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ASYMMETRY IN SINGULARITIES OF TANGENT SURFACES IN CONTACT-CONE LEGENDRE-NULL DUALITY

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Abstract. We give the generic classification on singularities of tangent surfaces to Legendre curves and to null curves by using the contact-cone duality between the contact 3-sphere and the Lagrange-Grassmannian with cone structure of a symplectic 4-space. As a consequence, we observe that the symmetry on the lists of such singularities is breaking for the contact-cone duality, compared with the ordinary projective duality.

1. Introduction

Let $V = (V, \Omega)$ be a real symplectic vector space of dimension 4 with a symplectic form Ω. We consider the Lagrange flag manifold $\mathcal{F} = \mathcal{F}^{\mathrm{Lag}}_{1,2}(V)$ consisting of pairs (ℓ, L) of lines ℓ and Lagrange planes L in V containing ℓ. Then there are natural projections $\pi_1 : \mathcal{F} \to P(V)$ to the projective 3-space and $\pi_2 : \mathcal{F} \to \mathrm{LG}(V)$ to the Grassmannian of Lagrange planes in V:

$$P(V) \xleftarrow{\pi_1} \mathcal{F} \xrightarrow{\pi_2} \mathrm{LG}(V).$$

Note that $\dim \mathcal{F} = 4, \dim P(V) = \dim \mathrm{LG}(V) = 3$ and both π_1 and π_2 are fibrations with S^1 as fibers.

There exist the projective Engel structure on $\mathcal{F}$, the projective contact structure on $P(V)$ and the projective indefinite conformal structure on $\mathrm{LG}(V)$ of signature $(1, 2)$, such that both π_1-fibers and π_2-fibers are projective lines in $\mathcal{F}$, and that each π_1-fiber (resp. π_2-fiber) projects to a projective line in $\mathrm{LG}(V)$ (resp. $P(V)$) by π_2 (resp. by π_1). We give precise coordinate charts on $\mathcal{F}$, $P(V)$ and $\mathrm{LG}(V)$ in §3. A projective Legendre line through $\ell \in P(V)$ is given by $\pi_1(\pi_2^{-1}(L))$ for some $L \in \mathrm{LG}(V)$. On the other hand, a null (lightlike) line through $L \in \mathrm{LG}(V)$ is given by $\pi_2(\pi_1^{-1}(\ell))$ for some $\ell \in P(V)$.

Let $f : I \to \mathcal{F}$ be an integral curve to the Engel structure of $\mathcal{F}$ from an open interval I. Then $\pi_1 \circ f : I \to P(V)$ is a Legendre curve and $\pi_2 \circ f : I \to \mathrm{LG}(V)$ is a null curve for the null cone field on $\mathrm{LG}(V)$.

For a curve $c : I \to M$ in a 3-dimensional space M with a projective structure, its *tangent surface* (or, *tangent developable*) is defined as the ruled surface by the tangent lines ([15],[16],[18],[10],[11]).

An *associated variety* to a curve in $P(V)$ (resp. $\mathrm{LG}(V)$) is the subset of $\mathrm{LG}(V)$ (resp. $P(V)$) consisting of $L \in \mathrm{LG}(V)$ (resp. $\ell \in P(V)$) corresponding to a Legendre line (resp. a null line) which intersects with the curve (cf. [8]). Then we see that the associated variety to $\pi_1 \circ f$ (resp. $\pi_2 \circ f$) is the tangent surface to $\pi_2 \circ f$ (resp. $\pi_1 \circ f$) if $\pi_2 \circ f$ (resp. $\pi_1 \circ f$) is an immersion. In fact it is given by $\pi_2(\pi_1^{-1}(\pi_1(f(I)))$ (resp. $\pi_1(\pi_2^{-1}(\pi_2(f(I))))$, see §2.

Key words: tangent developable, null curve, Legendre curve, Lagrangian-Grassmannian, projective structure, Engel structure.

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Then the main purpose of this paper is to prove the following result:

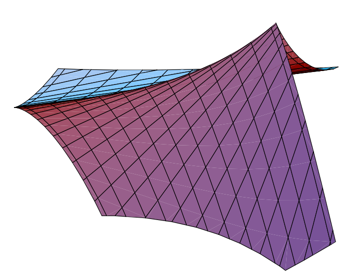
Theorem 1.1. *For a generic Engel integral curve $f : I \to \mathcal{F}$ from an open interval I to the Lagrange flag manifold $\mathcal{F}$ in C^∞ topology, we have that, for any $t_0 \in I$, the pair of singularities of tangent surfaces to $\pi_1 \circ f$ and to $\pi_2 \circ f$ is given by one of the following three cases:*

$$\begin{array}{ccl} \mathrm{I} & : & \text{(cuspidal edge, cuspidal edge)}, \\ \mathrm{II} & : & \text{(Mond surface, swallowtail)}, \\ \mathrm{III} & : & \text{(generic folded pleat, Shcherbak surface)}. \end{array}$$

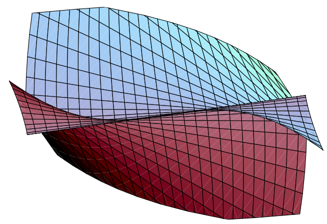
In fact, there exists a residual subset $\mathcal{R}$ in the space $C^\infty_E(I, \mathcal{F})$ of Engel integral curves with C^∞-topology, such that any $f \in \mathcal{R}$ enjoys the properties stated in Theorem 1.1. The usage of the C^∞ topology on an open interval is essential for our classification, see Remark 4.3.

The singularities appeared in Theorem 1.1 have the following parametric normal forms respectively, see Figure 1: A cuspidal edge (resp. Mond surface, swallowtail, generic folded pleat, Shcherbak surface) is locally *diffeomorphic* to the germ of parametrized surface $(\mathbf{R}^2, 0) \to (\mathbf{R}^3, 0)$ explicitly given by

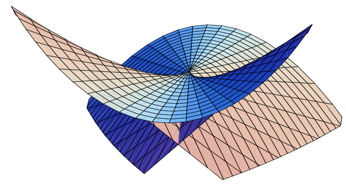
$$\begin{aligned} \text{cuspidal edge} : &\quad (x,t) \mapsto (x,\ -\tfrac{1}{2}t^2 + xt,\ \tfrac{1}{3}t^3 - \tfrac{1}{2}xt^2), \\ \text{Mond surface} : &\quad (x,t) \mapsto (x,\ -\tfrac{1}{3}t^3 + \tfrac{1}{2}xt^2,\ \tfrac{1}{4}t^4 - \tfrac{1}{3}xt^3), \\ \text{swallowtail} : &\quad (x,t) \mapsto (x,\ \tfrac{1}{6}t^3 - xt,\ -\tfrac{1}{4}t^4 + xt^2), \\ \text{generic folded pleat} : &\quad (x,t) \mapsto (x,\ -\tfrac{1}{6}t^3 + xt - \tfrac{1}{8}t^4 + \tfrac{1}{2}xt^2, \\ &\qquad\qquad \tfrac{1}{20}t^5 - \tfrac{1}{6}xt^3 + \tfrac{1}{24}t^6 - \tfrac{1}{8}xt^4), \\ \text{Shcherbak surface} : &\quad (x,t) \mapsto (x,\ \tfrac{1}{3}t^3 - \tfrac{1}{2}xt^2,\ -\tfrac{1}{5}t^5 + \tfrac{1}{4}xt^4). \end{aligned}$$



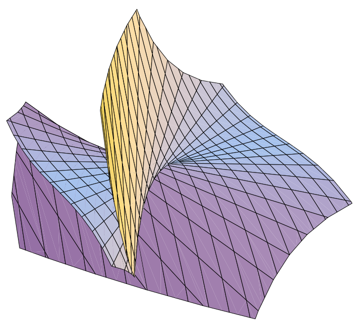
the cuspidal edge



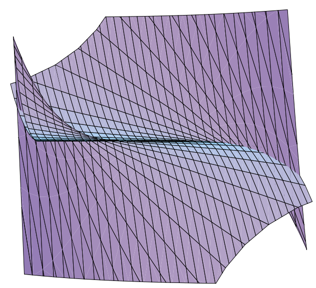
the Mond surface



the swallowtail



the generic folded pleat



the Shcherbak surface

Figure 1.

The above normal forms of singularities are written in the projective coordinates which are given in §3 and therefore they look different from, for example, those given in [10], [11]. Mond surfaces are called also *cuspidal beaks* and they appear as singularities on wave-fronts of codimension one, see for instance [1], [3].

Singularities of the tangent developable to a curve of type $(2, 3, 5)$ was called *folded pleats* in [12]. It was known that the local differential classes of folded pleats are not unique [10] while the local homeomorphism class of them is unique [11]. Any folded pleat is locally homeomorphic to the plane and it has singular locus along the original curve. In this paper, we show the folded pleat singularities form exactly two classes of local diffeomorphism equivalence and the folded pleat singularities arising from *generic* Engel integral curves have a unique diffeomorphism class, see §6. We call it the *generic folded pleat*.

The generic appearance of Shcherbak surfaces is observed in the classification of lightlike developables in Minkowski 3-space earlier in [6]. However the meaning of genericity of null curves in [6] is different from that of our paper.

In the context of the ordinary projective duality, the role of projective space and that of dual projective space are completely equal. Therefore the lists of singularities must be symmetric because of the symmetry on the underlying geometric structures. Compared with it, the contact-cone Legendre-null duality is naturally supposed to be asymmetric for the list of singularities on tangent surfaces, because of the asymmetry on the underlying geometric structures, see

Proposition 5.1. As we see clearly in Theorem 1.1, the list of singularities is never symmetric in fact.

The singularities of tangent surfaces to null curves are regarded as singularities of "null surfaces" in the Lagrange-Grassmannian $\mathrm{LG}(V)$. A surface in $\mathrm{LG}(V)$ is called a *null surface*, if it is tangent to the null-cone C_L at any point L of the surface. Typical examples of null surfaces in the Lagrange-Grassmannian $\mathrm{LG}(V)$ are given by Schubert varieties $S_L = \{L' \in \mathrm{LG}(V) \mid L \cap L' \neq \{0\}\}$ $(L \in \mathrm{LG}(V))$ and tangent surfaces to null curves. (Schubert varieties are called *trains* in [19]). They are associated varieties to Legendre curves in $\mathrm{Gr}(1, V)$. In fact any null surface in $\mathrm{LG}(V)$ is locally a part of the associated variety to a Legendre curve in $\mathrm{Gr}(1, V)$, see Proposition 2.3.

The double fibration treated in this paper is a prototype of various constructions appeared in twistor theory, where one geometric structure is related to another geometric structure via a double fibration. In our case, one is the contact structure and another is the conformal (or cone) structure. Moreover tangent surfaces and associated varieties to Legendre curves and to null curves turn out to be important objects in the geometric study of differential equations. For instance, the contact space $P(V)$ (resp. the Engel space $\mathcal{F}$) is regarded as the compactification of 1-jet space $J^1(\mathbf{R}, \mathbf{R}) = \mathbf{R}^3$ (resp. $J^2(\mathbf{R}, \mathbf{R}) = \mathbf{R}^4$), and tangent surfaces to Legendre curves appear naturally in the study on certain type of third order ordinary differential equations. Further, $\mathrm{LG}(V)$ can be identified with the compactification of $J^0(\mathbf{R}^2, \mathbf{R}) = \mathbf{R}^3$ and tangent surfaces to null curves appear as the first order partial differential equations called eikonal equations. See [7] as a related work. Furthermore, if we regard $\mathrm{LG}(V)$ as the compactification of the space of second derivatives (the space of 2 by 2 symmetric matrices), then tangent surfaces to null curves appear as second order partial differential equations associated with Lagrange cone fields. The cuspidal edge singularities of tangent surfaces were appeared in E. Cartan's classical work (see [13]). Therefore it is an interesting open problem to study the differential equations corresponding to the complicated generic singularities of tangent varieties, which we have classified in this paper, beyond the Cartan's case.

In §2, we introduce the Lagrange flag manifold and explain the duality between the projective contact 3-space and the Lagrange-Grassmannian of a symplectic 4-space. Mainly we provide the descriptions for the oriented case. Those for the non-oriented case can be obtained easily by just taking coverings or by the exactly same manner. In §3, we provide the exact projective coordinates of the Lagrange flag manifold, the contact 3-sphere and the Lagrange-Grassmannian, which are suitable to obtain normal forms of tangent surfaces. In §4, we formulate the transversality theorem in our case and prove it. It is necessary to make the meaning of the "generic" Engel integral curves clear. In §5, we introduce the notion of types for curves in a space with a projective structure and give the codimension formula and the duality formula for the set of Engel integral jets which have given types under the projections. In §6, we determine the diffeomorphism class of "generic" folded pleats and finally we give the proof of the main theorem.

2. The contact-cone Legendre-null duality

We explain the contact-cone, or, Legendre-null duality via the Lagrange flag manifold.

Let (V^4, Ω) be a symplectic 4-dimensional real vector space with a symplectic form Ω. See [3] on the symplectic geometry. Consider the *oriented Lagrange flag manifold* $\widetilde{\mathcal{F}} = \widetilde{\mathcal{F}}^{\mathrm{Lag}}_{1,2}(V)$ which consists of pairs (ℓ, L) of oriented lines ℓ and oriented Lagrange planes L containing ℓ in V :

$$\widetilde{\mathcal{F}} = \{(\ell, L) \mid \ell \subset L \subset V,\ \dim(\ell) = 1,\ \dim(L) = 2,\ \Omega|_L = 0,\ \ell, L \text{ are oriented}\}.$$

Note that $\widetilde{\mathcal{F}} \cong \mathrm{U}(2) \cong S^1 \times S^3$ via any isomorphism $V^4 \cong \mathbf{C}^2$ with the standard Hermitian form, see [2], [9]. Note that $\widetilde{\mathcal{F}}$ covers $\mathcal{F}$ in degree 4.

There are natural projections $\pi_1 : \widetilde{\mathcal{F}} \to \widetilde{\mathrm{Gr}}(1, V) \cong \mathrm{U}(2)/\mathrm{U}(1) \cong S^3$ and $\pi_2 : \widetilde{\mathcal{F}} \to \widetilde{\mathrm{LG}}(V) \cong \mathrm{U}(2)/\mathrm{SO}(2) \cong S^1 \times S^2$. Here $\widetilde{\mathrm{Gr}}(1, V)$ is the Grassmannian of oriented lines through 0 in V, the double cover of the projective 3-space $P(V)$, and $\widetilde{\mathrm{LG}}(V)$ is the Grassmannian of oriented Lagrange planes through 0 in V, the double cover of $\mathrm{LG}(V)$.

A point $(\ell, L) \in \widetilde{\mathcal{F}}$ defines an oriented projective line $[L] \subset \widetilde{\mathrm{Gr}}(1, V)$ through $\ell \in \widetilde{\mathrm{Gr}}(1, V)$, as well as an oriented line $[[L]] = T_\ell[L] \cong L/\ell$ in the tangent space $T_\ell\widetilde{\mathrm{Gr}}(1, V)$.

The contact distribution $D \subset T\widetilde{\mathrm{Gr}}(1, V)$ at $\ell \in \widetilde{\mathrm{Gr}}(1, V)$ is obtained by

$$D_\ell = [[\ell^s]] \subset T_\ell\widetilde{\mathrm{Gr}}(1, V),$$

where $\ell^s = \{v \in V \mid \Omega(v, w) = 0 \text{ for any } w \in \ell\}$. For $(\ell, L) \in \widetilde{\mathcal{F}}$, we have $[[L]] \subset D_\ell$. The canonical (or tautological) sub-bundle $E \subset T\widetilde{\mathcal{F}}$ over $\widetilde{\mathcal{F}}$ is defined by

$$E_{(\ell,L)} = \{\boldsymbol{v} \in T_{(\ell,L)}\widetilde{\mathcal{F}} \mid \pi_{1*}\boldsymbol{v} \in [[L]]\}.$$

Then E is an Engel distribution over $\widetilde{\mathcal{F}}$. In fact $\widetilde{\mathcal{F}}$ is identified with the manifold of oriented tangent lines in D, and E is obtained as the prolongation of the contact structure on $\widetilde{\mathrm{Gr}}(1, V) \cong S^3$ ([5]). Moreover, we have $E_{(\ell,L)} = T_{(\ell,L)}\pi_1^{-1}(\ell) \oplus T_{(\ell,L)}\pi_2^{-1}(L)$.

The natural structure on $\widetilde{\mathrm{LG}}(V)$ is not given by a vector sub-bundle of $T\widetilde{\mathrm{LG}}(V)$ but by a cone-bundle $C \subset T\widetilde{\mathrm{LG}}(V)$ which is defined as follows: For each $L \in \widetilde{\mathrm{LG}}(V)$, we consider the Schubert variety

$$S_L = \{L' \in \widetilde{\mathrm{LG}}(V) \mid L' \cap L \neq \{0\}\} = \pi_2(\pi_1^{-1}(\pi_1(\pi_2^{-1}(L)))).$$

Then the cone $C_L \subset T_L\widetilde{\mathrm{LG}}(V)$ is defined as the tangent cone of S_L at L. We regard the flag manifold $\widetilde{\mathcal{F}}$ as the oriented projective bundle $\widetilde{P}D = (D - Z)/\mathbf{R}_{>0}$, where Z is the zero-section, for the contact structure $D \subset T\widetilde{\mathrm{Gr}}(1, V)$ as well as $\widetilde{P}C$, the set of oriented lines in C, for the cone structure $C \subset T\widetilde{\mathrm{LG}}(V)$.

Note that, for any $\ell \in \widetilde{\mathrm{Gr}}(1, V)$, $\pi_1(\pi_2^{-1}(\pi_2(\pi_1^{-1}(\ell)))) \subset \widetilde{\mathrm{Gr}}(1, V)$ is the projective plane which is associated to $\ell^s \subset V$ and its tangent cone coincides with the contact plane $D_\ell \subset T_\ell\widetilde{\mathrm{Gr}}(1, V)$. Moreover, note that for the Engel structure $E \subset T\widetilde{\mathcal{F}}$, we can write as

$$\begin{aligned} E_{(\ell,L)} &= T_{(\ell,L)}\pi_1^{-1}(\ell) \oplus T_{(\ell,L)}\pi_2^{-1}(L) \\ &= T_{(\ell,L)}(\pi_2^{-1}(\pi_2(\pi_1^{-1}(\ell)))) = T_{(\ell,L)}(\pi_1^{-1}(\pi_1(\pi_2^{-1}(L)))). \end{aligned}$$

Let $E^2 = E + [E, E]$ be the derived system from the Engel structure E. Then E^2 is a sub-bundle of $T\widetilde{\mathcal{F}}$ of rank 3 and $E^2 = \pi_{1*}^{-1}(D)$ ([5]). Moreover, we have the following lemma.

Lemma 2.1. *Let $v \in T_{(\ell,L)}\widetilde{\mathcal{F}}$ for $(\ell, L) \in \widetilde{\mathcal{F}}$. Then $v \in (E^2)_{(\ell,L)}$ if and only if $\pi_{2*}(v) \in (T_L[\ell])^\perp \subset T_L(\widetilde{\mathrm{LG}}(V))$. Here $(T_L[\ell])^\perp$ means the pseudo-orthogonal space to $T_L[\ell]$ (the tangent line at L of the null line $[\ell]$ determined by ℓ) for the conformal structure defined by the null-cone field C.*

The proof is given in §3 by using a local coordinate.

A C^∞ map $f : I \to (\widetilde{\mathcal{F}}, E)$ is called an *Engel integral curve* if $f_*(TI) \subset E(\subset T\widetilde{\mathcal{F}})$. A C^∞ map $g : I \to (\widetilde{\mathrm{Gr}}(1,V), D)$ is called a *Legendre curve* if $g_*(TI) \subset D(\subset T\widetilde{\mathrm{Gr}}(1,V))$. A C^∞ map $h : I \to (\widetilde{\mathrm{LG}}(V), C)$ is called a *null curve* if $h_*(TI) \subset C(\subset T\widetilde{\mathrm{LG}}(V))$.

Lemma 2.2. *For any Engel integral curve f, the projection $\pi_1 \circ f$ by π_1 is a Legendre curve and the projection $\pi_2 \circ f$ by π_2 is a null curve.*

Proof: We have $(\pi_1 \circ f)_*(T_tI) \subseteq (\pi_1)_*(E_{f(t)}) \subseteq D_{(\pi_1\circ f)(t)}$ and

$$\begin{aligned}(\pi_2 \circ f)_*(T_tI) &\subseteq (\pi_2)_*(E_{f(t)})\\ &= (\pi_2)_*(T_{f(t)}(\pi_2^{-1}(\pi_2(\pi_1^{-1}(\pi_1(f(t))))))\\ &= T_{(\pi_2\circ f)(t)}(\pi_2(\pi_1^{-1}(\pi_1(f(t))))) \subseteq C_{(\pi_2\circ f)(t)}.\end{aligned}$$

□

There are natural classes of embedded Legendre curves in $\widetilde{\mathrm{Gr}}(1,V)$ and embedded null curves in $\widetilde{\mathrm{LG}}(V)$. Let $L \in \widetilde{\mathrm{LG}}(V)$. Then $\pi_1(\pi_2^{-1}(L))$ is a Legendre curve and is called a *Legendre straight line* or simply a *Legendre line* associated to L. Let $\ell \in \widetilde{\mathrm{Gr}}(1,V)$. Then $\pi_2(\pi_1^{-1}(\ell))$ is a null curve and is called a *null straight line* or simply a *null line* associated to ℓ. In fact we will give a projective structure on $\widetilde{\mathcal{F}}$ (resp. $\widetilde{\mathrm{Gr}}(1,V), \widetilde{\mathrm{LG}}(V)$) in §3. Then Legendre lines $\pi_1(\pi_2^{-1}(L))$, null lines $\pi_2(\pi_1^{-1}(\ell))$ and also $\pi_2^{-1}(L), \pi_1^{-1}(\ell)$ are actually "lines" for those projective structures. Same definitions are applied to $\mathcal{F}$ (non-oriented case).

Proposition 2.3. *Let $N \subset \widetilde{\mathcal{F}}$ be a null surface (see Introduction for the definition). Then locally (in a neighbourhood of any point of N), N is contained in the associated variety to a Legendre curve in $\widetilde{\mathrm{Gr}}(1,V)$.*

Proof: Let N be a null surface in $\widetilde{\mathrm{LG}}(V)$. Then N has the null direction field $C_L \cap T_LN$ $(L \in N)$ which lifts to a surface $\widetilde{N} \subset \widetilde{\mathcal{F}}$ via π_2. (If the direction field $C_L \cap T_LN$ $(L \in N)$ is not orientable, then $\pi_2|_{\widetilde{N}} : \widetilde{N} \to N$ is a double covering.) Then $\widetilde{N}$ is an integral surface to $E^2 = \pi_{1*}^{-1}(D)$. In fact, for any $\widetilde{x} \in \widetilde{N}$,

$$\pi_{2*}(T_{\widetilde{x}}\widetilde{N}) = T_{\pi_2(\widetilde{x})}N$$

is pseudo-orthogonal to the null direction $C_{\pi_2(\widetilde{x})} \cap T_{\pi_2(\widetilde{x})}N$, which is equal to $\pi_{2*}((E^2)_{\widetilde{x}})$ by Lemma 2.1. Since $\mathrm{Ker}(\pi_{2*}) \subset E$, we have

$$T_{\widetilde{x}}\widetilde{N} \subset (E^2)_{\widetilde{x}} + E_{\widetilde{x}} = (E^2)_{\widetilde{x}}.$$

Now $\pi_1|\widetilde{N}$ is an integral mapping to the contact distribution D. Therefore the rank of $\pi_1|\widetilde{N}$ is at most one, while at least one, hence the rank is identically one. Thus $\widetilde{N}$ is foliated by π_1-fibers. Take the local image γ of $\widetilde{N}$ by π_1. Then γ is a Legendre curve and, locally, $\widetilde{N} \subset \pi_1^{-1}(\gamma)$. Therefore we have $N \subset \pi_2(\pi_1^{-1}(\gamma))$, the associated variety to γ. □

Remark 2.4. The associated variety in $\widetilde{\mathrm{Gr}}(1,V)$ to a null curve in $\widetilde{\mathrm{LG}}(V)$ is characterized, in its smooth part, as a surface foliated by Legendre straight lines which lifts to an integral surface to the 3-dimensional cone field

$$\pi_{2*}^{-1}(C) = \{v \in T\widetilde{\mathcal{F}} \mid \pi_{2*}(v) \in C\}$$

on $\widetilde{\mathcal{F}}$. Typical examples are provided by tangent surfaces to Legendre curves and the "great spheres" given by

$$\widetilde{\mathrm{Gr}}(1,\ell^s) = \pi_1(\pi_2^{-1}(\pi_2(\pi_1^{-1}(\ell)))) \subset \widetilde{\mathrm{Gr}}(1,V),\ (\ell \in \widetilde{\mathrm{Gr}}(1,V)).$$

Thus naturally we are treating integral surfaces to derived systems $E^2 = \pi_{1*}^{-1}(D)$ or to $\pi_{2*}^{-1}(C)$ on the flag manifold $\widetilde{\mathcal{F}}$.

3. Projective Engel structure on the flag manifolds

We introduce systems of coordinates of $\widetilde{\mathcal{F}}$ which define the projective Engel structure on $\widetilde{\mathcal{F}}$. For projective structures, see [17] for instance.

Recall that (V, Ω) is a symplectic vector space of dimension 4 and $\widetilde{\mathcal{F}}$ the oriented Lagrange flag manifold consisting pairs (ℓ, L) of oriented lines ℓ and oriented Lagrangian planes $L \supset \ell$. Fix $(\ell_0, L_0) \in \widetilde{\mathcal{F}}$. Then the flag

$$\ell_0 \subset L_0 \subset {\ell_0}^s \subset V$$

is induced. Recall that ${\ell_0}^s$ denotes the skew-orthogonal space to ℓ_0 for Ω. We give a chart on the open subset

$$U = \{(\ell, L) \in \widetilde{\mathcal{F}} \mid L \cap L_0 = \{0\}, \ell \cap {\ell_0}^s = \{0\}\}.$$

Fix $(\ell_1, L_1) \in U$. Then we have the canonical direct sum decomposition

$$V = \ell_1 \oplus (L_1 \cap {\ell_0}^s) \oplus \ell_0 \oplus (L_0 \cap {\ell_1}^s).$$

Take a basis (e_1, e_2, f_1, f_2) of V such that

$$e_1 \in \ell_1,\ e_2 \in L_1 \cap {\ell_0}^s,\ f_1 \in \ell_0,\ f_2 \in L_0 \cap {\ell_1}^s,$$

and that

$$\Omega(e_1, f_1) = 1,\ \Omega(e_2, f_1) = 0,\ \Omega(e_1, f_2) = 0,\ \Omega(e_2, f_2) = 1.$$

Let $(\ell, L) \in U$. Since $L \cap L_0 = \{0\}$, there exists the unique basis g_1, g_2 of L of form

$$g_1 = e_1 + xf_1 + yf_2, \quad g_2 = e_2 + yf_1 + zf_2,$$

where $x, y, z \in \mathbf{R}$. Since $\ell \cap {\ell_0}^s = \{0\}$, there exists the unique basis h of ℓ of form $h = g_1 + \lambda g_2$, where $\lambda \in \mathbf{R}$. Then

$$h = e_1 + \lambda e_2 + (x + \lambda y)f_1 + (y + \lambda z)f_2.$$

Thus we have a chart $(\lambda, x, y, z) : U \to \mathbf{R}^4$. Then the Engel structure E on $\widetilde{\mathcal{F}}$ is described as follows: A curve $f(t) = (\lambda(t), x(t), y(t), z(t))$ in U through $(\ell, L) = (\lambda, x, y, z)$ at $t = 0$ defines a vector in $E_{(\ell,L)}$ if and only if the velocity vector $\frac{df}{dt}|_{t=0} \in L$. The condition is equivalent to that

$$\begin{pmatrix} 0 \\ \lambda' \\ (x+\lambda y)' \\ (y+\lambda z)' \end{pmatrix} = p \begin{pmatrix} 1 \\ 0 \\ x \\ y \end{pmatrix} + q \begin{pmatrix} 0 \\ 1 \\ y \\ z \end{pmatrix}$$

for some $p, q \in \mathbf{R}$. Then $p = 0$ and $q = \lambda'$. Therefore we have

$$(x + \lambda y)' = \lambda' y, \quad (y + \lambda z)' = \lambda' z.$$

Thus E is defined by the differential system

$$dx + \lambda dy = 0, \quad dy + \lambda dz = 0,$$

via the chart (λ, x, y, z).

In particular, any Engel integral curve $f(t) = (\lambda(t), x(t), y(t), z(t))$ in $U \subset \widetilde{\mathcal{F}}$ is given by

$$x(t) = \int \lambda(t)^2 z'(t)dt, \quad y(t) = -\int \lambda(t) z'(t)dt.$$

from any C^∞ functions $\lambda(t), z(t)$.

We describe the Engel structure E and its square E^2 in terms of frames (vector fields) on the coordinate neighbourhood $U \subset \widetilde{\mathcal{F}}$ introduced above. Moreover we give the coordinate expression of the cone field C and the conformal indefinite metric uniquely defined from C on the coordinate neighbourhood of $\widetilde{\mathrm{LG}}(V)$. Then we show the geometric interpretation of $\pi_{2*}(E_{(\ell,L)})$ and $\pi_{2*}(E^2_{(\ell,L)})$ for any $(\ell, L) \in U$, which shows Lemma 2.1.

Let $(\ell, L) \in \widetilde{\mathcal{F}}$. We fix an $(\ell_0, L_0) \in \widetilde{\mathcal{F}}$ satisfying $L \cap L_0 = \{0\}, \ell \cap \ell_0^s = \{0\}$, and, setting $(\ell_1, L_1) = (\ell, L)$, we consider the local coordinate system (λ, x, y, z) of $\widetilde{\mathcal{F}}$ centered at (ℓ, L) as above.

The local frame of $E \subset T\widetilde{\mathcal{F}}$ is given by

$$\Lambda = \frac{\partial}{\partial \lambda}, \quad X = \lambda^2 \frac{\partial}{\partial x} - \lambda \frac{\partial}{\partial y} + \frac{\partial}{\partial z},$$

under the coordinates λ, x, y, z. The square E^2 is spanned by Λ, X and $Y = 2\lambda \frac{\partial}{\partial x} - \frac{\partial}{\partial y}$. In terms of co-frame, E^2 is given by the 1-form

$$dx + 2\lambda dy + \lambda^2 dz = (dx + \lambda dy) + \lambda(dy + \lambda dz) = 0.$$

The condition that a Lagrange plane $\langle e_1 + x f_1 + y f_2, e_2 + y f_1 + z f_2 \rangle_{\mathbf{R}}$ belongs to the Schubert variety S_L is given by

$$S_L : xz - y^2 = 0.$$

The tangent cone C_L at L of S_L is given by

$$C_L : \xi\zeta - \eta^2 = 0.$$

for $v = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \zeta \frac{\partial}{\partial z} \in T_L \widetilde{\mathrm{LG}}(V)$. Using symmetric tensors, C is defined by $dxdz - (dy)^2 = 0$. The induced conformal metric g on $\widetilde{\mathrm{LG}}(V)$ is given by the bilinear form on $T_L\widetilde{\mathrm{LG}}(V)$ defined by

$$g(v_1, v_2) = \frac{1}{2}(\xi_1\zeta_2 + \xi_2\zeta_1) - \eta_1\eta_2,$$

for $v_i = \xi_i \frac{\partial}{\partial x} + \eta_i \frac{\partial}{\partial y} + \zeta_i \frac{\partial}{\partial z}$ $(i = 1, 2)$.

The projection $\pi_{2*}(E^2_{(\ell,L)})$ of the derived $E^2_{(\ell,L)}$ is given by the plane

$$\xi + 2\lambda\eta + \lambda^2\zeta = 0,$$

in $T_L\widetilde{\mathrm{LG}}(V)$ for a fixed λ. Regarding λ as a parameter, we have one-parameter family of planes, which envelopes C_L. The projection $\pi_{2*}(E_{(\ell,L)}) = T_L[\ell]$ of $E_{(\ell,L)}$ itself is given by the line

$$\xi + \lambda\eta = 0, \quad \eta + \lambda\zeta = 0,$$

while the null-vector $v = \pi_{2*}X = \lambda^2 \frac{\partial}{\partial x} - \lambda \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$ provides the direction of the null straight line $[\ell]$.

Proof of Lemma 2.1: Note that $\pi_{2*}((E^2)_{(\ell,L)})$ is spanned by v and $u = 2\lambda \frac{\partial}{\partial x} - \frac{\partial}{\partial y}$ and that $g(v, u) = 0$. Therefore, by counting the dimension, we see that $\pi_{2*}((E^2)_{(\ell,L)})$ coincides with the pseudo-orthogonal space to $T_L[\ell]$. $\square$

Remark 3.1. The contact structure D on $\widetilde{\mathrm{Gr}}(1, V)$ is expressed by

$$D : \ d\mu = \nu d\lambda - \lambda d\nu,$$

under the local coordinates $\lambda, \mu = x + \lambda y$ and $\nu = y + \lambda z$ of $\widetilde{\mathrm{Gr}}(1, V)$.

Remark 3.2. Let $J^1(\mathbf{R},\mathbf{R})$ be the projective contact manifold with coordinates t, u, p and the contact structure $D_0 : du - pdt = 0$. Then the projective contact structure (S^3, D) is not isomorphic to $(J^1(\mathbf{R},\mathbf{R}), D_0)$ as projective contact structures locally. In fact, there are just two Legendre straight lines through a given point (t_0, u_0, p_0) in $J^1(\mathbf{R},\mathbf{R})$:

$$(s + x_0, p_0 s + y_0, p_0), \quad (x_0, y_0, s + p_0),$$

up to right equivalence, s being the parameter of straight line. On the other hand, on (S^3, D), there exists a Legendre straight line though any point with any direction of D in S^3.

Let $J^2(\mathbf{R},\mathbf{R})$ be the projective Engel manifold with coordinates t, u, p, q and the Engel structure $E_0 : du - pdt = 0, dp - qdt = 0$. Then the projective Engel structure $(\widetilde{\mathcal{F}}, E)$ with coordinates λ, x, y, z is not isomorphic to $(J^2(\mathbf{R},\mathbf{R}), E_0)$ as projective Engel structures locally. In fact, there is just one Engel integral straight line $(t_0, u_0, p_0, s + q_0)$ through a given point (t_0, u_0, p_0, q_0) in $J^2(\mathbf{R},\mathbf{R})$, if $q_0 \neq 0$. On the other hand, on $(\widetilde{\mathcal{F}}, E)$, there exist exactly two Engel straight lines, the π_1-fiber and the π_2-fiber, through any given point of $\widetilde{\mathcal{F}}$.

For the projective coordinate neighbourhood U, there exists the explicit diffeomorphism between $(U, E|_U)$ and $(J^2(\mathbf{R},\mathbf{R}), E_0)$ of Engel manifolds, given by

$$\begin{aligned}
(\lambda, x, y, z) &\mapsto (t, u, p, q) = (\lambda,\ \tfrac{1}{2}\{(x + \lambda y) + \lambda(y + \lambda z)\},\ y + \lambda z,\ z),\\
(t, u, p, q) &\mapsto (\lambda, x, y, z) = (t,\ 2u - 2tp + t^2 q,\ p - tq,\ q),
\end{aligned}$$

the "Engel-Legendre transformation".

Remark 3.3. For any $p_0 = (\lambda_0, x_0, y_0, z_0) \in \mathbf{R}^4$, there is a linear Engel transformation $T : (\mathbf{R}^4, p_0) \to (\mathbf{R}^4, 0)$ defined by

$$T(\lambda, x, y, z) = (\lambda - \lambda_0, x + 2\lambda_0 y + \lambda_0^2 z - x_0 - 2\lambda_0 y_0 - \lambda_0^2 z_0, y + \lambda_0 z - y_0 - \lambda_0 z_0, z - z_0).$$

4. Engel integral jet space and transversality

We introduce the jet-spaces of Engel integral curves.

Let I be an open interval. In the jet-space $J^r(I, \widetilde{\mathcal{F}})$ we consider the Engel integral jet-space:

$$J^r_E(I, \widetilde{\mathcal{F}}) = \{j^r f(t_0) \mid t_0 \in I,\ f : (\mathbf{R}, t_0) \to \widetilde{\mathcal{F}} \text{ is Engel integral}\}.$$

Lemma 4.1. *$J^r_E(I, \widetilde{\mathcal{F}})$ is a subbundle of $J^r(I, \widetilde{\mathcal{F}})$ for the projection $\Pi : J^r(I, \widetilde{\mathcal{F}}) \to I \times \widetilde{\mathcal{F}}$ of codimension $2r$.*

Proof: By Remark 3.3, it is sufficient to show that

$$J^r_E(1, 4) = \{j^r f(0) \mid f : (\mathbf{R}, 0) \to (\mathbf{R}^4, 0) \text{ is Engel integral}\}$$

is a submanifold of $J^r(1, 4)$ of codimension $2r$. To show it, define the mapping $\Phi : J^r(1, 4) \to \Lambda_1^{r-1} \times \Lambda_1^{r-1} \cong \mathbf{R}^{2r}$ by

$$\Phi(j^r(\lambda, x, y, z)(0)) = (j^{r-1}(dx + \lambda dy)(0), j^{r-1}(dy + \lambda dz)(0)).$$

Here Λ_1^{r-1} denotes the $(r-1)$-jet space of 1-forms on $(\mathbf{R}, 0)$. Then Φ is a submersion. In fact any deformation $(B_1(t, s), B_2(t, s))$ with parameter s of the pair $(b_1(t), b_2(t)) = (x'(t) + \lambda(t)y'(t), y'(t) + \lambda(t)z'(t))$ is lifted to $(\lambda(t), x(t, s), y(t, s), z(t))$ by setting

$$x(t, s) = \int \{\lambda(t)^2 z'(t) + B_1(t, s)\}dt, \quad y(t, s) = \int \{-\lambda(t)z'(t) + B_2(t, s)\}dt,$$

$x(0,s) = 0, y(0,s) = 0$. Therefore $\Phi^{-1}(0) = J^r_E(1,4)$ is a submanifold of $J^r(1,4)$ of codimension $2r$. □

Proposition 4.2. (Engel transversality theorem on open intervals) *Let $Q \subset J^r_E(I, \widetilde{\mathcal{F}})$ be a submanifold. Then any Engel integral curve $f : I \to \widetilde{\mathcal{F}}$ is approximated in C^∞-topology by an Engel integral curve $f' : I \to \widetilde{\mathcal{F}}$ for which $j^r f' : I \to J^r_E(I, \widetilde{\mathcal{F}})$ is transverse to Q.*

Proof: For any open sub-interval $V \subset I$ and for any coordinate neighbourhood $U \subset \widetilde{\mathcal{F}}$ introduced in §3, we define a diffeomorphism

$$\varphi = \varphi_{(V,U)} : J^r_E(V,U) \to V \times U \times J^r(1,2)$$

by $\varphi(j^r f(t_0)) = (t_0, f(t_0), j^r((\lambda, z) \circ T \circ f(t + t_0))(0))$, using the linear Engel transformation T with $T(f(t_0)) = 0$.

Now let $f : I \to \widetilde{\mathcal{F}}$ be an Engel integral curve. Suppose, as a special case, $f(I)$ is in some projective coordinate neighbourhood U introduced in §3. Then, by the ordinary transversality theorem, (λ, z)-components of f are perturbed so that, for a perturbed f', $\varphi \circ j^r f'$ is transverse to $\varphi(Q \cap J^r_E(I,U)) \subset I \times U \times J^r(1,2)$. Then $j^r f'$ is transverse to Q.

In general case, there is a strictly increasing sequence $\{t_i\}_{i \in \mathbf{Z}}$ of points in I such that $f([t_i, t_{i+1}])$ is contained in some projective coordinate neighbourhood U_i. We set $K_i = [t_i, t_{i+1}]$ and take open intervals $W_i \supset K_i$ such that also $f(W_i) \subset U_i$ and that $W_i \cap W_j = \emptyset$ if $|i - j| \geq 2$.

First we perturb f over W_0 into an Engel integral curve $f_0 : W_0 \to \widetilde{\mathcal{F}}$ such that $j^r f_0$ is transverse to Q over W_0. In fact, similarly as in the special case, by the ordinary transversality theorem via $\varphi = \varphi_{(W_0,U_0)}$, (λ, z)-components of $f|_{W_0}$ are perturbed so that, for the perturbed f_0, $\varphi \circ j^r f_0$ is transverse to $\varphi(Q \cap J^r_E(W_0, U_0)) \subset W_0 \times U_0 \times J^r(1,2)$. Then $j^r f_0$ is transverse to Q over W_0.

Second we perturb f over $W_0 \cup W_1$ into an Engel integral curve $f_1 : W_0 \cup W_1 \to \widetilde{\mathcal{F}}$ such that $j^r f_1$ is transverse to Q and $f_1|_{K_0} = f_0|_{K_0}$. This is achieved, under the coordinates on U_1, by

$$x(t) = \int_{t_1}^{t} \lambda(t)^2 z'(t) dt + x(t_1), \quad y(t) = -\int_{t_1}^{t} \lambda(t) z'(t) dt + y(t_1),$$

perturbing $\lambda(t), z(t)$ over W_1 just outside of $K_0 \cap W_1$ and setting $f_1(t_1) = f_0(t_1)$.

Third we perturb f over $W_0 \cup W_1 \cup W_2$ into an Engel integral curve $f_2 : W_0 \cup W_1 \cup W_2 \to \widetilde{\mathcal{F}}$ such that $j^r f_2$ is transverse to Q and $f_2|_{K_0 \cup K_1} = f_1|_{K_0 \cup K_1}$. Thus, by continuing this procedure, we have a perturbation $f' : \cup_{0 \leq i} W_i \to \widetilde{\mathcal{F}}$ of f such that $j^r f'$ is transverse to Q.

Finally we perturb f backward to an Engel integral curve $f'' : I = \cup_{i \in \mathbf{Z}} W_i \to \widetilde{\mathcal{F}}$ such that $j^r f''$ is transverse to Q, by perturbing $\lambda(t), z(t)$ and using, for $i \leq 0$,

$$x(t) = -\int_{t}^{t_i} \lambda(t)^2 z'(t) dt + x(t_i), \quad y(t) = \int_{t}^{t_i} \lambda(t) z'(t) dt + y(t_i).$$

Note that, on any compact $K \subset \cup_{i \in \mathbf{Z}} W_i$, the perturbation is achieved just by a finite number of steps. Therefore we can take transversal perturbations of f to Q which are arbitrarily small in C^∞ topology. □

Remark 4.3. The transversality theorem does not hold for Engel integral curves by perturbations with *compact supports* (or for Engel integral curves on closed interval by perturbations with *fixed ends*). In fact it is known that the abnormal (singular) curves for Engel structures are rigid and have no essential perturbations with fixed ends ([5]).

5. Codimension formula, duality, and generic Engel integral curves

For the local coordinates (λ, x, y, z) of $\widetilde{\mathcal{F}}$ introduced in §3, the double fibration

$$\widetilde{\mathrm{Gr}}(1, V) \xleftarrow{\pi_1} \widetilde{\mathcal{F}} \xrightarrow{\pi_2} \widetilde{\mathrm{LG}}(V)$$

are given by

$$\pi_1(\lambda, x, y, z) = (\lambda, x + \lambda y, y + \lambda z), \quad \pi_2(\lambda, x, y, z) = (x, y, z).$$

Let $c : I \to M^3$ be a C^∞ curve in a 3-space with a projective structure. We say that c is *of finite type* at $t = t_0 \in I$ if there exists a local projective coordinates (x_1, x_2, x_3) of M centred at $c(t_0)$ such that

$$x_1 \circ c(t) = t^{a_1} + O(t^{a_1+1}),\ x_2 \circ c(t) = t^{a_2} + O(t^{a_2+1}),\ x_3 \circ c(t) = t^{a_3} + O(t^{a_3+1}),$$

for some increasing sequence of positive integers $1 \le a_1 < a_2 < a_3$. Then (a_1, a_2, a_3) is uniquely determined from the projective class of the germ of c at $t = t_0$, and we say that c is *of type* (a_1, a_2, a_3) at $t = t_0$. If we consider the Wronski matrices

$$W_i(t) = \begin{pmatrix} x_1'(t) & x_1''(t) & \cdots & x_1^{(i)}(t) \\ x_2'(t) & x_2''(t) & \cdots & x_2^{(i)}(t) \\ x_3'(t) & x_3''(t) & \cdots & x_3^{(i)}(t) \end{pmatrix}, \quad i = 1, 2, \ldots,$$

then we have

$$a_1 = \min\{i \mid \text{rank } W_i(t_0) = 1\}, \quad a_2 = \min\{i \mid \text{rank } W_i(t_0) = 2\},$$
$$a_3 = \min\{i \mid \text{rank } W_i(t_0) = 3\}.$$

Let $\mathbf{A} = (a_1, a_2, a_3)$ and $\mathbf{B} = (b_1, b_2, b_3)$ be increasing sequences of positive integers, $1 \le a_1 < a_2 < a_3$, $1 \le b_1 < b_2 < b_3$. We set, for a sufficiently large r,

$$\begin{aligned} \Sigma_{\pi_1,\mathbf{A}} &= \{j^r f(t_0) \in J^r_E(I, \widetilde{\mathcal{F}}) \mid \pi_1 \circ f : I \to \widetilde{\mathrm{Gr}}(1, V) \text{ is of type } \mathbf{A}\}, \\ \Sigma_{\pi_2,\mathbf{B}} &= \{j^r f(t_0) \in J^r_E(I, \widetilde{\mathcal{F}}) \mid \pi_2 \circ f : I \to \widetilde{\mathrm{LG}}(V) \text{ is of type } \mathbf{B}\}. \end{aligned}$$

Proposition 5.1.

(1) Codimension formula for π_1:
We have, for $r \ge a_3$, $\Sigma_{\pi_1,\mathbf{A}} \neq \emptyset$ if and only if $a_3 = a_1 + a_2$. Then we have $\Sigma_{\pi_1,\mathbf{A}}$ is a submanifold of $J^r_E(I, \widetilde{\mathcal{F}})$ of codimension $a_2 - 2$.

(2) Codimension formula for π_2:
We have, for $r \ge b_3$, $\Sigma_{\pi_2,\mathbf{B}} \neq \emptyset$ if and only if $b_3 = 2b_2 - b_1$. Then we have $\Sigma_{\pi_2,\mathbf{B}}$ is a submanifold of $J^r_E(I, \widetilde{\mathcal{F}})$ of codimension $b_2 - 2$.

(3) The duality formula:
Let $f : I \to \widetilde{\mathcal{F}}$ be an Engel integral curve of finite type. Then the type $\mathbf{A}$ of $\pi_1 \circ f$ and the type $\mathbf{B}$ of $\pi_2 \circ f$ are related by

$$(b_1, b_2, b_3) = (a_2 - a_1, a_2, a_3), \quad (a_1, a_2, a_3) = (b_2 - b_1, b_2, b_3).$$

Proof: Let $f : (\mathbf{R}, 0) \to (\mathbf{R}^4, 0)$, $f(t) = (\lambda(t), x(t), y(t), z(t))$ be an Engel integral curve-germ. If $\lambda(t)$ or $z(t)$ is infinitely flat at $t = 0$, then both $x(t)$ and $y(t)$ are infinitely flat at $t = 0$ by the Engel condition. Then both $\pi_1 \circ f$ and $\pi_2 \circ f$ are not of finite type. Now let $u = \mathrm{ord}\lambda(t) < \infty, v = \mathrm{ord} z(t) < \infty$. Here $\mathrm{ord}\varphi(t)$ denotes the order of a function $\varphi(t)$ at $t = 0$. Then

$$\mathrm{ord} y(t) = \mathrm{ord}\lambda(t) + \mathrm{ord} z(t) = u + v \quad \mathrm{ord} x(t) = \mathrm{ord}\lambda(t) + \mathrm{ord} y(t) = 2u + v.$$

Since

$$(x+\lambda y)'(t) = y(t)\lambda'(t), \quad (y+\lambda z)'(t) = z(t)\lambda'(t),$$

we have

$$\mathrm{ord}(x(t)+\lambda(t)y(t)) = 2u+v, \quad \mathrm{ord}(y(t)+\lambda(t)z(t)) = u+v.$$

Suppose the type of $\pi_1 \circ f$ at $t=0$ is $\mathbf{A} = (a_1, a_2, a_3)$. Then we have $a_1 = u, a_2 = u+v, a_3 = 2u+v$. This is realized for some $u, v \geq 1$ if and only if $a_3 = a_1 + a_2$. Then the codimension of $\Sigma_{\pi_1,\mathbf{A}}$ is given by $u+v-2 = a_2 - 2$. This shows (1). On the other hand, suppose the type of $\pi_2 \circ f$ at $t=0$ is $\mathbf{B} = (b_1, b_2, b_3)$. Then $b_1 = v, b_2 = v+u, b_3 = v+2u$. This is realized for some $v, u \geq 1$ if and only if $b_3 = 2b_2 - b_1$. Then the codimension of $\Sigma_{\pi_2,\mathbf{B}}$ is given by $u+v-2 = b_2-2$. This shows (2). Moreover $b_1 = v = a_2 - a_1, b_2 = v+u = a_2, b_3 = v+2u = a_3$. Thus we see (3). □

Remark 5.2. The conditions $\mathrm{ord}\lambda(t) = u$ and $\mathrm{ord}z(t) = v$ give a submanifold of $J^r(1,2)$ of codimension $(u-1)+(v-1) = u+v-2$.

Proposition 5.3. *For any generic Engel integral curve $f : I \to \widetilde{\mathcal{F}}$ and for any point $t_0 \in I$, the type of $\pi_1 \circ f : I \to \widetilde{\mathrm{Gr}}(1,V)$ is $(1,2,3), (1,3,4)$ or $(2,3,5)$. Moreover the type of $\pi_2 \circ f : I \to \widetilde{\mathrm{LG}}(V)$ is $(1,2,3), (2,3,4)$ or $(1,3,5)$ correspondingly.*

Proof: For a sufficiently large r, we set

$$\Sigma = \overline{(\cup_{a_2 \geq 4}\ \Sigma_{\pi_1,\mathbf{A}}) \cup (\cup_{b_2\geq 4}\ \Sigma_{\pi_2,\mathbf{B}})} \subset J^r_E(I, \widetilde{\mathcal{F}}).$$

Then Σ is fibered over $I \times \widetilde{\mathcal{F}}$ by a real algebraic set in $J^r_E(1,4)$ of codimension ≥ 2. In fact the fiber of Σ is defined in $J^r(1,4)$ by the vanishing of some minors of the Wronski matrices for the curves $\pi_1 \circ f$ and $\pi_2 \circ f$. Note that Σ contains curve-jets $j^r f(t_0)$ for which the type of $\pi_1 \circ f$ or $\pi_2 \circ f$ at t_0 is not determined by the jet $j^r f(t_0)$. However they form a subset of codimension $\geq r-2$, which does not affect the codimension calculus.

Let $\mathcal{R}$ be the set of $f \in C^\infty_E(I, \widetilde{\mathcal{F}})$ such that $j^r f : I \to \widetilde{\mathcal{F}}$ is transversal to all $\Sigma_{\pi_1,\mathbf{A}}$ with $a_2 \leq 3$ and to all $\Sigma_{\pi_2,\mathbf{B}}$ with $b_2 \leq 3$ and moreover to (all strata of a stratification of) Σ. By Proposition 4.2, $\mathcal{R}$ is dense in $C^\infty_E(I, \widetilde{\mathcal{F}})$ for the C^∞-topology. By Proposition 5.1, $f \in \mathcal{R}$ is equivalent to that $j^r f$ is transversal to $\Sigma_{\pi_1,\mathbf{A}}$ with $a_2 = 3$ and $\Sigma_{\pi_2,\mathbf{B}}$ with $b_2 = 3$ at isolated points in I and that $j^r f(I) \cap \Sigma = \emptyset$. Therefore $\mathcal{R}$ is residual in $C^\infty_E(I, \widetilde{\mathcal{F}})$ for the C^∞-topology. Let $f \in \mathcal{R}$ and $t_0 \in I$. Let $\mathbf{A}$ be the type of $\pi_1 \circ f$ and $\mathbf{B}$ the type of $\pi_2 \circ f$. Then we have $a_2 \leq 3$. So $a_1 \leq 2$. If $a_1 = 1$, then $(a_1, a_2, a_3) = (1,2,3)$ or $(1,3,4)$ by Propositions 5.1 (1). If $a_1 = 2$, then $(a_1, a_2, a_3) = (2,3,5)$. Then the rest is proved by the formula $(b_1, b_2, b_3) = (a_2 - a_1, a_2, a_3)$ of Proposition 5.1 (3). □

In particular we have:

Corollary 5.4. *Generic Engel integral curves are immersions. In fact, for any generic Engel integral curve $f : I \to \widetilde{\mathcal{F}}$, and for any point $t_0 \in I$, either $\pi_1 \circ f$ or $\pi_2 \circ f$ is an immersion.*

Remark 5.5. Under the ordinary projective duality of space curves, the duality formula between a space curve and its projective dual curve is given by

$$(b_1, b_2, b_3) = (a_3 - a_2, a_3 - a_1, a_3), \quad (a_1, a_2, a_3) = (b_3 - b_2, b_3 - b_1, b_3),$$

see [18]. Then the cuspidal edges, Mond surfaces and folded pleats are self-dual, the swallowtails are dual to the folded umbrellas (the cuspidal cross-caps), and the Shcherbak surfaces are dual to the butterflies as singularities of tangent surfaces, see the survey article [11].

6. Normal forms on singularities of tangent surfaces

First we show the procedure to obtain normal forms of tangent surfaces to space curves in $P(V)$ or in $\widetilde{\mathrm{Gr}}(1,V)$. Then we give the differential classification of tangent surfaces to curves of type $(2,3,5)$ and prove all statements in Theorem 1.1.

Let $f=(\lambda,x,y,z):(\mathbf{R},0)\to(\mathbf{R}^4,0)$ be an Engel integral curve satisfying $dx+\lambda dy=0$ and $dy+\lambda dz=0$ (see §3).

For example, let $\lambda=t, z=t$. Then

$$y=-\tfrac{1}{2}t^2,\ x=\tfrac{1}{3}t^3,\ x+\lambda y=-\tfrac{1}{6}t^3,\ y+\lambda z=\tfrac{1}{2}t^2.$$

Then

$$\pi_1(f(t))=(\lambda,x+\lambda y,y+\lambda z)=(t,-\tfrac{1}{6}t^3,\tfrac{1}{2}t^2),$$
$$\pi_2(f(t))=(x,y,z)=(\tfrac{1}{3}t^3,-\tfrac{1}{2}t^2,t).$$

The tangent surface in $\widetilde{\mathrm{Gr}}(1,V)$ is parametrized by

$$\begin{pmatrix} t \\ -\frac{1}{6}t^3 \\ \frac{1}{2}t^2 \end{pmatrix} + s\begin{pmatrix} 1 \\ -\frac{1}{2}t^2 \\ t \end{pmatrix} = \begin{pmatrix} t+s \\ -\frac{1}{6}t^3-\frac{1}{2}st^2 \\ \frac{1}{2}t^2+st \end{pmatrix}.$$

Introducing a new parameter $X=t+s$, we have the parametrization

$$(X,\ -\tfrac{1}{2}t^2+Xt,\ \tfrac{1}{3}t^3-\tfrac{1}{2}Xt^2)$$

of the tangent surface in $\widetilde{\mathrm{Gr}}(1,V)$ to a curve of type $(1,2,3)$.

In general, the velocity vector of $\pi_1\circ f$ is given by

$$(\lambda',(x+\lambda y)',(y+\lambda z)')=\lambda'(1,y,z).$$

Therefore the parametrization of the tangent surface to $\pi_1\circ f$ is diffeomorphic to

$$(\lambda,\ y+\lambda z,\ x+\lambda y)+s(1,z,y)=(\lambda+s,\ y+(\lambda+s)z,\ x+(\lambda+s)y).$$

If we set $X=\lambda+s$, then we have the parametrization

$$(X,t)\mapsto(X,\ y(t)+Xz(t),\ x(t)+Xy(t)).$$

Now for a given Engel integral curve, suppose that $\mathrm{ord}\lambda(t)=2$ and $\mathrm{ord}z(t)=1$ at $t=0$. Then after a re-parametrization of t, we may suppose that $\lambda=\frac{1}{2}t^2$ and $z=at+\frac{b}{2}t^2+O(t^3)$ for some $a,b\in\mathbf{R}, a\neq 0$. Then we have the parametrization

$$x=\tfrac{a}{20}t^5+\tfrac{b}{24}t^6+O(t^7),\quad y=-\tfrac{a}{6}t^3-\tfrac{b}{8}t^4+O(t^5).$$

The parametrization of $\pi_1\circ f$ is given by

$$(\tfrac{1}{2}t^2,\ \ \tfrac{a}{3}t^3+\tfrac{b}{8}t^4+O(t^5),\ \ -\tfrac{a}{30}t^5-\tfrac{b}{48}t^6+O(t^7)).$$

We obtain the parametrization $F:(\mathbf{R}^2,0)\to(\mathbf{R}^3,0)$, $(X,t)\mapsto(\lambda,\mu,\nu)$ of the tangent surface in $\widetilde{\mathrm{Gr}}(1,V)$ to the curve $\pi_1\circ f$ given in a form

$$\big(X,\ \ a(-\tfrac{1}{6}t^3+Xt)+b(-\tfrac{1}{8}t^4+\tfrac{1}{2}Xt^2)+\psi(X,t),$$
$$a(\tfrac{1}{20}t^5-\tfrac{1}{6}Xt^3)+b(\tfrac{1}{24}t^6-\tfrac{1}{8}Xt^4)+\rho(X,t)\big).$$

Here we give the natural weights $w(X)=2, w(t)=1$. Then the order of ψ (resp. ρ) is higher than 4 (resp. 6) with respect to the given weights. Moreover, we have that $\frac{\partial\psi}{\partial t}$ is a multiple of $-\frac{1}{2}t^2+X$ by some function, and that $\frac{\partial\rho}{\partial t}=-\frac{t^2}{2}\frac{\partial\psi}{\partial t}$.

Proposition 6.1. *If $b \neq 0$, then F is locally diffeomorphic to*
$$(X,\quad -\tfrac{1}{6}t^3 + Xt - \tfrac{1}{8}t^4 + \tfrac{1}{2}Xt^2,\quad \tfrac{1}{20}t^5 - \tfrac{1}{6}Xt^3 + \tfrac{1}{24}t^6 - \tfrac{1}{8}Xt^4).$$
If $b = 0$, then F is locally diffeomorphic to
$$(X,\quad -\tfrac{1}{6}t^3 + Xt,\quad \tfrac{1}{20}t^5 - \tfrac{1}{6}Xt^3).$$
The two map-germs are not diffeomorphic to each other.

Remark 6.2. Let $\Sigma'_{\pi_1,(2,3,5)}$ be set of jets $j^r f(t_0)$ such that $\pi_1 \circ f$ is of type $(2,3,5)$ at t_0 and $z \circ f(t + t_0) = f(t_0) + at + O(t^3)$, for some $a \neq 0$, in a projective chart introduced in §3. Then $\Sigma'_{\pi_1,(2,3,5)}$ has codimension ≥ 2. Therefore the Engel integral transversality theorem (Proposition 4.2) yields that generically we have $b \neq 0$.

Remark 6.3. The proof of Proposition 6.1 can be applied also to the differential classification of singularities for tangent developables to curves of type $(2,3,5)$: There exists exactly two diffeomorphism classes as in Proposition 6.1.

To show Proposition 6.1, we follow the standard infinitesimal method of singularity theory ([14], [4], [20]). Because we treat a specialized class of map-germs, we need also an additional algebraic method as in [10]. The proof goes similarly to that for the classification, for instance, in case $(1,3,5)$ of [10]. However, in our case $(2,3,5)$, the terms next to the leading terms turn to be regarded as well, and the proof must be modified accordingly.

Introducing an additional parameter s, we set
$$\begin{aligned} F_s(X,t) = {}^T\big(X,\quad & a(-\tfrac{1}{6}t^3 + Xt) + b(-\tfrac{1}{8}t^4 + \tfrac{1}{2}Xt^2) + s\psi, \\ & a(\tfrac{1}{20}t^5 - \tfrac{1}{6}Xt^3) + b(\tfrac{1}{24}t^6 - \tfrac{1}{8}Xt^4) + s\rho\big)\,. \end{aligned}$$
We are going to show that this family is trivialized under diffeomorphism equivalence (i.e. C^∞-right-left equivalence). Strictly we see that it is trivialized, preserving the tangent lines to the base point.

Proposition 6.4. *For any $s_0 \in \mathbf{R}$, we can solve the infinitesimal equation*
$$\begin{pmatrix} 0 \\ \psi \\ \rho \end{pmatrix} = \left(A\frac{\partial}{\partial X} + Bt\frac{\partial}{\partial t}\right)F_s - \begin{pmatrix} C(F_s) \\ D(F_s) \\ E(F_s) \end{pmatrix}$$
near $(0,0,s_0)$, for some C^∞ functions $A = A(X,t,s), B = B(X,t,s)$ and $C(\lambda,\mu,\nu), D(\lambda,\mu,\nu), E(\lambda,\mu,\nu)$ satisfying that
$$A(0,0,s) = 0, C(0,0,0) = D(0,0,0) = E(0,0,0) = 0.$$

Proof: The form of the vector field $A\frac{\partial}{\partial X} + Bt\frac{\partial}{\partial t}$ is essential to apply our algebraic method.

By the first row of the equation, necessarily we have $A = C(F_s)$.

We set $U = U(X,t) = a(-\frac{1}{6}t^3 + Xt) + b(-\frac{1}{8}t^4 + \frac{1}{2}Xt^2)$. First we solve the equation of second row:

$$\psi = (C(F_s))\frac{\partial(U + s\psi)}{\partial X} + Bt\,\frac{\partial(U + s\psi)}{\partial t} - D(F_s). \tag{1}$$

Lemma 6.5. *The equation (1) is solved for some $B(X,t,s), C(\lambda,\mu,\nu), D(\lambda,\mu,\nu)$ with the condition $C(0,0,0) = 0, D(0,0,0) = 0$.*

To show Lemma 6.5, we define, additionally, the map-germ

$$G : (\mathbf{R}^3, (0,0,s_0)) \to (\mathbf{R}^3, (0,0,s_0))$$

by $G(X,t,s) = (X, U(X,t) + s\psi(X,t), s)$, and we denote by $\mathcal{E}_{X,t,s}$ (resp. $\mathcal{E}_{\lambda,\mu,s}$) the algebra of function-germ $(\mathbf{R}^3, (0,0,s_0)) \to \mathbf{R}$ on the source (resp. target) of G and by $\mathfrak{m}_{X,t,s}$ (resp. $\mathfrak{m}_{\lambda,\mu,s}$) its maximal ideal. Moreover we set, for $\ell = 0,1,2,\dots$,

$$\mathfrak{m}^{(\ell)}_{X,t,s} = \{h \in \mathcal{E}_{X,t,s} \mid \mathrm{ord}(h) \geq \ell\},$$

with respect to the weights $w(t) = w(s) = 1, w(X) = 2$. Note that $\psi \in \mathfrak{m}^{(5)}_{X,t,s}$.

We define the $\mathcal{E}_{\lambda,\mu,s}$-submodule, for $r = 0,1,2,\dots$,

$$M^{(r)} := G^*\mathfrak{m}_{\lambda,\mu,s} + \frac{\partial(U+s\psi)}{\partial X} G^*\mathfrak{m}_{\lambda,\mu,s} + t\frac{\partial(U+s\psi)}{\partial t}\mathfrak{m}^{(r)}_{X,t,s}$$

of $\mathcal{E}_{X,t,s}$ via $G^* : \mathcal{E}_{\lambda,\mu,s} \to \mathcal{E}_{X,t,s}$.

Lemma 6.6. *If $\ell \geq 5$, then $\mathfrak{m}^{(\ell)}_{X,t,s} \subset M^{(\ell-3)}$.*

Proof: In fact, using the initial part of U, we obtain that, if $\ell \geq 5$, then

$$\mathfrak{m}^{(\ell)}_{X,t,s} \subset M^{(\ell-3)} + \mathfrak{m}^{(\ell+1)}_{X,t,s}.$$

For example, in the case $\ell = 5$, we have $t^5 + 2Xt^3 \equiv 0, Xt^3 + 2X^2t \equiv 0, -\frac{1}{6}Xt^3 + X^2t \equiv 0$ modulo $M^{(2)} + \mathfrak{m}^{(6)}_{X,t,s}$, which implies $t^5 \equiv Xt^3 \equiv X^2t \equiv 0$.

Note that G is a finite map-germ, namely that $\mathcal{E}_{X,t,s}$ is a finite $\mathcal{E}_{\lambda,\mu,s}$-module via G^*. Then, for any ℓ and for a sufficiently large N, we have

$$\mathfrak{m}^{(N)}_{X,t,s} \subset G^*\mathfrak{m}_{\lambda,\mu,s} \cdot \mathfrak{m}^{(\ell)}_{X,t,s}.$$

Therefore we have

$$\mathfrak{m}^{(\ell)}_{X,t,s} \subset M^{(\ell-3)} + G^*\mathfrak{m}_{\lambda,\mu,s} \cdot \mathfrak{m}^{(\ell)}_{X,t,s}.$$

Since $\mathfrak{m}^{(\ell)}_{X,t,s}$ is a finite $\mathcal{E}_{\lambda,\mu,s}$-module via G^*, we have $\mathfrak{m}^{(\ell)}_{X,t,s} \subset M^{(\ell-3)}$ by Nakayama's lemma. □

Proof of Lemma 6.5: Since $\psi \in \mathfrak{m}^{(5)}_{X,t,s}$, Lemma 6.6 implies Lemma 6.5. □

Since we can solve the infinitesimal equation for the first and second rows in Proposition 6.4, we have a diffeomorphism germ $\sigma : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ of form $\sigma(X,t) = (\sigma_1(X,t), t\sigma_2(X,t))$ and a diffeomorphism germ $\tau : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ such that $\tau \circ (X, U + \psi) \circ \sigma^{-1} = (X, U)$. This construction is needed just to guarantee the properties of the following algebraic objects.

As in [10], we set, for $k = 0,1,2,\dots$,

$$\begin{aligned}\mathcal{H}_k \quad &:= \quad \left\{ h \in t^k\mathcal{E}_{X,t,s} \;\middle|\; \frac{\partial h}{\partial t} \in t^k\frac{\partial U}{\partial t}\mathcal{E}_{X,t,s} \right\} \\ &= \quad \left\{ h \in t^k\mathcal{E}_{X,t,s} \;\middle|\; \frac{\partial h}{\partial t} \in t^k(-\frac{1}{2}t^2 + X)\mathcal{E}_{X,t,s} \right\}.\end{aligned}$$

Note that $G^*\mathcal{E}_{\lambda,\mu,s} \in \mathcal{H}_0$ and $\rho \in \mathcal{H}_4$. Also note that $\frac{\partial U}{\partial t} = (a+bt)(-\frac{1}{2}t^2 + X)$.

We have a sequence of $G^*\mathcal{E}_{\lambda,\mu,s}$-modules:

$$\mathcal{E}_{X,t,s} \supset \mathcal{H}_0 \supset \mathcal{H}_1 \supset \cdots \supset \mathcal{H}_k \supset \cdots.$$

Then we have

Lemma 6.7. (Lemma 2.3 of [10]) *Let a vector field of form $\xi = A\frac{\partial}{\partial X} + Bt\frac{\partial}{\partial t}$ satisfy $A \in \mathcal{H}_0$ and $\xi(U + s\psi) \in \mathcal{H}_0$. Then, for any $k \geq 0$ and for any $h \in \mathcal{H}_k$, we have $\xi h \in \mathcal{H}_k$.*

We set $U_k = \int_0^t \frac{t^k}{k!}\frac{\partial U}{\partial t}dt$. Then $U_k \in \mathcal{H}_k$. Note that the leading term of the third component of F_s is equal to $-U_2$. Moreover $U_k(0,t)$ is of order $k+3$. Then we have

Lemma 6.8. (1) *$\mathcal{H}_k$ is generated as $G^*\mathcal{E}_{\lambda,\mu,s}$-module by $U_k, U_{k+1}, U_{k+2}, U_{k+3}$.*
(2) *$\mathcal{H}_k$ is generated as $G^*\mathcal{E}_{\lambda,\mu,s}$-module by those elements generating the vector space $t^{k+3}\mathcal{E}_t/t^{k+7}\mathcal{E}_t$ over $\mathbf{R}$ via the inclusion $i : (\mathbf{R}, 0) \to (\mathbf{R}^3, (0, 0, s_0))$, $i(t) = (0, t, s_0)$.*

Proof: The proof is achieved by applying the method used in the proof of Lemma 2.4 of [10], to the case $m = 3$ and $U = a(-\frac{1}{6}t^3 + Xt) + b(-\frac{1}{8}t^4 + \frac{1}{2}Xt^2)$. Note that we need more generators in (1) than in the case treated in [10], since U may not be taken to be quasi-homogeneous in our case. □

To complete the proof of Proposition 6.1, we modify the vector field $\xi = A\frac{\partial}{\partial X} + Bt\frac{\partial}{\partial t}$ and $D(F_s)$ such that also the equation of third row holds, for some $E(F_s)$. Since $\rho, \xi(-U_2 + s\rho) \in \mathcal{H}_4$, it is sufficient, for the solvability of our infinitesimal equation, to find C_1, B_1, D_1, E_1 satisfying that $\xi = C_1(G)\frac{\partial}{\partial X} + B_1 t\frac{\partial}{\partial t}$ satisfies that $\xi(U + s\psi) - D_1(F_s) = 0$, and that $h = \xi(-U_2 + s\rho) - E_1(F_s)$ is of order $7, 8, 9, 10$ when restricted to $\{X = 0, s = s_0\}$, by Lemma 6.8.

Note that $h_{10} := (-U_2 + s\rho)^2 \in \mathcal{H}_4$ is a composite function of F_s and that $h_{10}(0, t, s_0)$ is of order 10. In fact any order ≥ 10 is realizable by a composite function of F_s which belongs to $\mathcal{H}_4$. Then we take it as $E_1(F_s)$ and set $C_1(G) = 0, B_1 = 0, D_1(G) = 0$.

To produce elements of order $7, 8, 9$, we use Lemma 6.6 again.

We choose $c_1, c_2, c_3, c_4 \in \mathbf{R}$ with $c_2 \neq 0$ such that the terms of weight 5 of

$$\theta_7 = c_1 X^2\frac{\partial(U + s\psi)}{\partial X} + (c_2 t^3 + c_3 Xt)\frac{\partial(U + s\psi)}{\partial t} + c_4 X(U + s\psi).$$

vanish and so that θ_7 belongs to $\mathfrak{m}_{X,t,s}^{(6)} \subset M^{(3)}$. Then we have, for some C_2, B_2, D_2,

$$\theta_7 = C_2(G)\frac{\partial(U + s\psi)}{\partial X} + B_2 t\frac{\partial(U + s\psi)}{\partial t} + D_2(G)$$

with $C_2(0) = D_2(0) = 0$ and $B_2 \in \mathfrak{m}_{X,t,s}^{(3)}$. We set

$$\xi_2 = (c_1 X^2 - C_2(G))\frac{\partial}{\partial X} + (c_2 t^3 + c_3 Xt - B_2 t)\frac{\partial}{\partial t},$$

and set $h_7 := \xi_2(-U_2 + s\rho)$. Then we see that $\xi_2(U + s\psi) - D_2'(F_s) = 0$ where $D_2'(F_s) = c_4 X(U + s\psi) - D_2(G)$. Moreover we have $h_7 \in \mathcal{H}_4$. By comparing orders, we see also that $h_7(0, t, s_0)$ is of order 7.

Similarly choose $c_1, c_2, c_3, c_4 \in \mathbf{R}$ with $c_2 \neq 0$ such that

$$\theta_8 = c_1 X(U + s\psi)\frac{\partial(U + s\psi)}{\partial X} + (c_2 t^4 + c_3 Xt^2)\frac{\partial(U + s\psi)}{\partial t} + c_4(U + s\psi)^2$$

belongs to $\mathfrak{m}_{X,t,s}^{(7)} \subset M^{(4)}$. Then we have, for some C_3, B_3, D_3,

$$\theta_8 = C_3(G)\frac{\partial(U + s\psi)}{\partial X} + B_3 t\frac{\partial(U + s\psi)}{\partial t} + D_3(G)$$

with $C_3(0) = D_3(0) = 0$ and $B_3 \in \mathfrak{m}_{X,t,s}^{(4)}$. We set

$$\xi_3 = (c_1 X(U + s\psi) - C_3(G))\frac{\partial}{\partial X} + (c_2 t^4 + c_3 Xt^2 - B_3 t)\frac{\partial}{\partial t},$$

and set $h_8 := \xi_3(-U_2+s\rho)$. Then $\xi_3(U+s\psi)-D'_3(F_s)=0$ where $D'_3(F_s)=c_4(U+s\psi)^2-D_3(G)$. Moreover we have $h_8 \in \mathcal{H}_4$ and $h_8(0,t,s_0)$ is of order 8.

Lastly choose c_1, c_2, c_3, c_4, c_5 with $c_1 \neq 0$ such that

$$\theta_9 = (c_1t^5 + c_2Xt^3 + c_3X^2t)\frac{\partial(U+s\psi)}{\partial t} + c_4X(-U_2+s\rho) + c_5X^2(U+s\psi).$$

belongs to $\mathfrak{m}^{(8)}_{X,t,s} \subset M^{(5)}$. Then we can write as

$$\theta_9 = C_4(G)\frac{\partial(U+s\psi)}{\partial X} + B_4t\frac{\partial(U+s\psi)}{\partial t} + D_4(G),$$

for some C_4, D_4, B_4 with $C_4(0)=D_4(0)=0$ and $B_4 \in \mathfrak{m}^{(5)}_{X,t,s}$. Then we set $\xi_4 = (c_1t^5+c_2Xt^3+c_3X^2t-B_4)t\frac{\partial}{\partial t}$ and $h_9 := \xi_4(-U_2+s\rho)$. Then we see that $\xi_4(U+s\psi)-D'_4(F_s)=0$, where $D'_4(G) = c_4X(-U_2+s\rho)+c_5X^2(U+s\psi)-D_4(G)$. Moreover we have $h_9 \in \mathcal{H}_4$ and $h_9(0,t,s_0)$ is of order 9.

By Lemma 6.8, we see that $h_7(0,t,s_0), h_8(0,t,s_0), h_9(0,t,s_0), h_{10}(0,t,s_0)$ from a basis of $t^7\mathcal{E}_t/t^{11}\mathcal{E}_t$ and therefore $1, h_7, h_8, h_9, h_{10}$ generate $\mathcal{H}_4$ as $G^*\mathcal{E}_{\lambda,\mu,s}$-module. Hence we have

$$\begin{aligned}\rho - \xi(-U_2+s\rho) = {} & A_1(G) + (A_2(G)\xi_2 + A_3(G)\xi_3 + A_4(G)\xi_4)(-U_2+s\rho) \\ & + A_5(G)(-U_2+s\rho)^2,\end{aligned}$$

for some A_1, A_2, A_3, A_4, A_5. We set $\widetilde{\xi} = \xi + A_2(G)\xi_2 + A_3(G)\xi_3 + A_4(G)\xi_4$, then we have

$$\rho = \widetilde{\xi}(-U_2+s\rho) + A_1(G) + A_5(G)(-U_2+s\rho)^2,$$

while

$$\psi = \widetilde{\xi}(U+s\psi) - (D(F_s) + A_2(G)D'_2(F_s) + A_3(G)D'_3(F_s) + A_4(G)D'_4(F_s)).$$

Thus we have solved the infinitesimal equation as required. This complete the proof of Proposition 6.4. □

Proof of Theorem 6.1: By Proposition 6.4, F_s is trivialized under the diffeomorphism equivalence. Hence we have that $F = F_1$ is diffeomorphic to $F_0 = F_{a,b}$ namely to

$$\left(X,\quad a(-\tfrac{1}{6}t^3 + Xt) + b(-\tfrac{1}{8}t^4 + \tfrac{1}{2}Xt^2),\quad a(\tfrac{1}{20}t^5 - \tfrac{1}{6}Xt^3) + b(\tfrac{1}{24}t^6 - \tfrac{1}{8}Xt^4)\right).$$

Then we easily see that $F_{a,b}$ is diffeomorphic to $F_{1,1}$ if $b \neq 0$ and to $F_{1,0}$ if $b=0$, by a linear change of coordinates.

Finally $F_{1,1}$ and $F_{1,0}$ are not diffeomorphic. In fact, for $F_{1,0}$, we see that the infinitesimal equation

$$\begin{pmatrix} 0 \\ -\frac{1}{8}t^4 + \frac{1}{2}Xt^2 \\ \frac{1}{24}t^6 - \frac{1}{8}Xt^4 \end{pmatrix} = \left(A\frac{\partial}{\partial X} + B\frac{\partial}{\partial t}\right)\begin{pmatrix} X \\ -\frac{1}{6}t^3 + Xt \\ \frac{1}{20}t^5 - \frac{1}{6}Xt^3 \end{pmatrix} - \begin{pmatrix} C(F) \\ D(F) \\ E(F) \end{pmatrix}$$

has no solution. This complete the proof of Proposition 6.1. □

Proof of Theorem 1.1: We combine Proposition 5.3 and the known results on singularities of tangent surfaces (tangent developables) ([10], [11], [12]). It was proved that the tangent surface to a curve of type $(1,2,3)$ (resp. $(1,3,4)$, $(2,3,4)$, $(1,3,5)$) is locally diffeomorphic to the *cuspidal edge* (resp. *Mond surface*, *swallowtail*, *Shcherbak surface*) respectively (Theorem 1 of [10]). Moreover it is known that the local differential types (resp. the local topological type) of the tangent surface to a curve of type $(2,3,5)$ are not unique (resp. is unique) ([10],[11]). Then by above Proposition 6.1 and Remark 6.2, generically the local differential type is unique and diffeomorphic to the *generic folded pleat*. Thus we complete the proof of Theorem 1.1. □

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