CURVATURE LINES OF A TRANSVERSAL EQUIAFFINE VECTOR FIELD ALONG A SURFACE IN 3-SPACE

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ABSTRACT. In this paper we discuss the behavior of the curvature lines of a transversal equiaffine vector field along a surface in 3-space at isolated umbilical points.

1. INTRODUCTION

Let M be a smooth surface and denote by $\mathfrak{X}(M)$ the space of smooth vector fields tangent to M. Given an immersion $f : M \to \mathbb{R}^3$ and an arbitrary transversal vector field $\xi : M \to \mathbb{R}^3$, write, for $X, Y \in \mathfrak{X}(M)$,

$$(1.1) \qquad D_X f_* Y = f_*(\nabla_X Y) + h(X, Y)\xi,$$

where ∇ is a torsion free connection and h a symmetric bilinear form. We shall assume that h is positive definite, which corresponds geometrically to the convexity of the surface $f(M)$. For $X \in \mathfrak{X}(M)$, write

$$(1.2) \qquad D_X \xi = -f_*(BX) + \tau(X)\xi.$$

where B is a $(1,1)$-tensor called the *shape operador* and τ a 1-form. The vector field ξ is called *equiaffine* if $\tau = 0$.

The class of equiaffine transversal vector fields includes the Euclidean normal and the Blaschke affine normal vector fields. The behavior of curvature lines at an umbilical point has a vast literature in the Euclidean case ([2],[4],[5]) and has also been studied in the affine Blaschke case ([1]). Curvature lines of immersions endowed with equiaffine transversal vector fields are closely related to asymptotic lines of surfaces in 4-space ([3]).

When ξ is equiaffine, B is self-adjoint with respect to the metric h, and so admit a pair of linearly independent eigenvectors at each point. The lines tangent to these eigenvectors of B are called *curvature lines*. Points $r_0 \in M$ where $B(r_0)$ is a multiple of the identity are called *umbilical* and at such points curvature lines are not defined.

Consider an isolated umbilical point $r_0 \in M$. In a fixed neighborhood $V \subset M$ of r_0, consider an h-orthonormal frame $\{X_1, X_2\}$, and denote by $B(r) = (b_{ij}(r))$ the matrix of the shape operator in this frame. Since B is self-adjoint with respect to h, we have $b_{12} = b_{21}$. Now consider the vector field on V given by

$$(1.3) \qquad \mathcal{B} = (b_{11} - b_{22})X_1 + 2b_{12}X_2.$$

It is easy to verify that r_0 is an isolated zero of $\mathcal{B}$ and the index of $\mathcal{B}$ at r_0 is twice the index of the curvature lines at r_0. We say that r_0 is *semi-homogeneous* of degree k if $\mathcal{B}$ has zero $(k-1)$-jet

and r_0 is an isolated zero of the k-jet of $\mathcal{B}$. We verify in section 3 that this definition does not depend on the choice of the h-orthonormal frame $\{X_1, X_2\}$.

In this paper we prove a version of Loewner's conjecture, which generalizes the well-known result for euclidean normal vector fields ([6],[9]).

Theorem 1.1. *Assume r_0 is a semi-homogeneous umbilical point of a pair (f, ξ), where $f : M \to \mathbb{R}^3$ is an immersion with positive definite metric and ξ is an equiaffine transversal vector field. Then the index of the curvature line foliation at r_0 is at most 1.*

Natural questions are whether or not the semi-homogeneous or the equiaffine conditions can be dropped in Theorem 1.1:

Question 1: Assume r_0 is an umbilical point of an equiaffine pair (f, ξ), not necessarily semi-homogeneous. Is it still true that the index of the curvature line foliation at r_0 is at most 1?

Question 2: Assume r_0 is an umbilical point of a pair (f, ξ), not necessarily equiaffine. Is it still true that the index of the curvature line foliation at r_0 is at most 1?

Loewner's type results always have consequences regarding Carathéodory's type conjectures, which states that a compact ovaloid should have at least 2 umbilical points. A related question is whether or not there exists an ovaloid with only two Blaschke umbilical points. We show in Section 5 that every compact rotational surface admit at least one umbilical parallel.

The paper is organized as follows: In Section 2, we discuss line congruences, which is related but not an essential tool in the proof of Theorem 1.1. In Section 3 we describe the main tools needed in the proof of Theorem 1.1, while in Section 4, we prove Theorem 1.1. In Section 5 ,we discuss the corresponding Carathéodory's type results.

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2. Line Congruences

The notion of line congruence is not needed for the proof of Theorem 1.1. Nevertheless, it is instructive to describe the concepts involved in this theorem in terms of line congruences.

A 3-dimensional line congruence is a 2-dimensional family of lines in $\mathbb{R}^3$. Given a 3-dimensional line congruence, consider smooth functions f, $\xi : M \to \mathbb{R}^3$ such that, for each $r \in M$, $f(r)$ is a point of the line at r and $\xi(r)$ is a non-zero vector in the direction of the line at r.

Principal lines of a congruence. Let $r(t)$ be a curve on M. Then we set the induced 1-parameter family of lines by $f(t) = f(r(t))$ and $\xi(t) = \xi(r(t))$. The ruled surface defined by this family is developable if and only if

$$[\xi, f_t, \xi_t] = 0. \tag{2.1}$$

The solutions of Equation (2.1) are called the *curvature lines* of the congruence.

In a local coordinate system (u, v) for M, we can write Equation (2.1) as

$$P\left(\frac{du}{dt}\right)^2 + 2Q\frac{du}{dt}\frac{dv}{dt} + R\left(\frac{dv}{dt}\right)^2 = 0, \tag{2.2}$$

where, denoting by $[\cdot, \cdot, \cdot]$ the determinant of 3 vectors,

$$P = [\xi, f_u, \xi_u], \quad 2Q = [\xi, f_v, \xi_u] + [\xi, f_u, \xi_v], \quad R = [\xi, f_v, \xi_v].$$

Writing

$$\begin{cases} \xi_u = b_{11}f_u + b_{21}f_v + \tau_1\xi \\ \xi_v = b_{12}f_u + b_{22}f_v + \tau_2\xi \end{cases},$$

Equation (2.2) becomes

$$b_{21}\left(\frac{du}{dt}\right)^2 + (b_{22} - b_{11})\frac{du}{dt}\frac{dv}{dt} - b_{12}\left(\frac{dv}{dt}\right)^2 = 0,$$

which implies that the curvature lines of the congruence are tangent to the eigenvectors of the shape operator B defined by Equation (1.2). On the other hand, we remark that the curvature lines of the congruence are independent of the choice of f or ξ.

Equiaffine vector fields. We say that ξ is equiaffine with respect to the reference surface f if $\tau = 0$, where τ is given by Equation (1.2).

Lemma 2.1. *There exists a vector field $\tilde{\xi} = \lambda\xi$ such that the pair $(f, \tilde{\xi})$ is equiaffine if and only if τ is exact.*

Proof. Assume that $\tau = d\mu$ and take $\tilde{\xi} = \exp(-\mu)\xi$. Then

$$\tilde{\xi}_*X = \exp(-\mu)(\xi_*X - d\mu(X)\xi) = -f_*(\exp(-\mu)BX),$$

thus proving that $\tilde{\xi}$ is equiaffine. Conversely, assume that $\tilde{\xi} = \exp(-\mu)\xi$ is equiaffine. Then the above calculations show that $\tau = d\mu$, thus proving that τ is exact.   □

We say that the pair (f, ξ) is *exact* when τ is exact. We have the following question:

Question 3: Given a line congruence (f, ξ), is there a reference surface $\tilde{f} = f + \lambda\xi$ such that the pair $(\tilde{f}, \xi)$ is exact?

3. Main Tools

In this section we describe the main tools needed in the proof of Theorem 1.1.

3.1. **Isothermal coordinates.** Consider a convex immersion $f : M \to \mathbb{R}^3$ together with an equiaffine transversal vector field ξ, and let h denote the (positive definite) affine metric defined by Equation (1.1). We say that (u, v) are isothermal coordinates on M if

$$h_{11} = h_{22} = \rho; \ \ h_{12} = 0.$$

If we change ξ, the metric h is multiplied by some scalar function and thus the isothermal property is preserved. Moreover, it is well-known that any convex surface can be covered by isothermal parameterizations ([7]). Let ν denote the co-normal vector field defined by

$$(3.1) \qquad\qquad \nu \cdot f_u = \nu \cdot f_v = 0, \ \ \nu \cdot \xi = 1.$$

Lemma 3.1. *We can write*

$$(3.2) \qquad\qquad \xi = \frac{1}{[\nu, \nu_u, \nu_v]}\nu_u \times \nu_v.$$

Moreover,

$$(3.3) \qquad\qquad \nu_u \cdot f_u = \nu_v \cdot f_v = -\rho, \ \ \nu_u \cdot f_v = \nu_v \cdot f_u = 0,$$

and

$$(3.4) \qquad\qquad f_u = \frac{\rho}{[\nu, \nu_u, \nu_v]}\nu \times \nu_v, \ \ f_v = -\frac{\rho}{[\nu, \nu_u, \nu_v]}\nu \times \nu_u.$$

Finally

$$(3.5) \qquad\qquad [f_u, f_v, \xi] = \frac{\rho^2}{[\nu, \nu_u, \nu_v]} \ .$$

Proof. From ξ equiaffine and $\nu \cdot \xi = 1$, we conclude (3.2). From $h_{11} = \rho$ we obtain $\nu \cdot f_{uu} = \rho$ and so $\nu_u \cdot f_u = -\rho$. With a similar reasoning, we conclude (3.3). Observe that $\nu_v \cdot f_u = 0$ and $\nu \cdot f_u = 0$ to obtain the first of Equations (3.4). The second one is obtained similarly. Equation (3.5) is obtained from Equations (3.4). $\square$

Remark 3.2. In case ξ is Blaschke, $[\nu, \nu_u, \nu_v] = [f_u, f_v, \xi] = \rho$ and the above formulas become much simpler (see [7, N4, p.208]).

Lemma 3.3. *The shape operator is given by*

$$B = -\frac{1}{\delta} \begin{bmatrix} \nu_{uu} \cdot \xi & \nu_{uv} \cdot \xi \\ \nu_{uv} \cdot \xi & \nu_{vv} \cdot \xi \end{bmatrix},$$

where

(3.6)
$$\delta = \frac{[f_u, f_v, \xi][\nu, \nu_u, \nu_v]}{\rho} .$$

Proof. Writing $\xi_u = -b_{11}f_u - b_{21}f_v$ we obtain

$$-b_{11}[f_u, f_v, \xi] = [\xi_u, f_v, \xi] = \frac{\rho}{[\nu, \nu_u, \nu_v]^2}[\nu_u, \nu_v, \nu_{uu}] = \frac{\rho}{[\nu, \nu_u, \nu_v]}\nu_{uu} \cdot \xi,$$

which proves the formula for b_{11}. The other formulas are obtained similarly. $\square$

3.2. **The vector field $\mathcal{B}$.** For an isolated umbilical point r_0, let $B = (b_{ij})$ be the matrix of the shape operator in an h-orthonormal frame $\{X_1, X_2\}$ and consider the vector field $\mathcal{B}$ defined by Equation (1.3).

The *order* of the umbilical point is the order of the first non-zero jet of $\mathcal{B}$. We call an umbilical point *simple* if it is of order 1. We say that an umbilical point r_0 is *semi-homogeneous* of degree k if it is an isolated zero of the first non-zero jet $J_k\mathcal{B}$ of $\mathcal{B}$.

Lemma 3.4. *The above definition does not depend on the choice of the h-orthonormal frame $\{X_1, X_2\}$.*

Proof. If $\{\tilde{X}_1, \tilde{X}_2\}$ is another h-orthonormal frame, we can write

$$\begin{cases} \tilde{X}_1 = \cos(\theta)X_1 - \sin(\theta)X_2 \\ \tilde{X}_2 = \sin(\theta)X_1 + \cos(\theta)X_2. \end{cases}$$

Straightforward calculations show that

$$\begin{cases} \tilde{b}_{11} - \tilde{b}_{22} = \cos(2\theta)(b_{11} - b_{22}) - \sin(2\theta)2b_{12} \\ 2\tilde{b}_{12} = \sin(2\theta)(b_{11} - b_{22}) + \cos(2\theta)2b_{12}, \end{cases}$$

and the same relation holds for the k-jet of $\mathcal{B}$. Thus r_0 is isolated for the k-jet of $\mathcal{B}$ if and only if it is isolated for the k-jet of $\tilde{\mathcal{B}}$. $\square$

Consider now isothermal coordinates (u, v). Then we can write $f_u = \rho X_1$, $f_v = \rho X_2$, for some h-orthonormal frame $\{X_1, X_2\}$. Since $\xi_u = \rho D_{X_1}\xi$, $\xi_v = \rho D_{X_2}\xi$, we conclude that the matrix of the shape operator B in coordinates (u, v) is the same as in the frame $\{X_1, X_2\}$. Thus we can use Lemma 3.4 also for the coordinate lines of an isothermal parameterization.

From now on we shall always be using an isothermal parameterization. In particular, $b_{12} = b_{21}$. The following lemma is well-known and relate the index of the curvature lines with the index of $\mathcal{B}$ at an umbilical point.

Lemma 3.5. *The index of the curvature lines of ξ at an umbilical point $r_0 = (u_0, v_0)$ is exactly one half of the index of the vector field $\mathcal{B}$.*

Proof. The equation of curvature lines is given by

$$(b_{11} - b_{22})dudv + b_{12}(dv^2 - du^2) = 0.$$

Writing $du = dr\cos(\eta)$, and $dv = dr\sin(\eta)$, we obtain

$$(b_{11} - b_{22})\sin(2\eta) - 2b_{12}\cos(2\eta) = 0,$$

which implies that $\mathcal{B}$ is a multiple of $(\cos(2\eta), \sin(2\eta))$. Thus the rotation of $\mathcal{B}$ is twice the rotation $[du : dv]$, which proves the lemma. $\qquad\qquad\square$

3.3. The support function and the vector field $\mathcal{P}$.

Consider the *support function* $p : M \to \mathbb{R}$ defined by

$$p(u, v) = \nu(u, v) \cdot (f(u, v) - q_0),$$

where $q_0 \in \mathbb{R}^3$ is a fixed point. Observe that

$$p_u = \nu_u \cdot (f - q_0), \ p_v = \nu_v \cdot (f - q_0).$$

Differentiating we obtain

$$p_{uu} = \nu_{uu} \cdot (f - q_0) - \rho; \ p_{vv} = \nu_{vv} \cdot (f - q_0) - \rho; \ p_{uv} = \nu_{uv} \cdot (f - q_0).$$

It follows that

$$(3.7) \qquad\qquad p_{uu} - p_{vv} = (\nu_{uu} - \nu_{vv}) \cdot (f - q_0); \ 2p_{uv} = 2\nu_{uv} \cdot (f - q_0).$$

Denote

$$(3.8) \qquad\qquad P_1 = p_{uu} - p_{vv}, \quad P_2 = 2p_{uv}.$$

and let $\mathcal{P}$ be the the vector field whose components are P_1 and P_2.

3.4. Loewner's conjecture for planar vector fields.

Consider a function $w : U \subset \mathbb{R}^2 \to \mathbb{R}$ defined in a neighborhood U of $(0, 0)$ and assume $(0, 0)$ is an isolated zero of the vector field $W = \frac{\partial^n w}{\partial \bar{z}^n}$. We are identifying here $\mathbb{R}^2$ with $\mathbb{C}$, $\frac{\partial}{\partial \bar{z}}$ means $\frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)$, and the domain of W in $\mathbb{C}$ is identified with $U \subset \mathbb{R}^2$. Loewner's conjecture states that the index of W is at most n.

This conjecture is proved in [9], but there are also some controversies concerning the proof. Nevertheless, assuming that W is semi-homogeneous, i.e., $(0, 0)$ is an isolated zero of its first non-zero jet, then there are no doubts that Loewner's conjecture holds ([6],[8],[9],[10]). Thus we can state the following theorem, considering $n = 2$:

Theorem 3.6. *Let $w : U \subset \mathbb{R}^2 \to \mathbb{R}$ be a function defined in a neighborhood U of $(0, 0)$ and assume $(0, 0)$ is an isolated zero of the semi-homogeneous vector field $W = (w_{uu} - w_{vv}, 2w_{uv})$. Then the index of W is at most 2.*

4. Proof of Theorem 1.1

The idea to prove Theorem 1.1 is to apply the Theorem 3.6 to $\mathcal{P}$ defined by Equation (3.8) and conclude results concerning $\mathcal{B}$.

4.1. **Simple umbilical points.** From now on we shall fix an isolated umbilical point $(u_0, v_0) = (0, 0)$ of an immersion $f : M \to \mathbb{R}^3$ together with an equiaffine transversal vector field ξ and isothermal coordinates (u, v). Denote $f_0 = f(0, 0)$, $\xi_0 = \xi(0, 0)$ and let $q_0 = f_0 + \lambda_0^{-1} \xi_0$, where

$$\lambda_0 = b_{11}(0, 0) = b_{22}(0, 0).$$

Denote also $\delta_0 = \delta(0, 0)$, where δ is defined by Equation (3.6).

From Equations (3.7) and (3.8), we obtain

$$\mathcal{P}(0, 0) = -\lambda_0^{-1} \left((\nu_{uu} - \nu_{vv}) \cdot \xi_0, 2\nu_{uv} \cdot \xi_0 \right).$$

Now Lemma 3.3 implies that

$$\mathcal{P}(0, 0) = \lambda_0^{-1} \delta_0 \mathcal{B}(0, 0).$$

Since $\mathcal{B}(0, 0) = 0$, we conclude that $\mathcal{P}(0, 0) = 0$.

Lemma 4.1. *We have that*

(4.1) $$f(u, v) + \lambda_0^{-1} \xi(u, v) = f_0 + \lambda_0^{-1} \xi_0 + O(2),$$

where $O(2)$ denotes terms of degree at least 2 in (u, v). Conversely, if equation (4.1) is satisfied, then $(0, 0)$ is an umbilical point and $\lambda_0 = b_{11}(0, 0) = b_{22}(0, 0)$.

Proof. The equation $f(u, v) + \lambda_0^{-1} \xi(u, v) = f_0 + \lambda_0^{-1} \xi_0$ holds up to order 1 if and only if

$$f_u + \lambda_0^{-1} \xi_u = f_v + \lambda_0^{-1} \xi_v = 0$$

at $(0, 0)$, which is equivalent to say that $(0, 0)$ is an umbilical point and

$$\lambda_0 = b_{11}(0, 0) = b_{22}(0, 0).$$

$\square$

Proposition 4.2. *Let $(0, 0)$ be an umbilical point and $q_0 = f_0 + \lambda_0^{-1} \xi_0$. Then*

$$J_1 \mathcal{P}(0, 0) = \lambda_0^{-1} \delta_0 J_1 \mathcal{B}(0, 0),$$

where $J_1 \mathcal{P}$ and $J_1 \mathcal{B}$ denote the first jet of the vector fields $\mathcal{P}$ and $\mathcal{B}$, respectively.

Proof. We shall verify the equality for the derivatives of P_i and B_i with respect to u and v. We shall verify the equality for the derivatives of P_1 and B_1 with respect to u, the other 3 cases being similar. Observe that

$$(P_1)_u = p_{uuu} - p_{vvu} = (\nu_{uuu} - \nu_{vvu}) \cdot (f - q_0) + (\nu_{uu} - \nu_{vv}) \cdot f_u.$$

At $(0, 0)$, $f_u = -\lambda_0^{-1} \xi_u$ and so

$$(P_1)_u = -\lambda_0^{-1} \left((\nu_{uuu} - \nu_{vvu}) \cdot \xi + (\nu_{uu} - \nu_{vv}) \cdot \xi_u \right).$$

On the other hand, since $B_1 = -\frac{1}{\delta}(\nu_{uu} - \nu_{vv}) \cdot \xi$, we have that

$$\delta(B_1)_u + \delta_u B_1 = -(\nu_{uuu} - \nu_{vvu}) \cdot \xi - (\nu_{uu} - \nu_{vv}) \cdot \xi_u.$$

Thus, at $(0, 0)$,

$$\delta_0 (B_1)_u = \lambda_0 (P_1)_u,$$

thus proving the desired result. $\square$

4.2. **Umbilical points of order k.**

Lemma 4.3. *Let $(0,0)$ be umbilical and $q_0 = f_0 + \lambda_0^{-1}\xi_0$. Then $(0,0)$ is umbilical of order $\geq k$ if and only if*

$$(4.2) \qquad f(u,v) + \lambda_0^{-1}\xi(u,v) = q_0 + O(k+1).$$

Proof. Write

$$(4.3) \qquad \begin{aligned} \xi_u &= -b_{11}f_u - b_{21}f_v \\ \xi_v &= -b_{12}f_u - b_{22}f_v \end{aligned}$$

If the $(k-1)$-jet of $\mathcal{B}$ at $(0,0)$ equals 0, differentiating equation (4.3) $(k-1)$ times we obtain that, at (u_0, v_0), the k-jet of ξ equals $-\lambda_0$ times the k-jet of f, thus proving formula (4.2).

Conversely, assume that equation (4.2) holds with $k \geq 2$. Differentiating Equation (4.3) and taking $(u,v) = (0,0)$, we obtain

$$(b_{11})_u f_u + (b_{21})_u f_v = 0, \quad (b_{11})_v f_u + (b_{21})_v f_v = 0,$$

which implies that

$$(b_{11})_u = (b_{21})_u = (b_{11})_v = (b_{21})_v = 0.$$

Similarly, we can prove that

$$(b_{12})_u = (b_{22})_u = (b_{12})_v = (b_{22})_v = 0,$$

which implies that the 1-jet of $\mathcal{B}$ vanishes at $(0,0)$. To prove that the $(k-1)$-jet of $\mathcal{B}$ vanishes at $(0,0)$, one can proceed by induction. $\qquad\square$

Proposition 4.4. *At an umbilical point of order $\geq k$,*

$$(4.4) \qquad J_k\mathcal{P} = \lambda_0^{-1}\delta_0 J_k\mathcal{B},$$

where $J_k\mathcal{P}$ and $J_k\mathcal{B}$ denote the k-jets of $\mathcal{P}$ and $\mathcal{B}$, respectively.

Proof. For $k = 2$, we must prove that the second derivatives of $\mathcal{P}$ and $\mathcal{B}$ are multiples at $(0,0)$. Let us consider $(P_1)_{uu}$ and $(B_1)_{uu}$, the other cases being similar. Observe that

$$(P_1)_{uu} = (\nu_{uuuu} - \nu_{vvuu}) \cdot (f - q_0) + 2(\nu_{uuu} - \nu_{vvu}) \cdot f_u + (\nu_{uu} - \nu_{vv}) \cdot f_{uu}.$$

At (u_0, v_0), by Lemma 4.3,

$$(P_1)_{uu} = -\lambda_0^{-1}\left((\nu_{uuuu} - \nu_{vvuu}) \cdot \xi + 2(\nu_{uuu} - \nu_{vvu}) \cdot \xi_u + (\nu_{uu} - \nu_{vv}) \cdot \xi_{uu}\right).$$

On the other hand,

$$\delta(B_1)_{uu} + 2\delta_u(B_1)_u + \delta_{uu}B_1 = -(\nu_{uuuu} - \nu_{vvuu}) \cdot \xi - 2(\nu_{uuu} - \nu_{vvu}) \cdot \xi_u - (\nu_{uu} - \nu_{vv}) \cdot \xi_{uu}.$$

At $(0,0)$,

$$\delta_0(B_1)_{uu} = \lambda_0(P_1)_{uu},$$

thus proving the claim. To prove Equation (4.4) for any k, one can proceed by induction. $\qquad\square$

4.3. Completing the proof.

Lemma 4.5. *If $(0,0)$ is an umbilical point, semi-homogeneous of degree k, the index of $\mathcal{B}$ at $(0,0)$ is the same index of $J_k\mathcal{B}$ at $(0,0)$.*

Proof. For $||(u,v)||$ sufficiently small,

$$||\mathcal{B} - J_k\mathcal{B}|| \leq \frac{1}{2}||J_k\mathcal{B}||,$$

which proves the lemma. $\square$

We can now prove Theorem 1.1.

Proof. If $(0,0)$ is an isolated umbilical point, semi-homogeneous of degree k, Proposition 4.4 implies that the index of $J_k\mathcal{B}$ at (u_0,v_0) is the same index of $J_k\mathcal{P}$. From the above lemma, they are also equal to the index of $\mathcal{B}$. By Theorem 3.6, any vector field of the form $\mathcal{P}_k$ has index ≤ 2. By Lemma 3.5, the index of the curvature lines is at most 1. $\square$

5. Carathéodory's type Results and Questions

5.1. A Carathéodory's type result.
In the Euclidean case, Carathéodory's conjecture states that any compact surface homeomorphic to the sphere admits at least 2 umbilical points. This conjecture is a consequence of Loewner's conjecture, which states that the index of the Euclidean curvature lines at umbilical points are at most 1.

In our case, Loewner's type Theorem 1.1 implies the following result:

Corollary 5.1. *Consider a convex centroaffine immersion $f : M^2 \to \mathbb{R}^3$, M^2 compact, and an equiaffine transversal vector field ξ. Assume that all umbilical points are semi-homogeneous. Then there are at least 2 umbilical points.*

Proof. Assuming all umbilical points are semi-homogeneous, their curvature lines indices are at most 1. By the index theorem of Euler-Poincaré, the sum of the indices must be the Euler characteristic of M, in this case 2 since f is convex. We conclude that there are at least 2 umbilical points. $\square$

If Question 1 in the Introduction is answered positively, then we can drop the semi-homogeneous condition in Corollary 5.1.

5.2. Rotational surfaces.
Concerning Corollary 5.1, it is natural to ask whether or not there exist compact surfaces in 3-space with only two umbilical points. If we are free to choose the equiaffine transversal vector field ξ, then the answer is positive, since we can take ξ to be the Euclidean unit normal and the surface to be a rotational ellipsoid. But what happens if we choose ξ to be the Blaschke vector field? Are there compact surfaces with only two Blaschke umbilical points?

To answer this question, it is natural to look at the rotational surfaces. But if we consider rotational surfaces with the Blaschke transversal vector field, then there exists at least one umbilical parallel. In this section we prove this surprising fact, and so the following question remains open:

Question 4: Is there an ovaloid in 3-space with 2 umbilical points with respect to the Blaschke transversal vector field?

5.2.1. *Shape operator of a rotation surface at the axis.* Any rotation surface is the graph a smooth function z of the form $z = h(x^2 + y^2)$, for some smooth function h. We shall assume that the linear part of h is non-zero so that the rotation surface becomes non-degenerate. Then, by an affine change of variables, the 4-jet of z at $(0,0)$ is given by

$$z = \frac{1}{2}(x^2 + y^2) + \frac{\alpha}{24}\left(x^2 + y^2\right)^2 + O(6),$$

which implies that $(0,0)$ is umbilical with respect to the Blaschke affine normal. We conclude that the axes points of a rotation surface are Blaschke umbilical.

5.2.2. *Shape operator of a rotation surface outside the axis.* Assume S is a smooth rotation surface generated by the convex planar arc $\gamma(t) = (x(t), y(t))$, $t \in [0, \pi]$, such that $x(t) > 0$ and $y'(t) > 0$, for $t \in (0, \pi)$. A parameterization of S is given by

$$\psi(t, \theta) = (x(t)\cos\theta, x(t)\sin\theta, y(t)).$$

To calculate the Blaschke shape operator, we calculate first the co-normal vector field ν and the Blaschke metric $h = (h_{ij})$. Straightforward calculations show that

$$\nu(t, \theta) = \frac{x(t)}{\phi(t)}\left(-y'(t)\cos\theta, -y'(t)\sin\theta, x'(t)\right),$$

$$h_{11} = \frac{x(t)}{\phi(t)}, \quad h_{12} = h_{21} = 0, \quad h_{22} = \frac{x(t)^2 y'(t)}{\phi(t)},$$

where $\phi^4(t) = x(t)^3 y'(t)$.

Lemma 5.2. *The Blaschke affine normal vector field ξ can be written as*

$$\xi = (a\cos\theta, a\sin\theta, b),$$

for certain $a = a(t)$, $b = b(t)$.

Proof. Observe that $\nu_\theta \cdot \xi = 0$. The conditions $\nu \cdot \xi = 1$ and $\nu_s \cdot \xi = 0$ are given by

$$\begin{cases} -ay' + bx' = \frac{\phi}{x} \\ -ay'' + bx'' = -\left(\frac{\phi}{x}\right)^2\left(\frac{x}{\phi}\right)' \end{cases}.$$

This system certainly has a solution $\xi = (a, b)$.   □

Write $g = \frac{\phi}{x}$. Then

$$\begin{cases} -ay' + bx' = g \\ -ay'' + bx'' = g' \end{cases}.$$

We conclude that $a'y' - b'x' = 0$, which implies that $\xi_t = \frac{b'}{y'}\psi_t$. Since $\xi_\theta = \frac{a}{x}\psi_\theta$, the parallels and meridians are affine curvature lines. Moreover, a parallel is umbilical if and only if $\frac{a}{x} = \frac{b'}{y'}$.

5.2.3. *A useful parameterization.* Any generator arc $\gamma(t) = (x(t), y(t))$ as above can be parameterized with $y'(t) = x(t)$. In fact, take

$$t = \int \frac{y'(s)}{x(s)}ds.$$

This parameterization simplifies a lot the calculations. In fact, assuming this parameterization, we obtain $g = 1$ and a parallel is umbilical if and only if $a = b'$. Moreover

$$\xi = (a, b) = \frac{1}{\Delta}\left(x''(t), y''(t)\right),$$

where $\Delta = [\gamma', \gamma''] = (y'')^2 - y'y'''$ which is positive, by hypothesis. Then a parallel t is umbilical if and only if

$$\frac{y'''}{\Delta} = \frac{y'''}{\Delta} - \frac{\Delta'y''}{\Delta^2} \ .$$

Thus if $y''(t) = 0$, the parallel t is umbilical. Since $y''(0) = x'(0) > 0$ and $y''(\pi) = x'(\pi) < 0$, there exists at least one $t_0 \in (0, \pi)$ such that $y''(t_0) = 0$. We conclude that there exists at least one umbilical parallel $t_0 \in (0, \pi)$.

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