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SOME NOTES ON THE LOCAL TOPOLOGY OF A DEFORMATION OF A FUNCTION-GERM WITH A ONE-DIMENSIONAL CRITICAL SET

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Abstract. The Brasselet number of a function f with nonisolated singularities describes numerically the topological information of its generalized Milnor fibre. In this work, we consider two function-germs f and g on a complex analytic space X such that f has a stratified isolated singularity at the origin and g has a stratified one-dimensional critical set. We use the Brasselet number to study the local topology of a deformation of g defined by adding a large power of f. As an application of this study, we present a new proof of the Lê-Yomdin formula for the Brasselet number.

Introduction

The Milnor number, defined in [14], is a very useful invariant associated to a complex function f with a stratified isolated singularity defined over an open neighborhood of the origin in $\mathbb{C}^N$. It gives numerical information about the local topology of the hypersurface $V(f)$ and computes the Euler characteristic of the Milnor fibre of f at the origin.

In the case where the function-germ has nonisolated singularity at the origin, the Milnor number is not well defined, but the Milnor fibre is, which led many authors ([19],[8],[3], [7], [13]) to study an extension for this number in more general settings. For example, if we consider a function with a one-dimensional critical set defined over an open subset of $\mathbb{C}^n$ and a generic linear form l over $\mathbb{C}^n$, Yomdin gave an algebraic proof (Theorem 3.2), in [19], of a relation between the Euler characteristic of the Milnor fibre of f and the Euler characteristic of the Milnor fibre of $f + l^N, N \gg 1$ and $N \in \mathbb{N}$, using properties of algebraic sets with one-dimensional critical locus. In [8], Lê proved (Theorem 2.2.2) this same relation in a more geometric approach and with a way to obtain the Milnor fibre of f by attaching a certain number of n-cells to the Milnor fibre of $f|_{\{l=0\}}$.

In [13], Massey worked with a function f with critical locus of higher dimension defined over a nonsingular space and defined the Lê numbers and cycles, which provides a way to numerically describe the Milnor fibre of this function with nonisolated singularity. Massey compared (Theorem II.4.5), using appropriate coordinates, the Lê numbers of f and $f + l^N$, where l is a generic linear form over $\mathbb{C}^n$ and $N \in \mathbb{N}$ is sufficiently large, obtaining a Lê-Yomdin type relation between these numbers. He also gave (Theorem II.3.3) a handle decomposition of the Milnor fibre of f, where the number of attached cells is a certain Lê number. Massey extended the concept of Lê numbers to the case of functions with nonisolated singularities defined over complex analytic spaces, introducing the Lê-Vogel cycles, and proved the Lê-Yomdin-Vogel formulas: the generalization of the Lê-Yomdin formulas in this more general sense.

The Brasselet number, defined by Dutertre and Grulha in [7], also describes the local topological behavior of a function with nonisolated singularities defined over an arbitrarily singular analytic space: if $f : (X, 0) \to (\mathbb{C}, 0)$ is a function-germ and $\mathcal{V} = \{\{0\}, V_1, \ldots, V_q\}$ is a good

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stratification of X relative to f (see Definition 1.8), the Brasselet number $B_{f,X}(0)$ is defined by

$$B_{f,X}(0) = \sum_{i=1}^{q} \chi(V_i \cap f^{-1}(\delta) \cap B_\epsilon)\mathrm{Eu}_X(V_i),$$

where B_ϵ is a closed ball centered at the origin and with radius ϵ in some local embedding of X and χ denotes the usual Euler characteristic. In [7], the authors proved several formulas about the local topology of the generalized Milnor fibre of a function germ f using the Brasselet number, like the Lê-Greuel type formula (Theorem 4.2 in [7]): $B_{f,X}(0) - B_{f,X^g}(0) = (-1)^{\dim_{\mathbb{C}} X} n$, where n is the number of stratified Morse critical points of a Morsification of $g|_{X\cap f^{-1}(\delta)\cap B_\epsilon}$ on $V_q \cap f^{-1}(\delta) \cap B_\epsilon$. In [5], Dalbelo and Pereira provided formulas to compute the Brasselet number of a function defined over a toric variety and in [1], Ament, Nuño-Ballesteros, Oréfice-Okamoto and Tomazella computed the Brasselet number of a function-germ with a stratified isolated singularity at the origin and defined over an isolated determinantal variety (IDS) and the Brasselet number of finite functions defined over a reduced curve. More recently, in [4], Dalbelo and Hartmann calculated the Brasselet number of a function-germ defined over a toric variety using combinatorical properties of the Newton polygons. In the global study of the topology of a function germ, Dutertre and Grulha defined, in [6], the global Brasselet numbers and the Brasselet numbers at infinity. In that paper, the authors compared the global Brasselet numbers of a function-germ f with the global Euler obstruction of the fibres of f, defined by Seade, Tibăr and Verjovsky in [16]. They also related the number of critical points of a Morsification of a polynomial function f on an algebraic set X to the global Brasselet numbers and the Brasselet numbers at infinity of f. Therefore, the Brasselet number has been a useful tool in the study of the topology of function-germs and it will be the main object in this work.

We consider analytic function-germs $f, g : (X, 0) \to (\mathbb{C}, 0)$, a Whitney stratification $\mathcal{W}$ of X, suppose that f has a stratified isolated singularity at the origin and g has a one-dimensional stratified critical set.

Consider the good stratification of X induced by f,

$$\mathcal{V} = \{W_\lambda \setminus X^f, W_\lambda \cap X^f \setminus \{0\}, \{0\}, W_\lambda \in \mathcal{W}\}$$

and suppose that g is tractable at the origin with respect to $\mathcal{V}$ (see Definition 1.12). Consider a decomposition of the critical locus $\Sigma_{\mathcal{W}} g$ into branches b_j, $\Sigma_{\mathcal{W}} g = \bigcup_{\alpha=1}^{q} \Sigma g|_{W_\alpha} \cup \{0\} = b_1 \cup \ldots \cup b_r$, where $b_j \subseteq W_\alpha$, for some $\alpha \in \{1, \ldots, q\}$. Let δ be a regular value of f, $0 < |\delta| \ll 1$, and let us write, for each $j \in \{1, \ldots, r\}$, $f^{-1}(\delta) \cap b_j = \{x_{i_1}, \ldots, x_{i_{k(j)}}\}$. So, in this case, the local degree m_{f,b_j} of $f|_{b_j}$ is $k(j)$. Let ϵ be sufficiently small such that the local Euler obstruction of X^g is constant on $b_j \cap B_\epsilon$. In this case, we denote by $\mathrm{Eu}_{X^g}(b_j)$ the local Euler obstruction of X at a point of $b_j \cap B_\epsilon$ and by $B_{g,X\cap f^{-1}(\delta)}(b_j)$ the Brasselet number of $g|_{X\cap f^{-1}(\delta)}$ at a point of $b_j \cap B_\epsilon$. For a deformation of g, $\tilde{g} = g + f^N$, $N \gg 1$, we prove (Proposition 2.5)

$$B_{g,X^f}(0) = B_{\tilde{g},X^f}(0) = B_{f,X^{\tilde{g}}}(0).$$

and for $0 < |\delta| \ll \epsilon \ll 1$ (Proposition 2.9),

$$B_{f,X^g}(0) - B_{f,X^{\tilde{g}}}(0) = \textstyle\sum_{j=1}^{r} m_{f,b_j}(\mathrm{Eu}_{X^g}(b_j) - B_{g,X\cap f^{-1}(\delta)}(b_j)).$$

As an application of these results, we compare the Brasselet numbers $B_{g,X}(0)$ and $B_{\tilde{g},X}(0)$, and obtain (Theorem 3.4) a topological proof of the Lê-Yomdin formula for the Brasselet number,

$$B_{\tilde{g},X}(0) = B_{g,X}(0) + N \sum_{j=1}^{r} m_{f,b_j} \mathrm{Eu}_{f,X\cap \tilde{g}^{-1}(\alpha')}(b_j),$$

where $0 \ll |\alpha'| \ll 1$ is a regular value of $\tilde{g}$. This formula generalizes the Lê-Yomdin formula for the Euler characteristic of the Milnor fibre in the case of a function with a stratified isolated singularity. We note that an algebraic proof can be obtained using the description (see [3]) of the defect of a function-germ f in terms of the Euler characteristic of vanishing cycles and the Lê-Vogel numbers associated to f.

In [18], Tibăr provided a bouquet decomposition for the Milnor fibre of $\tilde{g}$ and related it with the Milnor fibre of g. As a consequence of this strong result, Tibăr gave a Lê-Yomdin formula to compare the Euler characteristics of these Milnor fibres. In the last section of this work, we suppose f is a generic linear form l in some local embedding of X. In that case, m_{l,b_j} does not depend on l which allows us to write m_{b_j} (in fact, m_{b_j} is the algebraic multiplicity of b_j). We apply our results to give an alternative proof for this Lê-Yomdin formula (see Proposition 4.1):

$$\chi(X \cap \tilde{g}^{-1}(\alpha') \cap B_\epsilon) = \chi(X \cap g^{-1}(\alpha) \cap B_\epsilon) \quad + \quad N \sum_{j=1}^{r} m_{b_j}\big(1 - \chi(F_j)\big),$$

where $F_j = X \cap g^{-1}(\alpha) \cap H_j \cap D_{x_t}$ is the local Milnor fibre of $g|_{\{l=\delta\}}$ at a point of the branch b_j, H_j denotes the generic hyperplane $l^{-1}(\delta)$ passing through $x_t \in \{l = \delta\} \cap b_j$, for

$$\{l = \delta\} \cap b_j = \{x_{i_1} \dots, x_{i_{k(j)}}\}$$

and $\chi(F_j)$ does not depend on the choice of $x_t \in \{l = \delta\} \cap b_j$.

1. Preliminaries

In this section, we introduce the definitions and results needed to develop this paper.

Let $(X, 0) \subset (\mathbb{C}^n, 0)$ be an equidimensional reduced complex analytic germ of dimension d in a open set $U \subset \mathbb{C}^n$. Consider a complex analytic Whitney stratification $\mathcal{V} = \{V_\lambda\}$ of U adapted to X such that $\{0\}$ is a stratum. We choose a small representative of $(X, 0)$, denoted by X, such that 0 belongs to the closure of all strata. We write $X = \cup_{i=0}^{q} V_i$, where $V_0 = \{0\}$ and V_q is an open and dense subset in X_{reg}, where X_{reg} is the regular part of X. We suppose that $V_0, V_1, \dots, V_{q-1}$ are connected. We write $d_i = dim(V_i),\ i \in \{1, \dots, q\}$. Note that $d_q = d$. Let $G(d, n)$ be the Grassmannian manifold, $x \in X_{reg}$ and consider the Gauss map $\phi : X_{reg} \to U \times G(d, n)$ given by $x \mapsto (x, T_x(X_{reg}))$.

Let us denote by $\tilde{X}$ the Nash modification of X, by $\nu : \tilde{X} \to X$ its corresponding analytic projection map and by $\tilde{T}$ the Nash bundle defined over $\tilde{X}$.

Consider the extension of the tautological bundle $\mathcal{T}$ over $U \times G(d, n)$. Since $\tilde{X} \subset U \times G(d, N)$, we consider $\tilde{T}$ the restriction of $\mathcal{T}$ to $\tilde{X}$, called the **Nash bundle**, and $\pi : \tilde{T} \to \tilde{X}$ the projection of this bundle. We introduce now the local Euler obstruction, a singular invariant defined by MacPherson and used as one of the main tools in his proof of the Deligne-Grothendieck conjecture about the existence and uniqueness of Chern classes for singular varities.

Let $\varphi : U \times G(d, N) \to U$ denote the natural projection over U. If $||z|| = \sqrt{z_1\overline{z_1} + \cdots + z_n\overline{z_n}}$, the 1-differential form $w = d||z||^2$ over $\mathbb{C}^n$ defines a section in $T^*\mathbb{C}^n$ and its pullback $\varphi^* w$ is a 1- form over $U \times G(d, n)$. Denote by $\tilde{w}$ the restriction of $\varphi^* w$ over $\tilde{X}$, which is a section of the dual bundle $\tilde{T}^*$.

Choose ϵ small enough for $\tilde{w}$ to be a nonzero section over $\nu^{-1}(z), 0 < ||z|| \leqslant \epsilon$, let B_ϵ be the closed ball with center at the origin with radius ϵ and denote by

$$\mathrm{Obs}(\tilde{T}^*, \tilde{w}) \in \mathbb{H}^{2d}(\nu^{-1}(B_\epsilon), \nu^{-1}(S_\epsilon), \mathbb{Z})$$

the obstruction for extending $\tilde{w}$ from $\nu^{-1}(S_\epsilon)$ to $\nu^{-1}(B_\epsilon)$ and $O_{\nu^{-1}(B_\epsilon),\nu^{-1}(S_\epsilon)}$ the fundamental class in $\mathbb{H}_{2d}(\nu^{-1}(B_\epsilon), \nu^{-1}(S_\epsilon), \mathbb{Z})$.

Definition 1.1. ([10], p. 425) The **local Euler obstruction** of X at 0, $\mathrm{Eu}_X(0)$, is given by the evaluation

$$\mathrm{Eu}_X(0) = \langle \mathrm{Obs}(\tilde{T}^*, \tilde{w}), O_{\nu^{-1}(B_\epsilon),\nu^{-1}(S_\epsilon)} \rangle$$

In [2], Brasselet, Lê and Seade gave an alternative way to compute this number, using linear functions.

Theorem 1.2. *([2], Th. 3.1) For each generic linear form l, there exists ϵ_0 such that for all ϵ with $0 < \epsilon < \epsilon_0$ and $\delta \neq 0$ sufficiently small, the Euler obstruction of $(X, 0)$ is equal to*

$$\mathrm{Eu}_X(0) = \sum_{i=1}^{q} \chi(V_i \cap B_\epsilon \cap l^{-1}(\delta)) \cdot \mathrm{Eu}_X(V_i),$$

where χ is the Euler characteristic, $\mathrm{Eu}_X(V_i)$ is the Euler obstruction of X at a point of V_i, $i = 1, \ldots, q$ and $0 < |\delta| \ll \epsilon \ll 1$.

The right side sum in the last formula is related to the local Euler obstruction of X at the origin when one replaces the linear function l with a function with a stratified isolated singularity at the origin defined over X. Let $f : X \to \mathbb{C}$ be a holomorphic function with a stratified isolated singularity at the origin given by the restriction of a holomorphic function $F : U \to \mathbb{C}$ and denote by $\overline{\nabla} F(x)$ the conjugate of the gradient vector field of F in $x \in U$.

For all $x \in X \setminus \{0\}$, the projection $\hat{\zeta}_i(x)$ of $\overline{\nabla} F(x)$ over $T_x(V_i(x))$ is nonzero, where $V_i(x)$ is the stratum containing x. Using this projection, in [3], Brasselet, Massey, Parameswaran and Seade construct a stratified vector field over X, denoted by $\overline{\nabla} f(x)$. Let $\tilde{\zeta}$ be the lifting of $\overline{\nabla} f(x)$ as a section of the Nash bundle $\tilde{T}$ over $\tilde{X}$, without singularity over $\nu^{-1}(X \cap S_\epsilon)$. Let $\mathcal{O}(\tilde{\zeta}) \in \mathbb{H}^{2n}(\nu^{-1}(X \cap B_\epsilon), \nu^{-1}(X \cap S_{.\epsilon}))$ be the obstruction cocycle for extending $\tilde{\zeta}$ as a nonzero section of $\tilde{T}$ inside $\nu^{-1}(X \cap B_\epsilon)$.

Definition 1.3. The **local Euler obstruction of the function** f, $\mathrm{Eu}_{f,X}(0)$, is the evaluation of $\mathcal{O}(\tilde{\zeta})$ on the fundamental class $[\nu^{-1}(X \cap B_\epsilon), \nu^{-1}(X \cap S_\epsilon)]$.

The next theorem compares the Euler obstruction of a space X with the Euler obstruction of function defined over X.

Theorem 1.4. *([3], Th. 3.1) For $0 < |\delta| \ll \epsilon \ll 1$,*

$$\mathrm{Eu}_{f,X}(0) = \mathrm{Eu}_X(0) - \sum_{i=1}^{q} \chi(V_i \cap B_\epsilon \cap f^{-1}(\delta)).\mathrm{Eu}_X(V_i)\cdot$$

The Euler obstruction of a function is closely related to Morse Theory. However, we first introduce how we may understand this theory in the stratified sense. Let p be a point in a stratum V_β of $\mathcal{V}$. A degenerate tangent plane of $\mathcal{V}$ at p is an element T of some Grassmanian manifold such that $T = \lim_{p_i \to p} T_{p_i} V_\alpha$, where $p_i \in V_\alpha$, $V_\alpha \neq V_\beta$.

Definition 1.5. Let $\tilde{f} : (X, x) \to (\mathbb{C}, 0)$ be an analytic function, given by the restriction of an analytic function $\tilde{F} : (U, x) \to (\mathbb{C}, 0)$. Then x is said to be a **generic point** of $\tilde{f}$ if the hyperplane $Ker(d_x \tilde{F})$ is transverse in $\mathbb{C}^n$ to all degenerate tangent planes of the Whitney stratification at x.

Notice that the condition for x to be a generic point of $\tilde{f}$ is independent of the local embedding of X or the lifting function of $\tilde{f}$. Now, let us see the definition of a Morsification of a function.

Definition 1.6. A function germ $\tilde{f} : (X, x) \to (\mathbb{C}, 0)$ for a point x in a stratum W_i of X is said to be **Morse stratified** if x is a generic point of $\tilde{f}$ and $\tilde{f}|_{W_i} : W_i \to \mathbb{C}$ has a complex Morse point in x in case $\dim W_i \geq 1$. Note that a Morse stratified point is an isolated stratified critical point.

A **stratified Morsification** of a germ of an analytic function $f : (X, 0) \to (\mathbb{C}, 0)$ is a deformation $\tilde{f}$ such that $\tilde{f}$ only has Morse stratified critical points, so that locally there are only finitely many of them.

In [17], Seade, Tibăr and Verjovsky proved that the Euler obstruction of a function f can be computed by the number of Morse critical points of a stratified Morsification of f.

Proposition 1.7. *([17], Prop. 2.3) Let $f : (X, 0) \to (\mathbb{C}, 0)$ be a germ of an analytic function with a stratified isolated singularity at the origin. Then,*

$$\mathrm{Eu}_{f,X}(0) = (-1)^d n_{reg},$$

where n_{reg} is the number of Morse points in X_{reg} in a stratified Morsification of f.

In [7], Dutertre and Grulha studied the topology of the fibre of a function $f : X \to \mathbb{C}$ with nonisolated singularities. The good properties of the Whitney stratifications were not enough for their approach since, as we will see, a "good behaviour" between the strata of the stratification and the fibres of f is needed.

In [11], Massey not only presents this suitable stratification, but also explains its use in the study of functions with nonisolated singularities from the Morse Theory perspective.

Let X be a reduced complex analytic space (not necessarily equidimensional) of dimension d in an open set $U \subseteq \mathbb{C}^n$, let $f : (X, 0) \to (\mathbb{C}, 0)$ be an analytic map, and let $V(f) = X^f = f^{-1}(0)$.

Definition 1.8. A **good stratification of X relative to f** is a stratification $\mathcal{V}$ of X which is adapted to $V(f)$ such that $\{V_\lambda \in \mathcal{V}, V_\lambda \nsubseteq V(f)\}$ is a Whitney stratification of $X \setminus V(f)$ and such that for any pair (V_λ, V_γ) such that $V_\lambda \nsubseteq V(f)$ and $V_\gamma \subseteq V(f)$, the (a_f)-Thom condition is satisfied, that is, if $p \in V_\gamma$ and $p_i \in V_\lambda$ are such that $p_i \to p$ and $T_{p_i} V(f|_{V_\lambda} - f|_{V_\lambda}(p_i))$ converges to some $\mathcal{T}$, then $T_p V_\gamma \subseteq \mathcal{T}$.

If $f : X \to \mathbb{C}$ has a stratified isolated critical point and $\mathcal{V}$ is a Whitney stratification of X, then

$$\{V_\lambda \setminus X^f, V_\lambda \cap X^f \setminus \{0\}, \{0\}, V_\lambda \in \mathcal{V}\} \tag{1}$$

is a good stratification of X relative to f, called the good stratification induced by f.

Definition 1.9. The **critical locus of f relative to $\mathcal{V}$**, $\Sigma_{\mathcal{V}} f$, is given by the union

$$\Sigma_{\mathcal{V}} f = \bigcup_{V_\lambda \in \mathcal{V}} \Sigma(f|_{V_\lambda}).$$

Definition 1.10. If $\mathcal{V} = \{V_\lambda\}$ is a stratification of X, the **symmetric relative polar variety of f and g with respect to $\mathcal{V}$**, $\tilde{\Gamma}_{f,g}(\mathcal{V})$, is the union $\cup_\lambda \tilde{\Gamma}_{f,g}(V_\lambda)$, where $\Gamma_{f,g}(V_\lambda)$ denotes the closure in X of the critical locus of $(f, g)|_{V_\lambda \setminus (X^f \cup X^g)}$, $X^f = X \cap \{f = 0\}$ and $X^g = X \cap \{g = 0\}$.

Definition 1.11. Let $\mathcal{V}$ be a good stratification of X relative to a function $f : (X, 0) \to (\mathbb{C}, 0)$. A function $g : (X, 0) \to (\mathbb{C}, 0)$ is **prepolar with respect to $\mathcal{V}$ at the origin** if the origin is a stratified isolated critical point, that is, 0 is an isolated point of $\Sigma_{\mathcal{V}} g$.

Definition 1.12. A function $g : (X, 0) \to (\mathbb{C}, 0)$ is **tractable at the origin with respect to a good stratification $\mathcal{V}$ of X relative to** $f : (X, 0) \to (\mathbb{C}, 0)$ if $dim_0\, \tilde{\Gamma}^1_{f,g}(\mathcal{V}) \leq 1$ and, for all strata $V_\alpha \subseteq X^f$, $g|_{V_\alpha}$ has no critical points in a neighbourhood of the origin except perhaps at the origin itself.

We present now the definition of the Brasselet number. Let $f : (X, 0) \to (\mathbb{C}, 0)$ be a complex analytic function germ and let $\mathcal{V}$ be a good stratification of X relative to f. We denote by $V_1, \ldots, V_q$ the strata of $\mathcal{V}$ that are not contained in $\{f = 0\}$ and we assume that $V_1, \ldots, V_{q-1}$ are connected and that V_q is an open and dense subset of $X_{reg} \setminus \{f = 0\}$. Note that V_q could be not connected.

Definition 1.13. Suppose that X is equidimensional. Let $\mathcal{V}$ be a good stratification of X relative to f. The **Brasselet number** of f at the origin, $B_{f,X}(0)$, is defined by

$$B_{f,X}(0) = \sum_{i=1}^{q} \chi(V_i \cap f^{-1}(\delta) \cap B_\epsilon) \cdot \mathrm{Eu}_X(V_i),$$

where $0 < |\delta| \ll \epsilon \ll 1$.

Remark: If V_q^i is a connected component of V_q, $\mathrm{Eu}_X(V_q^i) = 1$.

Notice that if f has a stratified isolated singularity at the origin, then

$$B_{f,X}(0) = \mathrm{Eu}_X(0) - \mathrm{Eu}_{f,X}(0);$$

see Theorem 1.4.

1.1. **On the Brasselet number of germs with a one-dimensional critical set.** Let $f, g : (X, 0) \to (\mathbb{C}, 0)$ be complex analytic function-germs such that f has a stratified isolated singularity at the origin. Let $\mathcal{W}$ be the Whitney stratification of X and $\mathcal{V}$ be the good stratification of X induced by f. Suppose that $\Sigma_{\mathcal{W}} g$ is one-dimensional and that $\Sigma_{\mathcal{W}} g \cap \{f = 0\} = \{0\}$. In this context, let us recall the following definition introduced by Dutertre and Grulha. A partial Morsification of $g : X \cap f^{-1}(\delta) \cap B_\epsilon \to \mathbb{C}$ is a function $\tilde{g} : X \cap f^{-1}(\delta) \cap B_\epsilon \to \mathbb{C}$ which is a local Morsification of all isolated critical points of g in $X \cap f^{-1}(\delta) \cap \{g \neq 0\} \cap B_\epsilon$ and which coincides with g outside a small neighbourhood of these critical points.

In [15], the author presents several properties about the topology of a function-germ g with a stratified one-dimensional critical set using Brasselet numbers. The approach of that work is appropriate to our goal here. Let us compile some results we will use.

We start with the stratifications needed to compute the Brasselet numbers we are interested in. If $\mathcal{V}^f$ denote the set of strata of $\mathcal{V}$ contained in $\{f = 0\}$, by the First Stratification Lemma (Lemma 3.1 of [15]),

$$\mathcal{V}' = \{V_i \setminus \Sigma_{\mathcal{W}} g, V_i \cap \Sigma_{\mathcal{W}} g, V_i \in \mathcal{V}\} \cup \mathcal{V}^f$$

is a good stratification of X relative to f, such that $\mathcal{V}'^{\{g=0\}}$ is a good stratification of X^g relative to $f|_{X^g}$, where

$$\mathcal{V}'^{\{g=0\}} = \Big\{V_i \cap \{g = 0\} \setminus \Sigma_{\mathcal{W}} g, V_i \cap \Sigma_{\mathcal{W}} g, V_i \in \mathcal{V}\Big\} \cup \mathcal{V}^f \cap \{g = 0\},$$

and $\mathcal{V}^f \cap \{g = 0\}$ denotes the collection of strata of type $V^f \cap \{g = 0\}$, with $V^f \in \mathcal{V}^f$. Also, by the Second Stratification Lemma (Lemma 4.1 of [15]), the refinement of $\mathcal{V}$,

$$\mathcal{V}'' = \{V_i \setminus \{g = 0\}, V_i \cap \{g = 0\} \setminus \Sigma_{\mathcal{W}} g, V_i \cap \Sigma_{\mathcal{W}} g, V_i \in \mathcal{V}\} \cup \{0\}.$$

Then $\mathcal{V}''$ is a good stratification of X relative to g and $\mathcal{V}'' \cap X^f$, denoted by $\mathcal{V}''^{\{f=0\}}$,

$$\mathcal{V}''^{\{f=0\}} = \{V_i \cap \{f = 0\} \setminus \{g = 0\}, V_i \cap \{f = 0\} \cap \{g = 0\} \setminus \Sigma_{\mathcal{W}} g, V_i \in \mathcal{V}^f\} \cup \{0\},$$

is a good stratification of X^f relative to $g|_{X^f}$. Moreover, f is prepolar at the origin with respect to $\mathcal{V}''$ relative to g.

We write the set $\Sigma_{\mathcal{W}} g$ as a union of branches b_j, $\Sigma_{\mathcal{W}} g = \bigcup_{\alpha=1}^{q} \Sigma g|_{W_\alpha} \cup \{0\} = b_1 \cup \ldots \cup b_r$, where $b_j \subseteq W_\alpha$, for some $\alpha \in \{1, \ldots, q\}$. Let us now enunciate the results we will need.

Theorem 1.14. *([15], Th. 3.2) Suppose that g is tractable at the origin with respect to $\mathcal{V}$ relative to f. Then, for $0 < |\delta| \ll \epsilon \ll 1$,*

$$B_{f,X}(0) - B_{f,X^g}(0) - \sum_{j=1}^{r} m_{f,b_j}(\mathrm{Eu}_X(b_j) - \mathrm{Eu}_{X^g}(b_j)) = (-1)^{d-1} m,$$

where m is the number of stratified Morse critical points of a partial Morsification of $g : X \cap f^{-1}(\delta) \cap B_\epsilon \to \mathbb{C}$ appearing on $X_{reg} \cap f^{-1}(\delta) \cap \{g \neq 0\} \cap B_\epsilon$.

Theorem 1.15. *([15], Th. 4.8) Suppose that g is tractable at the origin with respect to $\mathcal{V}$ relative to f. Then, for $0 \ll |\delta| \ll \epsilon \ll 1$,*

$$B_{g,X^f}(0) = B_{f,X^g}(0) - \sum_{j=1}^{r} m_{f,b_j}(\mathrm{Eu}_{X^g}(b_j) - B_{g,X\cap f^{-1}(\delta)}(b_j)).$$

Theorem 1.16. *([15], Th. 4.10) Suppose that g is tractable at the origin with respect to the good stratification $\mathcal{V}$ of X induced by f. For $0 < |\alpha| \ll |\delta| \ll \epsilon \ll 1$,*

$$B_{g,X}(0) - B_{f,X}(0) = (-1)^{d-1}(n-m) - \sum_{j=1}^{r} m_{f,b_j}(\mathrm{Eu}_X(b_j) - B_{g,X\cap\{f=\delta\}}(b_j)),$$

where n (resp. m) is the number of stratified Morse critical points of a Morsification of $f : X \cap g^{-1}(\alpha) \cap B_\epsilon \to \mathbb{C}$ (resp. $g : X \cap f^{-1}(\delta) \cap B_\epsilon \to \mathbb{C}$) appearing on $T_s \cap g^{-1}(\alpha) \cap \{f \neq 0\} \cap B_\epsilon$ (resp. $V_t \cap f^{-1}(\delta) \cap \{g \neq 0\} \cap B_\epsilon$), where $0 < |\delta| \ll 1$ is a regular value of f and $0 < |\alpha| \ll 1$ is a regular value of g.

Lemma 1.17. *([15], Lemma 5.2) For a generic linear form l, the function g is tractable at the origin with respect to the good stratification $\mathcal{V}$ of X induced by l.*

Proposition 1.18. *([15], Prop. 5.6) If $H = l^{-1}(0)$ we have $B_{g,X\cap H}(0) \geq \mathrm{Eu}_{X^g}(0)$, if d is even, and $B_{g,X\cap H}(0) \leq \mathrm{Eu}_{X^g}(0)$, if d is odd.*

2. A deformation of a function-germ with one-dimensional critical set

Let $\tilde{g} : (X, 0) \to (\mathbb{C}, 0)$ be the function-germ given by $\tilde{g}(x) = g(x) + f^N(x), N \gg 1$. We begin this section with a discussion about the singular locus of the function $\tilde{g} = g + f^N$.

Proposition 2.1. *For a sufficiently large $N, \tilde{g}$ has a stratified isolated singularity at the origin with respect to the Whitney stratification $\mathcal{W}$ of X.*

Proof. Let x be a critical point of $\tilde{g}$, U_x be a neighborhood of x and G and F be analytic extensions of g and f to U_x, respectively. If $V(x)$ is a stratum of $\mathcal{W}$ containing $x \neq 0$,

$$d_x\tilde{G}|_{V(x)} = 0 \Leftrightarrow d_xG|_{V(x)} + N(F(x))^{N-1}d_xF|_{V(x)} = 0$$

If $d_xG|_{V(x)} = 0$, then $N(F(x))^{N-1}d_xF|_{V(x)} = 0$, hence $x \in \{F = 0\}$. Then

$$x \in \Sigma_{\mathcal{W}}g \cap \{f = 0\} = \{0\}.$$

If $d_xG|_{V(x)} \neq 0$, we have $G \neq 0$. Since $d_x\tilde{G}|_{V(x)} = 0$, by Proposition 1.3 of [11], $\tilde{G} = 0$, which implies that $F \neq 0$. On the other hand, if $d_xG|_{V(x)} \neq 0$, $d_xG|_{V(x)} = -N(F(x))^{N-1}d_xF|_{V(x)}$, and then $x \in \tilde{\Gamma}_{f,g}(V(x))$. Suppose that x is arbitrarily close to the origin. Since f has a stratified isolated singularity at the origin, we can define for the stratum $V(x)$, the function $\beta : (0, \epsilon) \to \mathbb{R}, 0 < \epsilon \ll 1$,

$$\beta(u) = \inf\left\{\frac{||d_zg|_{V(x)}||}{||d_zf|_{V(x)}||}; z \in \tilde{\Gamma}_{f,g}(V(x)) \cap \{|f|_{V(x)}(z)| = u, u \neq 0\}\right\},$$

where $||.||$ denotes the operator norm, (defined, for each linear transformation $T : V \to W$ between normed vector fields, by $sup_{v\in V,||v||=1}||T(v)||$). Notice that, for each stratum $W_i \in \mathcal{W}$,

$$\tilde{\Gamma}_{f,g}(W_i) = \tilde{\Gamma}_{f,g}(W_i \setminus \{f = 0\}).$$

Since g is tractable at the origin with respect to $\mathcal{V}, \dim_0 \tilde{\Gamma}_{f,g}(\mathcal{V}) \leq 1$. Therefore,

$$\dim_0 \tilde{\Gamma}_{f,g}(W_i) = \dim_0 \tilde{\Gamma}_{f,g}(W_i \setminus \{f = 0\}) \leq 1.$$

Hence $\tilde{\Gamma}_{f,g}(V(x)) \cap \{|f| = u, u \neq 0\}$ is a finite number of points and β is well defined.

Since the function β is subanalytic, $\alpha(R) = \beta(1/R)$, for $R \gg 1$, is subanalytic. Then, by [9], there exists $n_0 \in \mathbb{N}$ such that $\frac{1}{\alpha(R)} < R^{n_0}$, which implies $\beta(1/R) > (1/R)^{n_0}$, that is, $\beta(u) > u^{n_0}$. Hence, for $z \in \tilde{\Gamma}_{f,g}(V(x)) \cap \{|f(z)| = u\}$, $u \ll 1$, we have

$$\frac{||d_zg|_{V(x)}||}{||d_zf|_{V(x)}||} \geq \beta(u) > u^{n_0}, \text{ which implies, } ||d_zg|_{V(x)}|| > |f|_{V(x)}(z)|^{n_0}||d_zf|_{V(x)}||.$$

On the other hand, since N is sufficiently large, we can suppose $N > n_0$. Since $\tilde{g}(z) = g(z) + f^N(z)$, using the previous inequality that, for the critical point x of $\tilde{g}$,

$$N|f|_{V(x)}(x)|^{N-1}||d_x f|_{V(x)}|| = ||d_x g|_{V(x)}|| > |f|_{V(x)}(x)|^{n_0}||d_x f|_{V(x)}||,$$

which implies that $N|f|_{V(x)}(x)|^{N-1-n_0} > 1$. Since x was taken sufficiently close to the origin, $f|_{V(x)}(x)$ is close to zero. Hence, $|f|_{V(x)}(x)| \ll 1$, which implies that $N - 1 - n_0 < 0$. Therefore, $N \leq n_0$, which is a contradiction. So, there is no x sufficiently close to the origin such that $d_x\tilde{g} = 0$. Therefore, $\tilde{g}$ has a stratified isolated singularity at the origin. ■

We will now see how $\tilde{g}$ behaves with respect to the good stratification $\mathcal{V}$ of X induced by f.

Proposition 2.2. *If g is tractable at the origin with respect to the good stratification $\mathcal{V}$ of X induced by f, then $\tilde{g}$ is prepolar at the origin with respect to $\mathcal{V}$.*

Proof. By Proposition 2.1, $\tilde{g}$ is prepolar at the origin with respect to $\mathcal{V}$. So it is enough to verify that $\tilde{g}|_{V_i\cap\{f=0\}}$ is nonsingular or has a stratified isolated singularity at the origin, where V_i is a stratum from the Whitney stratification $\mathcal{V}$ of X. Suppose that $x \in \Sigma\tilde{g}|_{V_i\cap\{f=0\}}$. Then $d_x\tilde{g} = d_x g + Nf(x)^{N-1}d_x f = 0$, which implies that $d_x g = 0$. But g has no critical point on $V_i \cap \{f = 0\}$, since g is tractable at the origin with respect to $\mathcal{V}$. Therefore, $\tilde{g}$ is prepolar at the origin with respect to $\mathcal{V}$. ■

Corollary 2.3. *Let $\tilde{\mathcal{V}}$ be the good stratification of X induced by $\tilde{g}$. Then f is prepolar at the origin with respect to $\tilde{\mathcal{V}}$.*

Proof. Use Proposition 2.2 and Lemma 6.1 of [7]. ■

Using the previous results, we can relate the relative symmetric polar varieties $\tilde{\Gamma}_{f,\tilde{g}}(\mathcal{V})$ and $\tilde{\Gamma}_{f,g}(\mathcal{V})$.

Remark 2.4. Let us describe $\tilde{\Gamma}_{f,\tilde{g}}(\mathcal{V})$. Let $\Sigma(\tilde{g}, f) = \{x \in X; rk(d_x\tilde{g}, d_x f) \leq 1\}$. Since f is prepolar at the origin with respect to the good stratification induced by $\tilde{g}, f|_{W_i\cap\{\tilde{g}=0\}}$ is nonsingular, for all $W_i \in \mathcal{W}, i \neq 0$. Also $\tilde{g}$ is prepolar at the origin with respect to the good stratification induced by f, which implies that $\tilde{g}|_{W_i\cap\{f=0\}}$ is nonsingular, for all $W_i \in \mathcal{W}, i \neq 0$. Nevertheless, since f and $\tilde{g}$ have a stratified isolated singularity at the origin, $\Sigma_{\mathcal{W}}\tilde{g}\cup\Sigma_{\mathcal{W}}f = \{0\}$. Therefore, the map (f, g) has no singularities in $\{g = 0\}$ or in $\{f = 0\}$. Hence, $\Sigma(\tilde{g}, f) = \tilde{\Gamma}_{f,\tilde{g}}(\mathcal{V})$. So, it is sufficient to describe $\Sigma(\tilde{g}, f)$. Let $x \in \Sigma(\tilde{g}, f)$, then

$$\begin{aligned} rk(d_x\tilde{g}, d_x f) \leq 1 \quad &\Leftrightarrow \quad \big(d_x\tilde{g} = 0\big) \text{ or } \big(d_x f = 0\big) \text{ or } \big(d_x\tilde{g} = \lambda d_x f\big) \\ &\Leftrightarrow \quad \big(d_x\tilde{g} = 0\big) \text{ or } \big(d_x f = 0\big) \text{ or } \big(d_x g = (-Nf(x)^{N-1} + \lambda)d_x f\big) \end{aligned}$$

Since $x \notin \{f = 0\}, d_x f \neq 0$. And since $\tilde{g}$ has a stratified isolated singularity at the origin, $d_x\tilde{g} \neq 0$. If $-Nf(x)^{N-1} + \lambda = 0$, then $d_x g = 0$, that is, $x \in \Sigma_{\mathcal{W}}g$. If $-Nf(x)^{N-1} + \lambda \neq 0$, then $d_x g$ is a nonzero multiple of $d_x f$, that is, $x \in \tilde{\Gamma}_{f,g}(\mathcal{V})$. Therefore,

$$\Sigma(\tilde{g}, f) \subseteq \Sigma_{\mathcal{W}}g \cup \tilde{\Gamma}_{f,g}(\mathcal{V}).$$

On the other hand, if $x \in \Sigma_{\mathcal{W}}g$, then $d_x g = 0$, and

$$d_x\tilde{g} = d_x g + Nf(x)^{N-1}d_x f = Nf(x)^{N-1}d_x f.$$

So, $x \in \Sigma(\tilde{g}, f)$. If $x \in \tilde{\Gamma}_{f,g}(\mathcal{V}), d_x g = \lambda d_x f$, and

$$d_x\tilde{g} = d_x g + Nf(x)^{N-1}d_x f = (\lambda + N)f(x)^{N-1}d_x f,$$

which implies $x \in \Sigma(\tilde{g}, f)$. Therefore, $\tilde{\Gamma}_{f,\tilde{g}}(\mathcal{V}) = \Sigma(\tilde{g}, f) = \Sigma_{\mathcal{W}}g \cup \tilde{\Gamma}_{f,g}(\mathcal{V})$.

Proposition 2.5. *Suppose that g is tractable at the origin with respect to the good stratification $\mathcal{V}$ of X induced by f. Then, for $N \gg 1$,*

$$B_{g,X^f}(0) = B_{\tilde{g},X^f}(0) = B_{f,X^{\tilde{g}}}(0).$$

Proof. Since $\tilde{g} = g + f^N$, over $\{f = 0\}, \tilde{g} = g$. Therefore, $B_{g,X^f}(0) = B_{\tilde{g},X^f}(0)$. On the other hand, by Corollary 2.3, f is prepolar at the origin with respect to the good stratification $\tilde{\mathcal{V}}$ of X induced by $\tilde{g}$ and so is $\tilde{g}$ with respect to $\mathcal{V}$, by Proposition 2.2. Hence, by Corollary 6.3 of [7], $B_{f,X^{\tilde{g}}}(0) = B_{\tilde{g},X^f}(0)$. ■

Corollary 2.6. *Let l be a generic linear form in $\mathbb{C}^n$ and $H = l^{-1}(0)$. Then*

$$B_{g,X\cap H}(0) = B_{\tilde{g},X\cap H}(0) = \mathrm{Eu}_{X^{\tilde{g}}}(0).$$

Proof. By Lemma 1.17, g is tractable at the origin with respect to the good stratification $\mathcal{V}$ of X induced by l. Hence, the formula follows directly by Proposition 2.5, using the equality $B_{g,X^l}(0) = B_{\tilde{g},X^l}(0)$, and Corollary 6.6 of [7]. ■

Corollary 2.7. *Let $N \in \mathbb{N}$ be a sufficiently large number.*

(1) *If d is even,* $\mathrm{Eu}_{X^{\tilde{g}}}(0) \geq \mathrm{Eu}_{X^g}(0)$;
(2) *If d is odd,* $\mathrm{Eu}_{X^{\tilde{g}}}(0) \leq \mathrm{Eu}_{X^g}(0)$.

Proof. Use Proposition 1.18 and Corollary 2.6. ■

In order to compare the Brasselet numbers $B_{f,X^g}(0)$ and $B_{f,X^{\tilde{g}}}(0)$ we need to understand the stratified critical set of g. Let ϵ be sufficiently small such that the local Euler obstructions of X and of X^g are constant on $b_j \cap B_\epsilon$. Denote by $\mathrm{Eu}_X(b_j)$ (respectively, $\mathrm{Eu}_{X^g}(b_j)$) the local Euler obstruction of X (respectively, X^g) at a point of $b_j \cap B_\epsilon$.

Remark 2.8. If ϵ is sufficiently small and $x_l \in b_j, l \in \{i_1, \ldots, i_{k(j)}\}$, $B_{g,X\cap f^{-1}(\delta)}(x_l)$ is constant on $b_j \cap B_\epsilon$ (see Remark 4.5 of [15]). Then we denote $B_{g,X\cap f^{-1}(\delta)}(x_l)$ by $B_{g,X\cap f^{-1}(\delta)}(b_j)$. Since $B_{g,X\cap f^{-1}(\delta)}(x_l) = \mathrm{Eu}_{X\cap f^{-1}(\delta)}(x_l) - \mathrm{Eu}_{g,X\cap f^{-1}(\delta)}(x_l)$, we also denote $\mathrm{Eu}_{g,X\cap f^{-1}(\delta)}(x_l)$ by $\mathrm{Eu}_{g,X\cap f^{-1}(\delta)}(b_j)$.

Proposition 2.9. *Suppose that g is tractable at the origin with respect to the good stratification $\mathcal{V}$ of X relative to f. Then, for $0 < |\delta| \ll \epsilon \ll 1$,*

$$B_{f,X^g}(0) - B_{f,X^{\tilde{g}}}(0) = \textstyle\sum_{j=1}^r m_{f,b_j}(\mathrm{Eu}_{X^g}(b_j) - B_{g,X\cap f^{-1}(\delta)}(b_j)).$$

Proof. Use Theorem 1.15 and Proposition 2.5. ■

Corollary 2.10. *For $0 < |\delta| \ll \epsilon \ll 1$,*

$$\mathrm{Eu}_{X^g}(0) - \mathrm{Eu}_{X^{\tilde{g}}}(0) = \sum_{j=1}^r m_{b_j}(\mathrm{Eu}_{X^g}(b_j) - B_{g,X\cap l^{-1}(\delta)}(b_j)). \tag{2}$$

Proof. By Lemma 1.17, g is tractable at the origin with respect to the good stratification $\mathcal{V}$ of X induced by a generic linear form l. Hence, the formula follows directly from Proposition 2.9, using that $B_{l,X^g}(0) = \mathrm{Eu}_{X^g}(0)$ and that $B_{l,X^{\tilde{g}}}(0) = \mathrm{Eu}_{X^{\tilde{g}}}(0)$. ■

Remark 2.11. Since l is a generic linear form over $\mathbb{C}^n, l^{-1}(\delta)$ intersects $X\cap\{g = 0\}$ transversely and using Corollary 6.6 of [7], we have

$$\mathrm{Eu}_{X^g}(b_j) = \mathrm{Eu}_{X^g\cap l^{-1}(\delta)}(b_j \cap l^{-1}(\delta)) = B_{g,X\cap l^{-1}(\delta)\cap L}(b_j \cap l^{-1}(\delta)),$$

where L is a generic hyperplane in $\mathbb{C}^n$ passing through $x_l \in b_j \cap l^{-1}(\delta), j \in \{1, \ldots, r\}$ and $l \in \{i_1, \ldots, i_{k(j)}\}$. Denoting $B_{g,X\cap l^{-1}(\delta)\cap L}(b_j \cap l^{-1}(\delta))$ by $B'_{g,X\cap l^{-1}(\delta)}(b_j)$, the formula obtained in Corollary 2.10 can be written as

$$\mathrm{Eu}_{X^g}(0) - \mathrm{Eu}_{X^{\tilde{g}}}(0) = \textstyle\sum_{j=1}^r m_{b_j}(B'_{g,X\cap l^{-1}(\delta)}(b_j) - B_{g,X\cap l^{-1}(\delta)}(b_j)).$$

Let m be the number of stratified Morse points of a partial Morsification of $g|_{X\cap f^{-1}(\delta)\cap B_\epsilon}$ appearing on $X_{reg}\cap f^{-1}(\delta)\cap\{g\neq 0\}\cap B_\epsilon$ and $\tilde{m}$ be the number of stratified Morse points of a Morsification of $\tilde{g}|_{X\cap f^{-1}(\delta)\cap B_\epsilon}$ appearing on $X_{reg}\cap f^{-1}(\delta)\cap\{\tilde{g}\neq 0\}\cap B_\epsilon$. The next lemma shows how to compare m and $\tilde{m}$. In the following we keep the same description of $\Sigma_{\mathcal{W}}g$.

Corollary 2.12. *Suppose that g is tractable at the origin with respect to the good stratification $\mathcal{V}$ of X relative to f. Then*

$$\tilde{m}=(-1)^{d-1}\sum_{j=1}^{r}m_{f,b_j}\mathrm{Eu}_{g,X\cap f^{-1}(\delta)}(b_j)+m.$$

Proof. By Theorem 1.16,

$$B_{f,X}(0)-B_{f,X^g}(0)-\textstyle\sum_{j=1}^{r}m_{f,b_j}(\mathrm{Eu}_X(b_j)-\mathrm{Eu}_{X^g}(b_j))=(-1)^{d-1}m,$$

and by Proposition 2.2, $\tilde{g}$ is prepolar at the origin with respect to $\mathcal{V}$, by Theorem 4.4 of [7],

$$B_{f,X}(0)-B_{f,X^{\tilde{g}}}(0)=(-1)^{d-1}\tilde{m}.$$

Using Proposition 2.9, we obtain that

$$\tilde{m}=m+(-1)^{d-1}\sum_{j=1}^{r}m_{f,b_j}(\mathrm{Eu}_X(b_j)-B_{g,X\cap f^{-1}(\delta)}(b_j)).$$

Since f has a stratified isolated singularity at the origin, $f^{-1}(\delta)$ intersects each stratum of $\{f=0\}$ transversely. So, $\mathrm{Eu}_X(V_i)=\mathrm{Eu}_{X\cap f^{-1}(\delta)}(S)$, for each connected component of $V_i\cap f^{-1}(\delta)$. In particular, $\mathrm{Eu}_X(b_j)=\mathrm{Eu}_{X\cap f^{-1}(\delta)}(b_j)$. The formula holds by Theorem 1.4, $\mathrm{Eu}_X(b_j)-B_{g,X\cap f^{-1}(\delta)}(b_j)=\mathrm{Eu}_{g,X\cap f^{-1}(\delta)}(b_j)$. ■

Proposition 2.13. *Let $\tilde{\alpha}$ be a regular value of $\tilde{g}$ and let α_t be a regular value of f, $0<|\tilde{\alpha}|\ll|\alpha_t|\ll 1$. If g is tractable at the origin with respect to $\mathcal{V}$ relative to f, then $B_{g,X\cap f^{-1}(\alpha_t)}(b_j)=B_{f,X\cap\tilde{g}^{-1}(\tilde{\alpha})}(b_j)$.*

Proof. Let $x_t\in\{f=\alpha_t\}\cap b_j$, D_{x_t} be the closed ball with center at x_t and radius r_t, $0<|\alpha-\delta|\ll|\alpha_t|\ll r_t\ll 1$. We have

$$\begin{aligned}B_{g,X\cap f^{-1}(\alpha_t)}(x_t)&=\sum\chi(W_i\cap f^{-1}(\alpha_t)\cap g^{-1}(\alpha-\delta)\cap D_{x_t})\mathrm{Eu}_{X\cap f^{-1}(\alpha_t)}(W_i\cap f^{-1}(\alpha_t))\\&=\sum\chi(W_i\cap f^{-1}(\alpha_t)\cap g^{-1}(\alpha-\delta)\cap D_{x_t})\mathrm{Eu}_X(W_i).\end{aligned}$$

Let $g(x_t)=\alpha,\tilde{g}(x_t)=\alpha'$ and $f(x_t)=\alpha_t$. Then

$$\begin{aligned}p\in f^{-1}(\alpha_t)\cap g^{-1}(\alpha-\delta)&\Leftrightarrow g(p)=\alpha-\delta\ \text{ and }\ f(p)=\alpha_t\\&\Leftrightarrow g(p)=g(x_t)-\delta\ \text{ and }\ f(p)=\alpha_t\\&\Leftrightarrow g(p)+\alpha_t^N=\alpha+\alpha_t^N-\delta\ \text{ and }\ f(p)=\alpha_t\\&\Leftrightarrow g(p)+f^N(p)=g(x_t)+f^N(x_t)-\delta\ \text{ and }\ f(p)=\alpha_t\\&\Leftrightarrow \tilde{g}(p)=\tilde{g}(x_t)-\delta\ \text{ and }\ f(p)=\alpha_t\\&\Leftrightarrow \tilde{g}(p)=\alpha'-\delta\ \text{ and }\ f(p)=\alpha_t.\end{aligned}$$

Therefore, denoting $\tilde{\alpha}=\alpha'-\delta$,

$$\begin{aligned}B_{g,X\cap f^{-1}(\alpha_t)}(x_t)&=\sum\chi(W_i\cap f^{-1}(\alpha_t)\cap g^{-1}(\alpha-\delta)\cap D_{x_t})\mathrm{Eu}_X(W_i)\\&=\sum\chi(W_i\cap f^{-1}(\alpha_t)\cap \tilde{g}^{-1}(\tilde{\alpha})\cap D_{x_t})\mathrm{Eu}_{X\cap\tilde{g}^{-1}(\tilde{\alpha})}(W_i\cap\tilde{g}^{-1}(\tilde{\alpha}))\\&=B_{f,X\cap\tilde{g}^{-1}(\tilde{\alpha})}(x_t).\end{aligned}$$

■

An immediate consequence of the last proposition is the following.

Corollary 2.14. *Let $\tilde{\alpha}$ be a regular value of $\tilde{g}$ and α_t a regular value of f, $0 \ll |\tilde{\alpha}| \ll |\alpha_t| \ll 1$. If g is tractable at the origin with respect to $\mathcal{V}$, then $\mathrm{Eu}_{g,X\cap f^{-1}(\alpha_t)}(b_j) = \mathrm{Eu}_{f,X\cap\tilde{g}^{-1}(\tilde{\alpha})}(b_j)$.*

Proof. Let $x_t \in \{f = \alpha_t\} \cap b_j$, and D_{x_t} be the closed ball with center at x_t and radius $r_t, 0 < |\alpha'| \ll |\alpha_t| \ll r_t \ll 1$. The equality holds by Proposition 2.13. ■

3. Lê-Yomdin formula for the Brasselet number

Let $f, g : (X, 0) \to (\mathbb{C}, 0)$ be complex analytic function-germs such that f has a stratified isolated singularity at the origin. Let $\mathcal{W}$ be the Whitney stratification of X and $\mathcal{V}$ be the good stratification of X induced by f. Suppose that $\Sigma_{\mathcal{W}} g$ is one-dimensional and that

$$\Sigma_{\mathcal{W}} g \cap \{f = 0\} = \{0\}.$$

Suppose that g is tractable at the origin with respect to $\mathcal{V}$. By the First Stratification Lemma,

$$\mathcal{V}' = \Big\{V_i \setminus \Sigma_{\mathcal{W}} g, V_i \cap \Sigma_{\mathcal{W}} g, V_i \in \mathcal{V}\Big\} \cup \mathcal{V}^f$$

is a good stratification of X relative to f, where $\mathcal{V}^f$ denotes the collection of strata of $\mathcal{V}$ contained in $\{f = 0\}$ and

$$\mathcal{V}'' = \{V_i \setminus \{g = 0\}, V_i \cap \{g = 0\} \setminus \Sigma_{\mathcal{W}} g, V_i \cap \Sigma_{\mathcal{W}} g, V_i \in \mathcal{V}\} \cup \{0\},$$

is a good stratification of X relative to g. Let us denote by $\tilde{\mathcal{V}}$ the good stratification of X induced by $\tilde{g} = g + f^N, N \gg 1$.

Let α be a regular value of g, α' a regular value of $\tilde{g}$, $0 < |\alpha|, |\alpha'| \ll \epsilon \ll 1$, n be the number of stratified Morse points of a Morsification of $f|_{X\cap g^{-1}(\alpha)\cap B_\epsilon}$ appearing on

$$X_{reg} \cap g^{-1}(\alpha) \cap \{f \neq 0\} \cap B_\epsilon$$

and $\tilde{n}$ be the number of stratified Morse points of a Morsification of $f|_{X\cap\tilde{g}^{-1}(\alpha')\cap B_\epsilon}$ appearing on $X_{reg} \cap \tilde{g}^{-1}(\alpha') \cap \{f \neq 0\} \cap B_\epsilon$.

Proposition 3.1. *Suppose that g is tractable at the origin with respect to $\mathcal{V}$. Then,*

$$B_{g,X}(0) - B_{\tilde{g},X}(0) = (-1)^{d-1}(n - \tilde{n}).$$

Proof. By 1.16,

$$B_{g,X}(0) - B_{f,X}(0) = (-1)^{d-1}(n - m) - \textstyle\sum_{j=1}^{r} m_{f,b_j}(\mathrm{Eu}_X(b_j) - B_{g,X\cap\{f=\delta\}}(b_j)),$$

where m is the number of stratified Morse points of a Morsification of $g|_{X\cap f^{-1}(\delta)\cap B_\epsilon}$ appearing on $X_{reg} \cap f^{-1}(\delta) \cap \{g \neq 0\} \cap B_\epsilon$, for $0 < |\delta| \ll \epsilon \ll 1$.

By Lemma 2.2, $\tilde{g}$ is prepolar at the origin with respect to $\mathcal{V}$. So, by Corollary 6.5 of [7],

$$B_{\tilde{g},X}(0) - B_{f,X}(0) = (-1)^{d-1}(\tilde{n} - \tilde{m}),$$

where $\tilde{m}$ is the number of stratified Morse points of a Morsification of $\tilde{g}|_{X\cap f^{-1}(\delta)\cap B_\epsilon}$ appearing on $X_{reg} \cap f^{-1}(\delta) \cap \{\tilde{g} \neq 0\} \cap B_\epsilon$.

Using Corollary 2.12 and Theorem 1.4, we have the formula. ■

Lemma 3.2. *Suppose that g is tractable at the origin with respect to $\mathcal{V}$ relative to f. If $N \gg 1$ is bigger than the maximum gap ratio of all components of the symmetric relative polar curve $\tilde{\Gamma}_{f,g}(\mathcal{V})$ and Proposition 2.1 is satisfied, then*

$$\Big([\tilde{\Gamma}_{f,g}(\mathcal{V})].[V(g)]\Big)_0 = \Big([\tilde{\Gamma}_{f,g}(\mathcal{V})].[V(\tilde{g})]\Big)_0.$$

Proof. Since g is tractable at the origin with respect to $\mathcal{V}, \tilde{\Gamma}_{f,g}(\mathcal{V})$ is a curve. Let us write $[\tilde{\Gamma}_{f,g}(\mathcal{V})] = \sum_v m_v[v]$, where each component v of $\tilde{\Gamma}_{f,g}(\mathcal{V})$ is a reduced irreducible curve at the origin. Let $\alpha_v(t)$ be a parametrization of v such that $\alpha_v(0) = 0$. By page 974 of [11], each component v intersects $V(g - g(p))$ at a point $p \in v, p \neq 0$, sufficiently close to the origin and such that $g(p) \neq 0$. So,

$$\operatorname{codim}_X\{0\} = \operatorname{codim}_X V(g) + \operatorname{codim}_X v.$$

Also, each component (reduced irreducible curve at the origin) $\tilde{v}$ of $\tilde{\Gamma}_{f,\tilde{g}}(\mathcal{V})$ intersects $V(\tilde{g} - \tilde{g}(p))$ at such point $p \in \tilde{v}, p \neq 0$ and $\tilde{g}(p) \neq 0$. But since $\tilde{\Gamma}_{f,\tilde{g}}(\mathcal{V}) = \tilde{\Gamma}_{f,g}(\mathcal{V}) \cup \Sigma_{\mathcal{W}} g$, we also have that v intersects $V(\tilde{g} - \tilde{g}(p))$ at the point p, so

$$\operatorname{codim}_X\{0\} = \operatorname{codim}_X V(\tilde{g}) + \operatorname{codim}_X v.$$

Therefore, by A.9 of [13] ,

$$\begin{aligned} ([v].[V(g)])_0 &= \operatorname{mult}_t g(\alpha_v(t)) \\ ([v].[V(\tilde{g})])_0 &= \min\{\operatorname{mult}_t g(\alpha_v(t)), \operatorname{mult}_t f^N(\alpha_v(t))\} \end{aligned}$$

Now,

$$\operatorname{mult}_t f^N(\alpha_v(t)) = N\left([v].[V(f)]\right)_0 \text{ and } \operatorname{mult}_t g(\alpha_v(t)) = \left([v].[V(g)]\right)_0.$$

The gap ratio of v at the origin for g with respect to f is the ratio of intersection numbers $\frac{([v].[V(g)])_0}{([v].[V(f)])_0}$. So, if $N > \frac{([v].[V(g)])_0}{([v].[V(f)])_0}$, then $\operatorname{mult}_t f^N(\alpha_v(t)) > \operatorname{mult}_t g(\alpha_v(t))$.

Making the same procedure over each component v of $\tilde{\Gamma}_{f,g}(\mathcal{V})$ and using that N is bigger then the maximum gap ratio of all components v of $\tilde{\Gamma}_{f,g}(\mathcal{V})$ and Proposition 2.1 is satisfied, we conclude that

$$\left([\tilde{\Gamma}_{f,g}(\mathcal{V})].[V(g)]\right)_0 = \left([\tilde{\Gamma}_{f,g}(\mathcal{V})].[V(\tilde{g})]\right)_0.$$

■

Our next goal is give another proof for the Lê-Yomdin formula for the Brasselet number. For that we need to compare n and $\tilde{n}$.

Lemma 3.3. *If N is bigger than the maximum gap ratio of all components of the symmetric relative polar curve $\tilde{\Gamma}_{f,g}(\mathcal{V})$ and Proposition 2.1 is satisfied, if $0 < |\alpha|, |\alpha'| \ll \epsilon \ll 1$, then*

$$\tilde{n} = n + (-1)^{d-1} N \sum_{j=1}^{r} m_{f,b_j} \mathrm{Eu}_{f, X\cap \tilde{g}^{-1}(\alpha')}(b_j).$$

Proof. We start describing the critical points of $f|_{g^{-1}(\alpha)\cap B_\epsilon}$. We have

$$\begin{aligned} x \in \Sigma f|_{g^{-1}(\alpha)\cap B_\epsilon} \quad &\Leftrightarrow \quad x \in g^{-1}(\alpha) \cap B_\epsilon \ \text{ and } \ rk(d_x g, d_x f) \leq 1 \\ &\Leftrightarrow \quad x \in g^{-1}(\alpha) \cap B_\epsilon \ \text{ and } \ \big(d_x g = 0\big) \ \text{ or } \ \big(d_x f = 0\big) \ \text{ or } \ \big(d_x g = \lambda d_x f, \lambda \neq 0\big). \end{aligned}$$

Since f has a stratified isolated singularity at the origin and, by Proposition 1.3 of [11], $\Sigma_{\mathcal{W}} g \subset \{g = 0\}$, we have that $\Sigma f|_{g^{-1}(\alpha)\cap B_\epsilon} = g^{-1}(\alpha) \cap B_\epsilon \cap \tilde{\Gamma}_{f,g}(\mathcal{V})$. Therefore, n counts the number of Morse points of a Morsification of $f|_{g^{-1}(\alpha)\cap B_\epsilon}$ coming from $g^{-1}(\alpha) \cap B_\epsilon \cap \tilde{\Gamma}_{f,g}(\mathcal{V})$.

Now, let us describe $\Sigma f|_{\tilde{g}^{-1}(\alpha')\cap B_\epsilon}$.

$$x \in \Sigma f|_{\tilde{g}^{-1}(\alpha')\cap B_\epsilon} \Leftrightarrow x \in \tilde{g}^{-1}(\alpha') \cap B_\epsilon \ \text{ and } \ rk(d_x\tilde{g}, d_x f) \leq 1$$

$$\Leftrightarrow x \in \tilde{g}^{-1}(\alpha') \cap B_\epsilon \ \text{ and } \ \big(d_x\tilde{g} = 0\big) \ \text{ or } \ \big(d_x f = 0\big) \ \text{ or } \ \big(d_x\tilde{g} = \lambda' d_x f, \lambda' \neq 0\big).$$

Since f and $\tilde{g}$ has a stratified isolated singularity at the origin, we have that

$$\Sigma f|_{\tilde{g}^{-1}(\alpha')\cap B_\epsilon} = \tilde{g}^{-1}(\alpha') \cap B_\epsilon \cap \tilde{\Gamma}_{f,\tilde{g}}(\mathcal{V}).$$

Since $\tilde{\Gamma}_{f,\tilde{g}}(\mathcal{V}) = \tilde{\Gamma}_{f,g}(\mathcal{V}) \cup \Sigma_{\mathcal{W}} g$,

$$\Sigma f|_{\tilde{g}^{-1}(\alpha')\cap B_\epsilon} = \big(\Sigma_{\mathcal{W}} g \cap \tilde{g}^{-1}(\alpha') \cap B_\epsilon\big) \cup \big(\tilde{\Gamma}_{f,g}(\mathcal{V}) \cap \tilde{g}^{-1}(\alpha') \cap B_\epsilon\big).$$

Notice that, since $\Sigma_{\mathcal{W}} g \cap \{f=0\} = \{0\}, \Sigma_{\mathcal{W}} g \cap \tilde{g}^{-1}(\alpha') \cap B_\epsilon \subset \{f \neq 0\}$. Also, by definition, $\tilde{\Gamma}_{f,g}(\mathcal{V}) \setminus \{0\} \subset \{f \neq 0\}$ Therefore, $\tilde{n}$ counts the number of Morse points of a Morsification of $f|_{\tilde{g}^{-1}(\alpha')\cap B_\epsilon}$ coming from

$$\tilde{g}^{-1}(\alpha') \cap B_\epsilon \cap \Sigma_{\mathcal{W}} g \cap \{f \neq 0\} \cap \{g = 0\}$$

and from

$$\tilde{g}^{-1}(\alpha') \cap B_\epsilon \cap \tilde{\Gamma}_{f,g}(\mathcal{V}) \cap \{f \neq 0\} \cap \{g \neq 0\}.$$

By Lemma 3.2, the number of Morse points of a Morsification of $f|_{\tilde{g}^{-1}(\alpha')\cap B_\epsilon}$ appearing on $\tilde{g}^{-1}(\alpha') \cap B_\epsilon \cap \tilde{\Gamma}_{f,g}(\mathcal{V}) \cap \{f \neq 0\} \cap \{g \neq 0\}$ is precisely n. Let us describe the number of Morse points of a Morsification of $f|_{\tilde{g}^{-1}(\alpha')\cap B_\epsilon}$ appearing on $\tilde{g}^{-1}(\alpha') \cap B_\epsilon \cap \Sigma_{\mathcal{W}} g \cap \{f \neq 0\} \cap \{g = 0\}$. Using that $\Sigma_{\mathcal{W}} g \subset \{g = 0\}$,

$$\begin{aligned} x \in \tilde{g}^{-1}(\alpha') \cap B_\epsilon \cap \Sigma_{\mathcal{W}} g \quad &\Leftrightarrow \quad \tilde{g}(x) = \alpha' \ \text{ and } \ d_x g = 0 \\ &\Leftrightarrow \quad g(x) + f(x)^N = \alpha' \ \text{ and } \ d_x g = 0 \\ &\Leftrightarrow \quad f(x)^N = \alpha' \ \text{ and } \ d_x g = 0 \\ &\Leftrightarrow \quad f(x) \in \{\alpha_0, \ldots, \alpha_{N-1}\} \ \text{ and } \ d_x g = 0, \end{aligned}$$

where $\{\alpha_0, \ldots, \alpha_{N-1}\}$ are the N-th roots of α'. Therefore,

$$\tilde{g}^{-1}(\alpha') \cap B_\epsilon \cap \Sigma_{\mathcal{W}} g = \bigcup_{i=0}^{N-1} f^{-1}(\alpha_i) \cap B_\epsilon \cap \Sigma_{\mathcal{W}} g.$$

Since $\Sigma_{\mathcal{W}} g$ is one-dimensional, $f^{-1}(\alpha_i) \cap \Sigma_{\mathcal{W}} g$ is a finite set of critical points of $f|_{\tilde{g}^{-1}(\alpha')\cap B_\epsilon}$. Since $\tilde{\Gamma}_{f,\tilde{g}}(\mathcal{V}) = \Sigma_{\mathcal{W}} g \cup \tilde{\Gamma}_{f,g}(\mathcal{V})$, each branch b_j of $\Sigma_{\mathcal{W}} g$ is a component of $\tilde{\Gamma}_{f,\tilde{g}}(\mathcal{V})$. If $V_{i(j)}$ is the stratum of $\mathcal{V}''$ containing b_j, then $f|_{V_{i(j)}\cap\tilde{g}^{-1}(\alpha')}$ has a stratified isolated singularity at each point $x_l \in b_j \cap f^{-1}(\alpha_i) \cap \tilde{g}^{-1}(\alpha'), j \in \{1, \ldots, r\}$ and $l \in \{i_1, \ldots, i_{k(j)}\}$ (page 974, [11]). Using Proposition 1.7, we can count the number n_l of Morse points of a Morsification of $f|_{\tilde{g}^{-1}(\alpha')\cap B_\epsilon}$ in a neighborhood of each x_l,

$$\mathrm{Eu}_{f,X\cap\tilde{g}^{-1}(\alpha')}(x_l) = (-1)^{d-1} n_l.$$

Since the Euler obstruction of a function is constant on each branch b_j, so is the Euler obstruction of a function and we can denote $\mathrm{Eu}_{f,X\cap\tilde{g}^{-1}(\alpha')}(x_l)$ by $\mathrm{Eu}_{f,X\cap\tilde{g}^{-1}(\alpha')}(b_j)$, for all $x_l \in b_j \cap f^{-1}(\alpha_i) \cap \tilde{g}^{-1}(\alpha')$. Therefore, if $b_j \cap f^{-1}(\alpha_i) \cap \tilde{g}^{-1}(\alpha') = \{x_{j_1}, \ldots, x_{j_{m_{f,b_j}}}\}$, the number of Morse points of a Morsification of $f|_{\tilde{g}^{-1}(\alpha')\cap B_\epsilon}$ appearing on

$$\big(X_{reg} \setminus \{\tilde{g} = 0\}\big) \cap b_j \cap \{\tilde{g} = \alpha'\} \cap B_\epsilon \cap \{f = \alpha_i\}$$

is

$$n_{j_1} + \cdots + n_{j_{m_{f,b_j}}} \quad = \quad (-1)^{d-1} m_{f,b_j} \mathrm{Eu}_{f,X\cap\tilde{g}^{-1}(\alpha')}(x_l).$$

Making the same analysis over each $\alpha_i \in \sqrt[N]{\alpha'}$, the number of Morse points of a Morsification of $f|_{\tilde{g}^{-1}(\alpha')\cap B_\epsilon}$ appearing in $X_{reg} \setminus \{\tilde{g} = 0\} \cap \{g = 0\} \cap \{\tilde{g} = \alpha'\} \cap B_\epsilon$ is

$$(-1)^{d-1} N \sum_{j=1}^{r} m_{f,b_j} \mathrm{Eu}_{f,X\cap\tilde{g}^{-1}(\alpha')}(b_j).$$

■

Theorem 3.4. *Suppose that g is tractable at the origin with respect to $\mathcal{V}$. If $0 < |\alpha|, |\alpha'| \ll \epsilon$ and N is bigger than the maximum gap ratio of each component of the symmetric relative polar curve $\tilde{\Gamma}_{f,g}(\mathcal{V})$ and Proposition 2.1 is satisfied, then*

$$B_{\tilde{g},X}(0) = B_{g,X}(0) + N \sum_{j=1}^{r} m_{f,b_j} \mathrm{Eu}_{f,X\cap \tilde{g}^{-1}(\alpha')}(b_j).$$

Proof. It follows by Proposition 3.1 and Lemma 3.3. ■

This formula gives a way to compare the numerical data associated to the generalized Milnor fibre of a function g with a one-dimensional singular locus and and to the generalized Milnor fibre of the deformation $\tilde{g} = g + f^N$, for $N \gg 1$ sufficiently large. This is what Lê [8] and Yomdin [19] have done in the case where g is defined over a complete intersection in $\mathbb{C}^n$, g has a one-dimensional critical locus and f is a generic linear form over $\mathbb{C}^n$. Therefore, Theorem 3.4 generalizes this Lê-Yomdin formula.

For $X = \mathbb{C}^n$, let us consider $\mathcal{W} = \{\mathbb{C}^n \setminus \{0\}, \{0\}\}$ the Whitney stratification of $\mathbb{C}^n$. If f has a stratified isolated singularity at the origin, the good stratification $\mathcal{V}$ of $\mathbb{C}^n$ induced by f is given by $\mathcal{V} = \{\mathbb{C}^n \setminus \{f = 0\}, \{f = 0\} \setminus \{0\}, \{0\}\}$.

Corollary 3.5. *Suppose that g is tractable at the origin with respect to $\mathcal{V}$ relative to f. If α and α' are regular values of g and $\tilde{g}$, respectively, with $0 < |\alpha|, |\alpha'| \ll \epsilon$, then*

$$\chi(\tilde{g}^{-1}(\alpha') \cap B_\epsilon) = \chi(g^{-1}(\alpha) \cap B_\epsilon) + (-1)^{n-1} N \sum_{j=1}^{r} m_{f,b_j} \mu(g|_{f^{-1}(\delta_{j_i})}, b_j),$$

where $\mu(g|_{f^{-1}(\delta_{j_i})}, b_j)$ denotes the Milnor number of $g|_{X\cap f^{-1}(\delta_{j_i})\cap B_\epsilon}$ at a point x_{j_i} of the branch b_j, with $f(x_{j_i}) = \delta_{j_i}$.

Proof. By the Second Stratification Lemma, $\mathcal{V}' = \{\mathbb{C}^n \setminus \{f = 0\} \cup \Sigma_{\mathcal{W}} g, \{f = 0\} \setminus \{0\}, \Sigma_{\mathcal{W}} g, \{0\}\}$ is a good stratification of $\mathbb{C}^n$ relative to f. Also, $\mathcal{V}''$, given by

$$\Big\{\mathbb{C}^n \setminus \{f = 0\} \cup \{g = 0\}, \quad \{f = 0\} \setminus \{g = 0\}, \{g = 0\} \setminus \{f = 0\} \cup \Sigma_{\mathcal{W}} g,$$
$$\{f = 0\} \cap \{g = 0\} \setminus \Sigma_{\mathcal{W}} g, \Sigma_{\mathcal{W}} g, \{0\}\Big\},$$

is a good stratification of $\mathbb{C}^n$ relative to g.

By definition of the Brasselet number, if $0 < |\alpha| \ll \epsilon \ll 1$,

$$\begin{aligned}
B_{g,X}(0) &= \sum_{V_i \in \mathcal{V}''} \chi(V_i \cap g^{-1}(\alpha) \cap B_\epsilon) \mathrm{Eu}_{\mathbb{C}^n}(V_i) \\
&= \chi\big((\mathbb{C}^n \setminus \{f = 0\} \cup \{g = 0\}) \cap g^{-1}(\alpha) \cap B_\epsilon\big) \mathrm{Eu}_{\mathbb{C}^n}(\mathbb{C}^n \setminus \{f = 0\} \cup \{g = 0\}) \\
&+ \chi\big((\{f = 0\} \setminus \{g = 0\}) \cap g^{-1}(\alpha) \cap B_\epsilon\big) \mathrm{Eu}_{\mathbb{C}^n}(\{f = 0\} \setminus \{g = 0\}) \\
&= \chi\big((\mathbb{C}^n \setminus \{g = 0\}) \cap g^{-1}(\alpha) \cap B_\epsilon\big) \\
&= \chi(g^{-1}(\alpha) \cap B_\epsilon).
\end{aligned}$$

The good stratification of $\mathbb{C}^n$ induced by $\tilde{g}$ is $\tilde{\mathcal{V}} = \{\{\tilde{g} = 0\}, \mathbb{C}^n \setminus \{\tilde{g} = 0\}, \{0\}\}$ and then, if $0 < |\alpha'| \ll \epsilon \ll 1$,

$$B_{\tilde{g},X}(0) = \chi(\mathbb{C}^n \setminus \{\tilde{g} = 0\} \cap g^{-1}(\alpha) \cap B_\epsilon) \mathrm{Eu}_{\mathbb{C}^n}(\mathbb{C}^n \setminus \{0\}) = \chi(\tilde{g}^{-1}(\alpha') \cap B_\epsilon).$$

Since $f|_{\tilde{g}^{-1}(\alpha')\cap B_\epsilon}$ is defined over $\mathbb{C}^n$ and has a stratified isolated singularity at each $x_{j_i} \in b_j$, considering a small ball $B_\epsilon(x_{j_i})$ with radius ϵ and center at x_{j_i}, for $0 < |\delta| \ll \epsilon \ll 1$,

$$\begin{aligned}
\mathrm{Eu}_{f,\tilde{g}^{-1}(\alpha')}(x_{j_i}) &= (-1)^{n-1}\mu\big(f|_{\tilde{g}^{-1}(\alpha')}, x_{j_i}\big) \\
&= (-1)^{n-1}(-1)^{n-1}\big[\chi\big((f|_{\tilde{g}^{-1}(\alpha')})^{-1}(\delta)\cap B_\epsilon(x_{j_i})\big) - 1\big] \\
&= \chi\big(f^{-1}(\delta_{j_i}-\delta)\cap \tilde{g}^{-1}(\alpha')\cap B_\epsilon(x_{j_i})\big) - 1,\ f(x_{j_i}) = \delta_{j_i} \\
&\overset{*}{=} \chi\big(f^{-1}(\delta_{j_i})\cap \tilde{g}^{-1}(\alpha'-\delta)\cap B_\epsilon(x_{j_i})\big) - 1 \\
&= \chi\big(f^{-1}(\delta_{j_i})\cap g^{-1}(\alpha'-\delta_{j_i}^N-\delta)\cap B_\epsilon(x_{j_i})\big) - 1, g(x_{j_i}) = \alpha' - \delta_{j_i}^N \\
&= \chi\big((g|_{f^{-1}(\delta_{j_i})})^{-1}(\delta)\cap B_\epsilon(x_{j_i})\big) - 1 \\
&= (-1)^{n-1}\mu\big(g|_{f^{-1}(\delta_{j_i})}, x_{j_i}\big),
\end{aligned}$$

where the equality $(*)$ is justified by Proposition 6.2 of [7]. Therefore, applying Theorem 3.4, we obtain

$$\chi(\tilde{g}^{-1}(\alpha')\cap B_\epsilon) = \chi(g^{-1}(\alpha)\cap B_\epsilon) + (-1)^{n-1}N\sum_{j=1}^{r} m_{f,b_j}\mu(g|_{f^{-1}(\delta_{j_i})}, b_j).$$

■

Another consequence of Theorem 3.4 is a different proof for the Lê-Yomdin formula proved by Massey in [13] in the case of a function with a one-dimensional singular locus. For that we will need the definition of the Lê-numbers. We present here the case for functions defined over a nonsingular subspace of $\mathbb{C}^n$, and we recommend Part I of [13] for the general case. Let $h : (U,0) \subseteq (\mathbb{C}^n,0) \to (\mathbb{C},0)$ be an analytic function such that its critical locus Σh is a s-dimensional set. For $0 \le k \le n$, the **k-th relative polar variety** $\Gamma^k_{h,z}$ of h with respect to z is the scheme $V\left(\frac{\partial h}{\partial z_k}, \ldots, \frac{\partial h}{\partial z_n}\right)/\Sigma h$, where $z = (z_1, \ldots, z_n)$ are fixed local coordinates and the **k-th polar cycle** of h with respect to z is the analytic cycle $[\Gamma^k_{h,z}]$. The **k-th Lê cycle** $[\Lambda^k_{h,z}]$ of h with respect to z is the difference of cycles $\left[\Gamma^{k+1}_{h,z}\cap V(\frac{\partial h}{\partial z_k})\right] - [\Gamma^k_{h,z}]$.

Definition 3.6. The **k-th Lê number** of h in p with respect to z, $\lambda^k_{h,z}$, is the intersection number

$$(\Lambda^k_{h,z}.V(z_0 - p_0, \ldots, z_{k-1} - p_{k-1}))_p,$$

provided this intersection is purely zero-dimensional at p.

If this intersection is not purely zero-dimensional, the k-th Lê number of h at p with respect to z is said to be undefined.

Corollary 3.7. *Let $\mathcal{V}$ be the good stratification of an open set $(U,0) \subseteq (\mathbb{C}^{n+1},0)$ induced by a generic linear form l defined over $\mathbb{C}^{n+1}$. Let $N \ge 2, \mathbf{z} = (z_0 \ldots, z_n)$ be a linear choice of coordinates such that $\lambda^i_{g,\mathbf{z}}(0)$ is defined for $i = 0,1$, and $\tilde{\mathbf{z}} = (z_1 \ldots, z_n, z_0)$ be the coordinates for $\tilde{g} = g + l^N$ such that $\lambda^0_{\tilde{g},\tilde{\mathbf{z}}}$ is defined. If N is greater then the maximum gap ratio of each component of the symmetric relative polar curve $\tilde{\Gamma}_{f,g}(\mathcal{V})$ and such that Proposition 2.1 is satisfied, then*

$$\lambda^0_{\tilde{g},\tilde{\mathbf{z}}}(0) = \lambda^0_{g,\mathbf{z}}(0) + (N-1)\lambda^1_{g,\mathbf{z}}(0).$$

Proof. By 1.17, g is tractable at the origin with respect to the good stratification $\mathcal{V}$ induced by l. Without loss of generality, we can suppose that $l = z_0$. Let $F_{g,0}$ be the Milnor fibre of g at the origin and $F_{\tilde{g},0}$ be the Milnor fibre of $\tilde{g}$ at the origin. Since g has a one-dimensional critical set, the possibly nonzero Lê numbers are $\lambda^0_{g,\mathbf{z}}(0)$ and $\lambda^1_{g,\mathbf{z}}(0)$ and, since $\tilde{g}$ has a stratified isolated singularity at the origin, the only possibly nonzero Lê number is $\lambda^0_{\tilde{g},\tilde{\mathbf{z}}}(0)$. By Theorem 4.3 of [12],

$$\chi(F_{g,0}) = 1 + (-1)^n \lambda^0_{g,\mathbf{z}}(0) + (-1)^{n-1}\lambda^1_{g,\mathbf{z}}(0)$$

and

$$\chi(F_{\tilde{g},0}) = 1 + (-1)^n \lambda^0_{\tilde{g},\tilde{\mathbf{z}}}(0).$$

In [13], on page 49, Massey remarked that for $0 < |\delta| \ll \epsilon \ll 1$,

$$\lambda^1_{g,\mathbf{z}}(0) = \sum_{j=1}^{r} m_{b_j}\mu(g|_{l^{-1}(\delta)}, b_j).$$

Therefore, the formula holds by Corollary 3.5. ■

4. Applications to generic linear forms

Let $g : (X, 0) \to (\mathbb{C}, 0)$ be a complex analytic function-germ and l be a generic linear form in $\mathbb{C}^n$. Let $\mathcal{W} = \{\{0\}, W_1, \ldots, W_q\}$ be a Whitney stratification of X and $\mathcal{V}$ be the good stratification of X induced by l. Suppose that $\Sigma_{\mathcal{W}} g$ is one-dimensional.

Let $\mathcal{V}'$ be the good stratification of X relative to l, $\mathcal{V}''$ be the good stratification of X relative to g and $\tilde{\mathcal{V}}$ be the good stratification of X induced by $\tilde{g} = g + l^N, N \gg 1$, taken as in Section 4.

Let α be a regular value of g, α' a regular value of $\tilde{g}$, $0 < |\alpha|, |\alpha'| \ll \epsilon \ll 1$, n be the number of stratified Morse points of a Morsification of $l|_{X\cap g^{-1}(\alpha)\cap B_\epsilon}$ appearing on $X_{reg} \cap g^{-1}(\alpha) \cap \{l \neq 0\} \cap B_\epsilon$, n_i be the number of stratified Morse points of a Morsification of $l|_{W_i\setminus(\{g=0\}\cup\{l=0\})\cap g^{-1}(\alpha)\cap B_\epsilon}$ appearing on $W_i \cap g^{-1}(\alpha) \cap \{l \neq 0\} \cap B_\epsilon$, $\tilde{n}$ be the number of stratified Morse points of a Morsification of $l|_{X\cap\tilde{g}^{-1}(\alpha')\cap B_\epsilon}$ appearing on

$$X_{reg} \cap \tilde{g}^{-1}(\alpha') \cap \{l \neq 0\} \cap B_\epsilon$$

and $\tilde{n}_i$ be the number of stratified Morse points of a Morsification of $l|_{W_i\setminus\{\tilde{g}=0\}\cap\tilde{g}^{-1}(\alpha')\cap B_\epsilon}$ appearing on

$$W_i \cap \tilde{g}^{-1}(\alpha') \cap \{l \neq 0\} \cap B_\epsilon,$$

for each $W_i \in \mathcal{W}$.

As before, we write $\Sigma_{\mathcal{W}} g$ as a union of branches $b_1 \cup \ldots \cup b_r$ and we suppose that

$$\{l = \delta\} \cap b_j = \{x_{i_1}, \ldots, x_{i_{k(j)}}\}.$$

The number m_{l,b_j} of points in $\{l = \delta\} \cap b_j$ is equal to the algebraic multiplicity of b_j, that is, it does not depend on the generic linear form l. Hence, we write m_{b_j} instead of m_{l,b_j}. For each $t \in \{i_1, \ldots, i_{k(j)}\}$, let D_{x_t} be the closed ball with center at x_t and radius r_t,

$$0 < |\alpha|, |\alpha'| \ll |\delta| \ll r_t \ll \epsilon \ll 1,$$

be sufficiently small for the balls D_{x_t} to be pairwise disjoint and the union of balls $D_j = D_{x_{i_1}} \cup \ldots \cup D_{x_{i_{k(j)}}}$ to be contained in B_ϵ and ϵ sufficiently small such that the local Euler obstruction of X at a point of $b_j \cap B_\epsilon$ is constant.

In [18], Tibăr gave a bouquet decomposition for the Milnor fibre of $\tilde{g}$ in terms of the Milnor fibre of g. Let us denote by F_g the local Milnor fibre of g at the origin, $F_{\tilde{g}}$ the local Milnor fibre of $\tilde{g}$ at the origin and F_j the local Milnor fibre of $g|_{\{l=\delta\}}$ at a point of the branch b_j. Then there is a homotopy equivalence

$$F_{\tilde{g}} \overset{ht}{\simeq} (F_g \cup E) \bigvee\nolimits_{j=1}^{r} \bigvee\nolimits_{M_j} S(F_j),$$

where $\bigvee$ denotes the wedge sum of topological spaces, $M_j = Nm_{b_j} - 1, S(F_j)$ denotes the topological suspension over $F_j, E := \cup_{j=1}^{r} Cone(F_j)$ and $F_g \cup E$ is the attaching to F_g of one cone over $F_j \subset F_g$ for each $j \in \{1, \ldots, r\}$. As a consequence of this theorem, Tibăr proved a Lê-Yomdin formula for the Euler characteristic of these Milnor fibres.

In the following, we present a new proof for this formula using our previous results.

Proposition 4.1. *Suppose that g is tractable at the origin with respect to $\mathcal{V}$. If*

$$0 < |\alpha|, |\alpha'| \ll |\delta| \ll \epsilon \ll 1,$$

then

$$\chi(X \cap \tilde{g}^{-1}(\alpha') \cap B_\epsilon) - \chi(X \cap g^{-1}(\alpha) \cap B_\epsilon) \quad = \quad N \sum_{j=1}^{r} m_{b_j} \big(1 - \chi(F_j)\big),$$

where $F_j = X \cap g^{-1}(\alpha) \cap H_j \cap D_{x_t}$ is the local Milnor fibre of $g|_{\{l=\delta\}}$ at a point of the branch b_j and H_j denotes the generic hyperplane $l^{-1}(\delta)$ passing through $x_t \in b_j$, for $t \in \{i_1, \ldots, i_{k(j)}\}$.

Proof. For a stratum $V_i = W_i \setminus (\{g = 0\} \cup \{l = 0\})$ in $\mathcal{V}''$, $W_i \in \mathcal{W}$, let N_i be a normal slice to V_i at $x_t \in b_j$ (i.e. a complex submanifold germ in a local embedding intersecting V_i transversally and only in the point x_t), for $t \in \{i_1, \ldots, i_{k(j)}\}$ and D_{x_t} a closed ball of radius r_t centered at x_t. Considering the constructible function $\mathbf{1}_X$, the normal Morse index along V_i is by definition given by

$$\begin{aligned}
\eta(V_i, \mathbf{1}_X) &= \chi(W_i \setminus (\{g = 0\} \cup \{l = 0\}) \cap N_i \cap D_{x_t}) \\
&- \chi(W_i \setminus (\{g = 0\} \cup \{l = 0\}) \cap N_i \cap \{g = \alpha\} \cap D_{x_t}) \\
&= \chi(W_i \cap N_i \cap D_{x_t}) - \chi(W_i \cap N_i \cap \{g = \alpha\} \cap D_{x_t}) \\
&= 1 - \chi(l_{W_i}),
\end{aligned}$$

since $W_i \cap N_i \cap D_{x_t}$ is contractible and the Milnor fibre of $g|_{X \cap N_i}$ in x_t is a complex link l_{W_i} of X with respect to W_i.

For a stratum $\tilde{V}_i = W_i \setminus (\{\tilde{g} = 0\} \in \tilde{\mathcal{V}}$, $W_i \in \mathcal{W}$, let $\tilde{N}_i$ be a normal slice to $\tilde{V}_i$ at $x_t \in b_j$, for $t \in \{i_1, \ldots, i_{k(j)}\}$. Considering the constructible function $\mathbf{1}_X$, the normal Morse index along $\tilde{V}_i$ is by definition given by

$$\begin{aligned}
\eta(\tilde{V}_i, \mathbf{1}_X) &= \chi(W_i \setminus (\{\tilde{g} = 0\} \cup \{l = 0\}) \cap \tilde{N}_i \cap D_{x_t}) \\
&= \chi((W_i \setminus \{\tilde{g} = 0\}) \cap \tilde{N}_i \cap D_{x_t}) - \chi((W_i \setminus \{\tilde{g} = 0\}) \cap \tilde{N}_i \cap \{\tilde{g} = \alpha'\} \cap D_{x_t}) \\
&= \chi(W_i \cap \tilde{N}_i \cap D_{x_t}) - \chi(W_i \cap \tilde{N}_i \cap \{\tilde{g} = \alpha'\} \cap D_{x_t}) \\
&= 1 - \chi(l_{W_i}),
\end{aligned}$$

since $W_i \cap \tilde{N}_i \cap D_{x_t}$ is contractible and the Milnor fibre of $\tilde{g}|_{X \cap \tilde{N}_i}$ in x_t is a complex link l_{W_i} of X with respect to W_i.

Then applying Theorem 4.2 of [7] for $\mathbf{1}_X$, we obtain that

$$\chi(X \cap \tilde{g}^{-1}(\alpha') \cap B_\epsilon) - \chi(X \cap \tilde{g}^{-1}(\alpha') \cap l^{-1}(0) \cap B_\epsilon) = \textstyle\sum_{i=1}^{q} (-1)^{d_i - 1} \tilde{n}_i (1 - \chi(l_{W_i}))$$

and that

$$\chi(X \cap g^{-1}(\alpha) \cap B_\epsilon) - \chi(X \cap g^{-1}(\alpha) \cap l^{-1}(0) \cap B_\epsilon) = \textstyle\sum_{i=1}^{q} (-1)^{d_i - 1} n_i (1 - \chi(l_{W_i})),$$

where $d_i = \dim W_i$.

Therefore, since $\chi(X \cap \tilde{g}^{-1}(\alpha') \cap l^{-1}(0) \cap B_\epsilon) = \chi(X \cap g^{-1}(\alpha) \cap l^{-1}(0) \cap B_\epsilon)$,

$$\chi(X \cap \tilde{g}^{-1}(\alpha') \cap B_\epsilon) - \chi(X \cap g^{-1}(\alpha) \cap B_\epsilon) = \sum_{i=1}^{q} (-1)^{d_i - 1} (\tilde{n}_i - n_i)(1 - \chi(l_{W_i})).$$

Applying Lemma 3.3 and Corollary 2.14, we obtain, for each i,

$$\begin{aligned}\tilde{n}_i &= n_i + (-1)^{d_i-1} N \sum_{j=1}^{r} m_{b_j} \mathrm{Eu}_{l,\overline{W_i}\cap \tilde{g}^{-1}(\alpha')}(b_j)\\ &= n_i + (-1)^{d_i-1} N \sum_{j=1}^{r} m_{b_j} \mathrm{Eu}_{g,\overline{W_i}\cap H_j}(b_j),\end{aligned}$$

where H_j denotes the generic hyperplane $l^{-1}(\delta)$ passing through $x_t \in b_j$, for $t \in \{i_1, \dots, i_{k(j)}\}$.

Hence

$$\begin{aligned}\chi(X\cap \tilde{g}^{-1}(\alpha')\cap B_\epsilon) - \chi(X\cap g^{-1}(\alpha)\cap B_\epsilon) &= N\sum_{i=1}^{q}\Bigg(\sum_{j=1}^{r} m_{b_j}\mathrm{Eu}_{g,\overline{W_i}\cap H_j}(b_j)\Bigg)\big(1-\chi(l_{W_i})\big)\\ &= N\sum_{j=1}^{r} m_{b_j}\Big(1-\chi\big(X\cap g^{-1}(\alpha)\cap H_j\cap D_{x_t}\big)\Big)\\ &= N\sum_{j=1}^{r} m_{b_j}\big(1-\chi(F_j)\big),\end{aligned}$$

for $t \in \{i_1, \dots, i_{k(j)}\}$. ■

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