Journal *of* Singularities
Volume 25 (2022), 348-360

Proc. of 16th International Workshop on Singularities, São Carlos, 2020
DOI: 10.5427/jsing.2022.25p

INVARIANTS AND CLASSIFICATION OF SIMPLE FUNCTION GERMS WITH RESPECT TO LIPSCHITZ $\mathscr{A}$-EQUIVALENCE

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Abstract. In this note we discuss the classification of Lipschitz simple function germs. The full classification under Lipschitz right-equivalence has been established by M. Ruas and the authors. We consider Lipschitz left-right equivalence here and show that the classification for this equivalence coincides with the classification for Lipschitz right-equivalence.

1. Introduction

M. Ruas and the authors in [18] completely classified finitely determined simple complex analytic function germs under Lipschitz $\mathscr{R}$-equivalence ($\mathscr{R}$ stands for 'right'). In this article we show that the classification of Lipschitz $\mathscr{A}$-simple germs ($\mathscr{A}$ stands for 'left-right' equivalence) coincides with that of Lipschitz $\mathscr{R}$-simple germs. The idea comes from the classification presented in the above mentioned article which is quite technical in nature. For this reason, we present a less technical introduction to [18] for better accessibility to the reader.

That the classifications of simple germs coincide for smooth $\mathscr{R}$ and $\mathscr{A}$-equivalence follows from the fact that the smooth $\mathscr{A}$-classification agrees with smooth $\mathscr{R}$-classification for quasi-homogeneous germs. However, if the germs are not quasihomogenous the smooth $\mathscr{A}$-type and $\mathscr{R}$-type might differ. For example, the family $f_t(x,y) = x^5+y^5+tx^3y^3$ is not smoothly $\mathscr{R}$-trivial, for x^3y^3 does not belong to the $\mathscr{R}$-tangent space. The family however is smoothly $\mathscr{A}$-trivial for non-zero t. Since the list of Lipschitz $\mathscr{R}$-simple germs contains non-quasihomogenous singularities, the above argument cannot be applied directly to show that the classifications of Lipschitz $\mathscr{R}$-simple and Lipschitz $\mathscr{A}$-simple germs coincide. Our conclusion is an observation coming from the complete classification.

The paper is organized as follows. We first recall in Section 2 the classical idea of proving Lipschitz $\mathscr{R}$-triviality of a deformation, and then present a concrete example of a deformation in the real case which is Lipschitz trivial but has different Milnor numbers at two different parameter values. This shows that the Milnor number is not a Lipschitz $\mathscr{R}$-invariant. In fact, the Milnor number is also not a C^p-invariant for any $p \in \mathbb{N}$ as is shown by giving a counterexample.

In Section 3 we present some Lipschitz invariants related to finitely determined singularities. We first discuss the invariance of corank which is defined as the nullity of the Hessian of a germ at 0. This is interesting since it partially answers the question of Arnold whether corank is a topological invariant; see problem 1975-14 in Arnold [2]. Next we show the invariance of the multiplicty and singular locus of non-quadratic part of the germ obtained after applying the splitting lemma. The section ends with some open questions.

Section 4 discusses the notion of Lipschitz $\mathscr{R}$-simplicity and the idea behind the classification with least technical details as possible. In Section 5, we show that certain families of germs cannot deform to J_{10} using results of Kushnirenko [13] and Newton diagrams. Finally, in Section 6 we show that a germ is Lipschitz $\mathscr{A}$-simple if and only if it is Lipschitz $\mathscr{R}$-simple. The idea is

The authors would like to thank Maria Ruas for helpful discussions. The second author was supported by SERB-MATRICS Grant - MTR/2021/000043.

essentially the same as in the classification of simple germs under Lipschitz $\mathscr{R}$-equivalence. The key is to show that J_{10} is Lipschitz $\mathscr{A}$-modal family.

2. Milnor number and Triviality of deformations

2.1. **Milnor number.** Given a germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ $(n \geq 2)$ of a complex isolated hypersurface singularity, the *Milnor number* (or the *codimension*) $\mu(f)$ of f is defined to be the $\mathbb{C}$-vector space dimension of $\mathscr{E}_n/Jf$, where $\mathscr{E}_n$ is the local ring of complex analytic germs from $(\mathbb{C}^n, 0)$ to $\mathbb{C}$ (i.e. germs of complex analytic functions defined on a neighbourhood of 0 in $\mathbb{C}^n$) and Jf is the ideal of $\mathscr{E}_n$ generated by the partial derivatives of f. The Milnor number in the complex case has a purely topological description, namely the Milnor fiber of f has the homotopy type of a bouquet of spheres and the number of spheres in the bouquet is precisely the Milnor number; this is now known as the Milnor-Palamodov Theorem, see Milnor [16] and Palamodov [19]. This implies that the Milnor number is a topological invariant in the complex case.

One can define the Milnor number in the real case analogously. By abuse of notation, let $\mathscr{E}_n$ be the local ring of germs of C^∞ functions (also called smooth functions) on $\mathbb{R}^n$ at the origin, and $\mathfrak{m}_n$ be its maximal ideal (the germs that vanish at 0). Given $f \in \mathscr{E}_n$ define $\mu(f) = \dim_{\mathbb{R}} \mathscr{E}_n/Jf$, where Jf is the Jacobian ideal of f. It is then natural to ask if the Milnor number is a topological invariant in the real case. This is certainly not true, for the germs of $f(x) = x$ and $g(x) = x^3$ are topologically equivalent but they have different Milnor numbers. However, we know that:

1. The Milnor number $\mu(f)$ is a C^∞-invariant. That is, if f and g are C^∞-equivalent then $\mu(f) = \mu(g)$. A proof of this can be found in Trotman [23].

2. The Milnor number mod 2 is a topological invariant, see Wall [24]. That is, if f is topologically equivalent to g, then $\mu(f) \equiv \mu(g) \mod 2$. There are some simplifications of the result of Wall by Dudzinski et al. [7]. Wall, in the same article, also pointed out that the topological type of $f(x_1, \ldots, x_n) = x_1^{2i} + x_2^2 + \cdots + x_j^2 - x_{j+1}^2 - \cdots - x_n^2$ is determined by n and j, but the Milnor number of f is $2i - 1$.

It was first shown in [18] the Milnor number is not a Lipschitz $\mathscr{R}$-invariant by constructing a family of germs which is Lipschitz $\mathscr{R}$-trivial but the Milnor numbers are different for two distinct parameter values. In the following we describe how to prove the Lipschitz triviality of a given family.

2.2. **Triviality.** We begin with some definitions. Two germs $f, g \in \mathscr{E}_n$ are said to be *bi-Lipschitz $\mathscr{R}$-equivalent* if there exists a germ $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ of a bi-Lipschitz homeomorphism such that $f \circ h = g$. Recall that h is a bi-Lipschitz homeomorphism if there exists a real number $K \geq 1$ such that

$$\frac{1}{K}\|x - y\| \leq \|h(x) - h(y)\| \leq K\|x - y\|.$$

on a neighbourhood of 0. This is equivalent to say that h is a homeomorphism which is Lipschitz and whose inverse is also Lipschitz.

Given a germ $f_0 : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$, a smooth germ $F : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ such that $F(0, t) = 0$ for all t is said to be a *(one-parameter) deformation* of f_0 if $F(x, 0) = f_0(x)$. The deformation F is said to be smoothly trivial if there exists a germ of a diffeomorphism $H : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^n \times \mathbb{R}, 0)$ of the form $H(x, t) = (h(x, t), t)$ with $h(x, 0) = x$ and $h(0, t) = 0$ such that

$$F \circ H(x, t) = f_0(x). \tag{1}$$

As a consequence of the definition of a smoothly trivial deformation it follows that for every t, $F_t = F(\cdot, t)$ and f_0 are smoothly equivalent; see page 33-34 in Martinet [15]. If H is a bi-Lipschitz germ, F is said to be Lipschitz trivial. If H is only a germ of a homeomorphism, then F is said to be a topologically trivial deformation of f_0; see page 340 in Damon and Gaffney [6].

We can differentiate Equation (1) with respect to t to get

$$\sum_{i=1}^{n} \frac{\partial H_i}{\partial t}(x,t)\frac{\partial F}{\partial x_i}(H(x,t)) + \frac{\partial F}{\partial t}(H(x,t)) = 0. \tag{2}$$

Now, if we put $X(x,t) = \frac{\partial}{\partial t} + \sum_{i=1}^{n} X_i(x,t)\frac{\partial}{\partial x_i}$ where $X_i(x,t) = \frac{\partial H_i}{\partial t}(x,t)$, then Equation (2) can be rewritten as $X.F = 0$ (here x_i's are the coordinates of $\mathbb{R}^n$). This implies that if F is smoothly trivial, then there exists a vector field $X(x,t) = \frac{\partial}{\partial t} + \sum_{i=1}^{n} X_i(x,t)\frac{\partial}{\partial x_i}$ with $X_i(0,t) = 0$, $i = 1, \dots, n$, such that $X.F = 0$.

The converse of the above statement is also true with a weaker condition on the vector field, namely that the vector field be Lipschitz. More precisely, let $F : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ be a deformation of a function $f_0 : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$, i.e. $F(x, 0) = f_0(x)$. Suppose there exists a vector field $X(x,t)$ of the form

$$X(x,t) = \frac{\partial}{\partial t} + \sum_{i=1}^{n} X_i(x,t)\frac{\partial}{\partial x_i},$$

with each $X_i : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ a germ of a Lipschitz function, such that $X_i(0,t) = 0$, $i = 1, \dots, n$ and

$$X.F = \frac{\partial F}{\partial t} + \sum_{i=1}^{n} X_i(x,t)\frac{\partial F}{\partial x_i} = 0. \tag{3}$$

Think of X as a time dependent vector field. Suppose $H : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^n, 0)$ is the germ of the flow of X.

Since X is a Lipschitz vector field, H is a Lipschitz flow and for any fixed t, $H_t(x) = H(x,t)$ is a bi-Lipschitz map germ. Then,

$$\frac{\partial H}{\partial t}(x,t) = X(H(x,t),t) \tag{4}$$

$$= (X_1(H(x,t),t), \dots, X_n(H(x,t),t)) \tag{5}$$

$$= \left(\frac{\partial H_1}{\partial t}(x,t), \dots, \frac{\partial H_n}{\partial t}(x,t)\right) \tag{6}$$

From (1) and (2) we get:

$$\frac{\partial F}{\partial t}(H(x,t),t)) + \sum_{i=1}^{n} X_i(H(x,t),t)\frac{\partial F}{\partial x_i}(H(x,t),t) = 0. \tag{7}$$

From (5) we get:

$$\frac{\partial F}{\partial t}(H(x,t),t)) + \sum_{i=1}^{n} \frac{\partial H_1}{\partial t}(x,t)\frac{\partial F}{\partial x_i}(H(x,t),t) = 0. \tag{8}$$

Now, put $\Phi(x,t) = (H(x,t),t)$. Since H_t is bi-Lipschitz, Φ is also bi-Lipschitz. By the Chain Rule, (6) can be rewritten as:

$$\frac{\partial (F \circ \Phi)}{\partial t}(x,t) = 0 \tag{9}$$

But, since F is a deformation of f_0, $F(x,0) = f$, thus for all t:

$$F \circ \Phi(x,t) = f_0(x)$$

This implies that for every t, $F_t = F(\cdot, t)$ is Lipschitz equivalent to f_0. That is, the deformation F is Lipschitz trivial along the parameter t and the trivialization is given by integration of the Lipschitz vector field X. This is called the Thom-Levine Theorem and we state it as follows:

Theorem 2.1. *Let $F : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ be a deformation of a germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$. If there exists a Lipschitz vector field X of the form:*

$$X(x,t) = \frac{\partial}{\partial t} + \sum_{i=1}^{n} X_i(x,t) \frac{\partial}{\partial x_i},$$

such that $X_i(0,t) = 0$, $i = 1, \ldots, n$ and $X.F = 0$, then F is a Lipschitz trivial deformation.

The converse of the above theorem holds, as we have seen, if F is smoothly trivial. In the case of Lipschitz trivialization, the converse is still an open problem. It is for this reason that in some places a Lipschitz trivial deformation whose trivialization is given by integrating a Lipschitz vector field is termed a strongly Lipschitz trivial deformation.

2.3. **Lipschitz triviality.** To decide whether a given deformation is Lipschitz trivial, Theorem 2.1 suggests we seek a Lipschitz vector field which acts on the deformation trivially. But, this is not an easy task and requires some work which we will explain in this section.

So, given a deformation $F : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$, we want to find a Lipschitz vector field X such that $X.F = 0$. This is where Kuo vector fields (Kuo [11, 12]) are helpful. To understand the idea, let us restrict to 2-variables.

Given a one-parameter deformation $F : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ of $f : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$, put

$$\rho = \left(\frac{\partial F}{\partial x_1}\right)^2 + \left(\frac{\partial F}{\partial x_2}\right)^2.$$

Then,

$$\rho.\frac{\partial F}{\partial t} = \left(\frac{\partial F}{\partial t}\frac{\partial F}{\partial x_1}\right)\frac{\partial F}{\partial x_1} + \left(\frac{\partial F}{\partial t}\frac{\partial F}{\partial x_2}\right)\frac{\partial F}{\partial x_2} \tag{10}$$

Put $A = \left(\frac{\frac{\partial F}{\partial t}\frac{\partial F}{\partial x_1}}{\rho}\right)$ and $B = \left(\frac{\frac{\partial F}{\partial t}\frac{\partial F}{\partial x_2}}{\rho}\right)$. Then, Equation (10) can be rewritten as

$$\frac{\partial F}{\partial t} = A\frac{\partial F}{\partial x_1} + B\frac{\partial F}{\partial x_2}.$$

The function ρ is called a control function.

So, to prove that a deformation F is Lipschitz trivial, we attempt to show that the vector field

$$X = \frac{\partial}{\partial t} + A\frac{\partial}{\partial x_1} + B\frac{\partial}{\partial x_2},$$

called the Kuo vector field, is Lipschitz. This is equivalent to showing that A and B are Lipschitz functions, since a multivariable function is Lipschitz if and only if its components are Lipschitz. Observe that the idea can easily be generalized to any number of variables.

In the case where the deformation F of $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ is of the form

$$F(x,t) = f(x) + \theta(x,t)$$

and f is a quasihomogeneous germ, then we have a result of Fernandes and Ruas [8] giving sufficient conditions for Lipschitz triviality of F. We need some definitions for the statement of this result.

Given $\omega = (w_1, \dots, w_n)$ an n-tuple of non-negative integers in increasing order, the filtration of the monomial $x_1^{\alpha_1} \dots x_n^{\alpha_n}$ with respect to ω is defined to be $\sum_{i=1}^n \alpha_i w_i$. And, given a polynomial f the filtration of f with respect to ω is given by

$$fil(f) = \min\{fil(x^\alpha) | x^\alpha \text{ is a monomial of } f\}.$$

Then, the result of Fernandes and Ruas [8] says that:

Theorem 2.2. *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be the germ of a quasihomogeneous polynomial with isolated singularity of weight $(w_1, \dots, w_n)$ and total weight d such that $(w_1 \leq \dots \leq w_n)$. Let $f_t(x) = f(x) + t\theta(x, t)$ be a one-parameter deformation of f. If $fil(\theta) \geq d + w_n - w_1$, then f_t is a Lipschitz trivial deformation of f (for t sufficiently close to the origin).*

To prove the result, the authors show that the Kuo vector field associated to the deformation is Lipschitz and then use the Thom-Levine theorem to conclude the result.

Observe that, if t is not close to the origin in the above theorem, then the theorem fails to hold. Consider for example the deformation $f_t(x, y) = x^2 + y^2 + t(xy + y^3)$. This family satisfies the hypothesis of the above theorem. But

$$f_2(x, y) = (x + y)^2 + 2y^3 \qquad \text{and} \qquad f_1(x, y) = x^2 + y^2 + xy + y^3$$

cannot be Lipschitz equivalent because the tangent cones of their zero loci are different. It is known that the Lipschitz type of the tangent cone is a Lipschitz invariant of the variety; see Sampaio [21]. In [18], Theorem 7.9, we showed a global version of the above theorem by adding the condition that the initial part of f_t has isolated singularity for all t. The result holds also for the complex case. Let us recall the notion of initial part of a germ. Given a weight $w = (w_1, \dots, w_n)$ and a smooth germ f, we may write the Taylor expansion of f at 0 as $T_0 f = f_d + f_{d+1} + \dots$ where $f_d \neq 0$ and f_k is a weighted homogeneous polynomial of the total weight k. The polynomial f_d is called the *initial part of f* (with respect to the weight w).

Theorem 2.3 ([18], Theorem 7.9)**.** *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be the germ of a quasi homogeneous polynomial of weight $(w_1, \dots, w_n)$ and total weight d such that $(w_1 \leq \dots \leq w_n)$. A smooth family $f_t(x) = f(x) + t\theta(x, t)$ is Lipschitz trivial if the initial part of f_t has an isolated singularity for every t and $fil(\theta) \geq d + w_n - w_1$.*

2.4. **Non-invariance of Milnor number.** The following are examples showing that the Milnor number is not a Lipschitz invariant. In fact, it is not even a C^r-invariant for any finite r, in contrast with the C^∞ case.

Consider the one-parameter deformation

$$f_t(x, y) = x^4 + y^4 + t(x^2y^2 + y^6) \quad (t \geq 0)$$

and observe that this deformation satisfies the hypotheses of Theorem 2.3. Thus, this deformation is Lipschitz $\mathscr{R}$-trivial. However, for $t \neq 2$ the Milnor number of f_t is 9 while the Milnor number of f_2 is 13. This shows that the Milnor number is not a Lipschitz $\mathscr{R}$-invariant.

Furthermore, consider the deformation f_t given by

$$f_t(x, y) = x^4 + y^4 + 2x^2y^2 + ty^{4+k}, (k \geq 1).$$

This deformation is C^k-trivial since it can be easily shown that the Kuo vector field associated to f_t is C^k. However, the Milnor number $\mu(f_0) = \infty$ while $\mu(f_t) = 13$. Thus, the Milnor number is not a C^k-invariant for any finite k. This fact about C^p-non-invariance of Milnor number also follows from Takens [22]. Takens mentions in his article that the functions $(x_1^2 + x_2^2)^2 + x_1^{p+5}$ and $(x_1^2 + x_2^2)^2$ are C^1-equivalent but not C^{p+2}-equivalent and attributes the result to Kuiper [10]. The result follows by adding a parameter to the functions.

3. Lipschitz Invariance of Corank

Given a smooth function germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$, its Taylor expansion at 0 can be written as $T_0 f(x) = \sum_{i=k}^{\infty} f_k(x)$ where f_k is a homogeneous polynomial of degree k. The smallest k such that $f_k \neq 0$ is called the multiplicity of f, denoted by m_f, and H_k is called the lowest degree homogeneous polynomial of f, denoted by H_f. We denote by Σ_f the singular locus of f, i.e., $\Sigma_f = \{x \in \mathbb{R}^n : Df(x) = 0\}$.

It was shown in [18] that the corank of a function germ in $\mathfrak{m}_n^2$ is a Lipschitz $\mathscr{R}$-invariant. The proof of this result is based on the following important properties:

Lemma 3.1. *Let $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be two smooth germs. If f and g are Lipschitz $\mathscr{R}$-equivalent then:*

(1) The multiplicities $m_f = m_g$.
(2) The lowest degree homogeneous polynomials of f and g are Lipschitz $\mathscr{R}$-equivalent.
(3)

$$L^{-1}\|Df(h(x))\| \le \|Dg(x)\| \le L\|Df(h(x))\|$$

where $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ is a bi-Lipschitz homeomorphism such that $g = f \circ h$ and L is the Lipschitz constant of h.

The Lipschitz invariance of multiplicities in the complex case was proved by Risler and Trotman [20] and in the real case by Fernandes and Ruas [8].

The theorem is as follows:

Theorem 3.2. *Let f, g be finitely determined germs in $\mathfrak{m}_n^2$. If f and g are Lipschitz $\mathscr{R}$-equivalent then the corank of f is equal to the corank of g at 0.*

Let f, g be finitely determined germs in $\mathfrak{m}_{n+p}^2$. Assume that coranks of f and g are r and s respectively. By the Splitting Lemma, up to smooth change of coordinates we can write $f = \tilde{f}(x_1, \dots, x_r) + Q_f(x_{r+1}, \dots, x_n)$ and $g = \tilde{g}(x_1, \dots, x_s) + Q_g(x_{s+1}, \dots, x_n)$ where Q_f and Q_g are non-degenerate quadratic polynomials. By Theorem 3.2, if f and g are Lipschitz $\mathscr{R}$-equivalent, then $r = s$. The following is proved in [18] Theorem 5.1.

Theorem 3.3. *Suppose that f and g are Lipschitz $\mathscr{R}$-equivalent. Then,*

(i) $m_{\tilde{f}} = m_{\tilde{g}}$.
(ii) $\Sigma_{\tilde{f}}$ and $\Sigma_{\tilde{g}}$ are Lipschitz equivalent.

The proof of the above result also relies on properties in Lemma 3.1.

For Lipschitz $\mathscr{A}$-equivalence, it is not so difficult to prove that the results in Lemma 3.1 still hold. The multiplicity is also known as an invariant of a weaker equivalence relation called bi-Lipschitz contact equivalence (see for example Birbrair [3], Nguyen [17]). A proof for (2) can be found in [17]. Proof for (3) is similar to the case of Lipschitz $\mathscr{R}$-equivalence. Then, following the proof of Theorem 3.2 one can show that corank is invariant under Lipschitz $\mathscr{A}$-equivalence. Similarly, results in Theorem 3.3 are also true for this case. In conclusion, we have

Theorem 3.4. *Suppose that f and g are Lipschitz $\mathscr{A}$-equivalent. Then*

(1) $corank(f) = corank(g)$.
(2) $m_{\tilde{f}} = m_{\tilde{g}}$.
(3) $\Sigma_{\tilde{f}}$ and $\Sigma_{\tilde{g}}$ are Lipschitz equivalent.

Remark 3.5. All the above results are also valid for complex analytic germs. The proofs are similar.

Question 3.6. *Suppose that f and g are Lipschitz $\mathcal{R}$-equivalent (resp. Lipschitz $\mathcal{A}$-equivalent).*
1) Are the zero sets of $\tilde{f}$ and $\tilde{g}$ Lipschitz equivalent?
2) Are $\tilde{f}$ and $\tilde{g}$ Lipschitz $\mathcal{R}$-equivalent (resp. Lipschitz $\mathcal{A}$-equivalent)?

Remark 3.7. Arnold in 1975 (see Problem 1975-14 in [2]) asked if the corank of a germ is a topological invariant and remains unanswered to this date. Considering that the topological invariance of the corank in the complex case is an open problem, our result is the best known partial answer so far.

In the real case, analogous to the case of Milnor number, corank is not a topological invariant. Consider for example $f(x,y) = x^2 + y^2$ and $g(x,y) = x^4 + y^4$. We have $f \circ h = g$ where $h(x,y) = (x|x|, y|y|)$ which is a homeomorphism. It is clear that corank f is 0 while corank of g is 2.

Boardman symbols in general are not topological invariants as was proved by Loojienga [14]. Recently, the first author [17] has given an example showing that Boardman symbols are also not invariant under Lipschitz $\mathscr{R}$-equivalence for both the real and the complex cases.

4. Classifying Lipschitz simple $\mathscr{R}$-germs

The aim of this section is to recall the main result of [18] which is the classification of Lipschitz $\mathscr{R}$-simple complex analytic germs.

A finitely determined germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is said to be *Lipschitz $\mathscr{R}$-simple* if in the jet space $J^k(n,1)$ of sufficiently high-order k, there exists a neighbourhood of $j^k f(0)$ that intersects only finitely many Lipschitz $\mathscr{R}$-classes. Here by a finitely determined germ we mean a germ that is finitely determined with respect to the group $\mathscr{R}$. A germ that is not Lipschitz simple is called a Lipschitz modal germ. Replacing the Lipschitz $\mathscr{R}$-equivalence by the Lipschitz $\mathscr{A}$-equivalence we get the notion of *Lipschitz $\mathscr{A}$-simple* germs.

We denote again by $\mathcal{E}_n$ the ring of all complex analytic germs $(\mathbb{C}^n, 0) \to \mathbb{C}$ and by $\mathfrak{m}_n$ its maximal ideal. The space $\mathcal{E}_n$ can be considered as a bundle of copies of $\mathfrak{m}_n$, so it suffices to reduce the problem to classification of germs in $\mathfrak{m}_n$. Since germs in $\mathfrak{m}_n \setminus \mathfrak{m}_n^2$ are non-singular, they are obviously simple, therefore we only need to classify germs in $\mathfrak{m}_n^2$.

Thom's splitting lemma says that if $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is a finitely determined germ of corank c, then there exists a germ $g \in \mathfrak{m}_c^3$ such that

$$f(x_1, \ldots, x_n) \sim_{\mathscr{R}} g(x_1, \ldots, x_c) + x_{c+1}^2 + \cdots + x_n^2.$$

Moreover, this splitting of f is unique in the sense that if $g + Q \sim_{\mathscr{R}} h + Q$, then $g \sim_{\mathscr{R}} h$ where Q is a quadratic form in the rest of the variables. Owing to this result one can ignore the quadratic part in the classification under holomorphic $\mathscr{R}$-equivalence.

One of the key results used in the classification is the following:

Theorem 4.1 ([18], Theorem 6.4)**.** *The family $J_{10} : f_t(x) = x_1^3 + t x_1 x_2^4 + x_2^6 + Q(x_3, \ldots, x_n)$ is Lipschitz $\mathscr{R}$-modal.*

If the uniqueness property of the splitting lemma were true for the Lipschitz $\mathscr{R}$-equivalence then the above theorem would trivially follow from the result of Henry and Parusiński [9] that the Lipschitz $\mathscr{R}$-type of the family $g_t(x, y) = x^3 + txy^4 + x^6$ varies continuously. Unfortunately, such a property is still unknown.

By Theorem 4.1, J_{10} is a Lipschitz modal family, it then follows from the definition of modality that every germ that deforms to J_{10} is also Lipschitz modal. Here by saying a germ f deforms to a family $\mathcal{D}$ we mean that there is a continuous map $F : (\mathbb{C}^n \times \mathbb{C}, 0) \to (\mathbb{C}, 0)$ such that $F_0 = f$ and $F_t \in \mathcal{D}$ for every t near 0, $t \neq 0$.

In fact, it turns out that to classify Lipschitz simple germs, it is enough to check if a germ deforms to J_{10} or not, for we prove in [18] that:

Theorem 4.2. *The set of all finitely determined germs in $\mathfrak{m}_n^2$ can be split into two disjoint groups. The first one, denoted Ω_1, includes all germs deforming to J_{10}, the second one, denoted Ω_2, is the complement of the first one, which includes all smooth simple germs (A_k, D_k, E_6, E_7, E_8) and finitely many families of smooth unimodal germs as in Table 1.*

Table 1: List of germs in Ω_2

Name	*Normal form*	*Codimension*	*Corank*
$A_k, k \geq 1$	x^{k+1}	k	1
$D_k, k \geq 4$	$x^2y + y^{k-1}$	k	2
E_6	$x^3 + y^4$	6	
E_7	$x^3 + xy^3$	7	
E_8	$x^3 + y^5$	8	
X_9	$x^4 + y^4 + tx^2y^2$	9	2
$T_{2,4,5}$	$x^4 + y^5 + tx^2y^2$	10	
$T_{2,5,5}$	$x^5 + y^5 + tx^2y^2$	11	
Z_{11}	$x^3 + y^5 + txy^4$	11	
W_{12}	$x^4 + y^5 + tx^2y^3$	12	
$T_{p,q,r}, 3 \leq p \leq q \leq k \leq 5$	$x^p + y^q + z^r + txyz$	$p + q + r - 1$	3
Q_{10}	$x^3 + y^4 + yz^2 + txy^3$	10	
Q_{11}	$x^3 + y^2z + xz^3 + tz^5$	11	
S_{11}	$x^4 + y^2z + xz^2 + tx^3z$	11	
S_{12}	$x^2y + y^2z + xz^3 + tz^5$	12	

It is proved in [18] that all families in Table 1 are Lipschitz $\mathscr{R}$-trivial. Consequently, Ω_2 contains only finitely many Lipschitz $\mathscr{R}$-equivalence classes. Given a germ $f \in \Omega_2$, then f only deforms to germs in Ω_2, in other words, there is small neighborhood of f in $\mathfrak{m}_n^2$ which contains only germs in Ω_2, so by definition, f is Lipschitz simple.

To write down all the Lipschitz $\mathcal{R}$-classes of germs in Ω_2 it is necessary to distinguish the Lipschitz types of families in Table 1. It is shown in [18] that they are different by using several invariants including the Milnor number, the corank, the order and the singular locus of the non-quadratic part of a germ after applying Thom's splitting lemma, and also the zeta functions of the monodromy.

In short, we have the following result:

Theorem 4.3. *A germ $f \in \mathfrak{m}_n^2$ is Lipschitz $\mathscr{R}$ simple if and only if it is Lipschitz $\mathscr{R}$-equivalent to one of the germs in the table below:*

Table 2: List of Lipschitz $\mathscr{R}$-simple germs

Name	*Normal form*	*Codimension*	*Corank*
$A_k, k \geq 1$	x^{k+1}	k	1
$D_k, k \geq 4$	$x^2y + y^{k-1}$	k	
E_6	$x^3 + y^4$	6	2
E_7	$x^3 + xy^3$	7	
E_8	$x^3 + y^5$	8	
X_9	$x^4 + y^4 + x^2y^2$	9	
$T_{2,4,5}$	$x^4 + y^5 + x^2y^2$	10	
$T_{2,5,5}$	$x^5 + y^5 + x^2y^2$	11	2
Z_{11}	$x^3 + y^5 + xy^4$	11	
W_{12}	$x^4 + y^5 + x^2y^3$	12	
$T_{p,q,r}, 3 \leq p \leq q \leq k \leq 5$	$x^p + y^q + z^r + xyz$	$p + q + k - 1$	
Q_{10}	$x^3 + y^4 + yz^2 + xy^3$	10	
Q_{11}	$x^3 + y^2z + xz^3 + z^5$	11	3
S_{11}	$x^4 + y^2z + xz^2 + x^3z$	11	
S_{12}	$x^2y + y^2z + xz^3 + z^5$	12	

We would like to remark that although the definition seems easy, it is difficult in general to prove that a family of singularities is bi-Lipschitz trivial. In [18], we improved upon earlier results of Abderrahmane [1], Fernandes and Ruas [8] and Saia et al. [5] using Newton diagram to prove several of families in Table 1 are bi-Lipschitz trivial. It is worth mentioning that the bi-Lipschitz triviality of Q_{11} was rather tricky and did not follow from the improvements of results we just mentioned. The reader might want to look at the proof of Lipschitz triviality of Q_{11} for it could be applied to more general situations.

An interesting observation is that the notion of Lipschitz $\mathscr{R}$-simplicity is a Lipschitz $\mathscr{R}$-invariant in the sense that given two germs f and g which are Lipschitz $\mathscr{R}$- equivalent, if f is Lipschitz simple then g is also Lipschitz simple. This property is not clear from the definition, it is a consequence of the complete classification.

We would also like to mention that our method may be applied to get a classification of Lipschitz simple function germs in the real case, however, it would be more subtle since the family J_{10} is not completely Lipschitz modal in the real case, more precisely, germs in J_{10} are Lipschitz $\mathscr{R}$-modal if $t \leq 0$ and are Lipschitz $\mathscr{R}$-simple if $t > 0$; see Henry and Parusiński [9].

5. Certain adjacencies and non-adjacencies to J_{10}

The results of this section are not original and the proofs can also be found in Brieskorn [4]. However, during our investigation of Lipschitz simple germs we reproved some of Brieskorn's results and we thought it could be worthwhile to give an account of our method here. One must compare the proofs of Brieskorn with ours.

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a germ and D a class of singularities in $\mathfrak{m}_n$. Then, f is said to deform to D (or that f is adjacent to D) if there exists a deformation $F : (\mathbb{C}^n \times \mathbb{C}, 0) \to (\mathbb{C}, 0)$ such that $F(x, 0) = f(x)$ and $F(x, t) \in D$ for every $t \neq 0$. Recall that $T_{2,5,5}$ is a one modal family of germs given by $x^5 + y^5 + \lambda x^2y^2$ and J_{10} is $x^3 + y^6 + \lambda xy^4$.

We will prove that:

Lemma 5.1. *$T_{2,5,5}$ does not deform to J_{10}.*

Proof. The first thing to notice is that for $T_{2,5,5}$ to deform to J_{10}, the 3-jet of the deformation must be analytically equivalent to x^3. This is because J_{10} has 3-jet equal to x^3 (see Arnold's classification). Thus, if there were a deformation of $T_{2,5,5}$ to J_{10}, it should have the following general form:

$$F_\lambda = x^5 + y^5 + \lambda x^2y^2 + \epsilon(x - \beta y)^3 + p_4(x, y) + p_5(x, y) + R(x, y),$$

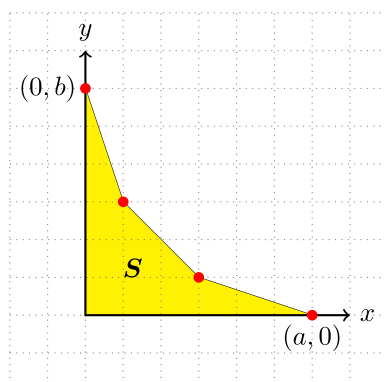
where ϵ is a complex number, p_4, p_5 are homogeneous polynomials of degree 4 and 5 respectively and R is a polynomial of degree ≥ 6. Under the change of coordinates $x \to x + \beta y$ and $y \to y$, we get:

$$F_\lambda = (x + \beta y)^5 + y^5 + \lambda(x + \beta y)^2y^2 + \epsilon x^3 + p_4(x, y) + p_5(x, y) + R(x, y).$$

Of course, p_4, p_5 and R change after the change of coordinates, but we can rearrange them to have the above form.

We will prove that such a deformation can have Milnor number at most 8. To see this we proceed as follows:

Recall first that if $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ is Newton non-degenerate in the sense of Kushnirenko (see Definition 1.19 in Kushnirenko [13]), then the Milnor number of f can be given in terms of the area under the Newton polyhedron of f and its x and y intercepts. It is a matter of simple calculations to verify that our germ and all that appear in the proof of the lemma, are Newton non-degenerate in the sense of Kushnirenko. More precisely if the Newton diagram of a Newton non-degenerate polynomial germ f looks like:



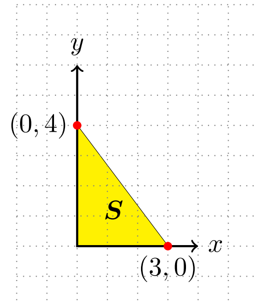
,

then the Milnor number of the germ f is equal to $2S - a - b + 1$, where S (in yellow) is the area below the Newton polyhedron of f, see Kushnirenko [13].

If $\beta \neq 0$, then F_λ can be rewritten as,

$$F_\lambda = \epsilon x^3 + (\lambda\beta^2 + \alpha)y^4 + (x + \beta y)^5 + y^5 + \lambda x^2y^2 + 2\lambda\beta xy^3 + \tilde{p}_4 + p_5 + R(x, y),$$

where $\tilde{p}_4$ is a homogeneous polynomial of degree 4 with coefficient of y^4 equals 0. Then, if $\lambda\beta^2 + \alpha \neq 0$ the Milnor number of F_λ is determined by its Newtonian principal part. The Newton polyhedron of F_λ is:



Thus, the Milnor number of F_λ is 6.

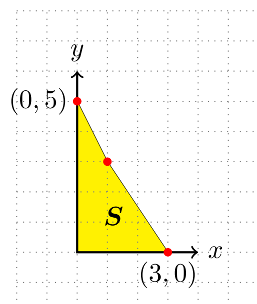
Now, suppose $\lambda\beta^2 = -\alpha$. Then, F_λ can be rewritten as,

$$F_\lambda = \epsilon x^3 + (2\lambda\beta + \gamma)xy^3 + (x + \beta y)^5 + \lambda x^2y^2 + y^5 + \tilde{\tilde{p}}_4 + \tilde{p}_5 + R(x, y).$$

But, since $\lambda\beta^2 = -\alpha$, we have,

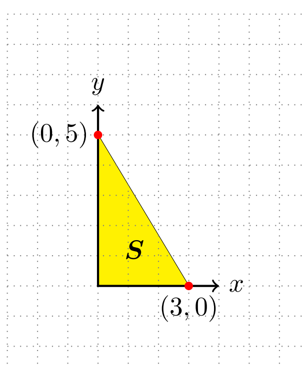
$$F_\lambda = \epsilon x^3 + \beta^{-1}(-2\alpha + \gamma\beta)xy^3 + (x + \beta y)^5 + \lambda x^2y^2 + y^5 + \tilde{\tilde{p}}_4 + \tilde{p}_5 + R(x, y).$$

If $-2\alpha + \gamma\beta \neq 0$. Then the Newton polyhedron of F_λ is:



And the Milnor number of F_λ in this case is 7.

Now, suppose $\gamma = 2\alpha\beta$. Then, the Newton polyhedron of F_λ is:



Then, the Milnor number of F_λ is 8. We have thus proved that the Milnor number of the deformation can be at most 8. Since J_{10} has Milnor number 10, this proves that $T_{2,5,5}$ cannot deform to J_{10}. $\square$

The class of singularities Z_{11} and W_{12} can also be shown similarly to not deform to J_{10}.

We next show that $T_{p,q,r}$ for $3 \leq p, q, r \leq 5$ also cannot deform to J_{10}. First notice that the Milnor number of $T_{p,q,r}$ is $p+q+r-1$. If $p+q+r-1 \leq 10$ then $T_{p,q,r}$ cannot deform to J_{10} for J_{10} has Milnor number more than or equal to $p+q+r-1$. Thus we consider the case when $p+q+r-1 > 10$. The first class in this set is $T_{3,4,5}$. We show:

Lemma 5.2. *The class of singularities $T_{3,4,5}$ cannot deform to J_{10}.*

Proof. Since $T_{3,4,5}$ and J_{10} have coranks 3 and 2 respectively, we first take an arbitrary small deformation $T_{3,4,5}$ of the form $F = T_{3,4,5} + \epsilon(x - by - cz)^2$. By the Splitting lemma this is equivalent to a corank 2 germ. With the change of coordinates of the form $x \to (x - by - cz)$ and keeping y and z fixed, the deformation is equivalent to:

$$F = (x + by + cz)^3 + y^4 + z^5 + t(x + by + cz)yz + x^2.$$

By the Splitting lemma this in turn is equivalent to $G = g(y, z) + x^2$ for some $g(y, z) \in m_2^3$. Then, the Milnor number of G is equal to that of g. Notice that after application of the Splitting lemma, $g(y, z)$ will still have the terms y^4+z^5, or, $g(y, z) = p_3(y, z)+y^4+z^5+p_4(y, z)+p_5(y, z)+R(y, z)$. The lemma follows from the fact that for a deformation of g to be equivalent to J_{10}, one must have $j^3g(0) = y^3$ or $j^3g(0) = z^3$. And, it follows from the Newton diagram of the deformation that in both the cases the Milnor number of g can at most be 8. □

The same analysis applies to singularities $T_{p,q,r}$ for any $p, q, r \leq 5$ and $p + q + r - 1 > 10$.

We also showed that all corank 4 germs deform to J_{10}. The calculations are lengthy and thus we only provided a sketch of this result in [18].

6. Lipschitz $\mathscr{A}$-classification

As we have seen in Section 4 all germs in Ω_2 are Lipschitz $\mathscr{R}$-simple germs, hence they are Lipschitz $\mathscr{A}$-simple germs. Since property (3) in Lemma 3.1 is valid also for the Lipschitz $\mathscr{A}$-equivalence, by similar arguments as in the proof of Theorem 4.1, one sees that

Theorem 6.1. *J_{10} is a Lipschitz $\mathscr{A}$-modal family.*

Consequently, all germs in Ω_1 are Lipschitz $\mathscr{A}$-modal because they deform to J_{10}. In conclusion, we have

Theorem 6.2. *A germ is Lipschitz $\mathscr{R}$-simple if and only if it is Lipschitz $\mathscr{A}$-simple.*

The invariants used to distinguish the Lipschitz $\mathscr{R}$-types of germs in the Table 1 are also invariants with respect to Lipschitz $\mathscr{A}$-equivalence (the Milnor number and the zeta function of the monodromy are topological invariants, hence Lipschitz $\mathscr{A}$-invariants, the others are by Theorem 3.4), so the lists of normal forms of Lipschitz simple germs with respect to Lipschitz $\mathscr{R}$-equivalence and Lipschitz $\mathscr{A}$-equivalence coincide.

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