Journal *of* Singularities
Volume 25 (2022), 393-402

Proc. of 16th International Workshop on Singularities, São Carlos, 2020
DOI: 10.5427/jsing.2022.25s

ON BI-LIPSCHITZ INVARIANCE AND THE UNIQUENESS OF TANGENT CONES

JOSÉ EDSON SAMPAIO AND EURÍPEDES CARVALHO DA SILVA

Abstract. In this note we present some remarks about tangent cones and their invariance under bi-Lipschitz homeomorphisms. In particular, we prove the bi-Lipschitz invariance of tangent cones of sets with unique tangent cone. We obtain also some characterizations for the uniqueness of the tangent cone of a set at a point, for example, the sets which satisfy the sequence selection property (SSP-sets for short) presented by Koike and Paunescu are just those sets which have unique tangent cones. The analogues versions at infinity of these results are also presented.

1. Introduction

The tangent cones at singular points generalize the notion of tangent spaces at smooth points and the geometry and topology of tangent cones of algebraic or analytic sets are very important in several questions of Singularity Theory. It is also a subject of interest in Lipschitz Geometry of Singularities and, in particular, the study on bi-Lipschitz invariance of tangent cones. For instance, for the subanalytic category, Bernig and Lytchak in [1] proved that if two subanalytic sets are subanalytically bi-Lipschitz homeomorphic, then their tangent cones are bi-Lipschitz homeomorphic. Koike and Paunescu proved in [5] that the dimension of tangent cones of subanalytic sets is invariant under bi-Lipschitz homeomorphisms and the first author of this article proved in [9] that if two subanalytic sets are bi-Lipschitz homeomorphic, then their tangent cones are bi-Lipschitz homeomorphic and with this result he obtained the Lipschitz Regularity Theorem, which says that if a germ of an analytic set is bi-Lipschitz homeomorphic to a smooth germ, then it is smooth itself. More recently, Koike and Paunescu proved in [7] that if two sets which satisfy the SSP-condition (see Definition 4.1) are bi-Lipschitz homeomorphic, then their tangent cones are bi-Lipschitz homeomorphic. The first author jointly with Fernandes in [3] proved that if two semialgebraic sets are bi-Lipschitz homeomorphic at infinity, then their tangent cones at infinity are bi-Lipschitz homeomorphic.

In this article, we prove the following result about bi-Lipschitz invariance of tangent cones without imposing any regularity on the sets.

Theorem 3.1. *Let $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^n$ be subsets. Let $\phi\colon (A,p) \to (B,q)$ be a bi-Lipschitz homeomorphism such that*

$$\frac{1}{K_1}\|x-y\| \le \|\phi(x)-\phi(y)\| \le K_2\|x-y\|, \quad \forall x,y \in A,$$

where $K_1, K_2 > 0$. Then, for any sequence of positive numbers $T = \{t_j\}_{j\in\mathbb{N}}$ such that $\lim t_j = 0$, there exist a subsequence $S = \{s_j\}_{j\in\mathbb{N}} \subset T = \{t_j\}_{j\in\mathbb{N}}$ and a global bi-Lipschitz homeomorphism

2010 *Mathematics Subject Classification.* 32B20; 32B25; 14P10; 14B05.
Key words and phrases. Lipschitz maps, Tangent cone, definable sets, Sequence selection property.
The first author was partially supported by CNPq-Brazil grant 303811/2018-8.

$d\phi\colon C^S(A,p) \to C^S(B,q)$ such that $d\phi(0) = 0$ and

$$\frac{1}{K_1}\|x-y\| \leq \|d\phi(x) - d\phi(y)\| \leq K_2\|x-y\|, \quad \forall x,y \in C^S(A,p).$$

As a consequence, we obtain that if additionally A (resp. B) has a unique tangent cone at p (resp. q) (see Definition 2.3), then $C(A,p)$ and $C(B,q)$ are bi-Lipschitz homeomorphic. A version at infinity of Theorem 3.1 is also presented (see Proposition 4.10).

Another subject studied in this article is the study about the uniqueness of the tangent cones of sets at a point or at infinity. In general, as it was already remarked in [3], it is not an easy task to verify whether unbounded subsets have a unique tangent cone at infinity, even in the case of some classes of analytic subsets, for instance, concerning to such a problem, there is a still unsettled conjecture by Meeks III ([8], Conjecture 3.15) stating that: *any properly immersed minimal surface in $\mathbb{R}^3$ of quadratic area growth has a unique tangent cone at infinity.*

In this article we present some characterizations of sets which have a unique tangent cone at some point and at infinity (see Proposition 4.14). In particular, it is shown that a set has a unique tangent cone at some point p if and only if that set satisfies the SSP-condition at p. More precisely, we have the following:

Theorem 4.2. *Let $X \subset \mathbb{R}^m$ be a subset such that $p \in \mathbb{R}^m$ is a non-isolated point of $\overline{X}$. Then the following statements are equivalents:*

(1) X has a unique tangent cone at p;
(2) $\operatorname{dist}(tv, X) = o(t)$ for all $v \in D(X,p)$;
(3) X satisfies SSP-condition at p.

As a consequence, we recover the main result proved in [7] on bi-Lipschitz invariance of tangent cones of sets that satisfy the SSP-condition.

2. Preliminaries

All the subsets of $\mathbb{R}^n$ are considered equipped with the induced Euclidean metric.

Definition 2.1. *Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$. A mapping $f\colon X \to Y$ is called* **Lipschitz** *if there exists $\lambda > 0$ such that*

$$\|f(x_1) - f(x_2)\| \leq \lambda\|x_1 - x_2\|$$

for all $x_1, x_2 \in X$. A Lipschitz mapping $f\colon X \to Y$ is called **bi-Lipschitz** *if its inverse mapping exists and is Lipschitz.*

Definition 2.2. *Let $X \subset \mathbb{R}^m$ be a subset such that $p \in \overline{X}$. Given a sequence of real positive numbers $\{t_j\}_{j\in\mathbb{N}}$ such that $t_j \to 0$ (resp. $t_j \to +\infty$), we say that $v \in \mathbb{R}^m$ is* **tangent to** *X* **at** *p (resp.* **infinity***)* **with respect to** *$\{t_j\}_{j\in\mathbb{N}}$ if there is a sequence of points $\{x_j\}_{j\in\mathbb{N}} \subset X$ such that $\lim\limits_{j\to+\infty} \frac{1}{t_j}(x_j - p) = v$ (resp. $\lim\limits_{j\to+\infty} \frac{1}{t_j}x_j = v$).*

Definition 2.3. *Let $X \subset \mathbb{R}^m$ be a subset such that $p \in \overline{X}$ and $T = \{t_j\}_{j\in\mathbb{N}}$ such that $t_j \to 0$ (resp. $t_j \to +\infty$). We denote the set of all vectors which are tangents to X at p (resp. infinity) w.r.t. T by $C^T(X,p)$ (resp. $C^T(X,\infty)$). We say X* **has a unique tangent cone at** *p (resp.* **infinity***), if $C^T(X,p) = C^S(X,p)$ (resp. $C^T(X,\infty) = C^S(X,\infty)$), for any two sequences of real positive numbers $S = \{s_j\}_{j\in\mathbb{N}}$ and $T = \{t_j\}_{j\in\mathbb{N}}$ such that $s_j \to 0$ and $t_j \to 0$ (resp. $s_j \to +\infty$ and $t_j \to +\infty$) and, in this case, we denote such $C^T(X,p)$ (resp. $C^S(X,\infty)$) by $C(X,p)$ (resp. $C(X,\infty)$) and we call $C(X,p)$ (resp. $C(X,\infty)$) the* **tangent cone of** *X* **at** *p (resp.* **infinity***).*

Let us remark that a tangent cone of a set X at p can be non-unique as we can see in the next example.

Example 2.4. *Let*

$$X = \{(x,y) \in \mathbb{R}^2; \sin(\log \frac{1}{x^2+y^2}) = 0\}.$$

For each $j \in \mathbb{N}$, we define $t_j = (e^{-j\pi})^{\frac{1}{2}}$ and $s_j = (e^{-j\pi+\pi/2})^{\frac{1}{2}}$. Thus, for $T = \{t_j\}_{j\in\mathbb{N}}$ and $S = \{s_j\}_{j\in\mathbb{N}}$, we have $(0,1) \in C^T(X,0) \setminus C^S(X,0)$ and, thus, $C^T(X,0) \neq C^S(X,0)$.

A tangent cone of a set X at infinity can also be non-unique as we can see in the next example.

Example 2.5. *Let $X = \{(x,y) \in \mathbb{R}^2; \sin(\log(x^2+y^2+1)) = 0\}$. For each $j \in \mathbb{N}$, we define $t_j = (e^{j\pi}-1)^{\frac{1}{2}}$ and $s_j = (e^{j\pi+\pi/2}-1)^{\frac{1}{2}}$. Thus, for $T = \{t_j\}_{j\in\mathbb{N}}$ and $S = \{s_j\}_{j\in\mathbb{N}}$, we have $(0,1) \in C^T(X,\infty) \setminus C^S(X,\infty)$ and, thus, $C^T(X,\infty) \neq C^S(X,\infty)$.*

Below we give several general examples of sets which have unique tangent cone at some point, to illustrate the richness of this class.

Example 2.6.
(1) If $A =$Cone$(L) = \{tv; v \in L$ and $t \in [0,+\infty)\}$, for some $L \subset \mathbb{S}^{n-1}$, then A has a unique tangent cone at 0;
(2) If A is subanalytic or definable in some o-minimal structure, then it has a unique tangent cone at each $p \in \overline{A}$. See [4] for the definition of subanalytic, and see [2] for the definitions of definable and o-minimal.
(3) If A is a finite union of sets, all of which have a unique tangent cone at p, then A has a unique tangent cone at p.
(4) If A is a C^1 submanifold of $\mathbb{R}^n$ such that $p \in A$, then it has a unique tangent cone at p and $C(A,p) = T_pA$, where T_pA denotes the tangent space of A at $p \in \mathbb{R}^n$.

3. bi-Lipschitz Invariance of the tangent cones

Theorem 3.1. *Let $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^n$ be subsets. Let $\phi\colon (A,p) \to (B,q)$ be a bi-Lipschitz homeomorphism such that*

$$\frac{1}{K_1}\|x-y\| \leq \|\phi(x)-\phi(y)\| \leq K_2\|x-y\|, \quad \forall x,y \in A,$$

where $K_1, K_2 > 0$. Then, for any sequence of positive numbers $T = \{t_j\}_{j\in\mathbb{N}}$ such that $\lim t_j = 0$, there exist a subsequence $S = \{s_j\}_{j\in\mathbb{N}} \subset T = \{t_j\}_{j\in\mathbb{N}}$ and a global bi-Lipschitz homeomorphism $d\phi\colon C^S(A,p) \to C^S(B,q)$ such that $d\phi(0) = 0$ and

$$\frac{1}{K_1}\|x-y\| \leq \|d\phi(x)-d\phi(y)\| \leq K_2\|x-y\|, \quad \forall x,y \in C^S(A,p).$$

Proof. This proof shares its structure with the proof of Theorem 2.19 in [3].

By taking translations, if necessary, we can assume that $p = 0$ and $q = 0$. By taking $\mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^n$ and doing the following identifications:

$$A \leftrightarrow A \times \{0\} \quad \text{and} \quad B \leftrightarrow \{0\} \times B$$

one can suppose that $A, B \subset \mathbb{R}^N$ and there exists a bi-Lipschitz map $\varphi\colon \mathbb{R}^N \to \mathbb{R}^N$ such that $\varphi(A) = B$ (see Lemma 3.1 in [9]). Let $K > 0$ be a constant such that

$$\frac{1}{K}\|x-y\| \leq \|\varphi(x)-\varphi(y)\| \leq K\|x-y\|, \quad \forall x,y \in \mathbb{R}^N. \tag{1}$$

Let $\{t_k\}_{k\in\mathbb{N}}$ be a sequence of positive numbers such that $\lim\limits_{k\to+\infty} t_k = 0$. For each $k \in \mathbb{N}$, let us define the mappings $\varphi_k, \psi_k : \mathbb{R}^N \to \mathbb{R}^N$ given by $\varphi_k(v) = \frac{1}{t_k}\varphi(t_kv)$ and $\psi_k(v) = \frac{1}{t_k}\varphi^{-1}(t_kv)$. For each integer $m \geq 1$, let us define $\varphi_{k,m} := \varphi_k|_{\overline{B}_m} : \overline{B}_m \to \mathbb{R}^N$ and $\psi_{k,m} := \psi_k|_{\overline{B}_{mK}} : \overline{B}_{mK} \to \mathbb{R}^N$, where $\overline{B}_r$ denotes the Euclidean closed ball of radius r and with center at the origin in $\mathbb{R}^N$. Since

$$\frac{1}{K}\|x-y\| \leq \|\varphi_{k,1}(x) - \varphi_{k,1}(y)\| \leq K\|x-y\|, \quad \forall x, y \in \overline{B}_1,\ \forall k \in \mathbb{N}$$

and

$$\frac{1}{K}\|u-v\| \leq \|\psi_{k,1}(u) - \psi_{k,1}(v)\| \leq K\|u-v\|, \quad u, v \in \overline{B}_K,\ \forall k \in \mathbb{N},$$

there exist a subsequence $\{k_{j,1}\}_{j\in\mathbb{N}} \subset \mathbb{N}$ and Lipschitz mappings $d\varphi_1 : \overline{B}_1 \to \mathbb{R}^N$ and $d\psi_1 : \overline{B}_K \to \mathbb{R}^N$ such that $\varphi_{k_{j,1},1} \to d\varphi_1$ uniformly on $\overline{B}_1$ and $\psi_{k_{j,1},1} \to d\psi_1$ uniformly on $\overline{B}_K$ (notice that $\{\varphi_{k,1}\}_{k\in\mathbb{N}}$ and $\{\psi_{k,1}\}_{k\in\mathbb{N}}$ have uniform Lipschitz constants). Furthermore, it is clear that

$$\frac{1}{K}\|u-v\| \leq \|d\varphi_1(u) - d\varphi_1(v)\| \leq K\|u-v\|, \quad \forall u, v \in \overline{B}_1$$

and

$$\frac{1}{K}\|z-w\| \leq \|d\psi_1(z) - d\psi_1(w)\| \leq K\|z-w\|, \quad \forall z, w \in \overline{B}_K.$$

Likewise as above, for each $m > 1$, we have

$$\frac{1}{K}\|x-y\| \leq \|\varphi_{k,m}(x) - \varphi_{k,m}(y)\| \leq K\|x-y\|, \quad x, y \in \overline{B}_m,\ \forall k \in \mathbb{N}$$

and

$$\frac{1}{K}\|u-v\| \leq \|\psi_{k,m}(u) - \psi_{k,m}(v)\| \leq K\|u-v\|, \quad u, v \in \overline{B}_{mK},\ \forall k \in \mathbb{N}.$$

Therefore, for each $m > 1$, there exist a subsequence $\{k_{j,m}\}_{j\in\mathbb{N}} \subset \{k_{j,m-1}\}_{j\in\mathbb{N}}$ and Lipschitz mappings $d\varphi_m : \overline{B}_m \to \mathbb{R}^N$ and $d\psi_m : \overline{B}_{mK} \to \mathbb{R}^N$ such that $\varphi_{k_{j,m},m} \to d\varphi_m$ uniformly on $\overline{B}_m$ and $\psi_{k_{j,m},m} \to d\psi_m$ uniformly on $\overline{B}_{mK}$ with $d\varphi_m|_{\overline{B}_{m-1}} = d\varphi_{m-1}$ and $d\psi_m|_{\overline{B}_{(m-1)K}} = d\psi_{m-1}$. Furthermore,

$$\frac{1}{K}\|u-v\| \leq \|d\varphi_m(u) - d\varphi_m(v)\| \leq K\|u-v\|, \quad \forall u, v \in \overline{B}_m \tag{2}$$

and

$$\frac{1}{K}\|z-w\| \leq \|d\psi_m(z) - d\psi_m(w)\| \leq K\|z-w\|, \quad \forall z, w \in \overline{B}_{mK}. \tag{3}$$

Let us define $d\varphi, d\psi : \mathbb{R}^N \to \mathbb{R}^N$ by $d\varphi(x) = d\varphi_m(x)$, if $x \in \overline{B}_m$ and $d\psi(x) = d\psi_m(x)$, if $x \in \overline{B}_{mK}$ and, for each $j \in \mathbb{N}$, let $s_j = k_{j,j}$.

Let $F \subset \mathbb{R}^N$ be a compact subset. Let us take $m \in \mathbb{N}$ such that $F \subset \overline{B}_m \subset \overline{B}_{mK}$. Thus, $\{s_j\}_{j>m}$ is a subsequence of $\{k_{j,m}\}_{j\in\mathbb{N}}$ and, since $\varphi_{k_{j,m},m} \to d\varphi_m$ uniformly on $\overline{B}_m$ and $\psi_{k_{j,m},m} \to d\psi_m$ uniformly on $\overline{B}_{mK}$, it follows that $\varphi_{s_j} \to d\varphi$ and $\psi_{s_j} \to d\psi$ uniformly on F. This shows that $\varphi_{s_j} \to d\varphi$ and $\psi_{s_j} \to d\psi$ uniformly on compact subsets of $\mathbb{R}^N$. Thus, it follows from inequalities (2) and (3) that $d\varphi, d\psi : \mathbb{R}^N \to \mathbb{R}^N$ are bi-Lipschitz homeomorphisms. Since $\psi_k \circ \varphi_k = \mathrm{id}_{\mathbb{R}^N}$ $\varphi_k \circ \psi_k = \mathrm{id}_{\mathbb{R}^N}$, for all $k \in \mathbb{N}$, we obtain that $d\psi = (d\varphi)^{-1}$.

Claim 1. $d\phi(C^S(A,0)) = C^S(B,0)$, where $d\phi := d\varphi|_{C^S(A,0)}$ and $S = \{s_j\}_{j\in\mathbb{N}}$.

By symmetry, it is enough to verify that $d\varphi(C^S(A,0)) \subset C^S(B,0)$. In order to do that, let $v \in C^S(A,0)$. Thus, there is a sequence $\{x_j\} \subset A$ such that $\lim\limits_{j\to+\infty} \frac{x_j}{s_j} = v$. Then, we obtain

$$\left\| \tfrac{\varphi(s_j v)}{s_j} - \tfrac{\varphi(x_j)}{s_j} \right\| \leq K\|v - \tfrac{x_j}{s_j}\| \to 0 \text{ as } j \to +\infty.$$

Therefore,

$$\lim_{j\to+\infty} \tfrac{\varphi(s_j v)}{s_j} = \lim_{j\to+\infty} \tfrac{\varphi(x_j)}{s_j} = d\varphi(v) = d\phi(v).$$

Since $\varphi|_{X\setminus K} = \phi$, we have

$$\lim_{j\to+\infty} \tfrac{\phi(x_j)}{s_j} = d\phi(v) \in C^S(B,0). \tag{4}$$

Therefore, $d\phi(C^S(A,0)) = C^S(B,0)$.

Claim 2. $\frac{1}{K_1}\|v-w\| \le \|d\phi(v) - d\phi(w)\| \le K_2\|v-w\|, \quad \forall v,w \in C^S(A,0).$

In fact, if $v,w \in C^S(A,0)$, there are sequences $\{x_j\},\{y_j\} \subset A$ such that $\lim\limits_{j\to+\infty} \frac{x_j}{s_j} = v$ and $\lim\limits_{j\to+\infty} \frac{y_j}{s_j} = w$. Thus, by the hypothesis of the theorem, we obtain

$$\tfrac{1}{K_1}\left\|\tfrac{x_j}{s_j} - \tfrac{y_j}{s_j}\right\| \le \left\|\tfrac{\phi(x_j)}{s_j} - \tfrac{\phi(y_j)}{s_j}\right\| \le K_2\left\|\tfrac{x_j}{s_j} - \tfrac{y_j}{s_j}\right\|.$$

Passing to the limit $j \to +\infty$ and using (4), we obtain

$$\frac{1}{K_1}\|v-w\| \le \|d\phi(v) - d\phi(w)\| \le K_2\|v-w\|.$$

□

As a corollary of Theorem 3.1, we have the generalised result of Theorem 3.2 in [9] to the case of sets with unique tangent cones.

Corollary 3.2. *Let $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^n$ be subsets such that A and B have a unique tangent cone at $p \in \mathbb{R}^m$ and $q \in \mathbb{R}^n$, respectively. If (A,p) and (B,q) are bi-Lipschitz homeomorphic, then $C(A,p)$ and $C(B,q)$ are bi-Lipschitz homeomorphic as well.*

Let us recall the notion of direction set.

Definition 3.3. *Let A be a set-germ at $p \in \mathbb{R}^n$ such that p is a non-isolated point of $\overline{A}$. We define the* **direction set** *$D(A,p)$ of A at $p \in \mathbb{R}^n$ by*

$$D(A,p) := \{v \in S^{n-1} \mid \exists \{x_i\} \subset A \setminus \{p\},\ x_i \to p \in \mathbb{R}^n \ s.t.\ \frac{x_i - p}{\|x_i - p\|} \to v,\ i \to \infty\}.$$

Here S^{n-1} denotes the unit sphere centered at $0 \in \mathbb{R}^n$.

Thus, we have the main result of [5].

Corollary 3.4 (Main Theorem in [5])**.** *Let $A,\ B \subset \mathbb{R}^n$ be subanalytic set-germs at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A} \cap \overline{B}$, and let $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a bi-Lipschitz homeomorphism. Suppose that $h(A),\ h(B)$ are also subanalytic. Then we have the equality of dimensions,*

$$\dim(D(h(A)) \cap D(h(B))) = \dim(D(A) \cap D(B)).$$

4. Characterizations of sets with a unique tangent cone

Definition 4.1. *Let A be a set-germ at $p \in \mathbb{R}^n$ such that p is a non-isolated point of $\overline{A}$. We say that A satisfies* **SSP-condition at** *p, if for any sequence of points $\{a_m\}$ of $\mathbb{R}^n$ tending to $p \in \mathbb{R}^n$, such that $\lim_{m\to\infty} \frac{a_m - p}{\|a_m - p\|} \in D(A,p)$, there is a sequence of points $\{b_m\} \subset A$ such that,*

$$\|a_m - b_m\| \ll \|a_m - p\|,\ \|b_m - p\|,$$

i.e. $\lim_{m\to\infty} \frac{\|a_m - b_m\|}{\|a_m - p\|} = \lim_{m\to\infty} \frac{\|a_m - b_m\|}{\|b_m - p\|} = 0.$

Recall that for $X \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, $\mathrm{dist}(x, X) = \inf_{y \in X} \|x - y\|$ and the Hausdorff distance between two bounded sets $A, B \subset \mathbb{R}^n$ is given by

$$d_H(A, B) = \max\{\sup_{x \in A} \mathrm{dist}(x, B), \sup_{y \in B} \mathrm{dist}(y, A)\}.$$

For a set-germ A at $p \in \mathbb{R}^n$ such that p is a non-isolated point of $\overline{A}$, we put

$$LD(A, p) := \{ta \in \mathbb{R}^n \mid a \in D(A, p),\ t \geq 0\},$$

and call it the **real tangent cone** of A at $p \in \mathbb{R}^n$.

Theorem 4.2. *Let $X \subset \mathbb{R}^m$ be a subset such that $p \in \mathbb{R}^m$ is a non-isolated point of $\overline{X}$. Then the following statements are equivalents:*

(1) X has a unique tangent cone at p;
(2) $\mathrm{dist}(tv, X) = o(t)$ for all $v \in D(X, p)$;
(3) X satisfies SSP-condition at p.

Remark 4.3. *The equivalence between (1) and (2) of Theorem 4.2 was already pointed out by Koike and Paunescu in* [6, Remark 2.8] *and* [7, Proposition 2.4]. *However, for convenience, we present the proofs here.*

Proof of Theorem 4.2. Without loss of generality, we assume $p = 0$.

We are going to show that: (1) $\Rightarrow$ (2). Let us remark that this proof resembles with the proof of Proposition 2.7 in [6]. Suppose by contradiction that X has a unique tangent cone at 0 and $\mathrm{dist}(tv, X) \neq o(t)$, so there exists $v \in D(X, 0)$, $\epsilon > 0$ and a sequence of real numbers $T = \{t_k\}_{k \in \mathbb{N}}$ such that

$$\frac{\mathrm{dist}(t_n v, X)}{t_n} \geq \epsilon > 0, \quad \forall n.$$

Since X has a unique tangent cone at 0, we have $v \in D(X, 0)$ implies $v \in C^T(X, 0)$, so there exists a sequence $\{x_k\}_{k \in \mathbb{N}} \subset X$ such that $\lim_{n \to +\infty} \frac{x_n}{t_n} = v$.

On the other hand, we have

$$\frac{\mathrm{dist}(t_n v, X)}{t_n} \leq \frac{\|t_n v - x_n\|}{t_n},$$

and note that $\lim_{n \to +\infty} \frac{t_n v - x_n}{t_n} = 0$, consequently

$$\epsilon \leq \lim_{n \to +\infty} \frac{\mathrm{dist}(t_n v, X)}{t_n} = 0,$$

which gives a contradiction. Therefore, $\mathrm{dist}(tv, X) = o(t)$ for all $v \in D(X, 0)$.

Now we are going to show that: (2) $\Rightarrow$ (3). So, assume $\mathrm{dist}(tv, X) = o(t)$, for all $v \in D(X, 0)$. Let $\{a_n\}$ be a sequence of points of $\mathbb{R}^m$ tending to $0 \in \mathbb{R}^m$, such that $\lim_{n \to \infty} \frac{a_n}{\|a_n\|} = v \in D(X, 0)$. For each n, we define $t_n = \|a_n\|$ and let $b_n \in X$ such that

$$\mathrm{dist}(t_n v, X) > \|t_n v - b_n\| - \|t_n\|^2. \tag{5}$$

We have also the following

$$\frac{\|a_n - b_n\|}{\|a_n\|} \leq \frac{\|a_n - t_n v\|}{t_n} + \frac{\|t_n v - b_n\|}{t_n}.$$

Since by assumption $\mathrm{dist}(tv, X) = o(t)$, it follows from inequality (5) and the choice of a_n that

$$\lim_{n \to +\infty} \frac{\|a_n - b_n\|}{\|a_n\|} = 0.$$

Now note that $\frac{\|a_n-b_n\|}{\|b_n\|}=\frac{\|a_n\|}{\|b_n\|}\frac{\|a_n-b_n\|}{\|a_n\|}$.

Claim. The sequence $\left\{\frac{\|a_n\|}{\|b_n\|}\right\}_{n\in\mathbb{N}}$ is a bounded sequence.

Indeed, by taking a subsequence, if necessary, assume that $\lim\limits_{n\to+\infty}\frac{\|a_n\|}{\|b_n\|}=\infty$ or equivalently $\lim\limits_{n\to+\infty}\frac{\|b_n\|}{\|a_n\|}=0$. Then,

$$\lim_{n\to+\infty}\frac{\operatorname{dist}(t_nv,X)}{t_n}\geq\lim_{n\to+\infty}\|v-\frac{b_n}{t_n}\|-\lim_{n\to+\infty}t_n=\|v\|>0,$$

which generates a contradiction, since $\operatorname{dist}(tv,X)=o(t)$.

Thus, $\left\{\frac{\|a_n\|}{\|b_n\|}\right\}_{n\in\mathbb{N}}$ is a bounded sequence, which implies $\lim_{n\to+\infty}\frac{\|a_n-b_n\|}{\|b_n\|}=0$.

Therefore X satisfies the SSP-condition at 0.

Finally, we are going to show that: (3) $\Rightarrow$ (1). Assume that X satisfies SSP-condition at 0. Let $T=\{t_n\}_{n\in\mathbb{N}}$ and $S=\{s_n\}_{n\in\mathbb{N}}$ be sequences of positive numbers tending to 0. Take $v\in C^T(X,0)\setminus\{0\}$, so there exists a sequence of points $\{x_n\}_{n\in\mathbb{N}}\subset X$ such that $\lim_{n\to+\infty}\frac{x_n}{t_n}=v$. Now we define the sequence $\{a_n\}$ given by $a_n=s_nv$ and note that $\lim_{n\to\infty}a_n=0$ and

$$\lim_{n\to\infty}\frac{a_n}{\|a_n\|}=\lim_{n\to\infty}\frac{a_n/s_n}{\|a_n/s_n\|}=\frac{v}{\|v\|}\in D(X,0).$$

Since X satisfies SSP-condition at 0, there exists a sequence $\{b_n\}\subset X$ such that

$$\lim_{n\to+\infty}\frac{\|a_n-b_n\|}{\|a_n\|}=0,$$

then

$$\lim_{n\to+\infty}\frac{b_n}{\|a_n\|}=\frac{v}{\|v\|}$$

and consequently

$$\lim_{n\to+\infty}\frac{b_n}{s_n}=v\in C^S(X,0).$$

Thus, $C^T(X,0)\subset C^S(X,0)$. Analogously, we can also prove $C^T(X,0)\subset C^S(X,0)$, which implies that X has a unique tangent cone at 0. □

As a consequence of Corollary 3.2 and Theorem 4.2, we have obtained the following results proved in [7].

Corollary 4.4 (Theorem 3.6 in [7])**.** *Let A, $B\subset\mathbb{R}^n$ be set-germs at $0\in\mathbb{R}^n$ such that $0\in\overline{A}\cap\overline{B}$, and let $\phi\colon A\to B$ be a bi-Lipschitz homeomorphism. If both A,B satisfy SSP-condition at 0, then $d\tilde{\tilde{\phi}}(LD(A,0))=LD(B,0)$.*

Corollary 4.5 (Theorem 3.12 in [7])**.** *Let A, $B\subset\mathbb{R}^n$ be set-germs at $0\in\mathbb{R}^n$ such that $0\in\overline{A}\cap\overline{B}$, and let $h:(\mathbb{R}^n,0)\to(\mathbb{R}^n,0)$ be a bi-Lipschitz homeomorphism. Suppose that A, B, $h(A)$, $h(B)$ satisfy SSP-condition at 0. Then we have the equality of dimensions,*

$$\dim(D(h(A),0)\cap D(h(B),0))=\dim(D(A,0)\cap D(B,0)).$$

Definition 4.6. *We say that $X\subset\mathbb{R}^m$ has a* **unique directional cone at** $p\in\overline{X\setminus\{p\}}$ *if for any sequence of positive numbers $\{t_j\}_{j\in\mathbb{N}}$ tending to 0 and $X_{t_j}=[\frac{1}{t_j}(X-p)]\cap\mathbb{S}^{m-1}$, we have that $\lim\limits_{j\to+\infty}d_H(X_{t_j},D(X,p))=0$.*

Thus, we have also the following.

Proposition 4.7. *Let $X \subset \mathbb{R}^m$ be a subset such that $p \in \mathbb{R}^m$ is a non-isolated point of $\overline{X}$. If X has a unique directional cone at p then X has a unique tangent cone at p.*

Proof. We may assume that $p = 0$. Let $\{t_j\}_{j\in\mathbb{N}}$ be a sequence of positive numbers tending to 0. It is enough to prove that $C^T(X,0) = LD(X,0)$. Indeed, it is obvious that $C^T(X,0) \subset LD(X,0)$. Now, let $w \in LD(X,0) \setminus \{0\}$. Then there exist $v \in D(X,0)$ and $\lambda > 0$ such that $w = \lambda v$. For each j, define $s_j = \lambda t_j$. By assumption $\lim\limits_{j\to+\infty} d_H(X_{s_j}, D(X,0)) = 0$, hence there exists a sequence of points $\{y_j\}_{j\in\mathbb{N}} \subset X$ with $\|y_j\| = s_j$ and $\lim\limits_{j\to+\infty} \frac{y_j}{s_j} = v$. Thus,

$$\frac{y_j}{t_j} = \lambda \frac{y_j}{s_j} \to \lambda v = w.$$

Therefore $LD(X,0) \subset C^T(X,0)$, which proves that $C^T(X,0) = LD(X,0)$. □

Surprisingly, the converse of Proposition 4.7 does not hold true, as we can see in the next examples.

Example 4.8. *Let $X = (\mathbb{R}^2 \setminus \bigcup\limits_{n=1}^{\infty} \mathbb{S}^2(0;1/n)) \cup \{(x,y); y = 0\}$. We have that X has a unique tangent cone at 0, but $\lim\limits_{j\to+\infty} d_H(X_{1/j}, D(X,0)) \neq 0$, where $X_{1/j} = (jX) \cap \mathbb{S}^1 = \{(\pm 1, 0)\}$.*

In fact, we can also obtain an example with X being a closed set.

Example 4.9. *For each $j \in \mathbb{N}$, let*

$$R_j = \{(r\cos\theta, r\sin\theta) \in \mathbb{R}^2; \theta \in (-\pi/4, \pi/4)$$

and $r \in (1/j - 1/j^3, 1/j + 1/j^3)\}$. Let $R = \bigcup\limits_{j=1}^{\infty} R_j$ and $X = \mathbb{R}^2 \setminus R$. We have that X has a unique tangent cone at 0, which is $\mathbb{R}^2$. In fact, let $T = \{t_j\}_{j\in\mathbb{N}}$ be a sequence of positive numbers tending to 0 and let $v = (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$. We define

$$a_j = \begin{cases} t_j v, & \text{if } t_j v \notin R, \\ (t_j + \frac{2}{m^3\|v\|})v, & \text{if } t_j v \in R_m. \end{cases}$$

Thus, $\{a_j\} \subset X$ and $\lim \frac{a_j}{t_j} = v$. This shows that $C^T(X,0) = \mathbb{R}^2$. However,

$$\lim_{j\to+\infty} d_H(X_{1/j}, D(X,0)) \neq 0,$$

where $X_{1/j} = (jX) \cap \mathbb{S}^1$.

4.1. **Looking at infinity.** With an easy adaptation of the proof of Theorem 3.1, we obtain also the following:

Proposition 4.10. *Let $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^n$ be non-empty subsets. Assume that there exist compact sets $K \subset \mathbb{R}^m$ and $\tilde{K} \subset \mathbb{R}^n$ $\phi \colon A \setminus K \to B \setminus \tilde{K}$ be a bi-Lipschitz homeomorphism such that*

$$\frac{1}{K_1}\|x - y\| \leq \|\phi(x) - \phi(y)\| \leq K_2\|x - y\|, \quad \forall x, y \in A \setminus K,$$

where $K_1, K_2 > 0$. Then, for any sequence of positive numbers $T = \{t_j\}_{j\in\mathbb{N}}$ such that $\lim t_j = +\infty$, there exist a subsequence $S = \{s_j\}_{j\in\mathbb{N}} \subset T = \{t_j\}_{j\in\mathbb{N}}$ and a global bi-Lipschitz homeomorphism $d\phi \colon C^S(A,\infty) \to C^S(B,\infty)$ such that $d\phi(0) = 0$ and

$$\frac{1}{K_1}\|x - y\| \leq \|d\phi(x) - d\phi(y)\| \leq K_2\|x - y\|, \quad \forall x, y \in C^S(A,\infty).$$

As a corollary of Proposition 4.10, we have the generalized result of Theorem 2.19 in [3] to the case of sets with unique tangent cones.

Corollary 4.11. *Let $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^n$ be non-empty subsets which have unique tangent cone at infinity. Assume that there exist compact sets $K \subset \mathbb{R}^m$ and $\tilde{K} \subset \mathbb{R}^n$ such that $A \setminus K$ and $B \setminus \tilde{K}$ are bi-Lipschitz homeomorphic, then $C(A, \infty)$ and $C(B, \infty)$ are bi-Lipschitz homeomorphic as well.*

Definition 4.12. *Let A be a set-germ at $p \in \mathbb{R}^n$ such that p is a non-isolated point of $\overline{A}$. We define the* **direction set of A at infinity**, *$D(A, \infty)$, by*

$$D(A, \infty) := \{v \in S^{n-1} \mid \exists \{x_i\} \subset A \text{ s.t. } \|x_i\| \to +\infty \text{ and } \frac{x_i}{\|x_i\|} \to v,\ i \to \infty\}.$$

We put $LD(A, \infty) := \{ta \in \mathbb{R}^n \mid a \in D(A, \infty),\ t \geq 0\}$.

Definition 4.13. *Let $A \subset \mathbb{R}^n$ be an unbounded set. We say that A satisfies* **SSP-condition at infinity**, *if for any sequence of points $\{a_m\}$ of $\mathbb{R}^n$ tending to infinity such that*

$$\lim_{m \to \infty} \frac{a_m}{\|a_m\|} \in D(A, \infty),$$

there is a sequence of points $\{b_m\} \subset A$ such that,

$$\lim_{m \to \infty} \frac{\|a_m - b_m\|}{\|a_m\|} = \lim_{m \to \infty} \frac{\|a_m - b_m\|}{\|b_m\|} = 0.$$

Thus, with easy adaptations of the proofs of Theorem 4.2 and Proposition 4.7, we obtain also the following results at infinity.

Proposition 4.14. *Let $X \subset \mathbb{R}^m$ be an unbounded subset. Then the following statements are equivalents:*

(1) X has a unique tangent cone at infinity;
(2) $dist(tv, X) = o_\infty(t)$ for all $v \in D(X, \infty)$, where $g(t) = o_\infty(t)$ means $\lim\limits_{t \to +\infty} \frac{g(t)}{t} = 0$;
(3) X satisfies the SSP-condition at infinity.

Proposition 4.15. *Let $X \subset \mathbb{R}^m$ be an unbounded closed subset. Assume that for any sequence of positive numbers $\{t_j\}_{j \in \mathbb{N}}$ tending to $+\infty$ and $X_{t_j} = (\frac{1}{t_j} X) \cap \mathbb{S}^{m-1}$, we have that $\lim\limits_{j \to +\infty} d_H(X_{t_j}, D(X, \infty)) = 0$. Then X has a unique tangent cone at infinity.*

We finish this article with an example which shows that the converse of Proposition 4.15 does not hold.

Example 4.16. *For each $j \in \mathbb{N}$, let*

$$R_j = \{(r\cos\theta, r\sin\theta) \in \mathbb{R}^2; \theta \in (-\pi/4, \pi/4)$$

and $r \in (j - 1/j^3, j + 1/j^3)\}$. Let $R = \bigcup\limits_{j=1}^{\infty} R_j$ and $X = \mathbb{R}^2 \setminus R$. We have that X has a unique tangent cone at infinity, which is $\mathbb{R}^2$. However, $\lim\limits_{j \to +\infty} d_H(X_j, D(X, \infty)) \neq 0$, where $X_j = (\frac{1}{j} X) \cap \mathbb{S}^1$.

Acknowledgements. The authors would like to thank the anonymous referee for the useful comments and suggestions.

References

[1] Bernig, A. and Lytchak, A. *Tangent spaces and Gromov-Hausdorff limits of subanalytic spaces* J. Reine Angew. Math., vol. 608 (2007), 1–15. DOI: 10.1515/crelle.2007.050

[2] Coste, M. *An introduction to o-minimal geometry*, Dottorato di Ricerca in Matematica, Dip. Mat. Pisa. Instituti Editoriali e Poligrafici Internazionali, 2000.

[3] Fernandes, A. and Sampaio, J. E. *On Lipschitz rigidity of complex analytic sets*. The Journal of Geometric Analysis, vol. 30 (2020), 706–718. DOI: 10.1007/s12220-019-00162-x

[4] Hironaka, H. *Subanalytic sets*, Number Theory, Algebraic Geometry and Commutative Algebra, in honor of Yasuo Akizuki, pp. 453–493, Kinokuniya, Tokyo, 1973. DOI: 10.1007/978-4-431-67947-9_53

[5] Koike, S. and Paunescu, L. *The directional dimension of subanalytic sets is invariant under bi-Lipschitz homeomorphisms*. Annales de l'Institut Fourier, Tome 59 (2009) no. 6, 2445–2467. DOI: 10.5802/aif.2496

[6] Koike, S. and Paunescu, L. *On the geometry of sets satisfying the sequence selection property.* J. Math. Soc. Japan, vol. 67 (2015), no. 2, 721–751. DOI: 10.2969/jmsj/06720721

[7] Koike, S. and Paunescu, L. *Applications of the sequence selection property to bi-Lipschitz geometry*. European Journal of Mathematics, vol. 5 (2019), 1202-–1211. DOI: 10.1007/s40879-018-0291-4

[8] Meeks III, W. H. *Global problems in classical minimal surface theory.* In Global theory of minimal surfaces, pages 453–470. American Mathematical Society, Providence, RI, for the Clay Mathematics Institute, Cambridge, MA, edited by D. Hoffman, 2005.

[9] Sampaio, J. Edson *Bi-Lipschitz homeomorphic subanalytic sets have bi-Lipschitz homeomorphic tangent cones*. Selecta Math. (N.S.), vol. 22 (2), 553–559 (2016). DOI: 10.1007/s00029-015-0195-9

J. Edson Sampaio, Departamento de Matemática, Universidade Federal do Ceará, Rua Campus do Pici, s/n, Bloco 914, Pici, 60440-900, Fortaleza-CE, Brazil

Email address: edsonsampaio@mat.ufc.br

E. Carvalho da Silva, Departamento de Matemática, Instituto Federal de Educação, Ciência e Tecnologia do Ceará, Av. Parque Central, 1315, Distrito Industrial I, 61939-140, Maracanaú-CE, Brazil

Email address: euripedes.carvalho@ifce.edu.br