(1;\ k)^2$ is stably reticular $\mathcal{C}$-equivalent to one of the following function germs:*

$$B_l : x^l \ (l \geq 2), \quad C_l^{\varepsilon} : xy + \varepsilon y^l \ (\varepsilon^{l-1} = 1, l \geq 3), \quad F_4 : x^2 + y^3.$$

The relation between reticular Lagrangian maps and their generating families under the caustic-equivalence are given as follows:

**Proposition 2.4.** *Let $\pi \circ i_j$ be reticular Lagrangian maps with generating families $F_j$ for $j = 1, 2$. If $F_1$ and $F_2$ are stably reticular $\mathcal{P}$-$\mathcal{C}$-equivalent then $\pi \circ i_1$ and $\pi \circ i_2$ are caustic-equivalent.*

*Proof.* The function germ $F_2$ may be written as $F_2(x, y, q) = a(q)F_3(x, y, q)$, where $a$ is a unit and $F_1$ and $F_3$ are stably reticular $\mathcal{P}$-$\mathcal{R}^+$-equivalent. Then the reticular Lagrangian map $\pi \circ i_3$ given by $F_3$ and $\pi \circ i_1$ are Lagrangian equivalent and the caustic of $\pi \circ i_2$ and $\pi \circ i_3$ coincide with each other. ■

This proposition shows that it is enough to classify function germs under stable reticular $\mathcal{P}$-$\mathcal{C}$-equivalence in order to classify reticular Lagrangian maps under caustic-equivalence. We give here the following classification list:

**Theorem 2.5.** (see [5, p.592]) *Let $f \in \mathfrak{M}(2; k)^2$ have reticular $\mathcal{C}$-codimension $\leq 3$. Then $f$ is stably reticular $\mathcal{C}$-equivalent to one in the following list.*

| $k$ | *Normal form* | *codim* | *Conditions* | *Notation* |
|---|---|---|---|---|
| 0 | $x_1^2 \pm x_1x_2 + ax_2^2$ | 3 | $0 < a < \frac{1}{4}$ | $B_{2,2,a}^{\pm,+,1}$ |
| | $x_1^2 \pm x_1x_2 + ax_2^2$ | 3 | $a > \frac{1}{4}$ | $B_{2,2,a}^{\pm,+,2}$ |
| | $x_1^2 \pm x_1x_2 + ax_2^2$ | 3 | $a < 0$ | $B_{2,2,a}^{\pm,-}$ |
| | $x_1^2 \pm x_2^2$ | 3 | | $B_{2,2}^{\pm,0}$ |
| | $(x_1 \pm x_2)^2 \pm x_2^3$ | 3 | | $B_{2,2,3}^{\pm,\pm}$ |
| | $x_1^2 \pm x_1x_2 \pm x_2^3$ | 3 | | $B_{2,3}^{\pm,\pm}$ |
| | $x_1^3 \pm x_1x_2 \pm x_2^2$ | 3 | | $B_{3,2}^{\pm,\pm}$ |
| 1 | $\pm y^3 + x_1y \pm x_2y + x_2^2$ | 3 | | $C_{3,2}^{\pm,\pm}$ |

We remark that the stable reticular $\mathcal{C}$-equivalence class $B_{2,3}^{+,+}$ of $x_1^2 + x_1x_2 + x_2^3$ consists of the union of the stable reticular $\mathcal{R}$-equivalence classes of $x_1^2 + x_1x_2 + ax_2^3$ and $-x_1^2 - x_1x_2 - ax_2^3$ for $a > 0$. Since $x_1\frac{\partial f}{\partial x_1}, x_2\frac{\partial f}{\partial x_2}, f$ are linear independent, this means that the $\mathcal{C}$-equivalence class of $f$ is simple. The same things hold for $B_{2,2,3}^{\pm,\pm}$, $B_{2,3}^{\pm,\pm}$, $B_{3,2}^{\pm,\pm}$, $C_{3,2}^{\pm,\pm}$.

We also remark that the classification list looks like that of D.Siersma [4, p.126]. But our equivalence relation differs from his.

## 3. Caustic-stability

We define *the caustic-stability* of reticular Lagrangian maps and reduce our investigation to finite-dimensional jet spaces of symplectic diffeomorphism germs.

We say that a reticular Lagrangian map $\pi \circ i$ is *caustic-stable* if the following condition holds: For any extension $S \in S(T^*\mathbb{R}^n, 0)$ of $i$ and any representative $\tilde{S} \in S(U, T^*\mathbb{R}^n)$ of $S$, there exists a neighborhood $N_{\tilde{S}}$ of $\tilde{S}$ such that for any $\tilde{S}' \in N_{\tilde{S}}$ the reticular Lagrangian map $\pi \circ \tilde{S}'|_{\mathbb{L}}$ at $x_0$ and $\pi \circ i$ are caustic-equivalent for some $x_0 = (0, \ldots, 0, p_{r+1}^0, \ldots, p_n^0) \in U$.

**Definition 3.1.** Let $\pi \circ i$ be a reticular Lagrangian map and $l$ be a non-negative number. We say that $\pi \circ i$ is *caustic $l$-determined* if the following condition holds: For any extension $S$ of

$i$, the reticular Lagrangian map $\pi \circ S'|_{\mathbb{L}}$ and $\pi \circ i$ are caustic-equivalent for any symplectic diffeomorphism germ $S'$ on $(T^*\mathbb{R}^n, 0)$ satisfying $j^l S(0) = j^l S'(0)$.

**Lemma 3.2.** *Let $\pi \circ i : (\mathbb{L}, 0) \to (T^*\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a reticular Lagrangian map. If a generating family of $\pi \circ i$ is reticular $\mathcal{P}$-$\mathcal{C}$-stable then $\pi \circ i$ is caustic $(n+2)$-determined.*

*Proof.* This is proved in a manner analogous to that of [6, Theorem 5.3]. We give a sketch of the proof. Let $S$ be an extension of $i$. Then we may assume that there exists a function germ $H(Q,p)$ such that the canonical relation $P_S$ has the form:

$$P_S = \{(Q, -\frac{\partial H}{\partial Q}(Q,p), -\frac{\partial H}{\partial p}(Q,p), p) \in (T^*\mathbb{R}^n \times T^*\mathbb{R}^n, (0,0))\}.$$

Then the function germ $F(x,y,q) = H_0(x,y) + \langle y, q\rangle$ is a reticular $\mathcal{P}$-$\mathcal{C}$-stable generating family of $\pi \circ i$, and $H_0$ is reticular $\mathcal{R}$-$(n+3)$-determined, where $H_0(x,y) = H(x,0,y)$. Let a symplectic diffeomorphism germ $S'$ on $(T^*\mathbb{R}^n, 0)$ satisfying $j^{n+2}S(0) = j^{n+2}S'(0)$ be given. Then there exists a function germ $H'(Q,p)$ such that the canonical relation $P_{S'}$ has the same form for $H'$ and the function germ $G(x,y,q) = H_0'(x,y) + \langle y, q\rangle$ is a generating family of $\pi \circ S'|_{\mathbb{L}}$. Then it holds that $j^{n+3}H_0(0) = j^{n+3}H_0'(0)$. There exists a function germ $G'$ such that $G$ and $G'$ are reticular $\mathcal{P}$-$\mathcal{R}$-equivalent and $F$ and $G'$ are reticular $\mathcal{P}$-$\mathcal{C}$-infinitesimal versal unfoldings of $H_0(x,y)$. It follows that $F$ and $G$ are reticular $\mathcal{P}$-$\mathcal{C}$-equivalent by Theorem 1.2. Therefore $\pi \circ i$ and $\pi \circ S'|_{\mathbb{L}}$ are caustic-equivalent. ■

Let $S^l(n)$ be the Lie group which consists of $l$-jets of symplectic diffeomorphisms on $(T^n\mathbb{R}, 0)$.

**Orbits of the caustic-equivalence classes** $B^{\pm,\pm}_{2,2,3}$, $B^{\pm,\pm}_{2,3}$ $B^{\pm,\pm}_{3,2}$, $C^{\pm,\pm}_{3,2}$: Let $[S_X]_c$ be the caustic-equivalence class of $S_X \in S(T^*\mathbb{R}^3, 0)$ for each $X = B^{\pm,\pm}_{2,2,3}$, $B^{\pm,\pm}_{2,3}$ $B^{\pm,\pm}_{3,2}$, $C^{\pm,\pm}_{3,2}$ in Theorem 2.5 such that $\pi \circ S_X|_{\mathbb{L}}$ has a generating family which is a reticular $\mathcal{P}$-$\mathcal{C}$-stable unfolding of $X$. Since the above singularities are reticular $\mathcal{C}$-simple, this means that $[j^l S_X(0)]_c (:= j^l[S_X(0)]_c)$ are immersed manifolds in $S^l(3)$ for $l \geq 2$.

## 4. Weak Caustic-equivalence

There exist modalities in the classification list of Theorem 2.5. This means that caustic-equivalence is still too strong for a generic classification of caustics on a corner. In order to obtain a generic classification, we need to admit weak caustic-equivalence and the corresponding equivalence relation on generating families.

We say that a reticular Lagrangian map $\pi \circ i$ is *weakly caustic-stable* if the following condition holds: For any extension $S \in S(T^*\mathbb{R}^n, 0)$ of $i$ and any representative $\tilde{S} \in S(U, T^*\mathbb{R}^n)$ of $S$, there exists a neighborhood $N_{\tilde{S}}$ of $\tilde{S}$ such that for any $\tilde{S}' \in N_{\tilde{S}}$ the reticular Lagrangian map $\pi \circ \tilde{S}'|_{\mathbb{L}}$ at $x_0$ and $\pi \circ i$ are weakly caustic-equivalent for some $x_0 = (0, \ldots, 0, p^0_{r+1}, \ldots, p^0_n) \in U$.

We say that two function germs in $\mathfrak{M}(r; k+n)^2$ are *weakly reticular $\mathcal{P}$-$\mathcal{C}$-equivalent* if they are generating families of weakly caustic-equivalent reticular Lagrangian maps. We say that two function germs in $\mathfrak{M}(r; k)^2$ are *weakly reticular $\mathcal{C}$-equivalent* if they have unfoldings which are weakly reticular $\mathcal{P}$-$\mathcal{C}$-equivalent. We define the *stable* weakly reticular $(\mathcal{P}$-$)\mathcal{C}$-equivalence in the usual way.

We say that a function germ $F(x,y,q) \in \mathfrak{M}(r; k+n)$ is *weakly reticular $\mathcal{P}$-$\mathcal{C}$-stable* if the following condition holds: For any neighborhood $U$ of $0$ in $\mathbb{R}^{r+k+n}$ and any representative $\tilde{F} \in C^\infty(U, \mathbb{R})$ of $F$, there exists a neighborhood $N_{\tilde{F}}$ of $\tilde{F}$ in the $C^\infty$-topology such that for any element $\tilde{G} \in N_{\tilde{F}}$ the germ $\tilde{G}|_{\mathbb{H}^r \times \mathbb{R}^{k+n}}$ at $(0, y_0, q_0)$ is weakly reticular $\mathcal{P}$-$\mathcal{C}$-equivalent to $F$ for some $(0, y_0, q_0) \in U$.

**Orbits of weak caustic-equivalence classes** $B^{\pm,+,1}_{2,2,a}$, $B^{\pm,+,2}_{2,2,a}$, $B^{\pm,-}_{2,2,a}$: We investigate here the weak reticular $\mathcal{C}$-equivalence classes $B^{+,+,2}_{2,2,a}$ of function germs. The same methods are valid for the classes $B^{\pm,+,1}_{2,2,a}$, $B^{\pm,+,2}_{2,2,a}$, $B^{\pm,-}_{2,2,a}$. So we discuss only the classes $B^{+,+,2}_{2,2,a}$.

We consider the reticular Lagrangian maps $\pi \circ i_a : (\mathbb{L}, 0) \to (T^*\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ with the generating families $F_a(x_1, x_2, q_1, q_2) = x_1^2 + x_1x_2 + ax_2^2 + q_1x_1 + q_2x_2$ $(a > \frac{1}{4})$. We give the caustics of $\pi \circ i_a$ and $\pi \circ i_b$ for $\frac{1}{4} < a < b$. In these figures $Q_{1,I_2}, Q_{2,I_2}, Q_{\emptyset,2}$ are in the same positions.

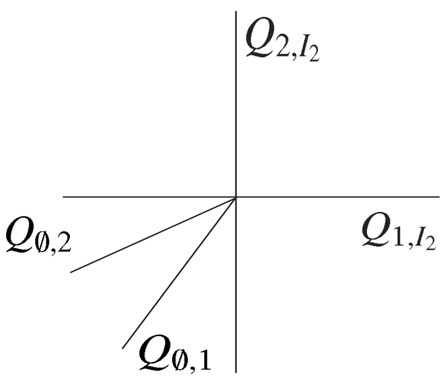


FIGURE 3. the caustics of $\pi \circ i_a$

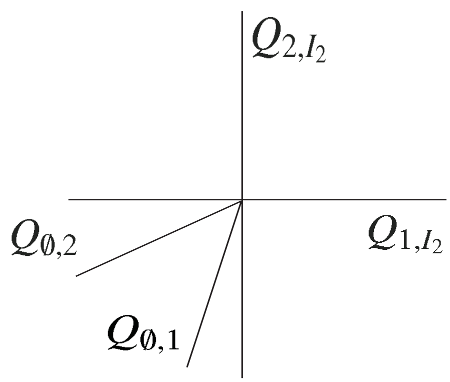


FIGURE 4. the caustics of $\pi \circ i_b$

Suppose that there exists a diffeomorphism germ $g$ on $(\mathbb{R}^2, 0)$ such that $Q_{1,I_2}, Q_{2,I_2}, Q_{\emptyset,2}$ are invariant under $g$. Then $g$ can not map $Q_{\emptyset,1}$ from one to the other. This implies that caustic-equivalence is too strong for generic classifications. But these caustics are equivalent under weak caustic-equivalence. The homeomorphism germ $\Phi^b_a$ on $(\mathbb{R}^2, 0)$ is given as follows: We consider the unit circle $U$ with center 0 and let $U_a, U_b$ be the intersection of $U$ and the caustics of $\pi \circ i_a$, $\pi \circ i_b$ respectively. Then there exists a diffeomorphism $\phi^b_a$ on $U$ such that $\phi^b_a(U_a) = U_b$ and $\phi^b_a$ depends smoothly on $a, b$. We extend naturally the source space of $\phi^b_a$ to $\mathbb{R}^2 - \{0\}$ and define

$$\Phi^b_a(x) = \begin{cases} \phi^b_a(x) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Then the map germ $\Phi^b_a$ at 0 gives a weak caustic-equivalence of $\pi \circ i_a$ and $\pi \circ i_b$. We remark that $\Phi^b_a$ is smooth and depends smoothly on $a, b$ except at the origin. This means that the weak caustic-equivalence $\Phi^1_a$ is naturally extended for weak caustic equivalence from the (caustic) stable reticular Lagrangian map with the generating families

$$F'_a(x_1, x_2, q_1, q_2, q_3) = x_1^2 + x_1x_2 + ax_2^2 + q_1x_1 + q_2x_2 + q_3x_2^2$$

to $F'(x_1, x_2, q_1, q_2, q_3) = x_1^2 + x_1x_2 + x_2^2 + q_1x_1 + q_2x_2$. The figure of the corresponding caustic is given in Figure 8. We also remark that the functions $x_1^2 + x_1x_2 + x_2^2 + q_1x_1 + q_2x_2$ and $x_1^2 + x_1x_2 + \frac{1}{5}x_2^2 + q_1x_1 + q_2x_2$ are not weakly reticular $\mathcal{P}$-$\mathcal{C}$-equivalent because $Q_{\emptyset,1}$ and $Q_{\emptyset,1}$ of their caustics are in the opposite positions to each other.

Then we have that the function germs $f_a(x) = x_1^2 + x_1x_2 + ax_2^2 (a > \frac{1}{4})$ are all weakly reticular $\mathcal{C}$-equivalent to each other.

We define the weak reticular $\mathcal{C}$-equivalence class $[f_a]_w$ of $f_a$ by $\bigcup_{a>\frac{1}{4}}[f_a]_c$. We also define the weak reticular $\mathcal{C}$-equivalence class $[j^l f_a(0)]_w$ of $j^l f_a(0)$ by $\bigcup_{a>\frac{1}{4}} j^l([f_a]_c)$.

Since $x_1\frac{\partial f_a}{\partial x_1}, x_2\frac{\partial f_a}{\partial x_2}, x_2^2$ are linear independent and span the tangent space of the weak reticular $\mathcal{C}$-equivalence class $[f_a]_w$, we have that $[f_a]_w$ is an immersed manifold in $J^3(2,1)$.

We classify function germs in $\mathfrak{M}(2;k)^2$ with respect to the reticular $\mathcal{C}$-equivalence and weak reticular $\mathcal{C}$-equivalence with the codimension$\leq 3$. Then we have the following list:

| $k$ | Normal form | codim | Notation |
|---|---|---|---|
| 0 | $x_1^2 \pm x_1x_2 + \frac{1}{5}x_2^2$ | 2 | $B_{2,2}^{\pm,+,1}$ |
| | $x_1^2 \pm x_1x_2 + x_2^2$ | 2 | $B_{2,2}^{\pm,+,2}$ |
| | $x_1^2 \pm x_1x_2 - x_2^2$ | 2 | $B_{2,2}^{\pm,-}$ |
| | $x_1^2 \pm x_2^2$ | 3 | $B_{2,2}^{\pm,0}$ |
| | $(x_1 \pm x_2)^2 \pm x_2^3$ | 3 | $B_{2,2,3}^{\pm,\pm}$ |
| | $x_1^2 \pm x_1x_2 \pm x_2^3$ | 3 | $B_{2,3}^{\pm,\pm}$ |
| 1 | $\pm y^3 + x_1y \pm x_2y + x_2^2$ | 3 | $C_{3,2}^{\pm,\pm}$ |

We need to show the following proposition:

**Proposition 4.1.** *Let $\pi\circ i_a : (\mathbb{L},0) \to (T^*\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$ be the reticular Lagrangian map with the generating family $x_1^2 + x_1x_2 + ax_2^2 + q_1x_1 + q_2x_2 (B_{2,2}^{+,+,2})$. Let $S_a \in S(T^*\mathbb{R}^2,0)$ be extensions of $i_a$. Then the weak caustic-equivalence class*

$$[j^l S_1(0)]_w := \bigcup_{a>\frac{1}{4}} [j^l S_a(0)]_c$$

*is an immersed manifold in $S^l(2)$ for $l \geq 1$.*

*Proof.* Let $f(x_1,x_2) = x_1^2 + x_1x_2 + ax_2^2$. The tangent space of $[j^l f(0)]_w$ is spanned by $j^l(x_1\frac{\partial f}{\partial x_1})(0), j^l(x_2\frac{\partial f}{\partial x_2})(0), j^l(x_2^2)(0)$ and these are linearly independent for $l \geq 2$. This means that $[j^l f(0)]_w$ is an immersed manifold of $J^l(2,1)$ for $l \geq 2$. This means that $[j^l S_1(0)]_w$ is an immersed manifold of $S^l(2)$ for $l \geq 1$. ∎

We consider the (caustic) stable reticular Lagrangian map

$$\pi\circ i_a : (\mathbb{L},0) \to (T^*\mathbb{R}^3,0) \to (\mathbb{R}^3,0)$$

with the generating family

$$x_1^2 + x_1x_2 + ax_2^2 + q_1x_1 + q_2x_2 + q_3x_2^2$$

and take an extension $S'_a \in S(T^*\mathbb{R}^2,0)$ of $i_a$, then we have by the analogous method that:

**Corollary 4.2.** *Let $S'_a$ be as above. Then the weak caustic-equivalence class*

$$[j^l S'_1(0)]_w := \bigcup_{a>\frac{1}{4}} [j^l S'_a(0)]_c$$

*is an immersed manifold in $S^l(3)$ for $l \geq 1$.*

**Theorem 4.3.** *The function germ $F(x_1,x_2,q_1,q_2) = x_1^2 + x_1x_2 + x_2^2 + q_1x_1 + q_2x_2$ is a weakly reticular $\mathcal{P}$-$\mathcal{C}$-stable unfolding of $f(x_1,x_2) = x_1^2 + x_1x_2 + x_2^2$*

*Proof.* We define $F' \in \mathfrak{M}(2;3)^2$ by $F'(x_1,x_2,q_1,q_2,q_3) = F(x_1,x_2,q_1,q_2) + q_3x_2^2$ Then $F'$ is a reticular $\mathcal{P}$-$\mathcal{R}^+$-stable unfolding of $f$. It follows that for any neighborhood $U'$ of 0 in $\mathbb{R}^5$ and any representative $\tilde{F'} \in C^\infty(U,\mathbb{R})$, there exists a neighborhood $N_{\tilde{F'}}$ such that for any $\tilde{G}' \in N_{\tilde{F}'}$ the function germ $\tilde{G}'|_{\mathbb{H}^2\times\mathbb{R}^3}$ at $p'_0$ is reticular $\mathcal{P}$-$\mathcal{R}^+$-equivalent to $F'$ for some $p'_0 = (0,0,q_1^0,q_2^0,q_3^0) \in U'$.

Let a neighborhood $U$ of 0 in $\mathbb{R}^4$ and a representative $\tilde{F} \in C^\infty(U,\mathbb{R})$ be given. We set the open interval $I = (-0.5, 0.5)$ and set $U' = U \times I$. Then there exists $N_{\tilde{F}'}$ for which the above condition holds. We can choose a neighborhood $N_{\tilde{F}}$ of $\tilde{F}$ such that for any $\tilde{G} \in N_{\tilde{F}}$ the function $\tilde{G} + q_3x_2^2 \in N_{\tilde{F}'}$. Let a function $\tilde{G} \in N_{\tilde{F}}$ be given. Then the function germ $G' = (\tilde{G} + q_3x_2^2)|_{\mathbb{H}^2\times\mathbb{R}^3}$ at $p'_0$ is reticular $\mathcal{P}$-$\mathcal{R}^+$-equivalent to $F'$ for some

$$p'_0 = (0,0,q_1^0,q_2^0,q_3^0) \in U'.$$

We define $G \in \mathfrak{M}(2;2)^2$ by $\tilde{G}|_{\mathbb{H}^2\times\mathbb{R}^2}$ at $p_0 = (0,0,q_1^0,q_2^0) \in U$. Then it holds that

$$G'(x,q) = G(x,q_1,q_2) + (q_3 + q_3^0)x_2^2,$$

and $G'|_{q=0} = G(x,0) + q_3^0x_2^2$ is reticular $\mathcal{R}$-equivalent to $f$. Let $(\Phi, a)$ be the reticular $\mathcal{P}$-$\mathcal{R}^+$-equivalence from $G'$ to $F'$. We write $\Phi(x,q) = (x\phi_1(x,q), \phi_1^2(q), \phi_2^2(q), \phi_3^2(q))$. By shrinking $U$ if necessary, we may assume that the map germ

$$(q_1,q_2) \mapsto (\phi_1^2(q_1,q_2,0), \phi_2^2(q_1,q_2,0)) \text{ on } (\mathbb{R}^2,0)$$

is a diffeomorphism germ. Then $F$ is reticular $\mathcal{P}$-$\mathcal{R}^+$-equivalent to $G_1 \in \mathfrak{M}(2;2)^2$ given by $G_1(x,q) = G(x_1,x_2,q_1,q_2) + (\phi_3^2(q_1,q_2,0) + q_3^0)x_2^2$. It follows that the reticular Lagrangian maps defined by $F$ and $G_1$ are Lagrangian equivalent. We have that

$$j^2(G + q_3^0x_2^2)(0) = j^2G_1(0), \quad q_3^0 > -0.5.$$

This means that the caustic of $G_1$ is weakly caustic-equivalent to the caustic of $G$ because the reticular Lagrangian maps of $G_1$ and $F$ are the same weak caustic-equivalence class that is 1-determined under weak caustic-equivalence. This means that $F$ and $G$ are weakly reticular $\mathcal{P}$-$\mathcal{C}$-equivalent. Therefore $F$ is weakly reticular $\mathcal{P}$-$\mathcal{C}$-stable. ∎

By the above consideration, we have that: For each singularity $B_{2,2}^{\pm,+,1}$, $B_{2,2}^{\pm,+,2}$, $B_{2,2}^{\pm,-}$, if we take the symplectic diffeomorphism germ $S_a(S'_a)$ as the above method, then the weak caustic-equivalence class $[j^lS_a(0)]_w([j^lS'_a(0)]_w)$ is one class and immersed manifold in $S^l(2)(S^l(3))$ for $l \geq 1$ respectively. Since the caustics of $\pi \circ S_a|_{\mathbb{L}}$ and $\pi \circ S'_a|_{\mathbb{L}}$ are determined by their linear parts, this means that the reticular Lagrangian maps are weakly caustic 1-determined.

We now start to prove the main theorem: We choose the weakly caustic-stable reticular Lagrangian maps $\pi \circ i_X : (\mathbb{L},0) \to (T^*\mathbb{R}^n,0) \to (\mathbb{R}^n,0)$ for

$$X = B_{2,2}^{\pm,+,1}, B_{2,2}^{\pm,+,2}, B_{2,2}^{\pm,-}. \tag{3}$$

We also choose the caustic-stable reticular Lagrangian maps $\pi\circ i_X : (\mathbb{L},0) \to (T^*\mathbb{R}^3,0) \to (\mathbb{R}^3,0)$ for

$$X = B_{2,2}^{\pm,0}, B_{2,2,3}^{\pm,\pm}, B_{2,3}^{\pm,\pm}, B_{3,2}^{\pm,\pm}, C_{3,2}^{\pm,\pm}. \tag{4}$$

Then other reticular Lagrangian maps are not caustic-stable since other singularities have reticular $\mathcal{C}$-codimension $> 3$. We choose extensions $S_X \in S(T^*\mathbb{R}^n, 0)$ of $i_X$ for all $X$. We define

$$O'_1 = \{\tilde{S} \in S(U, T^*\mathbb{R}^n) \mid j_0^1\tilde{S} \text{ is transversal to } [j^1S_X(0)]_w \text{ for all } X \text{ in (3)}\},$$

$$O'_2 = \{\tilde{S} \in S(U, T^*\mathbb{R}^n) \mid j_0^{n+2}\tilde{S} \text{ is transversal to } [j^{n+2}S_X(0)]_c \text{ for all } X \text{ in (4)}\},$$

where $j_0^l\tilde{S}(x) = j^l\tilde{S}_x(0)$. Then $O_1'$ and $O_2'$ are residual sets. We set

$$Y = \{j^{n+2}S(0) \in S^{n+2}(n) \mid \text{the codimension of } [j^{n+2}S(0)]_L > 8\}.$$

Then $Y$ is an algebraic set in $S^{n+2}(n)$ by [6, Theorem 6.6 (a')]. Therefore we can define

$$O'' = \{\tilde{S} \in S(U, T^*\mathbb{R}^n) \mid j_0^{n+2}\tilde{S} \text{ is transversal to } Y\}.$$

For any $S \in S(T^*\mathbb{R}^n, 0)$ with $j^{n+2}S(0) \in Y$ and any generating family $F$ of $\pi \circ S|_{\mathbb{L}}$, the function germ $F|_{q=0}$ has reticular $\mathcal{R}^+$-codimension $> 4$. This means that $F|_{q=0}$ has reticular $C$-codimension $> 3$. It follows that $j^{n+2}S(0)$ does not belong to the above equivalence classes. Then $Y$ has codimension $> 6$ because all elements in $Y$ are adjacent to one of the list (4) which are caustic-simple. Then we have that

$$O'' = \{\tilde{S} \in S(U, T^*\mathbb{R}^n) \mid j_0^{n+2}\tilde{S}(U) \cap Y = \emptyset\}.$$

We define $O = O_1' \cap O_2' \cap O''$. Since all $\pi \circ i_X$ for $X$ in (3) are weak caustic 1-determined, and all $\pi \circ i_X$ in (4) are caustic 5-determined by Lemma 3.2. Then $O$ has the required condition. ■

$B_{2,2}^{*;*}$ in $2D$

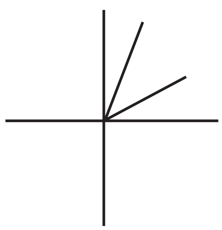

FIGURE 5. $B_{2,2}^{+;+,1}, B_{2,2}^{+;+,2}$

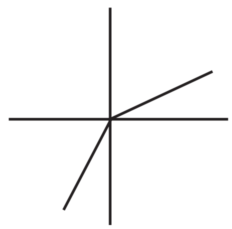

FIGURE 6. $B_{2,2}^{-;+,1}, B_{2,2}^{-;+,2}$

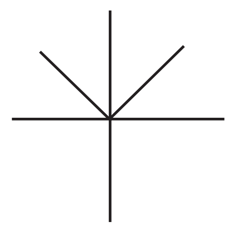

FIGURE 7. $B_{2,2}^{+;-}, B_{2,2}^{-;-}$

$B_{2,2}^{*;*}$ in $3D$

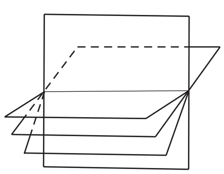

FIGURE 8. $B_{2,2}^{+;+,1}, B_{2,2}^{+;+,2}$

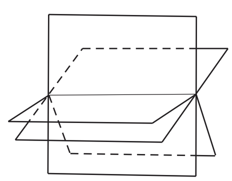

FIGURE 9. $B_{2,2}^{-;+,1}, B_{2,2}^{-;+,2}$

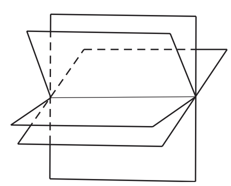

FIGURE 10. $B_{2,2}^{+;-}, B_{2,2}^{-;-}$

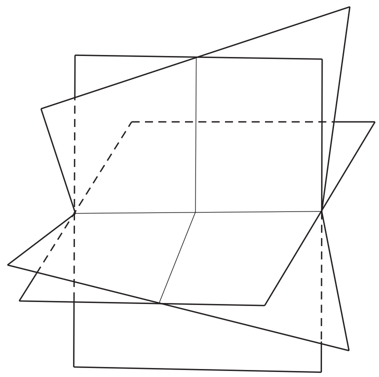

Figure 11. $B_{2,2}^{+,0}$

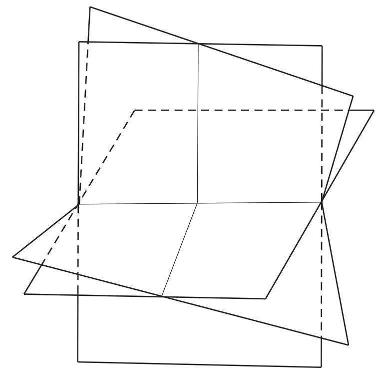

Figure 12. $B_{2,2}^{-,0}$

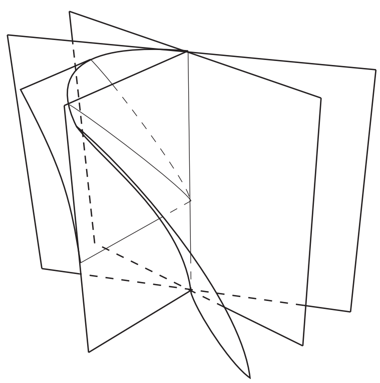

Figure 13. $B_{2,2,3}^{+,+}$

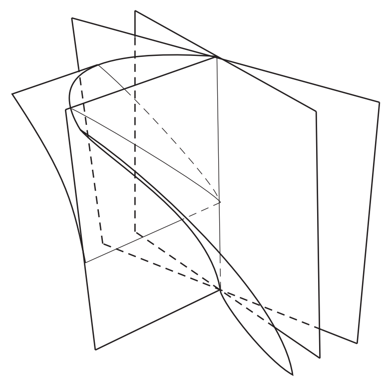

Figure 14. $B_{2,2,3}^{+,-}$

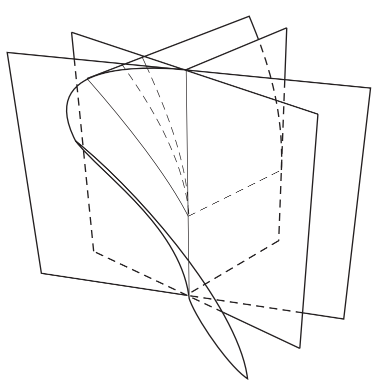

Figure 15. $B_{2,2,3}^{-,+}$

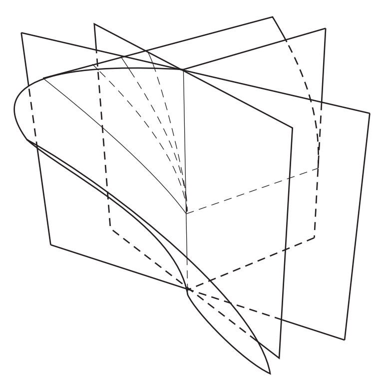

Figure 16. $B_{2,2,3}^{-,-}$

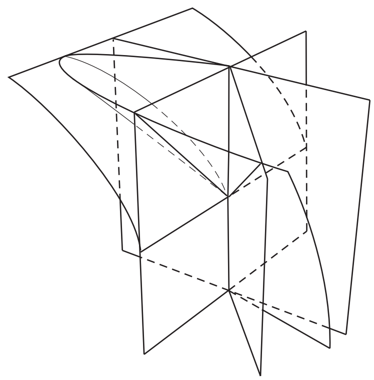

FIGURE 17. $B_{2,3}^{+,+}$

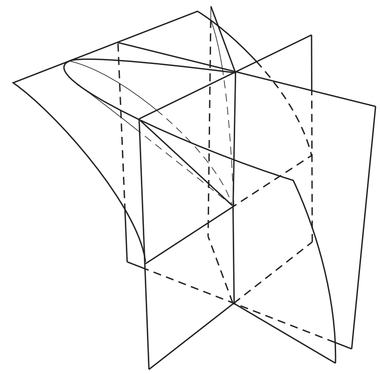

FIGURE 18. $B_{2,3}^{+,-}$

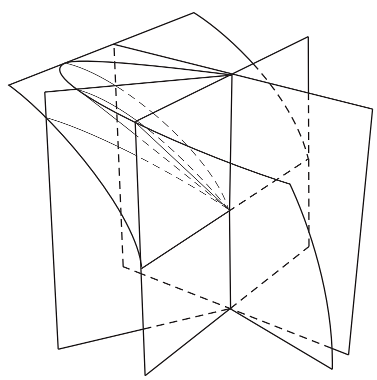

FIGURE 19. $B_{2,3}^{-,+}$

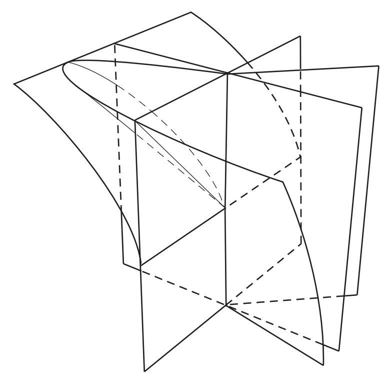

FIGURE 20. $B_{2,3}^{-,-}$

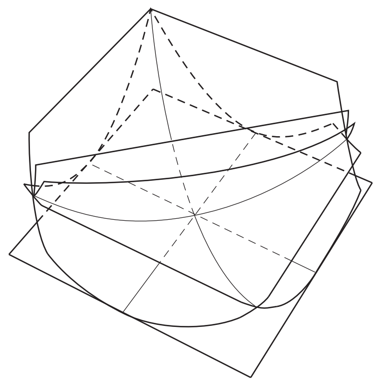

FIGURE 21. $C^{+;+}_{3,2}$

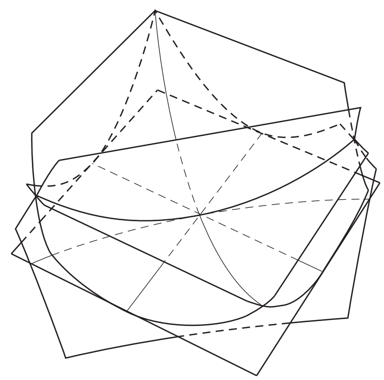

FIGURE 22. $C^{+;-}_{3,2}$

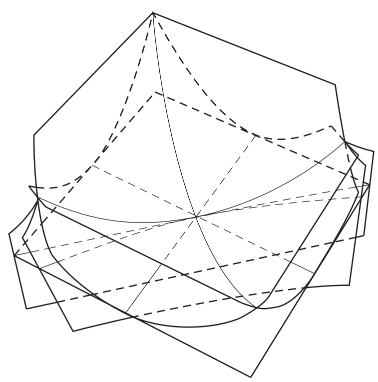

FIGURE 23. $C^{-;+}_{3,2}$

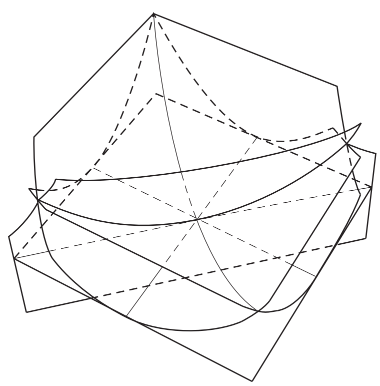

FIGURE 24. $C^{-;-}_{3,2}$

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COLLEGE OF HUMANITIES & SCIENCES, DEPARTMENT OF MATHEMATICS, NIHON UNIVERSITY, HIGASHIJUJO 3-1-16 KITA-KU, TOKYO 114-0001 JAPAN

*E-mail address*: tsukada@math.chs.nihon-u.ac.jp

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