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ON BI-LIPSCHITZ STABILITY OF FAMILIES OF FUNCTIONS

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Abstract. We focus on the Lipschitz stability of families of functions. We introduce a stability notion, called fiberwise bi-Lipschitz equivalence, which preserves the metric structure of the level surfaces of functions and show that it does not admit continuous moduli in the framework of semialgebraic geometry. We trivialize semialgebraic families of Lipschitz functions by constructing triangulations of their generic fibers which contain information about the metric structure of the sets.

0. Introduction

We study the metric stability of semialgebraic families of functions. In [S1], M. Shiota showed that a semialgebraic family of continuous functions $f_t : \mathbb{R}^n \to \mathbb{R}$, $t \in \mathbb{R}^p$, is generically topologically trivial. It means that we can find a partition of $\mathbb{R}^p$ and two semialgebraic families of homeomorphisms ϕ_t and h_t such that $\phi_t^{-1} \circ f_t \circ h_t$ is constant with respect to t on every element of this partition (see also [C, S2]). The fibers f_t are then said topologically equivalent. The main result of this paper is a partial Lipschitz counterpart of this theorem (Theorem 6.4).

The study of metric stability of analytic sets was initiated by T. Mostowski in his fundamental paper [M]. It was then developed, mainly by A. Parusiński [P1, P2], L. Birbrair [B], and the author of the present paper [V1, V2, V3]. The description of the metric structure of singularities provides a more accurate information than the description of their topology, valuable for applications [V5, V4]. The Lipschitz category can be considered as an intermediate category in between the C^1 category, too restricted to investigate singularities (C^1 equivalence admits continuous moduli), and the C^0 category, which often provides too vague information on the singularity.

The notion of semialgebraic bi-Lipschitz triviality of functions (Definition 6.1) is defined in the same way as the notion of topological triviality above, except that ϕ_t and h_t are required to be bi-Lipschitz. If many results about the topology have their counterpart in the framework of Lipschitz geometry [M, P1, P2, V1], it is however known that bi-Lipschitz equivalence of functions admits continuous moduli, in the sense that semialgebraic families of functions are not always generically bi-Lipschitz trivial. A counterexample was found by J.-P. Henry and A. Parusiński [H-P] (example 6.3 below). It was however shown in [RV] that bi-Lipschitz $\mathcal{K}$-equivalence does not admit continuous moduli.

We show in this paper that a slightly weaker equivalence notion than bi-Lipschitz equivalence does not admit continuous moduli for semialgebraic families of Lipschitz functions. This notion is stronger than C^0 equivalence since it preserves the metric structure of the level surfaces of the functions. Studying the stability of families of functions amounts to investigate triviality of foliations since the levels of the functions provide a singular foliation. In our equivalence

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relation, called *fiberwise bi-Lipschitz triviality* (see Definition 6.1), the homeomorphism is bi-Lipschitz on every level surface of the function, with *the same Lipschitz constant.* The Lipschitz condition may only fail for two points of two different fibers. The trivialization has however to vary continuously when we pass from one level of the function to one another.

Topological triviality of families of functions is proved in [BCR, C] by triangulating the generic fibers of semialgebraic families of functions. Triangulating and trivializing are thus two very related problems. In [V1], the author introduces the notion of *Lipschitz triangulation.* These are triangulations which provide information not only on the topology of the considered object but also on its metric structure. The metric type of a singularity is thus enclosed in finitely many combinatoric data in the sense that two singularities having the same Lipschitz triangulation are bi-Lipschitz homeomorphic. This is very convenient to describe the metric properties of semialgebraic sets or to prove finiteness properties regarding the metric structure of semialgebraic singularities [V2, V3]. Henry and Parusiński's example nevertheless shows that it is impossible to construct a triangulation of a semialgebraic function which would be a Lipschitz triangulation in sense of [V1] (since this would entail that bi-Lipschitz triviality of families of functions holds for generic parameters).

We prove generic fiberwise bi-Lipschitz triviality (Theorem 6.4) by showing that we can triangulate the generic fiber of a semialgebraic family of Lipschitz functions (Theorem 2.4). The triangulation that we construct satisfies a condition similar to the one required in the definition of the Lipschitz triangulations introduced in [V1], but just on points lying in the same fiber.

Our triviality theorem is thus, as in [C], derived from a triangulation theorem. Doing so, we have to work in an arbitrary real closed field (rather than in $\mathbb{R}$), since the generic fiber of the considered family lies in an extension of $\mathbb{R}$. We wish to emphasize here that even the study of semialgebraic functions of $\mathbb{R}^p \times \mathbb{R}^n$ requires, if one wants to use this kind of technique, to deal with an arbitrary real closed field. This kind of technique is classical and, although not completely elementary, has the significant advantage to get rid of the parameters during the best part of the proof. It is also worthy of notice that in this way we get two theorems (one showing triangulability and a second establishing triviality), both of their own interest. Noteworthy, these two theorems provide semialgebraic homeomorphisms. Semialgebraic mappings have nice properties. For instance, M. Shiota and Yoccoi established in [SY] a version of the Hauptvermutung for these mappings (see also [S2]). Semialgebraic bi-Lipschitz mappings have also nice differentiability properties used by the author of the present paper in [V4, V5] so as to study differential forms.

Content of the paper. In the first section we recall the known results on C^0 stability. This is useful so as to emphasize the close interplay between triangulations and trivializations. Indeed, the proof of the main theorem (Theorem 6.4) will make use of the same argument as the one used in the proof of Theorem 1.6. In section 2, we recall the notion of Lipschitz triangulation and state our triangulation theorem for functions (Theorem 2.4). The next sections are devoted to the proof of this theorem. Section 3 recalls some required results of [V1] and proves a parameterized version of the main tool used there, constructing "families of regular systems of hypersurfaces" for one parameter families of semialgebraic sets. Section 5 proves Theorem 2.4. The last section introduces the notion of fiberwise bi-Lipschitz triviality and yields it for semialgebraic families of Lipschitz functions, for generic parameters.

Notations 0.1. We write $\mathbb{Q}_+$ for the positive rational numbers. Let R be a real closed field. Given $A \subset R^n$ we denote by $int(A)$ the interior of A, $cl(A)$ the closure of A, and by $\delta(A)$ the topological boundary of A, $cl(A) \setminus int(A)$. We shall write $|.|$ for the Euclidean norm and $B(\lambda, r)$ for the ball of radius r centered at λ (for all the considered metric spaces R^n, $S^n, \dots$).

We denote by $e_1, \dots, e_n$ the canonical basis of R^n and by $\mathbb{G}_n^k$ the Grassmanian of k-dimensional vector spaces of R^n. We set $\mathbb{G}_n := \cup_{k=1}^{n-1} \mathbb{G}_n^k$. We denote by $\tau(A)$ the closure in the Grassmanian of the set of all the tangent spaces to A_{reg}, where A_{reg} stands for the set constituted by the points near which the set A is a C^1 manifold (of dimension $\dim A$ or smaller).

We shall denote by $d(\cdot,\cdot)$ the Euclidean distance in R^n. Given $x \in R^n$ and $P \subset R^n$, we write $d(x,P)$ for the distance to the subset P (defined by $\inf_{y\in P} d(x,y)$). Given a subset C of $\mathbb{G}_n$ we also set $d(x,C) := \inf_{P\in C} d(x,P)$.

A **Lipschitz function** is a function $f : A \to R$ satisfying for some $L \in R$ and all x and x' in A

$$|f(x) - f(x')| \le L|x - x'|.$$

The function may be said **L-Lipschitz** is one wants to specify the constant. It is said **$\mathbb{Q}$-Lipschitz** if it is L-Lipschitz with $L \in \mathbb{Q}$.

Given a couple of functions ξ_1 and ξ_2 on A, we write $\xi_1 \sim_K \xi_2$ if there exist C in K such that $\xi_1 \le C\xi_2$ and $\xi_2 \le C\xi_1$ (here $K \subset R$). We denote by $[\xi_1, \xi_2]$ the set $\{(x,y) \in A \times R : \xi_1(x) \le y \le \xi_2(x)\}$.

1. Topological stability

1.1. **Triangulations of functions.** Let R be a real closed field.

Simplicial complexes will be finite and may have open simplices (and hence will not always be compact). An **open simplex** is a simplex from which the proper faces have been taken off.

We will denote by $\widetilde{R^n}$ the real spectrum of the polynomial ring $R[X_1,\ldots,X_n]$ (see [BCR, C]). Given $\alpha \in \widetilde{R^n}$ we shall write $k(\alpha)$ for the corresponding extension of R.

Definition 1.1. Let X be a semialgebraic set. A **triangulation of** X is the data of a finite simplicial complex K, and a semialgebraic homeomorphism $h : |K| \to X$.

Let $f : X \to R$ be a semialgebraic function. A **triangulation of** f is the data of a triangulation h of X together with a homeomorphism $\varphi : R \to R$ such that $\varphi^{-1} \circ f \circ h$ is piecewise linear.

Theorem 1.2. [S1] *Every continuous bounded semialgebraic function admits a C^0-triangulation. The vertices of the simplicial complex may be assumed to have coordinates in $\mathbb{Q}$.*

Remark 1.3. If we do not require that the vertices lie in $\mathbb{Q}^n$ then the map φ (see definition 1.1) may be required to be the identity.

1.2. **Topological triviality of semialgebraic families of functions.**

Definition 1.4. A semialgebraic family of sets of $R^p \times R^n$ is a semialgebraic subset of $R^p \times R^n$, the first p variables being considered as parameters. Let X be a semialgebraic family of sets of $R^p \times R^n$. A **semialgebraic family of functions** is a semialgebraic mapping $f : X \to R^p \times R$, of type $X \ni (t,x) \mapsto (t, f_t(x))$, the first p variables being considered as parameters.

For a parameter t in R^p, we call the function f_t the **fiber** at t of this family. Given $\alpha \in \widetilde{R^p}$, we denote by f_α the generic fiber at α (see [BCR, C]).

Given a semialgebraic family of functions, it is a natural problem to compare the fibers f_t with each other.

Definition 1.5. We say that a semialgebraic family $f : X \to R^p \times R$ is **semialgebraically C^0 trivial along** $W \subset R^p$ if there exist two semialgebraic families of homeomorphisms $h : W \times R^n \to W \times R^n$ and $\phi : W \times R \to W \times R$ such that for any $t \in W$:

$$h_t(X_{t_0}) = X_t, \quad \phi_t \circ f_t \circ h_t = f_{t_0}, \quad t_0 \in W.$$

The fibers f_t are then said to be **semialgebraically C^0 equivalent**.

Semialgebraic families of functions are generically semialgebraically topologically trivial:

Theorem 1.6. *(Shiota) Let $f : X \to R^p \times R$ be a semialgebraic family of continuous functions. There exist a semialgebraic partition $V_1, \dots, V_m$ of R^p such that for every i, f is semialgebraically topologically trivial along V_i.*

Proof. We first check that we can assume, without loss of generality, that f_t is bounded on X. Indeed, the function $u(y) := \frac{y}{1+|y|}$ is a homeomorphism from R onto $(-1;1)$. If we prove the result for $\hat{f} := u \circ f$, we are done. Let us assume that f_t is bounded for any t without changing notations.

Let $\alpha \in \widetilde{R^p}$. By Theorem 2.3, there exist semialgebraic homeomorphisms $h : |K| \to X_\alpha$ and $\varphi : k(\alpha) \to k(\alpha)$, with K finite simplicial complex, such that $\varphi^{-1} \circ f_\alpha \circ h$ is a piecewise linear function on every simplex. The simplicial complex K may be assumed to have vertices in $\mathbb{Q}^n$. As a matter of fact, $|K|$ is indeed the generic fiber of a constant family $U \times |K|$, with $U \in \alpha$ (see [BCR, C] for more details).

The homeomorphisms h and φ respectively give rise to families of semialgebraic homeomorphisms:

$$\theta : U \times |K| \to U \times X,$$

and $\gamma : U \times R \to U \times R$.

As $\gamma_\alpha^{-1} \circ f_\alpha \circ \theta_\alpha$ is piecewise linear, $\gamma_t^{-1} \circ f_t \circ \theta_t$ is constant with respect to t. If we set $H_t := \theta_t \theta_{t_0}^{-1}$ and $\phi_t(x) := \gamma_t \gamma_{t_0}^{-1}$, we have $\phi_t^{-1} \circ f_t \circ H_t = f_{t_0}$. This shows that f is trivial along U. By compactness of $\widetilde{R^p}$, we have the desired finite covering. □

2. Lipschitz triangulations

2.1. **Lipschitz triangulation of semialgebraic sets.** We recall in this section the results proved in [V1]. We will adapt these techniques to families of functions.

Given a point $q \in R^n$, we write $q_1, \dots, q_n$ for the coordinates of q in the canonical basis and $\pi_i : R^n \to R^i$ for the canonical projection.

Definition 2.1. Let $\sigma \subset R^n$ be an open simplex. A **tame system of coordinates of** σ is a homeomorphism (onto its image) $(\psi_1, \dots, \psi_n) : \sigma \to R^n$ of the following form:

$$\psi_i(q) = \frac{q_i - \theta_i(\pi_{i-1}(q))}{\theta_i(\pi_{i-1}(q)) - \theta_i'(\pi_{i-1}(q))}, \tag{2.1}$$

(and 0 whenever $\theta_i \circ \pi_{i-1}(q) = \theta_i' \circ \pi_{i-1}(q)$) where θ_i and θ_i' are piecewise linear functions on R^{i-1}. A **standard simplicial function** on σ is a function given by finitely many iterations of sums, powers (possibly negative), and products of distances to faces.

Standard simplicial functions will sometimes be defined on $\sigma \times \sigma$ since they will be functions of two variables q and q', being given by finite iterations of sums, products, and powers of functions of type $q \mapsto d(q, \tau)$ and $q' \mapsto d(q', \tau)$ with τ face of σ.

Definition 2.2. A **Lipschitz triangulation of** R^n is the data of a finite simplicial complex K together with a semialgebraic homeomorphism $h : |K| \to R^n$, such that for every $\sigma \in K$ there exist $\varphi_{\sigma,1}, \dots, \varphi_{\sigma,n}$, standard simplicial functions over $\sigma \times \sigma$ satisfying for any q and q' in σ:

$$|h(q) - h(q')| \sim_R \sum_{i=1}^{n} \varphi_{\sigma,i}(q;q') \cdot |q_{i,\sigma} - q'_{i,\sigma}|, \tag{2.2}$$

where $(q_{1,\sigma}, \dots, q_{n,\sigma})$ is a tame system of coordinates of R^n. Let $A_1, \dots, A_k$ be subsets of R^n. A **Lipschitz triangulation of** $A_1, \dots, A_k$ is a Lipschitz triangulation of R^n such that each $h^{-1}(A_i)$ is a union of open simplices.

With this definition two semialgebraic subsets admitting the same simplicial complex as semi-algebraic triangulation, with $\sim_R$ functions φ_σ and same tame systems of coordinates are semi-algebraically bi-Lipschitz homeomorphic. As a matter of fact, simultaneous Lipschitz triangulations of the fibers of a family provide bi-Lipschitz trivializations.

Theorem 2.3. [V1] *Every finite collection of semialgebraic sets admits a Lipschitz triangulation.*

2.2. **Lipschitz triangulations of functions.** The theorem below gives a version of Theorem 2.3 for functions. Unfortunately, it is not possible to construct a triangulation of a function which would be a Lipschitz triangulation (see example 6.3). We prove something somewhat weaker: we show that we can triangulate every semialgebraic bounded Lipschitz function in such a way that (2.2) holds for couples of points of the same fiber (with a constant independent of the fiber).

Theorem 2.4. *Let $f : X \to R$ be a semialgebraic bounded Lipschitz function, $X \subset R^n$. There exists a triangulation (K, ϕ, ψ) of f, with $K \subset R^{n+1}$, such that on every open simplex σ of K, we can find standard simplicial functions $\varphi_{\sigma,1}, \dots, \varphi_{\sigma,n+1}$ with:*

$$|\psi(q) - \psi(q')| \sim_R \sum_{i=1}^{n+1} \varphi_{\sigma,i}(q;q') \cdot |q_{i,\sigma} - q'_{i,\sigma}|, \tag{2.3}$$

on the set

$$\{(q, q') \in \sigma \times \sigma : f(\psi(q)) = f(\psi(q'))\},$$

where $(q_{1,\sigma}, \dots, q_{n+1,\sigma})$ is a tame system of coordinates of σ. Moreover, the vertices of the simplicial complex K lie in $\mathbb{Q}^{n+1}$ and ϕ is bi-Lipschitz.

Furthermore, given finitely many semialgebraic subsets $A_1, ..., A_k$ of X, we may choose the triangulation in such a way that each A_i is a union of images of open simplices of K.

This theorem will be proved in section 5.

3. Regular lines

We recall that, given a subset C of $\mathbb{G}_n$, we have set $d(x, C) := \inf_{P \in C} d(x, P)$, where d stands for the Euclidian distance of R^n (see Notation 0.1).

Definition 3.1. Let A be a semialgebraic set of R^n. An element λ of S^{n-1} is said to be **regular for the set** A if there is $\alpha \in \mathbb{Q}_+$ such that:

$$d(\lambda; \tau(A)) \geq \alpha.$$

We say that $\lambda \in S^{n-1}$ is **regular for a semialgebraic family** X of $R \times R^n$ if there exists $\alpha \in \mathbb{Q}_+$ such that for any parameter $t \in R$:

$$d(\lambda; \tau(X_t)) \geq \alpha.$$

A **subset** $C \subset S^{n-1}$ **is regular** for a set (resp. family) X if all the elements of $cl(C)$ are regular for the set (resp. family) X.

Remark 3.2. Of course, if a line is regular for a family then it is regular for all the fibers of this family. But it is indeed much stronger since, when a line is regular for a family, the angle between this line and the tangent spaces to the fibers is bounded below away from zero by a constant α *independent of* t.

Proposition 3.3. [V1] *Let A be a semialgebraic subset of R^n of empty interior. There exists a semialgebraic $\mathbb{Q}$-bi-Lipschitz homeomorphism $h : R^n \to R^n$ such that $h(A)$ has a regular vector.*

We will need a parameterized version of this proposition. More precisely, we shall establish the following proposition.

Proposition 3.4. *Let A be a semialgebraic family of $R \times R^n$ such that A_t has empty interior for every $t \in R$. There exists a continuous semialgebraic family of mappings $h : R \times R^n \to R \times R^n$ and $C \in \mathbb{Q}$ such that:*

(1) *h_t C-bi-Lipschitz for any t.*
(2) *e_n is regular for the family $h(A)$.*

We will prove Proposition 3.4 by generalizing to families the techniques introduced in [V1] for subsets of R^n.

3.1. **Some preliminary lemmas.** We need to recall some results which were already used in [V1].

Lemma 3.5. [K] *Given $\nu \in \mathbb{N}$, there exists a strictly positive constant $\sigma \in \mathbb{Q}_+$ such that for any $P_1, \dots, P_\nu$ in $\mathbb{G}_n$ there exists $P \in S^{n-1}$ such that for any i we have:*

$$d(P; P_i) \geq \sigma.$$

The second lemma we need was proved by the author of the present paper in [V1].

Lemma 3.6. *There exists $\{\lambda_1, \dots, \lambda_N\} \subset S^{n-1}$ such that for any semialgebraic sets $A_1, \dots, A_m$ of R^n, there exists a cell decomposition $(C_i)_{i \in I}$ of R^n adapted to all the A_k's and such that for each open cell C_i, we may find $\lambda_{j(i)}$, $1 \leq j(i) \leq N$, regular for δC_i.*

Given $\lambda \in S^{n-1}$, we denote by π_λ the projection along the line generated by λ onto the vector space N_λ, normal to this line. Given $q \in R^n$, we write q_λ for the Euclidean inner product $< q, \lambda >$.

The third result we shall recall is the preparation theorem, so called because it can be considered as a Weierstrass preparation theorem for semialgebraic functions.

Theorem 3.7. *(Preparation Theorem)* [vDS, LR, V1, P3] *Let $\xi : R^{n+1} \to R$ be a semialgebraic function. Then there exists a finite semialgebraic partition $(V_i)_{i \in I}$ of R^{n+1} such that for any V_i there exist semialgebraic continuous functions a, $\theta : \pi_{e_{n+1}}(V_i) \to R$, and $r \in \mathbb{Q}$ such that for $q = (x; q_{n+1}) \in V_i$:*

$$\xi(q) \sim_{\mathbb{Q}} (q_{n+1} - \theta(x))^r \, a(x). \tag{3.4}$$

Definition 3.8. The subset $A \subset R^n$ is **the graph for** $\lambda \in S^{n-1}$ of the function $\xi : E \to R$, where $E \subset N_\lambda$, if

$$A = \{q \in \pi_\lambda^{-1}(E) : q_\lambda = \xi(\pi_\lambda(q))\}.$$

If A is the graph for λ of the function $\xi : N_\lambda \to R$, we denote by

$$E(A, \lambda) := \{q \in R^n : q_\lambda \leq \xi(\pi_\lambda(q))\}.$$

If A is a family of $R \times R^n$ such that A_t is the graph for λ of the function $\xi_t : N_\lambda \to R$ for every t, then $E(A_t, \lambda)$, $t \in R$, is a semialgebraic family of sets of $R \times R^n$. Indeed, since $S^{n-1} \subset S^n$, $E(A, \lambda)$ is also well defined, and is the semialgebraic family of sets whose fiber at t is $E(A_t, \lambda)$.

When dealing with families of $R \times R^n$, we will also write π_λ for the (constant) family of mappings $\pi_\lambda : R \times R^n \to R \times R^n$ given by $\pi_{\lambda,t}(x) := \pi_\lambda(x)$ for $(t, x) \in R \times R^n$.

The next proposition is a consequence of the preparation theorem that will be of service for us.

Proposition 3.9. [V1] *Let $\xi : R^n \to R$ be a nonnegative semialgebraic function. There exists a finite semialgebraic partition of R^n such that over each element of this partition, the function ξ is $\sim_R$ to a product of powers of distances to semialgebraic subsets of R^n.*

Proposition 3.10. [V1] *Let B be a connected subset of S^{n-1}, $\lambda_0 \in B$, and let $\xi : N_{\lambda_0} \to R$ be a continuous semialgebraic function. Let H be the graph of ξ for λ_0. Suppose that B is regular for H. Then, for any $\lambda \in B$ the set H is the graph of a function $\xi^\lambda : N_\lambda \to R$. Moreover the set $E(H;\lambda)$ is independent of $\lambda \in B$.*

We now formulate some elementary observations that we shall need and which are taken from [V1].

Observations. Let $\lambda \in S^{n-1}$ and $r \in \mathbb{Q}_+$.

(1) If A is a union of graphs for λ of some $\mathbb{Q}-$Lipschitz functions then there exists $r \in \mathbb{Q}_+$ such that $B(\lambda; r)$ is regular for A. Also, if $B(\lambda; r) \subseteq S^{n-1}$ is regular for the semialgebraic set $A \subseteq R^n$, then A is the union of the graphs for λ of some $\mathbb{Q}-$Lipschitz functions. Moreover, if A is the graph for λ of a Lipschitz function $\xi : N_\lambda \to R$ then ξ is C-Lipschitz with $C \leq \frac{1}{d(\lambda;\tau(A))}$.
(2) Every semialgebraic $C-$Lipschitz function ξ defined over a subset A of R^n may be extended to a semialgebraic $C-$Lipschitz function $\widehat{\xi}$ defined over the whole of R^n.
(3) If A is the union of the graphs for λ of some semialgebraic functions $\theta_1, \dots, \theta_k$ over N_λ we may find an ordered family of semialgebraic functions $\xi_1 \leq \dots \leq \xi_k$ such that A is the union of the graphs of these functions for λ.
(4) Given a family of Lipschitz functions $f_{1,t} \dots, f_{k,t}$, $t \in R$, defined over $R \times R^{n-1}$, we can find some Lipschitz families of functions $\xi_{1,t} \leq \dots \leq \xi_{l,t}$, $t \in R$, and a cell decomposition $\mathcal{D}$ of $R \times R^{n-1}$ such that for every cell $D \in \mathcal{D}$, the functions $|q_n - f_{i,t}(x)|$ (where $q = (t, x; q_n)$) are comparable with each other (for relation $\leq$) and comparable with the functions $f_{i,t} \circ \pi_{e_n}$ on the cell $[\xi_{i|D_t}; \xi_{i+1|D_t}]$.

3.2. **Regular systems of hypersurfaces.** We now adapt the techniques of [V1] to families in order to prove Theorem 3.4.

The main tool of the proof of Proposition 3.3 is the notion of regular systems of hypersurfaces. We shall generalize it to one parameter families, introducing the notion of *families of* regular systems of hypersurfaces.

Definition 3.11. A **family of regular systems of hypersurfaces** of $R \times R^n$ is a family $H = (H_k; \lambda_k)_{1 \leq k \leq b}$ with $b \in \mathbb{N}$, of semialgebraic families H_k of $R \times R^n$ together with elements of $\lambda_k \in S^{n-1}$ such that the following properties hold for each $k < b$:

(i) For every $t \in R$, the sets $H_{k,t}$ and $H_{k+1,t}$ are the respective graphs for λ_k of two functions $\xi_{k,t}$ and $\xi'_{k,t}$ such that $\xi_{k,t} \leq \xi'_{k,t}$.
(ii) The functions $\xi_{k,t}$ and $\xi'_{k,t}$ are C-Lipschitz with $C \in \mathbb{Q}$ (independent of t) and vary continuously with respect to t.
(iii) For every t we have:

$$E(H_{k+1,t}; \lambda_k) = E(H_{k+1,t}; \lambda_{k+1})$$

Let A be a semialgebraic family of $R \times R^n$. We say that the family H is **compatible** with A, if $A \subset \bigcup_{k=1}^{b} H_k$. An **extension** of H is a family of regular systems of hypersurfaces H' compatible with the set $\bigcup_{k=1}^{b} H_k$.

Observe that H_k is by definition the graph of the function $(x,t) \mapsto \xi_{k,t}(x)$ for $\lambda_k \in S^{n-1} \subset S^n$. Hence, $E(H_{k,t}; \lambda_k)$ is the fiber at t of the semialgebraic family $E(H_k; \lambda_k)$.

Given a positive integer $k < b$, we set:

$$G_k(H) := E(H_{k+1}; \lambda_k) \setminus int(E(H_k; \lambda_k)).$$

We shall write $\Lambda_k(H)$ for the connected component of

$$\{\lambda \in S^{n-1} : \lambda \text{ is regular for the family } H_k \cup H_{k+1}\}$$

which contains λ_k. Note that by Proposition 3.10, the family $G_k(H)$ may be defined using any $\lambda \in \Lambda_k(H)$.

We will say that another family of regular systems H' **coincides with H outside** $G_k(H)$ if for each j either $H'_j \subset G_k(H)$ or there exists j' such that $H'_j = H_{j'}$.

Remark 3.12. It is always possible to assume that the $G_k(H)$'s are of nonempty interior. Indeed if $int(G_k(H)) = \emptyset$ then $H_k = H_{k+1}$ and in this case we may remove $(H_k; \lambda_k)$ from the sequence.

Given $\lambda \in S^n$, we define $\widetilde{\pi}_\lambda : S^n \setminus \{\pm\lambda\} \to S^n \cap N_\lambda$ by $\widetilde{\pi}_\lambda(u) := \frac{\pi_\lambda(u)}{|\pi_\lambda(u)|}$.

Remark 3.13. Suppose $B \subset S^{n-2}$ to be regular for a subset $A \subset R^{n-1}$. Then, for any $a \in \mathbb{Q}_+$ the set

$$\widetilde{\pi}_{e_n}^{-1}(B) \cap \{\lambda \in S^{n-1} : d(\lambda; \{\pm e_n\}) \geq a\}$$

is regular for $\pi_{e_n}^{-1}(A)$. Furthermore, if A is the graph of a $\mathbb{Q}-$Lipschitz function for $\lambda \in B$, and if B is connected, then $\pi_{e_n}^{-1}(A)$ is the graph of a $\mathbb{Q}-$Lipschitz function for any λ' in

$$\widetilde{\pi}_{e_n}^{-1}(B) \cap \{\lambda' \in S^{n-1} / d(\lambda'; \{\pm e_n\}) \geq a\},$$

for any $a \in \mathbb{Q}_+$ (by Proposition 3.10). Moreover, in this case the following holds:

$$E(\pi_{e_n}^{-1}(A); \lambda') = \pi_{e_n}^{-1}(E(A; \lambda)).$$

3.3. **Some preliminary Lemmas.** We want to prove that every semialgebraic one-parameter family $A \subset R \times R^n$ with $\dim A_t < n$ for every $t \in R$, admits a family of regular systems compatible with it (Proposition 3.19). For this purpose, we prove some lemmas.

The following lemma says that we will be able to assume that the interiors of the $G_k(H)$'s are connected.

Lemma 3.14. *Let H be a family of regular systems of hypersurfaces. There exists an extension $\widehat{H}$ of H such that all the sets $int(G_k(\widehat{H}))$ are connected.*

Proof. Let $1 \leq m \leq b-1$. Suppose that $int(G_m(H))$ is not connected. Let $A_1, \dots, A_\nu$ be the connected components of $int(G_m(H))$. Set $A'_i = \pi_{\lambda_m}(A_i)$. For $t \in R$, the fiber $A_{i,t}$ is of the form:

$$\{q \in A'_{i,t} \oplus \lambda_m \cdot R \; / \; \xi_{m,t}(\pi_{\lambda_m}(q)) < q_{\lambda_m} < \xi'_{m,t}(\pi_{\lambda_m}(q))\}.$$

Clearly $\xi_{m,t} = \xi'_{m,t}$ on the boundary of $A'_{i,t}$. We thus may define some Lipschitz functions η_i, $1 \leq i \leq \nu-1$, as follows. We set over $A'_{j,t}$, $\eta_{i,t} := \xi'_{m,t}$, when $1 \leq j \leq i$, and $\eta_{i,t} := \xi_{m,t}$ whenever $i < j$. Extend the function $\eta_{i,t}$ by setting $\eta_{i,t} := \xi_{m,t} = \xi'_{m,t}$ on $N_{\lambda_m} \setminus \pi_{\lambda_m}(int(G_m(H)))$.

Therefore, we have that $\eta_{1,t} \leq \cdots \leq \eta_{(\nu-1),t}$. Now, it suffices to

- let $\widehat{H}_k := H_k$ and $\widehat{\lambda}_k := \lambda_k$ if $k \leq m$
- let $\widehat{H}_{k,t}$ be the graph of $\eta_{k-m,t}$ for λ_m (for every $t \in R$) and $\widehat{\lambda}_k := \lambda_m$ for $m+1 \leq k \leq m+\nu-1$
- let $\widehat{H}_k := H_{k-\nu+1}$ and $\widehat{\lambda}_k := \lambda_{k-\nu+1}$ if $m+\nu \leq k \leq b+\nu-1$.

This is clearly a family of regular systems of hypersurfaces. Note that the $int(G_k(\widehat{H}))$, $m \leq k < m+\nu$, are the connected components of $int(G_m(H))$. □

Given a family of regular systems of hypersurfaces (of $R \times R^n$) H, it will be convenient to extend the notations in the following way. Set for any $t \in R$: $H_{0,t} := \{-\infty\}$ and $H_{b+1,t} := \{+\infty\}$. By convention, all the elements of S^{n-1} will be regular for these two families. We will also consider that these two families of sets as the respective graphs of two functions which take $-\infty$ and $+\infty$ as constant values (for any λ). Define also $\lambda_0 := \lambda_1$, $\lambda_{b+1} := \lambda_b$, as well as $E(H_0; \lambda_0) := \emptyset$, $G_0(H) := E(H_1; \lambda_1)$, $G_b(H) := R \times R^n \setminus int(E(H_b; \lambda_b))$, as well as $E(H_{b+1}, \lambda_{b+1}) = R \times R^n$. Remark that now $R \times R^n = \bigcup_{k=0}^{b} G_k(H)$.

Lemma 3.15. *Let $H = (H_k; \lambda_k)_{1 \le k \le b}$ be a family of regular systems of hypersurfaces and let $j \in \{0, \dots, b\}$. Let X be a semialgebraic family of subsets of $G_j(H)$ such that λ_j is regular for X. Then H can be extended to a family of regular systems of hypersurfaces H' compatible with X, which coincides with H outside $G_j(H)$.*

Proof. By property (i) of Definition 3.11, for every t, the sets $H_{j,t}$ and $H_{j+1,t}$ are the respective graphs for λ_j of two functions $\xi_{j,t}$ and $\xi'_{j,t}$. By Observations (1) and (2), the sets X_t, $t \in R$, may be included in a finite number of graphs for λ_j of functions, say $\theta_{1,t}, \dots, \theta_{\nu,t}$, continuous with respect to t and C-Lipschitz, with $C \in \mathbb{Q}$ independent of t. Furthermore, by Observation (3), these families of functions can be assumed to be ordered and satisfy $\xi_{j,t} \le \theta_{i,t} \le \xi'_{j,t}$, for every t. Now,

- let $H'_k := H_k$ and $\lambda'_k := \lambda_k$ whenever $1 \le k \le j$,
- let $H'_{k,t}$ be the graph of $\theta_{k-j,t}$ for λ_j and $\lambda'_k := \lambda_j$ for $j < k \le j + \nu$, $t \in R$,
- let $H'_k := H_{k-\nu}$ and $\lambda'_k := \lambda_{k-\nu}$, whenever $j + 1 + \nu \le k \le b + \nu$.

Properties (i), (ii) and (iii) clearly hold by construction. □

Lemma 3.16. *Let $U_1, \dots, U_m$ be semialgebraic families covering $R \times R^n$. There exist finitely many semialgebraic families $V_1, \dots, V_p$ covering $R \times R^n$ such that:*

1. *For every $i \le p$, there are j and j' such that $V_i \subset U_j \cup U_{j'}$.*
2. *For every $i \le p$ and $t \in R$, the fiber $(\delta V_i)_t$ has empty interior in R^n (see Notations 0.1 for δ).*

Proof. Let $f : R \times R^n \to R$ be the projection onto the x_1-axis. Consider a C^0 triangulation $h : |K| \to R \times R^n$ of f such that the families $U_1, \dots, U_m$ are unions of images of simplices (up to a homeomorphism we may assume that the domain of f is bounded). Let $\sigma \in K$ be of dimension $(n+1)$. The set $\delta h(\sigma)$ is the union of the images of the faces of σ of dimension $< n+1$. Thus, $\delta h(\sigma)_t$ is of dimension n if and only if a face τ of σ of dimension n lies in the fiber σ_t. In this case there must be another simplex $l(\sigma)$ of which τ is also a face. The face τ is clearly always unique.

If the fiber $(\delta h(\sigma))_t$ is of dimension less than n for any t then set $l(\sigma) := \sigma$. Let $V_\sigma := cl(h(\sigma) \cup h(l(\sigma)))$. The family V_σ, $\sigma \in K$, has the required properties. □

Lemma 3.17. *Let $A \subset R \times R^n$ be a semialgebraic family of sets with $int(A_t) = \emptyset$ for any $t \in R$. There exists an integer ν such that for any $\varepsilon > 0$ we can find a finite semialgebraic partition $(A_i)_{i \in I}$ of $R \times R^{n-1}$ such that for every i the set*

$$\cup_{t \in R} \ \tau(\pi_{e_n}^{-1}(A_{i,t}) \cap A_t)$$

is included in ν balls of radius ε (in $\mathbb{G}_n$).

Proof. We can cover the Grassmanian by finitely many balls of radius ε. This gives rise to a covering $U_1, \dots, U_k$ of A (via the Gaussian mappings $A_{t,reg} \ni x \mapsto T_x A_{t,reg}$). Consider a cell decomposition of $R \times R^n$ compatible with $U_1, \dots, U_k$. The images of the cells under the canonical projection onto $R \times R^{n-1}$ constitute a covering having the desired property. □

Remark 3.18. The integer ν is indeed bounded by the maximal number of connected components of the fibers of the restriction of π_{e_n} to A.

3.4. **Existence of regular families.** We are ready to associate a family of regular systems of hypersurfaces to every semialgebraic family of nowhere dense sets.

Theorem 3.19. *Given a semialgebraic family of sets A of $R \times R^n$ such that every fiber A_t is of empty interior, there exists a family of regular systems of hypersurfaces of $R \times R^n$ compatible with A.*

Proof. Actually, we are going to prove by induction on n that there exists a family of regular systems of hypersurfaces of $R \times R^n$ compatible with a given semialgebraic family A of $R \times R^n$ (whose fibers have positive codimension) such that all the λ_k's can be chosen in a given ball $B(\lambda;\eta)$ in S^{n-1}, for $\eta \in \mathbb{Q}_+$.

For $n = 1$ the result is clear. So, we assume that it is true for $(n-1)$. Let A be a semialgebraic family of $R \times R^n$ such that A_t has empty interior for every t and consider a ball $B(\lambda;\eta) \subset S^{n-1}$, $\eta \in \mathbb{Q}_+$. We split the induction step into several steps.

Step 1. There exists a family of regular systems of hypersurfaces $H = (H_k;\lambda_k)_{1\le k\le b}$ with $\lambda_k \in B(\lambda;\frac{\eta}{2})$ and such that for every k the family $G_k(H) \cap A$ has a regular vector $P \in S^{n-1} \setminus B(\pm\lambda, \frac{\eta}{2})$.

Take $e \in S^{n-1}$ such that $\pm e \notin B(\lambda;\eta)$ (we may assume η small).

By Lemma 3.17, for any $\sigma \in \mathbb{Q}_+$, there exists a finite semialgebraic partition $(A_i)_{i\in I}$ of $R \times N_e$ such that, for each $i \in I$, the set $\bigcup_{t\in R} \tau(\pi_e^{-1}(cl(A_{i,t})) \cap A_t)$ is included in the union of ν balls in $\mathbb{G}_n$ of radius $\frac{\sigma}{2}$. Consider such a partition for the σ given by Lemma 3.5. By Lemma 3.16, we may assume that $(\delta A_i)_t$ has empty interior for every t. Changing η, we may assume that $\eta \le \frac{\sigma}{4}$.

Choose $\eta' \in \mathbb{Q}_+$ such that we have in S^{n-2}:

$$B(\widetilde{\pi}_e(\lambda);\eta') \subset \widetilde{\pi}_e(B(\lambda;\frac{\eta}{2})), \tag{3.5}$$

Apply the induction hypothesis (identify $R \times N_e$ with $R \times R^{n-1}$) to the families δA_i to get a family of regular systems of $R \times R^{n-1}$, $\overline{H} = (\overline{H}_k;\overline{\lambda}_k)_{k\le b}$, such that all the $\overline{\lambda}_k$'s belong to $B(\widetilde{\pi}_e(\lambda);\eta')$.

By lemma 3.14, up to a refinement, we may assume that each $int(G_k(\overline{H}))$ is connected. We may also assume it to be of nonempty interior (see remark 3.12).

We claim that for each j and k, either $int(G_k(\overline{H}))$ is disjoint from A_j or $int(G_k(\overline{H})) \subset A_j$. To see this, observe that, as $\overline{H}$ is compatible with the δA_j's, all the sets $A_j \cap int(G_k(\overline{H}))$ are open and of empty (topological) boundary in $int(G_k(\overline{H}))$. Hence, if nonempty, these are connected components of $int(G_k(\overline{H}))$. But, as $int(G_k(\overline{H}))$ is connected, this entails that $A_j \cap int(G_k(\overline{H}))$ is either the empty set or $int(G_k(\overline{H}))$ itself, as claimed.

We turn to define the family of regular systems H claimed in step 1. For $1 \le k \le b$, let:

$$H_k := \pi_e^{-1}(\overline{H}_k).$$

Since $\overline{\lambda}_k \in B(\widetilde{\pi}_e(\lambda);\eta')$, by (3.5), we have $\overline{\lambda}_k \in \widetilde{\pi}_e(B(\lambda;\frac{\eta}{2}))$. Choose some $\lambda_k \in \widetilde{\pi}_e^{-1}(\overline{\lambda}_k) \cap B(\lambda;\frac{\eta}{2})$.

As $\lambda_k \in B(\lambda;\frac{\eta}{2})$ and neither e nor $-e$ belongs to $B(\lambda;\eta)$ we have:

$$d(\lambda_k;\pm e) \ge \frac{\eta}{2}, \qquad \forall\, k \le b.$$

So, by Remark 3.13 (identify again $R \times N_e$ with $R \times R^{n-1}$), the set $H_{k,t}$ is the graph of a semialgebraic Lipschitz function. Moreover, as $\overline{H}$ satisfies $(i - iii)$, again by Remark 3.13, conditions $(i - iii)$ are clearly fulfilled by $H := (H_k;\lambda_k)_{k\le b}$.

By Lemma 3.5 and our choice of σ, for all m, the family $A \cap int(G_m(H))$ is the union of finitely many semialgebraic families having a common regular element $P \in S^{n-1}$ (since we have seen that each $int(G_m(\overline{H}))$ is included in A_j for some j). Moving slightly P, we may assume that $d(P, \pm\lambda) \geq \eta$ (we have assumed $\eta \leq \frac{\sigma}{4}$).

This completes the first step.

The flaw of the first step is that the regular vector that we get for $G_m(H) \cap A$ might not be in $\Lambda_m(H)$. If it belongs to this set, Lemma 3.15 is enough to conclude. The next step provides another system $\widehat{H}$. We will then have to find (in Step 3) a common refinement of H and $\widehat{H}$, obtained at step 1 and 2 respectively.

Step 2. Fix $m \leq b$. There exists a family of regular systems of hypersurfaces $\widehat{H}$ such that for every k, $\widehat{\lambda}_k$ belongs to $\Lambda_m(H)$ and is regular for $G_m(H) \cap G_k(\widehat{H}) \cap A$.

Note that as λ_m is regular for the semialgebraic family of sets $H_m \cup H_{m+1}$, there exists $r \in \mathbb{Q}_+$ such that $B(\lambda_m; r)$ is regular for $H_m \cup H_{m+1}$. Taking r smaller if necessary, we may assume that $r \leq \frac{\eta}{2}$.

Let $r' \in \mathbb{Q}_+$ be such that we have in S^{n-2}:

$$B(\widetilde{\pi}_P(\lambda_m); r') \subset \widetilde{\pi}_P(B(\lambda_m; \frac{r}{2})). \tag{3.6}$$

To complete the proof of step 2 we need a lemma.

Lemma 3.20. *Let l in S^{n-1}, $r \in \mathbb{Q}_+$ and $\mu \in \mathbb{N}$. Let C be a subset of $\mathbb{G}_n$ and $P \in S^{n-1}$ such that:*

$$d(P; C) \geq \sigma, \tag{3.7}$$

with $\sigma \in \mathbb{Q}_+$. There exists $\alpha \in \mathbb{Q}_+$ such that for any $P_1, \dots, P_\mu$ in C and any $y \in \widetilde{\pi}_P(B(l; \frac{r}{2}))$ there exists $\widehat{\lambda} \in B(l; r) \cap \widetilde{\pi}_P^{-1}(y)$ such that:

$$d(\widehat{\lambda}; \cup_{i=1}^{\mu} P_i) \geq \alpha.$$

The proof of this lemma is postponed. We first see why it is enough to carry out the proof of step 2. Let μ be the maximal number of points of a finite fiber of the restriction of π_P to $A \cap G_m(H)$. Applying this lemma with this integer μ, with $C = \cup_{t \in R} \tau(A_t \cap G_m(H))$ and $l = \lambda_m$, we get a positive constant α.

Applying Lemma 3.17 to $G_m(H) \cap A$ (identify π_P with π_{e_n}) provides a finite covering $(A'_i)_{i \in I'}$ of $R \times N_P$ such that for any $i \in I'$ and any t:

$$\tau(\pi_P^{-1}(A'_{i,t}) \cap G_m(H)_t \cap A_t) \subset \bigcup_{j=1}^{\mu} B(P_j; \frac{\alpha}{2}),$$

for some $P_1, \dots, P_\mu$ (depending on $i \in I'$) in $\tau(A \cap G_m(H))$. By Lemma 3.16, we may assume that $(\delta A'_i)_t$ has empty interior for every t and i.

By Lemma 3.20, for any $i \in I'$ and any $y \in \widetilde{\pi}_P(B(\lambda_m; \frac{r}{2}))$, there exists $\widehat{\lambda} \in B(\lambda_m; r) \cap \widetilde{\pi}_P^{-1}(y)$ such that for any $t \in R$:

$$d\big(\widehat{\lambda}\,;\, \tau(\pi_P^{-1}(A'_{i,t}) \cap G_m(H)_t \cap A_t)\big) \geq \frac{\alpha}{2}. \tag{3.8}$$

Apply the induction hypothesis to get a family of regular systems of hypersurfaces H'' of $R \times N_P$ (identify N_P with R^{n-1}) compatible with the $\delta A'_i$'s. Do it in such a way that all the associated lines λ''_k are elements of $B(\widetilde{\pi}_P(\lambda_m); r')$ (where r' is given by (3.6)).

Define now:

$$\widehat{H}_{k,t} := \pi_P^{-1}(H''_{k,t}). \tag{3.9}$$

The compatibility with the sets $\delta A'_i$ implies that every $int(G_k(H''))$ is included in A'_i for some i (by the same argument that the one we used in Step 1 for $G_k(H)$ and the partition $(A_i)_{i\in I}$).

As a matter of fact, according to (3.8) for $y = \lambda''_k$, we know that for every integer $k \le b''$ there exists $\widehat{\lambda}_k \in B(\lambda_m; r) \cap \widetilde{\pi}_P^{-1}(\lambda''_k)$ such that for any $t \in R$:

$$d\big(\widehat{\lambda}_k\,;\,\tau(\pi_P^{-1}(G_k(H'')_t) \cap G_m(H)_t \cap A_t)\big) \ge \alpha. \tag{3.10}$$

Let us check that $\widehat{H} := (\widehat{H}_k; \widehat{\lambda}_k)_{k \le \widehat{b}}$ (where $\widehat{b} := b''$) is the desired family of regular systems of hypersurfaces. For this purpose, observe that, since neither P nor $-P$ belongs to $B(\lambda; \eta)$, we have for each k (recall that $r \le \frac{\eta}{2}$):

$$d(\widehat{\lambda}_k; \pm P) \ge \frac{r}{2}.$$

By construction and Remark 3.13, as $\widehat{\lambda}_k \in \widetilde{\pi}_P^{-1}(\lambda''_k)$, this implies that the family $\widehat{H}$ fulfills the three conditions of definition 3.11.

Furthermore, as $B(\lambda_m; r) \subset B(\lambda; \eta)$ (since $r \le \frac{\eta}{2}$ and $\lambda_m \in B(\lambda, \frac{\eta}{2})$), all the $\widehat{\lambda}_k$'s belong to $B(\lambda; \eta)$. Note also that as $B(\lambda_m; r)$ is regular for $H_m \cup H_{m+1}$, the vector $\widehat{\lambda}_k$ belongs to $\Lambda_m(H)$. This completes the proof of the second step.

The inconvenient of Step 2 is that the provided vector is regular for the family $A \cap G_m(H) \cap G_k(\widehat{H})$ (instead of $A \cap G_k(\widehat{H})$). If $\widehat{H}$ were an extension of the family H constructed in Step 1, this would be no problem since in this case we would have $G_k(\widehat{H}) \subset G_m(H)$ (or $int(G_k(\widehat{H})) \cap int(G_m(H)) = \emptyset$). Thus, we will have to find a common extension $\widetilde{H}$ of H and $\widehat{H}$ given by steps 1 and 2 respectively. This is what is carried out in the third step.

Step 3. There exists an extension $\widetilde{H} = (\widetilde{H}_k, \widetilde{\lambda}_k)_{k \le \widetilde{b}}$ of H which coincides with H outside $G_m(H)$ and such that $\widetilde{\lambda}_k$ is regular for the family $A \cap G_k(\widetilde{H}) \cap G_m(H)$ for all k.

Let $k \le \widehat{b}$ be an integer. Since $\widehat{\lambda}_k \in \Lambda_m(H)$, by Proposition 3.10, the sets H_m and H_{m+1} are respectively the graphs for $\widehat{\lambda}_k$ of two functions μ_k and μ'_k. Moreover, the set $\widehat{H}_k$ is also the graph for $\widehat{\lambda}_k$ of a function $\widehat{\xi}_k$. Define:

$$\eta_k := \min(\max(\mu_k; \widehat{\xi}_k); \mu'_k)$$

in order to get a function whose graph is included in $G_m(H)$. Now we define the desired regular family $(\widetilde{H}_k; \widetilde{\lambda}_k)_{1 \le k \le \widetilde{b}}$ as follows.

- Let $\widetilde{H}_k := H_k$ and $\widetilde{\lambda}_k := \lambda_k$ if $k < m$.
- Let $\widetilde{H}_m := H_m$ and $\widetilde{\lambda}_m := \widehat{\lambda}_1$.
- Let $\widetilde{H}_k$ be the graph of η_{k-m} for $\widehat{\lambda}_{k-m}$, and let $\widetilde{\lambda}_k := \widehat{\lambda}_{k-m}$, whenever $m+1 \le k \le m+\widehat{b}$.
- And finally let $\widetilde{H}_k := H_{k-\widehat{b}}$ and $\widetilde{\lambda}_k := \lambda_{k-\widehat{b}}$ if $m + \widehat{b} + 1 \le k \le b + \widehat{b}$.

We shall check that the properties $(i - iii)$ hold for the family $\widetilde{H}$ in every case.

For $k < m-1$, or $k \ge m+\widehat{b}+1$, the result is clear since the family $\widetilde{H}$ is indeed the family H.

For $k = m-1$, properties $(i - iii)$ follow from $(i - iii)$ for H and Proposition 3.10 since we have assumed $\widehat{\lambda}_1 \in \Lambda_m(H)$.

It remains to check $(i - iii)$ for $\widetilde{H}_{k+m}$, with $0 < k \le \widehat{b}$. Let us check (i) in this case.

By (i) for $\widehat{H}$, the set $\widehat{H}_{k+1}$ is the graph for $\widehat{\lambda}_k$ of a function $\widehat{\xi}'_k$ such that $\widehat{\xi}_k \leq \widehat{\xi}'_k$. Define now:

$$\eta'_k = \min(\max(\mu_k; \widehat{\xi}'_k); \mu'_k).$$

Claim. The graph of η'_k for $\widehat{\lambda}_k$ is that of η_{k+1} for $\widehat{\lambda}_{k+1}$.

To see this, note that the graph of η'_k (resp. η_k) matches with $\widehat{H}_{k+1}$ over $E(H_{m+1}; \widehat{\lambda}_k) \setminus E(H_m; \widehat{\lambda}_k)$ (resp. $\widehat{\lambda}_{k+1}$). But, by Proposition 3.10, the sets $E(H_m; l)$ and $E(H_{m+1}; l)$ do not depend on $l \in \Lambda_m(H)$. As $\widehat{\lambda}_k$ and $\widehat{\lambda}_{k+1}$ both belong to $\Lambda_m(H)$, this already shows that the two graphs involved in the above claim match over $int(G_m(H))$.

The graph of η'_k (resp. η_{k+1}) for $\widehat{\lambda}_k$ (resp. $\widehat{\lambda}_{k+1}$) is also constituted by the points of $H_m \setminus int(E(\widehat{H}_{k+1}, \widehat{\lambda}_k))$ (resp. $\widehat{\lambda}_{k+1}$) on the one hand and by the points of $H_{m+1} \cap E(\widehat{H}_{k+1}, \widehat{\lambda}_k)$ (resp. $\widehat{\lambda}_{k+1}$) on the other hand. By (iii) for $\widehat{H}$, the claim ensues.

This claim proves that $\widetilde{H}_{m+k+1}$ is the graph of η'_k for $\widehat{\lambda}_k$. Therefore, to check $(i-iii)$, we just have to prove that $\eta_k \leq \eta'_k$. But, as $\widehat{\xi}_k \leq \widehat{\xi}'_k$, this immediately comes down from the respective definitions of η'_k and η_k. This establishes (i) and (ii) (for $\widetilde{H}_{k+m}$, $k \leq \widehat{b}$).

Let us check property (iii) for $\widetilde{H}_{k+m}$, $k \leq \widehat{b}$. If $k = \widehat{b}$ it is a consequence of Proposition 3.10 since we have assumed that $\widehat{\lambda}_k$ belongs to $\Lambda_m(H)$.

Let k be such that $0 \leq k \leq \widehat{b} - 1$. First note that by (iii) for $\widehat{H}$ we have:

$$E(\widehat{H}_{k+1}; \widehat{\lambda}_k) = E(\widehat{H}_{k+1}; \widehat{\lambda}_{k+1}).$$

But, $E(\widetilde{H}_{k+m+1}; \widehat{\lambda}_k)$ (resp. $\widehat{\lambda}_{k+1}$) coincides with $E(\widehat{H}_{k+1}; \widehat{\lambda}_k)$ (resp. $\widehat{\lambda}_{k+1}$) over $int(G_m(H))$. It is also constituted by the points of $E(H_m, \widehat{\lambda}_k)$ (resp. $\widehat{\lambda}_{k+1}$) and the points of $E(H_{m+1}, \widehat{\lambda}_k) \cap E(\widehat{H}_{k+1}; \widehat{\lambda}_k)$ (resp. $\widehat{\lambda}_{k+1}$). As $\widehat{\lambda}_{k+1}$ and $\widehat{\lambda}_k$ both belong to $\Lambda_m(H)$, this establishes (iii).

To complete the proof of Step 3, it remains to make sure that for every $k \leq \hat{b}$ the line $\widetilde{\lambda}_{k+m}$ is regular for $G_{k+m}(\widetilde{H}) \cap G_m(H) \cap A$. By construction we have $\widetilde{\lambda}_m = \widehat{\lambda}_1$, $\widetilde{\lambda}_{k+m} = \widehat{\lambda}_k$ and:

$$G_{k+m}(\widetilde{H}) \subset G_k(\widehat{H}) \cap G_m(H), \tag{3.11}$$

for each $0 \leq k \leq \widehat{b}$.

As for any k the vector $\widehat{\lambda}_k$ is regular for $A \cap G_k(\widehat{H}) \cap G_m(H)$, this implies that for each $k \leq \widehat{b}$, the vector $\widetilde{\lambda}_{k+m}$ is regular for $G_{k+m}(\widetilde{H}) \cap A$. This completes the third step.

Finally, let us show why Step 3 is enough to conclude. By Lemma 3.15 (applied to $\widetilde{H}$ of Step 3), we may extend $\widetilde{H}$ to a family compatible with the set

$$G_m(H) \cap \cup_{k=0}^{\widetilde{b}} G_k(\widetilde{H}) \cap A = G_m(H) \cap A.$$

Since all the extensions coincide with H outside $G_m(H)$, we may carry out the construction on all the $G_m(H)$'s successively. This provides the desired family. □

It remains to prove Lemma 3.20. The lemma below describes a property of $\widetilde{\pi}_P$ that we need for this purpose.

Lemma 3.21. *Let λ and P in S^{n-1}, $T \in \mathbb{G}_n$ and $x \in T \cap \widetilde{\pi}_P^{-1}(\lambda)$. Let v be a unit vector tangent at x to the curve $\widetilde{\pi}_P^{-1}(\lambda)$. Then:*

$$d(P; T) \leq d(v; S^{n-1} \cap T).$$

Proof. Let w be the vector of $S^{n-1} \cap T$ which realizes $d(v; S^{n-1} \cap T)$. Remark that the vectors x, P, and v are in the same two dimensional vector space. Moreover $(x; v)$ is an orthonormal basis of this plane. Let $P = \alpha x + \beta v$ with $\alpha^2 + \beta^2 = 1$. Then, as x and w both belong to T we have

$$d(P;T) \leq |P - (\alpha x + \beta w)| = |\beta| \cdot |v - w| \leq d(v; S^{n-1} \cap T).$$

□

Proof of Lemma 3.20. We will work up to a ("projective") coordinate system of S^{n-1} defined as follows. Let U_i^+ (resp. U_i^-) denote

$$\{x \in S^{n-1}/x_i \geq \epsilon\}$$

(resp. $x_i \leq -\epsilon$) with $\epsilon \in \mathbb{Q}_+$ small enough. Define then: $h_i : U_i \to R^{n-1}$ by $h_i(x_1; \dots; x_n) = (\frac{x_1}{x_i}; \dots; \widehat{\frac{x_i}{x_i}}; \dots; \frac{x_n}{x_i})$. Note that h_i is a $\mathbb{Q}$-bi-Lipschitz homeomorphism.

Through such a chart, the set $S^{n-1} \cap N_P$ is a vector subspace and $\widetilde{\pi}_P$ becomes an orthogonal projection along a line, say Q. By Lemma 3.21, hypothesis (3.7) implies that there exists $u \in \mathbb{Q}_+$ such that:

$$d(Q;T) \geq u,$$

for any $T \in C \subset \mathbb{G}_{n-1}$.

It is then an easy exercise of elementary geometry to derive from this that for any $x \in Q$ and any $P_1, \dots, P_\mu$ in C:

$$d\big(x; \cup_{i=1}^{\mu} P_i \cap Q\big) \leq \frac{1}{u} \cdot d\big(x\,; \cup_{i=1}^{\mu} P_i\big). \tag{3.12}$$

For any $y \in \widetilde{\pi}_P(B(l; \frac{r}{2}))$ the length of the line segment $\widetilde{\pi}_P^{-1}(y) \cap B(l;r)$ is bounded below away from zero by a strictly positive rational number α_0.

Let α be the rational number $\frac{\alpha_0 u}{4\mu}$. Then, using (3.12) one can easily see that if the conclusion of the lemma failed for some $y \in \widetilde{\pi}_P(B(l; \frac{r}{2}))$, we could cover the segment $\widetilde{\pi}_P^{-1}(y) \cap B(l;r)$ by μ segments of length less than $\frac{\alpha_0}{2\mu}$. This contradicts the fact that the length of this segment is not less than α_0. □

3.5. **Proof of Proposition 3.4.**

Proof. By Proposition 3.19 there exists a family of regular systems of hypersurfaces $H = (H_k; \lambda_k)_{1 \leq k \leq b}$ compatible with A. We shall define h over $E(H_k; \lambda_k)$, by induction on k, in such a way that $h(E(H_k; \lambda_k)) = E(F_k; e_n)$ (so that $h(H_k) = F_k$) where F_k is the graph of a function $\eta_k : R \times R^{n-1} \to R$ for e_n.

For $k = 1$ choose an orthonormal basis of N_{λ_1} and set $h(q) := (x_{\lambda_1}; q_{\lambda_1})$ where x_{λ_1} are the coordinates of $\pi_{\lambda_1}(q)$ in this basis. Let $k \geq 1$. By (i) of Definition 3.11, the sets H_k and H_{k+1} are the graphs for λ_k of two functions ξ_k and ξ'_k. For $q \in E(H_{k+1}; \lambda_k) \setminus E(H_k; \lambda_k)$ define $h(q)$ as the element:

$$h(\pi_{\lambda_k}(q) + \xi_k(\pi_{\lambda_k}(q)) \cdot e_n) + (q_{\lambda_k} - \xi_k(\pi_{\lambda_k}(q)))e_n.$$

Thanks to the property (iii) of Definition 3.11 we have $E(H_{k+1}; \lambda_{k+1}) = E(H_{k+1}; \lambda_k)$, and hence h is actually defined over $E(H_{k+1}; \lambda_{k+1})$. Since $\xi_{k,t}$ is C-Lipschitz with $C \in \mathbb{Q}$, h_t is a family of bi-Lipschitz homeomorphisms. Note also that the image is $E(F_{k+1}; e_n)$ where F_{k+1} is the graph (for e_n) of the family of Lipschitz functions on $R \times R^{n-1}$:

$$\eta_{k+1}(x) := \eta_k(x) + (\xi'_k - \xi_k) \circ \pi_{\lambda_k} \circ h^{-1}(x; \eta_k(x)).$$

This gives h over $E(H_b; \lambda_b)$. To extend h to the whole of $R \times R^n$ do it similarly as in the case $k = 1$. □

4. On families of semialgebraic functions

Let $k(0_+)$ be the extension of R corresponding to the ultrafilter 0_+, constituted by all the semialgebraic sets of R containing a right-hand-side neighborhood of the origin (see [BCR]). The field $k(0_+)$ is the real closure of the fraction field of the ring $R[Y]$ endowed with the order relation that makes the indeterminate Y smaller than any element of R. We shall denote by $Y(0_+)$ the indeterminate regarded in $k(0_+)$.

Lemma 4.1. *For any $u \in k(0_+)$ there is a rational number ν such that:*

$$u \sim_R Y(0_+)^\nu.$$

Proof. There exists a semialgebraic function $\xi : (0, \varepsilon) \to R$ such that $\xi(Y(0_+)) = u$ (see [BCR]). By the preparation theorem, there exist a and b in R and $\nu \in \mathbb{Q}$ such that:

$$\xi(x) \sim_{\mathbb{Q}} b \cdot (x - a)^\nu.$$

Thus, $\xi(Y(0_+)) \sim_R Y(0_+)^\nu$ if $a = 0$ and $\xi \sim_R 1$ otherwise. □

Proposition 4.2. *Let $\xi : k(0_+)^n \to k(0_+)$ be a nonnegative semialgebraic function. There exists a cell decomposition of $k(0_+)^n$ such that over every cell:*

$$\xi(x) \sim_R Y(0_+)^r \cdot d(x, W_1)^{r_1} \cdots d(x, W_k)^{r_k} \tag{4.13}$$

where the W_i's are semialgebraic subsets of $k(0_+)^n$ and r as well as the r_i's are rational numbers.

Proof. We prove it by induction on n. The case $n = 0$ follows from Lemma 4.1.

Assume that the lemma is true for $(n-1)$ and apply the preparation theorem to the function ξ. Let $n \geq 1$ and let $\lambda_1, \dots, \lambda_N$ be the elements of S^{n-1} given by Lemma 3.6. Applying the preparation theorem (Theorem 3.7) to $\xi \circ A_i$, where A_i is an orthogonal linear mapping of $k(0_+)^n$ sending the vector e_n onto λ_i for $i \in \{1, \dots, N\}$, and taking a common refinement of the images under the A_i^{-1} of all the obtained partitions we get a semialgebraic partition $(V_j)_{j \in J}$ of $k(0_+)^n$. Therefore, over each V_j and for each i we can find continuous functions $a, \theta : \pi_{\lambda_i}(V_j) \to k(0_+)$ and $r \in \mathbb{Q}$ such that:

$$\xi(q) \sim_{\mathbb{Q}} (q_{\lambda_i} - \theta(x_{\lambda_i}))^r \, a(x_{\lambda_i}), \tag{4.14}$$

for $q = x_{\lambda_i} + q_{\lambda_i} \lambda_i \in \pi_{\lambda_i}(V_j) \oplus k(0_+) \cdot \lambda_i$.

Apply Proposition 3.6 to the family constituted by all the sets of the partition $(V_j)_{j \in J}$ and the zero locus of ξ. This gives rise to a partition $(V'_j)_{j \in J'}$ such that each V'_j which is open is of the form

$$\{q \in \pi_{\lambda_i}(V'_j) \ \oplus k(0_+) \cdot \lambda_i : \quad \xi_1(\pi_{\lambda_i}(q)) < q_{\lambda_i} < \xi_2(\pi_{\lambda_i}(q))\},$$

for some $i \in \{1, \dots, N\}$, where $\xi_\nu : \pi_{\lambda_i}(V'_j) \to k(0_+)$, $\nu = 1, 2$, are $\mathbb{Q}$−Lipschitz functions, and such that the function ξ is of the form (4.14) on V'_j for each vector λ_i.

Thanks to the induction hypothesis (identify N_{λ_i} with $k(0_+)^{n-1}$) it is sufficient to prove the result for the function $|q_{\lambda_i} - \theta(\pi_{\lambda_i}(q))|$.

Fix $j \in J'$. Due to the compatibility of the partition with the zero locus of ξ, we have, for every x, either $\theta(x) \leq \xi_1(x)$ or $\theta(x) \geq \xi_2(x)$. Up to a subpartition we may assume that only one case occurs over V'_j, for instance $\theta \leq \xi_1$. Writing

$$q_{\lambda_i} - \theta(\pi_{\lambda_i}(q)) = (q_{\lambda_i} - \xi_1(\pi_{\lambda_i}(q)) + (\xi_1(\pi_{\lambda_i}(q)) - \theta(\pi_{\lambda_i}(q))),$$

we see that (up to a refinement we may assume that these functions are comparable) $|q_{\lambda_i} - \theta(\pi_{\lambda_i}(q))|$ is $\sim_{\mathbb{Q}}$ either to $|q_{\lambda_i} - \xi_1(\pi_{\lambda_i}(q))|$ or to $|\xi_1(\pi_{\lambda_i}(q)) - \theta(\pi_{\lambda_i}(q))|$. For the former function, since ξ_1 is Lipschitz, $|q_{\lambda_i} - \xi_1(x_{\lambda_i})|$ is $\sim_{\mathbb{Q}}$ to the distance to the graph of ξ_1 for λ_i. For the latter one, this is a consequence of the induction hypothesis. For the V'_j's having positive codimension, one may deduce the result from the induction hypothesis. □

5. Proof of Theorem 2.4

Proof. We first check that we can assume, without loss of generality, that the mapping $f : X \to R$ is the projection on the first coordinate. Indeed, if we replace X with

$$\hat{X} := \{(y, x) \in R \times X : y = f(x)\},$$

and prove the result for $\hat{f} : \hat{X} \to R$, defined by $\hat{f}(y, x) := y$, we are done.

We shall establish a stronger result, proving by induction on n the following facts:

$(\mathcal{P}_n)$. Let $f : [-M, M] \times R^n \to R$, $M > 0$, be defined by $f(y, x) := y$. Let $A_1, \dots, A_k$ be semialgebraic subfamilies of $[-M, M] \times R^n$ and let $\eta_1, \dots, \eta_l$ be semialgebraic families of nonnegative functions on $[-M, M] \times R^n$. There is a triangulation (K, ϕ, ψ) of f such that:

(1) (2.3) holds.
(2) The A_i's are union of images (by ψ) of simplices.
(3) The functions $\eta_i \circ \psi$ are $\sim_R$ to standard simplicial functions.

For $n = 0$ the result is clear. Assume that it is true for some $n \geq 0$ and let us check it for $(n + 1)$. We denote by $\pi : R \times R^{n+1} \to R \times R^n$ the canonical projection.

We claim that there is a cell decomposition of $R \times R^{n+1}$ such that for every cell C, we can find some semialgebraic families $W_1, \dots, W_c$ of $R \times R^{n+1}$ as well as, for each i, some rational numbers $r, r_1, \dots, r_c$, and $y_0 \in R$, such that for $(y, x) \in C \subset [-M, M] \times R^{n+1}$:

$$\eta_{i,y}(x) \sim_R |y - y_0|^r d(x, W_{1,y})^{r_1} \cdots d(x, W_{c,y})^{r_c}, \tag{5.15}$$

where the constants of this equivalence are independent of y (below all the constants will be independent of the parameter y).

Let $\alpha \in \widetilde{[-M; M]}$ and denote by $k(\alpha)$ the corresponding extension of R. If α has a specialization then, by Proposition 4.2, we can find $U \in \alpha$ such that (5.15) holds true for the restriction of the η_i's to $U \times R^{n+1}$. If α has no specialization then every element of $k(\alpha)$ is bounded by an element of R. Hence, in this case (5.15) follows from Proposition 3.9 (applied to $\eta_{i,\alpha}$). In any case we thus find an element $U \in \alpha$ along which the desired equivalence may be established. By compactness of the real spectrum, we may extract a finite covering of $[-M, M]$. Taking a common refinement of all the corresponding cell decompositions, we get a cell decomposition $\mathcal{E}$ having the required property (5.15). We may assume that this cell decomposition is compatible the A_i's.

By Proposition 3.4, up to a family of bi-Lipschitz maps (that we will identify with the identity), we may assume that all the cells of this cell decomposition which are graphs (i.e. which are not bands) as well as the (topological) boundaries of the $W_{j,y}$'s (see (5.15)) are included in the union of a finite number of graphs of families of Lipschitz functions $\theta_{1,y} \leq \cdots \leq \theta_{\mu,y}$ (continuous with respect to y).

Applying Observation (4) to the θ_i's and to the functions $(y, x) \mapsto d(x; \pi(\delta W_{i,y}))$, we see that there exist a cell decomposition $\mathcal{D}$ of $R \times R^n$ and finitely many families of Lipschitz functions $\xi_{1,y} \leq \cdots \leq \xi_{m,y}$ whose graphs contain the graphs for the $\theta_{i,y}$'s, such that for every $D \in \mathcal{D}$, all the functions $|q_{n+1} - \theta_{\nu,y}(\pi(q))|$ are comparable (for $\leq$) with each other and comparable with the functions $d(x; \pi(\delta W_{i,y} \cap \Gamma_{\theta_{\nu,y}}))$ on the set $[\xi_{i,y|D_y}; \xi_{i+1,y|D_y}]$.

Consider a semialgebraic cell decomposition of $R \times R^{n+1}$ adapted to the graphs of the families of functions ξ_i, the cells of $\mathcal{D}$ and $\mathcal{E}$, as well as the sets W_j. Let $X_1, \dots, X_s$ be the images of the cells under π. Refining this partition, we may assume that the functions $d(x; \pi(\delta W_{i,y} \cap \Gamma_{\theta_{\nu},y}))$ are comparable with respect to each other on the cells. Apply the induction hypothesis to get a triangulation (K, ϕ, ψ) of f (restricted to $[-M, M] \times R^n$) such that the X_i's are unions of images of open simplices. Moreover, by (3) of the induction hypothesis, we may do it in such a way that over each simplex, each function $|\xi_j - \theta_i| \circ \psi$ as well as all the functions

$(y,x) \mapsto d(\psi_y(x);\pi(\delta W_{j,y} \cap \Gamma_{\theta_{i,y}}))$, and $\eta_{k,y}(\psi_y(x), \xi_{i,y}(\psi_y(x)))$, are $\sim$ to standard simplicial functions.

Let $\zeta_1 \leq \cdots \leq \zeta_m$ be piecewise linear functions over $|K|$ such that $\zeta_i \equiv \zeta_{i+1}$ on the set $\{\xi_i \circ \psi = \xi_{i+1} \circ \psi\}$ (this set is a subcomplex of K). Let also $\zeta_0 := \zeta_1 - 1$ and $\zeta_{m+1} := \zeta_m + 1$. Let

$$N = \{(y,x,q_{n+1}) \in R \times R^n \times R \,:\, \zeta_{0,y}(x) \leq q_{n+1} \leq \zeta_{m+1,y}(x)\}.$$

We obtain a polyhedral decomposition of N by taking the respective inverse images by $\pi_{|N}$ of the simplices of K of dimension n on the one hand, and by taking all the images of the simplices of $|K|$ by the mappings $x \to (x;\zeta_i(x))$ on the other hand. After a barycentric subdivision of this polyhedra we get a simplicial complex L.

Let $\widetilde{K}$ be the union of the open simplices σ included in

$$\{(y,x,q_{n+1}) \in |K| \times R : \zeta_{0,y}(x) < q_{n+1} < \zeta_{m+1,y}(x)\}.$$

Define now for $y \in R$ over $\widetilde{K}_y$ the desired family of homeomorphisms $\widetilde{\psi}_y$ in the following way:

$$\widetilde{\psi}_y(x; t\,\zeta_{i,y}(x) + (1-t)\zeta_{i+1,y}(x)) = (\psi_y(x); t\,\xi_{i,y}(\psi_y(x)) + (1-t)\xi_{i+1,y}(\psi_y(x)))$$

for $1 \leq i \leq m-1$, $x \in R^n$ and $t \in [0;1]$. Define also:

$$\widetilde{\psi}_y(x; t\,\zeta_{0,y}(x) + (1-t)\,\zeta_{1,y}(x)) = (\psi_y(x); \xi_1(\psi_y(x)) - \frac{t}{1-t})$$

and

$$\widetilde{\psi}_y(x; t\,\zeta_{m+1,y}(x) + (1-t)\,\zeta_{m,y}(x)) = (\psi_y(x); \xi_{m,y}(\psi_y(x)) + \frac{t}{1-t})$$

for $t \in [0;1)$. This defines a family of homeomorphisms $\widetilde{\psi} : |\widetilde{K}| \to [-M,M] \times R^{n+1}$.

We shall check that over each simplex σ the mapping $\widetilde{\psi}$ fulfills (2.3). Let $\sigma \subset [\zeta_i, \zeta_{i+1}]$ be a simplex of $\widetilde{K}$, q and q' two points of σ_y, $y \in R$ fixed. The points q and q' may be expressed $q = (x; t\zeta_i(x) + (1-t)\zeta_{i+1}(x))$ and $q' = (x'; t'\zeta_i(x') + (1-t')\zeta_{i+1}(x'))$ for some $0 \leq i \leq m$ and some $(t;t')$ in $[0;1]^2$. Then define

$$q'' := (x; t'\zeta_i(x) + (1-t')\zeta_{i+1}(x)).$$

We begin with the case where $1 \leq i \leq m-1$. Let $p = \widetilde{\psi}_y(q)$, $p' = \widetilde{\psi}_y(q')$ and $p'' = \widetilde{\psi}_y(q'')$. We may consider x, x', p, p' and p'' as functions of q and q'. As $\xi_{i,y}$ and $\xi_{i+1,y}$ are Lipschitz functions we have over $\sigma \times \sigma$:

$$|p - p'| \sim |p - p''| + |\psi_y(x) - \psi_y(x')|. \tag{5.16}$$

Let σ' be the simplex of K containing $\pi(\sigma)$. Thanks to the induction hypothesis, we may find some functions $\varphi_{\sigma',1}, \ldots, \varphi_{\sigma',n}$ and a tame system of coordinates $(x_{1,\sigma'};\ldots;x_{n,\sigma'})$ such that for any x and x' in σ'_y:

$$|\psi_y(x) - \psi_y(x')| \sim \sum_{l=1}^{n} \varphi_{\sigma',l}(x;x')|x_{l,\sigma'} - x'_{l,\sigma'}|. \tag{5.17}$$

The result is therefore clear if $\zeta_i = \zeta_{i+1}$ on σ'. Otherwise, as $\pi(q) = \pi(q'')$, by construction we have:

$$|p_{n+1} - p''_{n+1}| \sim |q_{n+1} - q''_{n+1}| \cdot \frac{\xi_{i+1,y}(\psi_y(x)) - \xi_{i,y}(\psi_y(x))}{\zeta_{i+1,y}(x) - \zeta_{i,y}(x)}.$$

Recall that we have constructed the triangulation (K,ϕ,ψ) in such a way that for every i, $(\xi_{i+1} - \xi_i) \circ \psi$ is $\sim$ to a standard simplicial function of K, say ω_i. The composite $\omega_i \circ \pi$ gives a standard simplicial function of $\widetilde{K}$. The functions ζ_i and ζ_{i+1} define a tame coordinate on R^{n+1} that we will denote by $q_{n+1,\sigma}$. By the preceding estimation, we have:

$$|p - p''| \sim |q_{n+1,\sigma} - q'_{n+1,\sigma}| \cdot \varphi_{\sigma,n+1}(q;q') \tag{5.18}$$

for a standard simplicial function $\varphi_{\sigma,n+1}$ (which here actually depends only on q).

Define for $j < n+1$:

$$\varphi_{\sigma,j}(q;q') = \varphi_{\sigma',j}(\pi(q);\pi(q')).$$

Then by (5.18), (5.17) and (5.16) we get the desired equivalence (in the case $1 \leq i \leq m-1$).

The case $i = 0$ and m are dealt in an analogous way (see [V1] for details). This proves that $\widetilde{\psi}_y$ satisfies (2.3). By construction, the A_j's are images of open simplices.

It remains to check that the functions $\eta_j \circ \widetilde{\psi}$ are $\sim$ to standard simplicial functions over any simplex σ. Let $\sigma \in \widetilde{K}$; if the set $\widetilde{\psi}(\sigma)$ is included in the graph of ξ_i for some i, the result follows by induction. So, assume that it sits in $]\xi_i;\xi_{i+1}[$, for some $1 \leq i \leq m-1$. By construction, on $\widetilde{\psi}(\sigma)$, the $\eta_{j,y}$'s are $\sim_R$ to a product of powers of distances to the $W_{j,y}$'s (see (5.15)).

Therefore, it suffices to show the result for the functions $q \mapsto d(\widetilde{\psi}_y(q);W_{j,y})$. As $(\widetilde{\psi};\widetilde{K})$ is also a triangulation of the sets W_j, for each j, either $\widetilde{\psi}(\sigma)_y$ is included in $W_{j,y}$ or the distance to $W_{j,y}$ is $\sim$ to the distance to its boundary. In the former case the result is obvious since the function $q \mapsto d(\widetilde{\psi}_y(q);W_{j,y})$ is zero over σ. By construction, the boundary δW_j is included in the union of the $\Gamma_{\theta_{\nu,y}}$'s.

Moreover, we have for any $\nu \in \{1,\dots,\mu\}$:

$$d(q;\delta W_{i,y} \cap \Gamma_{\theta_{\nu,y}}) \sim |q_{n+1} - \theta_{\nu,y}(x)| + d(x;\pi(\delta W_{i,y} \cap \Gamma_{\theta_{\nu,y}})) \tag{5.19}$$

where $q = (x;q_{n+1})$ in $\widetilde{\psi}_y(\sigma_y) \subset R^n \times R$.

As both terms of the right-hand-side are positive, the sum is $\sim$ to the max of these two terms that is to say is $\sim$ to one of them since they are comparable over $\widetilde{\psi}(\sigma)$. Note that clearly $d(q;\delta W_{i,y}) = \min_{1\leq\nu\leq\mu} d(q;\delta W_{i,y} \cap \Gamma_{\theta_{\nu,y}})$. But as by construction the functions $g_{\nu,y} := d(\pi(q);\pi(\delta W_{i,y} \cap \Gamma_{\theta_{\nu,y}}))$ are comparable with each other and comparable with all the functions $|q_{n+1} - \theta_{\nu,y}(x)|$, the function $d(q;\delta W_{i,y})$ is equivalent over $\tilde{\psi}_y(\sigma_y)$ to one of the functions $g_{\nu,y}$ or to some function $|q_{n+1} - \theta_{\nu,y}(x)|$.

Recall that we have required the triangulation $(\psi;K)$ to be such that

$$(y,x) \mapsto d(\psi_y(x);\pi(\delta W_{j,y} \cap \Gamma_{\theta_{\nu,y}}))$$

is $\sim$ to a standard simplicial function of K. Hence, by (5.19), it suffices to prove that the function $(y,q) \mapsto |\psi_{n+1,y}(q) - \theta_{\nu,y}(\pi(\psi_y(q)))|$ is $\sim$ over σ to a standard simplicial function of $\widetilde{K}$. Assume that $\sigma \subset [\zeta_i;\zeta_{i+1}]$. We may write for $p = (y,x,p_{n+1}) \in \sigma \subset R \times R^n \times R$:

$$|p_{n+1} - \theta_\nu \circ \psi| = p_{n+1} - \xi_i \circ \psi + (\xi_i \circ \psi - \theta_\nu \circ \psi)$$

if $\theta_\nu \leq \xi_i$ on $\pi(\tilde{\psi}(\sigma))$, and

$$|p_{n+1} - \theta_\nu \circ \psi| = \xi_{i+1} \circ \psi - p_{n+1} + (\theta_\nu \circ \psi - \xi_{i+1} \circ \psi)$$

if $\theta_\nu \geq \xi_{i+1}$ (with the convention $\xi_0 = -\infty$, $\xi_{m+1} = \infty$). By (5.18), we have over σ for $q = \tilde{\psi}^{-1}(p) = (y,z,q_{n+1})$:

$$p_{n+1} - \xi_{i,y}(\psi_y(x)) \sim |q_{n+1} - \zeta_{i,y}(z)| \,.\, \varphi_{\sigma,n+1}(q;q').$$

The function $|q_{n+1} - \zeta_{i,y}(x)|$ is $\sim$ to a standard simplicial function. As all the $|\xi_{i,y} \circ \psi_y - \theta_{\nu,y} \circ \psi_y|$ have been assumed to be equivalent to standard simplicial functions, the theorem is proved. $\square$

6. bi-Lipschitz triviality of families of functions

Definition 6.1. We say that a semialgebraic family of functions $f : X \to \mathbb{R}^p \times \mathbb{R}$ is **fiberwise semialgebraically bi-Lipschitz trivial along** $W \subset \mathbb{R}^p$ if there exist two families of semialgebraic homeomorphisms $h : W \times \mathbb{R}^n \to W \times \mathbb{R}^n$ and $\phi : W \times \mathbb{R} \to W \times \mathbb{R}$, such that for any $t \in W$:

(1) $h_t(X_{t_0}) = X_t$ and $\phi_t^{-1} \circ f_t \circ h_t = f_{t_0}$, $t_0 \in W$.
(2) ϕ_t is bi-Lipschitz.
(3) There is a constant $C_t \in \mathbb{R}$ such that the restriction of h_t to every fiber $f_t^{-1}(y)$ is C_t-bi-Lipschitz.

In the case where h_t is bi-Lipschitz (i.e. not only the restriction to the fibers but h_t itself), we say that it is **semialgebraically bi-Lipschitz trivial** along W.

Remark 6.2. It is worthy of notice that, in the definition of fiberwise bi-Lipschitz triviality, the mapping h_t is not only assumed to be C-bi-Lipschitz on every fiber: it is a homeomorphism.

The flaw of bi-Lipschitz triviality of functions is that it admits continuous moduli: the Lipschitz counterpart of Theorem 1.6 is not true, even for families as simple as two variable polynomials. The counterexample is due to A. Parusiński and J.-P. Henry.

Example 6.3. In [H-P] J-P. Henry and A. Parusiński gave the following example: $f_t(x, y) := x^3 + y^6 + 3t^2xy^4$. They proved by exhibiting some metric invariants for functions that there is no interval W of R along which this family is semialgebraically bi-Lipschitz trivial. As bi-Lipschitz triviality can be derived from triangulability (see proofs of Theorems 1.6 and 6.4), this example shows that in Theorem 2.4 we could not require (2.3) to hold for all couples (q, q') (not necessarily in the same fiber).

Nevertheless, fiberwise bi-Lipschitz triviality does *not* admit continuous moduli. This is the main theorem of this article.

Theorem 6.4. *Given a semialgebraic family of Lipschitz functions $f : X \to \mathbb{R}^p \times \mathbb{R}$ there exists a semialgebraic partition $V_1, \dots, V_m$ of $\mathbb{R}^p$ such that for every i, f is fiberwise semialgebraically bi-Lipschitz trivial along V_i.*

Proof. We apply exactly the same argument as in the proof of Theorem 1.6, replacing Theorem 1.2 with Theorem 2.4. As in the proof of the latter theorem, possibly replacing f_t with $u \circ f_t$ where $u(y) := \frac{y}{1+|y|}$, we may assume that f is bounded (if $\phi : R \to R$ is bi-Lipschitz and $\phi([-1, 1]) = [-1, 1]$ then $u^{-1} \circ \phi \circ u$ is bi-Lipschitz). By (2.3), the homeomorphisms h_t (at the end of the proof of Theorem 1.6) are C_t-bi-Lipschitz on the fibers $f_t^{-1}(y)$ with C_t independent of y. □

Remark 6.5. In the above theorem, we could also require the homeomorphism h_t (see Definition 6.1) to satisfy

$$d(h_t(x), f_t^{-1}(0)) \sim d(x, f_{t_0}^{-1}(0)).$$

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