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BI-LIPSCHITZ AND DIFFERENTIABLE SUFFICIENCY OF WEIGHTED JETS

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Abstract. In this work we study bi-Lipschitz and differential sufficiency in the set of weighted jets of map germs of weighted degree. Our main result improves the degree of sufficiency of jets in the classical jet space obtained by Takens and by Martins-Fávaro. Moreover, we give a condition for the bi-Lipschitz sufficiency in both the weighted and non-weighted cases. Bi-Lipschitz sufficiency was not considered in those previously cited works.

1. Introduction

Let $E_{[k]}(n,p)$ be the set of all C^k map germs from $(\mathbb{R}^n,0)$ to $(\mathbb{R}^p,0)$, for $0 \leq k \leq \infty$. Given an element f of $E_{[k]}(n,p)$, it is natural to ask when we can truncate f without affecting the local topological picture determined by f. This problem concerns the property of sufficiency of jets. The sufficiency of jets was introduced by René Thom in order to establish structural stability theory. So, this subject gained prominence in the development of Singularity Theory and several authors investigated this problem. A pioneer work, among all, was done by T.-C. Kuo, in [8].

We denote by $J^r(n,p)$ the set of r-jets of map germs $f:(\mathbb{R}^n,0)\to(\mathbb{R}^p,0)$ of class C^k $(k \geq r)$ and by $j^rf(0)$ the r-jet of f at the origin. Each $g \in E_{[k]}(n,p)$ such that

$$j^rg(0) = j^rf(0) = z \in J^r(n,p)$$

is called a C^k*-realization* of z in $J^r(n,p)$.

Two map germs $f,g \in E_{[k]}(n,p)$ are:

- SV*-equivalent* if and only if there exists a germ of a homeomorphism

  $$h:(\mathbb{R}^n,0)\to(\mathbb{R}^n,0)$$

  such that $h(f^{-1}(0)) = g^{-1}(0)$.
- V*-equivalent* if and only if $f^{-1}(0)$ is homeomorphic to $g^{-1}(0)$, as germs at $0 \in \mathbb{R}$.
- C^0*-equivalent* if and only if there exists a germ of a homeomorphism

  $$h:(\mathbb{R}^n,0)\to(\mathbb{R}^n,0)$$

  such that $f = g\circ h$.

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- *bi-Lipschitz equivalent* if and only if there exists a germ of a bi-Lipschitz homeomorphism
$$h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$$
such that $f = g \circ h$.
- C^i*-equivalent* $(1 \leq i \leq \infty)$ if and only if there exists a germ of a C^i-diffeomorphism
$$h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$$
such that $f = g \circ h$.

An r-jet $z \in J^r(n,p)$ is said to be *SV-sufficient*, *V-sufficient*, C^0*-sufficient*, *bi-Lipschitz sufficient* or C^i*-sufficient* $(1 \leq i \leq \infty)$ in C^k $(k \geq r)$ if and only if any two C^k-realizations f and g of z are SV-equivalent, V-equivalent, C^0-equivalent, bi-Lipschitz equivalent or C^i-equivalent $(1 \leq i \leq \infty)$, respectively.

The characterization of C^0-sufficiency of jets in $J^r(n,p)$ was investigated by several authors, including [8, 9, 3, 10, 12]. In $J^r(n,1)$, the characterization of C^0-sufficiency can be characterized by the called *Kuiper-Kuo condition*. In fact, it follows from Bochnak-Lojasiewicz [4], Kuo [11] and Kuiper [7] that for every r-jet $z \in J^r(n,1)$ the following conditions are equivalent:

(1) z is V-sufficient in $E_{[r]}(n,1)$;
(2) z is C^0-sufficient in $E_{[r]}(n,1)$;
(3) the Kuiper-Kuo condition is satisfied, that is, there exists $C, \epsilon > 0$ such that $|\nabla f(x)| \geq C|x|^{r-1}$, holds in some neighbourhood of $0 \in \mathbb{R}^n$.

It is worth mentioning that sufficiency in $E_{[r+1]}(n,1)$ does not imply sufficiency in $E_{[r]}(n,1)$ (see [4]).

On the other hand, R. Thom formulated another condition, called the *Thom condition*, which implies C^0-sufficiency. Bekka and Koike [2] showed that for an isolated singularity the Kuiper-Kuo condition is equivalent to the Thom condition.

T.-C. Kuo in [9] also investigated V-sufficiency of jets. He proved that an r-jet z is V-sufficient if and only if there exists $C, \epsilon, \delta > 0$ such that $|\nabla z(x)| \geq C|x|^{r-\delta}$ for
$$x \in \{x \in \mathbb{R}^n \,;\, |f(x)| \leq \epsilon |x|^{r+1}\}$$
(a horn-neighborhood of the zero set of f).

L. Paunescu [13] studied the V-sufficiency from the weighted point of view, extending Kuo's results given in [9]. He obtained a version of Kuo's criterion for V-sufficiency for the case of weighted jets in $J^r_w(n,p)$. Later, O. Abderrahmane [1] gave a characterization of Kuo's V-sufficiency using Newton filtration.

Notice that C^0-equivalence implies V-equivalence and SV-equivalence. However the converse is not true. An important result in this converse sense was obtained by King in [6].

With respect to C^i-sufficiency $(i \geq 1)$ of jets in $J^r(n,1)$, this subject was studied by F. Takens [15] and N. Kuiper [7], among others. When $p \geq 2$, Fávaro and Martins [5] showed that Takens's result is also valid in $J^r(n,p)$, with $p \geq 2$, without any change in the Takens's condition for $p = 1$. In [5], the authors applied results of T.-C. Kuo [8, 9] and Bochnak and Kucharz [3].

The main result of this paper is a criterion for C^0, bi-Lipschitz and C^i-sufficiency $(i \geq 1)$ of weighted jets in $J^r_w(n,p)$. Our result improves the degree of C^i-sufficiency given by Takens in [15] (for $p = 1$) and by Fávaro-Martins in [5], for $p \geq 2$. Our condition in the weighted case is a generalization of the condition given by Fávaro-Martins, and they coincide if the weights of each variable are equal to 1. Moreover, we also show that the characterization of C^0-sufficiency in

$J^r(n,p)$ given in [5] and also by Bochnak and Kucharz in [3, Theorem 1] is in fact bi-Lipschitz. We remark that the bi-Lipschitz sufficiency case in $J^r(n,p)$ was not considered in any of these previous works.

2. Sufficiency from the weighted point of view

Our target now is to describe conditions for bi-Lipschitz and C^i-sufficiency $(0 \leq i \leq \infty)$ in the set of weighted r-jets, denoted by $J^r_w(n,p)$, for some fixed set of weights $w = (w_1, \dots, w_n)$.

First we recover the result given by Fávaro-Martins in [5] for C^i-sufficiency, $i \geq 0$, in the canonical jet space $J^r(n,p)$. To do this, we need to define some numbers $r(z)$ and $r_0(z)$ that are fundamental.

Given p vectors $v_1, \dots, v_p \in \mathbb{R}^n$, let d_i denote the distance from v_i to the subspace of $\mathbb{R}^n$ generated by vectors v_j, where $j \neq i$. In the case that each $v_i = v_i(x)$ depends on a variable x, we write $d_i(x)$ for the distance. The d_i's can be expressed in terms of the v_i's as follows

$$d_i(x) = \inf_{y_j \in \mathbb{R}} \{ \, |v_i(x) - \sum_{\substack{j=1 \\ j \neq i}}^{p} y_j v_j(x)| \}.$$

Following [15, 5], call $d(v_1, \dots, v_p) = \min\{d_1, \dots, d_p\}$ and for all z in $J^r(n,p)$ define:

$r(z) = min\{\ell \in \mathbb{N} \mid \exists c > 0 \text{ and } \delta > 0 \text{ with } d(\nabla z_1(x), \cdots, \nabla z_p(x)) \geq c|x|^{\ell - 1} \text{ for } |x| < \delta\}$

and

$r_0(z) = max\{\ell \in \mathbb{N} \mid \exists c > 0 \text{ and } \delta > 0 \text{ with } |z(x)| \leq c|x|^{\ell} \text{ for } |x| < \delta\}$.

Theorem 2.1. [5, Teorema, p. 4] *Let $z \in J^r(n,p)$. Suppose $r(z) \leq r$ and*

$$r - r(z) + 1 - i[r(z) - r_0(z) + 1] \geq 0$$

for some $i \geq 0$. Then z is C^i-sufficient in C^r.

Remark 2.2. i). When $p = 1$, we have exactly the result of Takens (see [15, Theorem, p. 225]), with the same condition of Theorem 2.1.

ii). Notice that $r - r(z) + 1 \geq 0$ is always true because we are taking $r \geq r(z)$. Then, by [3, Theorem 1], z is at least C^0-sufficient in C^r.

For the weighted jet space, $J^r_w(n,p)$, the generalization of these numbers $r(z)$ and $r_0(z)$ is the first challenge. It is done in the next subsection.

2.1. Bi-Lipschitz and C^i-sufficiency from the weighted point of view.

In order to fix the notation, we consider a set of n positive integers $w = (w_1, \dots w_n)$ with $1 \leq w_1 \leq w_2 \leq \dots \leq w_n$, called *weights*.

We recall that $f \colon (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ is weighted homogeneous of type $(w_1, \dots, w_n; k_1, \dots, k_p)$, with $w_i,\ k_j \in \mathbb{Z}$, if for all $\lambda \in \mathbb{R} - \{0\}$,

$$f(\lambda^{w_1} x_1, \lambda^{w_2} x_2, \dots, \lambda^{w_n} x_n) = (\lambda^{k_1} f_1(x), \lambda^{k_2} f_2(x), \dots, \lambda^{k_p} f_p(x)).$$

The numbers k_j are called *weighted degrees* of f_j, $j = 1, \dots, p$.

For any germ of function g we consider its decomposition according to the corresponding weighted degrees of its monomials, or $g := g_{k_1} + \cdots + g_{k_\ell} + ...$, with $k_1 \leq k_2 \leq \cdots \leq k_\ell < ...$ where each monomial of g_{k_j} has weighted degree k_j. The number $k_1 = \min\{k_j\}$ is called the *filtration* of the germ g, and it is denoted by $\text{fil}(g)$.

Fixing a set of weights w, we denote by $J^r_w(n,p)$, the space of weighted jets of weighted degree r; here an r-weighted jet, denoted by $j^r_w f(0)$, is a polynomial map such that all monomials of its coordinate functions have weighted degree smaller than or equal to r. In this case, each germ g such that $j^r_w g(0) = j^r_w f(0) = z \in J^r_w(n,p)$ is called a *w-C^k-realization* of z in $J^r_w(n,p)$.

To compare the sets $J^r(n,p)$ and $J^r_w(n,p)$, consider for instance the following germs of functions $f(x,y) = x^6 - y^3$ and $g(x,y) = x^6 - y^3 + y^4$. These germs have different 6-jets in $J^6(2,1)$ and then it is not possible to apply Theorem 2.1. However if we consider the set of weights $w = (1,2)$, the germs f and g have the same 6-weighted jet in $J^6_{(1,2)}(2,1)$, since the monomial y^4 has weighted degree 8 for this set of weights $w = (1,2)$. Then this example motivates us to search for a similar result to Takens and Fávaro-Martins but now from the weighted point of view.

The analogous definitions of weighted sufficiency are given below.

Definition 2.3. An r-jet $z \in J^r_w(n,p)$ is said to be *w-SV-sufficient*, *w-V-sufficient* *w-C^0-sufficient*, *w-bi-Lipschitz sufficient* or *w-C^i-sufficient* $(1 \leq i \leq \infty)$ in C^k, $(k \leq r)$ if any two w-C^k-realizations f and g of z are SV-equivalent, V-equivalent, C^0-equivalent, bi-Lipschitz equivalent or C^i-equivalent for $1 \leq i \leq \infty$, respectively.

We denote by $\rho(x)$ the *control function* associated to the set of weights w defined by

$$\rho(x) = \left(x_1^{2s_1} + x_2^{2s_2} + \ldots + x_n^{2s_n}\right)^{\frac{1}{2q}},$$

where $s_i = q/w_i$ and $q = \text{l.c.m. } \{w_i\}$, $i = 1, \ldots, n$.

Since $s_1 \geq s_n$, then in a neighborhood of the origin, the following inequalities are valid,

$$(**) \quad x_1^{2s_1} + x_2^{2s_1} + \ldots + x_n^{2s_1} \leq x_1^{2s_1} + x_2^{2s_2} + \ldots + x_n^{2s_n} \leq x_1^{2s_n} + x_2^{2s_n} + \ldots + x_n^{2s_n}.$$

For any weighted r-jet $z \in J^r_w(n,p)$, we define the numbers $R_w(z)$ and $R^0_w(z)$ associated to z:

$R_w(z) = min\{\ell \in \mathbb{N} \mid \exists c > 0,\ \delta > 0 \text{ with } d(\nabla z_1(x), \cdots, \nabla z_p(x)) \geq c\rho(x)^{\ell - w_n}, \forall |x| < \delta\}$,

and $\quad R^0_w(z) = max\{\ell \in \mathbb{N} \mid \exists c > 0 \text{ and } \delta > 0 \text{ with } |z(x)| \leq c\rho(x)^\ell \text{ for } |x| < \delta\}$.

First we show the Lipschitz version of the Theorem 2.12 from the weighted point of view, as following:

Theorem 2.4. *Fix a set of weights w. Suppose that $R_w(z) \leq r$ and $r > 1$, then z is, at least, w-bi-Lipschitz sufficient in C^r.*

Notice that in Theorem 2.1 there is an analogous condition $r(z) \leq r$ which implies directly that z is, at least, C^0-sufficient in C^r (see Remark 2.2 ii)).

For instance, the germ $f(x,y) = x^6 - y^3$ is $(1,2)$-bi-Lipschitz sufficient in C^6. Then, since $g(x,y) = x^6 - y^3 + y^4$ has the same weighted 6-jet of f (i.e., $j^6_{(1,2)} g(x,y) = x^6 - y^3 = j^6_{(1,2)} f(x,y)$), we see that g is, at least, $(1,2)$-bi-Lipschitz equivalent to f.

Proof of Theorem 2.4.

The proof of Theorem 2.4 follows the strategy of constructing a Kuo type vector field to obtain the desired bi-Lipschitz homeomorphim, which is given by integrating this vector field. In fact, the proof is a consequence of the following results that are given below.

Consider a C^r map germ $\theta : \mathbb{R}^n \to \mathbb{R}^p$ with $j^r_w \theta = 0$ and $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^p$, given by $F(x,t) = z(x) + t\theta(x)$. Call $F = (F_1, \ldots, F_p)$. Then each $F_k(x,t) = z_k(x) + t\theta_k(x)$ and $\nabla F_k(x,t) = (\nabla z_k(x) + t\nabla\theta_k(x), \theta_k(x)) \in \mathbb{R}^{n+1}$, $k = 1, \ldots, p$.

Lemma 2.5. *(see [9, Lemma 3.2, p. 118]) Considering $\nabla F_1(x,t),\dots,\nabla F_p(x,t)$ linearly independent p vectors in $\mathbb{R}^{n+1}$, denote by $P_i(x,t)$ the projection of $\nabla F_i(x,t)$ in the subspace generated by $\nabla F_j(x,t)$ with $j\neq i$ and by $n_i(x,t)=\nabla F_i(x,t)-P_i(x,t)$ the orthogonal vector to the subspace generated by $\nabla F_j(x,t)$ with $j\neq i$. Then, the vector field*

$$\begin{aligned} X(x,t) &= \sum_{i=1}^{p}\frac{\langle e_{n+1},\nabla F_i(x,t)\rangle}{|n_i(x,t)|^2}\cdot n_i(x,t)\\ &= \sum_{i=1}^{p}\frac{\theta_i(x)}{|n_i(x,t)|^2}\cdot n_i(x,t)\end{aligned}$$

is the projection of e_{n+1} to the subspace generated by $\nabla F_i(x,t)$, $i=1,\dots,p$.

Lemma 2.6. *Fix a set of weights $w=(w_1,\dots,w_n)$. Let $f,g:(\mathbb{R}^n,0)\to(\mathbb{R}^p,0)$ be C^k map germs with $j^\ell_w f(0)=j^\ell_w g(0)$. If there exist $c>0$ and $\delta>0$ such that*

$$d(\nabla f_1(x),\cdots,\nabla f_p(x))\geq c\rho(x)^{\ell-w_n}$$

for all $0<|x|<\delta$, then there exist $c_1>0$ and $\delta_1>0$ such that

$$d(\nabla g_1(x),\cdots,\nabla g_p(x))\geq c_1\rho(x)^{\ell-w_n},\ \text{ for }\ 0<|x|<\delta_1.$$

Proof of Lemma 2.6.

Since $j^\ell_w f(0)=j^\ell_w g(0)$, then

$$\lim_{|x|\to 0}\frac{|f_i(x)-g_i(x)|}{\rho(x)^\ell}=0 \text{ and} \tag{1}$$

$$\lim_{|x|\to 0}\frac{|\nabla f_i(x)-\nabla g_i(x)|}{\rho(x)^{\ell-w_n}}=0. \tag{2}$$

Let $y=(y_1,\cdots,y_p)\neq 0$. For each x and k fixed, suppose $y_k\neq 0$, then

$$\begin{aligned} \frac{|y_k(\nabla g_k(x)-\nabla f_k(x))|}{|\sum_{i=1}^{p}y_i\nabla f_i(x)|} &= \frac{|y_k(\nabla g_k(x)-\nabla f_k(x))|}{|y_k\nabla f_k(x)+\sum_{\substack{i=1\\ i\neq k}}^{p}y_i\nabla f_i(x)|}=\\ \frac{|\nabla g_k(x)-\nabla f_k(x)|}{|\nabla f_k(x)+y_k^{-1}\sum_{\substack{i=1\\ i\neq k}}^{p}y_i\nabla f_i(x)|} &\leq \frac{|\nabla g_k(x)-\nabla f_k(x)|}{d_k(x)}\leq\\ \frac{|\nabla g_k(x)-\nabla f_k(x)|}{d(\nabla f_1(x),\cdots,\nabla f_p(x))} &\leq \frac{|\nabla g_k(x)-\nabla f_k(x)}{c\rho(x)^{\ell-w_n}}\leq\frac{\epsilon}{c},\ \text{ for all } \epsilon>0.\end{aligned}$$

If $y_k=0$, the above inequality is trivially satisfied. Then we conclude that

$$\lim_{x\to 0}\frac{|\sum_{i=1}^{p}y_i(\nabla g_i(x)-\nabla f_i(x))|}{|\sum_{i=1}^{p}y_i\nabla f_i(x)|}=0. \tag{3}$$

On the other hand, from the triangle inequality

$$|\sum_{i=1}^{p}y_i\nabla g_i(x)|\geq|\sum_{i=1}^{p}y_i\nabla f_i(x)|-|\sum_{i=1}^{p}y_i(\nabla g_i(x)-\nabla f_i(x))|.$$

Hence,

$$\frac{|\sum_{i=1}^{p} y_i \nabla g_i(x)|}{|\sum_{i=1}^{p} y_i \nabla f_i(x)|} \geq 1 - \frac{|\sum_{i=1}^{p} y_i (\nabla g_i(x) - \nabla f_i(x))|}{|\sum_{i=1}^{p} y_i \nabla f_i(x)|}.$$

From (3) and since $y = (y_1, \cdots, y_p)$ is arbitrary,

$$|\sum_{i=1}^{p} y_i \nabla g_i(x)| \geq \frac{1}{2} |\sum_{i=1}^{p} y_i \nabla f_i(x)|.$$

Then, for x small enough

$$d(\nabla g_1(x), \cdots, \nabla g_p(x)) \geq \frac{1}{2} d(\nabla f_1(x), \cdots, \nabla f_p(x)) \geq \frac{c}{2} \rho(x)^{\ell - w_n}$$

and Lemma 2.6 is proved. □

Coming back to the proof of Theorem 2.4, since $R_w(z) \leq r$ by hypothesis and z and $j^r_w F(0)$ have the same weighted r-jet, then by Lemma 2.6 we obtain that:

$$d(\nabla F_1(x,t), \cdots, \nabla F_p(x,t)) \geq \frac{1}{2} d(\nabla z_1(x), \cdots, \nabla z_p(x)) \geq \frac{c}{2} \rho(x)^{R_w(z) - w_n}$$

for $|x| \leq \delta$, $t \in [0,1]$. As a consequence,

1. For (x,t) as above, since $\nabla F_1(x,t), \cdots, \nabla F_p(x,t)$ are linearly independent vectors then $|n_i(x,t)| = d_i(x,t) > 0$, $1 \leq i \leq p$.
2. Since F is of class C^r, then ∇F_i is of class C^{r-1}. Hence X is C^{r-1} for $x \neq 0$.

Now we show that the vector field X can be extended to the origin as a Lipschitz map.

Lemma 2.7. *Let B be a small ball centered at the origin in $\mathbb{R}^n$ and redefine*

$$X : B \times [0,1] \subset \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$$

by

$$X(x,t) = \begin{cases} \displaystyle\sum_{i=1}^{p} \frac{\langle e_{n+1}, \nabla F_i(x,t) \rangle}{|n_i(x,t)|^2} \cdot n_i(x,t), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases} \tag{4}$$

Then X, including the origin, is a Lipschitz map.

Proof of Lemma 2.7.

This follows from the filtration associated to the functions involved as follows:

i. $fil(\theta_i(x)) \geq r + 1$ since $j^r_w \theta = 0$;

ii. $fil(n_i(x,t)) \leq R_w(z) - w_n$ since

$$|n_i(x,t)| = d_i(x,t) \geq d(\nabla F_1(x,t), \cdots, \nabla F_p(x,t)) \geq \frac{c}{2} \rho(x)^{R_w(z) - w_n};$$

iii. since $|X(x,t)| \leq \sum \dfrac{|\theta_i(x)|}{|n_i(x,t)|}$ and

$$fil\left(\frac{\theta_i(x)}{n_i(x,t)}\right) \geq (r+1) - (R_w(z) - w_n) = r - R_w(z) + w_n + 1 \geq w_n + 1,$$

we conclude that $fil(X) \geq fil\left(\dfrac{\theta_i(x)}{n_i(x,t)}\right) \geq w_n + 1$, then $fil(X') \geq 1$ (remember that $r > 1$). Therefore $\lim_{x\to 0} X'(x,t) = 0$, that is, X' is bounded, hence X is Lipschitz and Lemma 2.7 is proved. $\square$

Lemma 2.8. *Let $Y(x,t) = e_{n+1} - X(x,t)$. Then $\lim_{|x|\to 0} \dfrac{Y(x,t) - Y(0,t)}{|x|} = 0$ and $Y(x,t)$ is Lipschitz.*

Proof of Lemma 2.8.

In fact, since $\dfrac{|Y(x,t) - Y(0,t)|}{|x|} = \dfrac{|X(x,t) - X(0,t)|}{|x|}$ then the filtration of this quotient is $(r - R_w(z) + w_n + 1) - 1 \geq 1$.

Then Y is Lipschitz (at the variable x) because its derivative is bounded. Since at the variable t, the filtration of θ is at least greater than r, then we conclude that Y is Lipschitz and Lemma 2.8 is proved. $\square$

Conclusion of the proof of Theorem 2.4:

Since $\frac{d}{dy}F_k(Y(y)) = 0$, for all $k = 1,\ldots,p$, then $Y(x,t)$ is tangent to the level surface $F(x,t) = c$ on each (x,t).

From Picard's Theorem, the differential equation

$$\begin{cases} Y'(y) = Y(y) \\ Y(0) = y_0 \end{cases}$$

with $y = (x,t), y_0 = (x_0,t_0) \in B \times [0,1]$ has a unique solution which depends continuously on y_0, and therefore F is constant along its trajectories.

From the integration of the vector field Y we obtain the map $h : B\times\{0\} \subset \mathbb{R}^n\times\{0\} \to \mathbb{R}^n\times\{1\}$ that for each $(x,0)$ associates $(h(x),1) =$ the point where the trajectory of the field Y intercepts $\mathbb{R}^n \times \{1\}$, which is bi-Lipschitz and also $h(0) = 0$.

Hence,

$$z(x) = F(x,0) = F(h(x),1) = z(h(x)) + \theta(h(x)) = [(z+\theta)\circ h](x),$$

with h bi-Lipschitz and we conclude that z is w-bi-Lipschitz sufficient in C^r. $\square$

Applying Theorem 2.4 for the set of weights $w = (1,\ldots,1)$ then $J^r_w(n,p) = J^r(n,p)$ and then we obtain the following corollary in the language of Theorem 2.1:

Corollary 2.9. *Suppose $r(z) \leq r$ and $r > 1$. Then z is, at least bi-Lipschitz sufficient in C^r.*

Remark 2.10. In the particular case that $r = 1$, the method used in Lemma 2.7 (and consequently in the proof of Theorem 2.4) does not apply. Therefore only the result of Bochnak-Kucharz in [3] holds, that is, z is at least C^0-sufficient in C^1.

When $r > 1$, Corollary 2.9 is stronger than Remark 2.2 ii), as a consequence of Theorem 2.1. In fact, bi-Lipschitz equivalence implies C^0-equivalence. However the authors in [15, 5] did not consider the bi-Lipschitz case at that time.

2.2. **The weighted C^i-sufficiency,** $i \geq 1$**.**

Now we show the weighted C^i-version of Theorem 2.1. Notice that when the set of weights is $w = (1, \ldots, 1)$, the weighted jet space $J^r_{(1,\ldots,1)}(n,p)$ coincides with $J^r(n,p)$ and the next Theorem 2.12 also coincides with Theorem 2.1 of Fávaro-Martins.

First, we show a technical lemma.

Lemma 2.11. *Given $z \in J^r_w(n,p)$, then $R^0_w(z) \leq R_w(z)$, for $|x|$ small enough.*

Proof of Lemma 2.11.

From the Definition of $R^0_w(z)$, $|z(x)| \leq c\rho(x)^{R^0_w(z)}$, then $|\nabla z_i(x)| \leq c_1\rho(x)^{R^0_w(z)-w_n}$, for all $i = 1, \ldots, p$. On the other hand, from the definition of $R_w(z)$,

$$c\rho(x)^{R_w(z)-w_n} \leq d(\nabla z_1(x), \ldots, \nabla z_p(x)) \leq d_i(x) \leq \sum_{j=1}^{p} |y_j \nabla z_j(x)|\} \leq$$

$$\leq p|\nabla z_{j_0}(x)| \leq pc_1\rho(x)^{R^0_w(z)-w_n},$$

where the penultimate inequality is valid because the $|y_j(x)|$ can be taken ≤ 1, for all j (see [9, Lemma 3.1]), for $|x|$ small enough.

Then, $R^0_w(z) \leq R_w(z)$ for $|x|$ small enough and Lemma 2.11 is proved. $\square$

Theorem 2.12. *Fix a set of weights $w = (w_1, \ldots, w_n)$. Let $z \in J^r_w(n,p)$, suppose $R_w(z) \leq r$ and for some $i \geq 0$,*

$$r - R_w(z) + w_n - i\left[R_w(z) - R^0_w(z) + w_n\right] \geq 0.$$

Then z is w-C^i-sufficient in C^r.

Proof. From the hypothesis, $R_w(z) \leq r$. By Lemma 2.11, for $|x|$ small enough, we can assume $R^0_w(z) \leq R_w(z) \leq r$.

Consider the vector field $X(x,t) = \sum_{s=1}^{p} \theta_s(x).n_s(x,t).|n_s(x,t)|^{-2}$, at a point $(x,t) \neq (0,t)$, where $s = 1, \ldots, p$. Denote by

$$H_s(x,t) = \theta_s(x).n_s(x,t).|n_s(x,t)|^{-2}, \;\; s = 1, \ldots, p.$$

The derivative of order $\alpha \leq r-1$ of H_s at a point $(x,t) \neq (0,t)$ is

$$H_s^{(\alpha)}(x,t) = \sum_{j=0}^{\alpha} \binom{\alpha}{j} n_s^{(\alpha-j)}(x,t) \left(\sum_{k=0}^{j} \binom{j}{k} \theta_s^{(j-k)}(x) \left[|n_s(x,t)|^{-2}\right]^{(k)} \right).$$

Since $fil(\theta_s(x)) > r = fil(\rho(x)^r)$, then $fil(\theta_s^{(j-k)}(x)) > r - w_n(j-k) = fil(\rho(x)^{r-w_n(j-k)})$.

We also have

$$|n_s(x,t)| = d_s(x,t) = |\nabla F_s(x,t) - \sum_{j \neq s} y_j \nabla F_j(x,t)| \leq M \sum_{j=1}^{p} |\nabla F_j(x,t)| \leq$$

$$\leq M\left[\sum_{j=1}^{p} |\nabla z_j(x)| + t|\nabla\theta_j(x)|\right],$$

where $M > 0$ and the first inequality above holds because the y_j can be taken less than or equal to 1, for all j (see [9, Lemma 3.1]).

From the definition of $R_w^0(z)$, we have $|z(x)| \leq \rho(x)^{R_w^0(z)}$, hence $fil(z(x)) \geq R_w^0(z)$ and consequently, $fil(\nabla z_j(x)) \geq R_w^0(z) - w_n$, for all $j = 1, \dots, p$. Then by previous inequalities, it follows that

$$fil(n_s(x,t)) \geq \min\{fil(\nabla z_j(x)), fil(\nabla\theta_j(x))\} = fil(\nabla z_j(x)) \geq R_w^0(z) - w_n, \forall\, j = 1, \dots, p.$$

Then, $fil(n_s^{(\alpha-j)}(x,t)) \geq R_w^0(z) - w_n - (\alpha - j)w_n$, for all $j = 1, \dots, p$.

On the other hand, we have seen in the proof of Lemma 2.7 that

$$|n_s(x,t)| = d_s(x,t) \geq d\left(\nabla F_1(x,t), \dots, \nabla F_p(x,t)\right) \geq \frac{c}{2}\rho(x)^{R_w(z)-w_n}.$$

Finally, $|n_s(x,t)|^{-2} \leq C_1\rho(x)^{-2R_w(z)+2w_n}$ and $[|n_s(x,t)|^{-2}]^{(k)} \leq C_2\rho(x)^{-2R_w(z)+2w_n-kw_n}$, where $C_1, C_2 > 0$. Then, from the sum of the corresponding filtrations, we conclude that

$$|X^{(\alpha)}(x,t)| \leq \sum_{s=1}^{p} |H_s^{(\alpha)}(x,t)| < \rho(x)^{r-2R_w(z)+R_w^0(z)+(1-\alpha)w_n}.$$

Now we remember that $\alpha \geq 1$, then $(1-\alpha)(R_w(z) - R_w^0) \leq 0$, therefore

$$\begin{aligned} r - 2R_w(z) + R_w^0(z) + (1-\alpha)w_n &= r - R_w(z) + w_n - (R_w(z) - R_w^0(z)) - \alpha w_n \\ &\geq r - R_w(z) + w_n - \alpha(R_w(z) - R_w^0(z)) - \alpha w_n. \end{aligned}$$

Hence we obtain

$$|X^{(\alpha)}(x,t)| \leq \sum_{s=1}^{p} |H_s^{(\alpha)}(x,t)| < \rho(x)^{r-R_w(z)+w_n-\alpha[R_w(z)-R_w^0(z)+w_n]}, \tag{5}$$

and thus $fil(X^{(\alpha)}) > r - R_w(z) + w_n - \alpha[R_w(z) - R_w^0(z) + w_n]$.

Now, in the particular case when $\alpha = i$, $\lim_{x\to 0} |X^{(i)}(x,t)| = 0$ uniformly for all $t \in [0,1]$. Then

$$X(0,t) = X'(0,t) = \cdots = X^{(i)}(0,t) = 0$$

and hence X is of class C^i.

If $\alpha < i$, then $r - R_w(z) + w_n - \alpha[R(z) - R_w^0(z) + w_n] > 0$, hence the derivative of $|X^{(\alpha)}(x,t)|$ exists and it is continuous in $(0,t)$ because of (5). Then X is of class C^α.

Then we can conclude that X is of class C^i. Since $Y(x,t) = e_{n+1} - X(x,t)$, we also obtain that the vector field Y is of class C^i and then z is w-C^i-sufficient in C^r. □

3. Examples

Example 3.1. Let $z(x,y) = (x^2 - y^2, 2xy)$. This example was given by Fávaro-Martins in [5]. It is a homogeneous map of degree 2 in both coordinate functions and they proved that z is C^1-sufficient in C^2.

Now, if we consider this example in the language of w-C^i-sufficiency, we fix the set of weights $w = (1,1)$ and then $z \in J^2_{(1,1)}(2,2)$. In this case we also obtain that z is $(1,1)$-C^1-sufficient in C^2, since $R_w(z) = R_w^0(z) = 2$.

Example 3.2. Let $z(x,y) = (z_1, z_2) = (xy, x^4 - y^2)$.

This germ is weighted homogeneous of type $(1,2;3,4)$, as $fil(|z(x,y)|) = \min\{xy, x^4 - y^2\} = 3$ then $|z(x,y)| \leq c_1 \rho(x,y)^3$, for some constant $c_1 > 0$. Hence $R^0_w(z) = 3$.

Now, since $\nabla z_1 = (y,x)$ and $\nabla z_2 = (4x^3, -2y)$, the values of $d_1(x,y)$ and $d_2(x,y)$ can be calculated using Linear Algebra via the Gram matrix $G(x,y)$ as follows:

$$
\begin{aligned}
G(x,y) &= \begin{bmatrix} \langle \nabla z_1, \nabla z_1 \rangle & \langle \nabla z_1, \nabla z_2 \rangle \\ \langle \nabla z_2, \nabla z_1 \rangle & \langle \nabla z_2, \nabla z_2 \rangle \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{\partial z_1}{\partial x}\right)^2 + \left(\frac{\partial z_1}{\partial y}\right)^2 & \frac{\partial z_1}{\partial x} \cdot \frac{\partial z_2}{\partial x} + \frac{\partial z_1}{\partial y} \cdot \frac{\partial z_2}{\partial y} \\ \frac{\partial z_2}{\partial x} \cdot \frac{\partial z_1}{\partial x} + \frac{\partial z_2}{\partial y} \cdot \frac{\partial z_1}{\partial y} & \left(\frac{\partial z_2}{\partial x}\right)^2 + \left(\frac{\partial z_2}{\partial y}\right)^2 \end{bmatrix}
\end{aligned}
$$

$$
d_1(x,y) = \frac{\sqrt{\det G(x,y)}}{\sqrt{\langle \nabla z_2, \nabla z_2 \rangle}} = \frac{\frac{\partial z_1}{\partial x} \cdot \frac{\partial z_2}{\partial y} - \frac{\partial z_1}{\partial y} \cdot \frac{\partial z_2}{\partial x}}{\left[\left(\frac{\partial z_2}{\partial x}\right)^2 + \left(\frac{\partial z_2}{\partial y}\right)^2\right]^{\frac{1}{2}}}
$$

and

$$
d_2(x,y) = \frac{\sqrt{\det G(x,y)}}{\sqrt{\langle \nabla z_1, \nabla z_1 \rangle}} = \frac{\frac{\partial z_1}{\partial x} \cdot \frac{\partial z_2}{\partial y} - \frac{\partial z_1}{\partial y} \cdot \frac{\partial z_2}{\partial x}}{\left[\left(\frac{\partial z_1}{\partial x}\right)^2 + \left(\frac{\partial z_1}{\partial y}\right)^2\right]^{\frac{1}{2}}}.
$$

Analysing the corresponding filtrations of $d_1(x,y)$ and $d_2(x,y)$, we conclude that

$$d_2(x,y) = \min\{d_1(x,y), d_2(x,y)\}$$

and $fil(d_2(x,y)) = 3$. Then,

$$
\begin{aligned}
d(\nabla z_1, \nabla z_2) &= d_2(x,y) \\
&\geq c_2 \rho^3(x,y) \\
&= c_2 \rho^{5-2}(x,y) \\
&= c_2 \rho^{5-w_2}(x,y), \quad c_2 > 0.
\end{aligned}
$$

Hence, $R_w(z) = 5$.

Therefore, the hypothesis of Theorem 2.12 is reduced to

$$r - (R_w(z) - w_n) - i[R_w(z) - (R^0_w(z) - w_n)] = r - 3 - 4i \geq 0. \tag{6}$$

Notice that if $i = 0$ we should take $r \geq 3$ to obtain the $(1,2)$-C^0-sufficiency in C^r. Or more generally, we should take $r \geq 3 + 4i$ to obtain $(1,2)$-C^i-sufficiency in C^r.

For instance, if we consider

$$\theta(x,y) = (0, \frac{x^{19} + x^9 y^5}{x^4 + y^2})$$

which is a map germ of class C^7 with $j^7_{(1,2)}\theta = 0$, then z and $z + \theta$ can be seen as two $(1,2)$-C^7-realizations of z in $J^7_{(1,2)}(2,2)$ and $7 \geq 3 + 4$ by expression (6). As a consequence of Theorem 2.12 we obtain that $z + \theta$ is C^1-equivalent to z.

Moreover, if we consider $f(x,y) = z(x,y) + \theta(x,y)$ and $g(x,y) = (xy, x^4 - y^2 + y^4) + \theta(x,y)$ these germs have the same weighted 7-jet, and are two $(1,2)$-C^7-realizations of z.

Then we can apply Theorem 2.12 to conclude that f is C^1-equivalent to g, consequently, g is also C^1-equivalent to z. But as f and g do not have the same 4-jet in $J^4(2,2)$, we cannot apply Theorem 2.1.

Example 3.3. Let $z(x,y) = (z_1, z_2) = (x^4 - y^2, x^3 y)$.

This germ is weighted homogeneous of type $(1,2;4,5)$. Proceeding in the same way as in Example 3.2, we obtain in this case $R_w^0(z) = 4$ and $R_w(z) = 6$.

Then the hypothesis of Theorem 2.12 is reduced to

$$r - (R_w(z) - w_n) - i[R_w(z) - (R_w^0(z) - w_n)] = r - 4 - 4i \geq 0.$$

Therefore, if $r \geq 4 + 4i$ we obtain $(1,2)$-C^i-sufficiency in C^r.

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