Journal *of* Singularities

Volume 2

Proceedings of Singularities in Aarhus, 17-21 August 2009,
in honor of Andrew du Plessis
on the occasion of his sixtieth birthday

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www.journalofsing.org

Journal

of

Singularities

Volume 2
2010

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ISSN: # 1949-2006

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Andrew du Plessis

This volume contains proceedings of the international workshop *Singularities in Aarhus* held in honor of Andrew du Plessis to celebrate his sixtieth birthday. The workshop took place at the Department of Mathematical Sciences of Aarhus University, Denmark in the week of August 17-21, 2009. Its main theme was singularity theory, both of varieties and mappings. The meeting was attended by about sixty participants from all over the world.

The papers in this volume cover a variety of subjects discussed at the workshop. All the manuscripts have been carefully peer-reviewed. We would like to express our gratitude to the authors for their contributions as well as to the referees for the high quality job.

We also thank all the participants – especially the speakers – who made the meeting successful and fruitful. Last but not least, we are very grateful for the financial support received from the Department of Mathematical Sciences of Aarhus University, from the grant "Symmetry and Moduli Problems in Topology" allocated by the Danish Agency for Science, Technology and Innovation, and from the Center for Topology and Quantization of Moduli spaces (CTQM). The CTQM funding was allocated from the Niels Bohr Visiting Professorship Grant provided by the Danish National Research Foundation.

December 2010
The editors

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Andrew du Plessis: the story so far

It is a great pleasure to celebrate the 60^{th} birthday of Andrew, my long time friend and collaborator. My personal association with Andrew goes back to 1970, when he arrived in Liverpool as my research student, having just completed a first degree at Cambridge. Andrew's father was also a mathematician, an analyst, then at the University of Newcastle.

Andrew's arrival coincided with the end of our year long Liverpool Singularities Symposium. Among the striking new developments reported that year (by Haefliger) was a technique due to Gromov, dubbed 'homotopy integration', for constructing examples of geometric structures.

Andrew set to work to apply this new idea to problems in singularity theory, and in due course wrote an excellent thesis doing this, which led to his first 3 publications [1, 2, 4][1]. During this period we had close contact, and I came to regard Andrew as friend and collaborator more than just as student, with several common interests.

When his SERC grant ran out, Andrew obtained a research assistantship at Bangor. This conveniently allowed him time to complete writing up his work, to explore the mountains of Snowdonia, and also to visit Liverpool every couple of weeks to participate in our Singularities Seminar. It was a particularly noteworthy seminar that year, working through a proof of Mather's topological stability theorem, and led by Eduard Looijenga: and a year in which we all learned a lot. The final notes [3] of the seminar remain a key reference in this whole area.

From Bangor, Andrew moved (in 1977) to Aarhus. I was told later that within his first 6 months he had explored the life of the city and had learned to speak, and to lecture in, fluent Danish; and it was fairly soon that he and Annie got together. Perhaps understandably, there is a slight gap in his publications at this point.

But then he began a wonderfully productive period, with a series of great ideas. His next paper [5] obtained the first effective estimates of orders of determinacy of map-germs for right-left equivalence. The techniques were developed and extended in later papers of Andrew and collaborators [7, 12, 18], and led to effective classifications of germs of low codimensions, several of which were published. Unfortunately, the lists available now seem shorter than those that existed 25 years ago: some may still be buried in piles of paper in Andrew's office.

In his paper [6], Andrew made ingenious use of known methods to develop a new technique to study the family of maps with a fixed $k-$jet: here he proved that all germs except for those in a subset of infinite codimension are topologically finitely determined.

In [8] he found the conditions ('semi-nice dimensions') necessary and sufficient for map-germs to be finitely $C^\infty-$determined (for right-left equivalence) in general, and extended this in [10] to a global result. Outside these dimensions, he gives a map not homotopic to a $C^\infty-$stable map, and even one not homotopic to a map with all germs finitely determined. He also combined this with his own early work to find in favourable cases sufficient conditions. In [13] these ideas are extended to give general results for C^1-stability (nice dimensions) and for finite and even for $\infty-$ C^1-determinacy (semi-nice dimensions).

Next he began a collaboration with Leslie Wilson and others producing a series of beautiful papers on right equivalence [11, 14, 16, 19], showing (under mild conditions on f) that:

f is $J^2_f - \mathcal{R}-$determined,

f is determined up to $\mathcal{R}-$equivalence by Σ_f and $f|\Sigma_f$,

the group of $\mathcal{R}-$symmetries of f is homotopically trivial,

$f|\Sigma_f$ is a normalisation of Δ_f, and hence:

f is determined up to right equivalence by Δ_f.

This suggests a big challenge of finding reasonable conditions under which the homeomorphism

[1]The references are to the list of Andrew's publications which folllows.

type of Δ_f determines f up to topological right equivalence.

Andrew continued thinking about topological stability, and in the mid 1980's came up with a brilliant idea (disruptive germ classes) for obtaining necessary conditions for stability. This began our period of close collaboration, which lasted about a dozen years and led to our book [17] on stability. It was a most enjoyable period, exciting mathematically, with congenial companionship, (ir)regular meetings at exotic locations, and of course numerous visits to each other at Aarhus and Liverpool.

The usual pattern was that Andrew and I would talk together, often seeking a way round a problem, then separate and each try to write something, then discuss what we had written. When we were not together Andrew would rarely answer letters promptly, but would then send a huge package of handwritten manuscript which I would write (or later type) out, editing and modifying it as I went.

At first we had planned a series of related papers: on the whole, I was doing classifications, Andrew was producing geometrical ideas, and I was typing them up. But once Andrew had built on Jim Damon's ideas to obtain a more general argument for sufficiency, it was clear we should put the work together as a book. The process had its frustrations: every time I thought we had finished and could send the manuscript off for publication, Andrew came up with another brilliant idea, which took one or two years to write up, and added a hundred pages to the length of the manuscript. The book took nearly all our research output for 10 years. Filling in extra points, and finding a number of applications of the book's results or ideas, led to several more years' work and numerous papers: [15] was an advance summary, papers [20, 24, 25, 28, 32] all arise directly from topics in the book; [30] is an application of the main result, and another idea of Andrew's led to the sequence [21-23, 26, 27, 29, 35-38].

I must mention also Andrew's more recent collaboration [31, 33] with David Trotman, with work on stratified transversality, and on a tantalising conjecture that would resolve a number of problems and strengthen the main results in the book; and there are other significant projects at various stages of completion.

I conclude with my very best wishes to Andrew for the future.

Terry Wall

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Journal *of* Singularities
Volume 2 (2010), 1-18

Proc. of Singularities in Aarhus
received 5 February 2010
DOI: 10.5427/jsing.2010.2a

ON THE CONNECTION BETWEEN FUNDAMENTAL GROUPS AND PENCILS WITH MULTIPLE FIBERS

ENRIQUE ARTAL BARTOLO AND JOSÉ IGNACIO COGOLLUDO-AGUSTÍN

Introduction

The study of the topology of complex projective (or quasiprojective) smooth varieties depends strongly on the knowledge of the topology of the complement of hypersurfaces in a projective space. Considering a projection, any smooth projective variety is a covering of a projective space of the same dimension ramified along a hypersurface. These coverings are measured by (finite index subgroups of) the fundamental group of the complement of the hypersurface. Using Lefschetz-Zariski theory, if we take a *generic* plane section the fundamental group of the complement does not change. As a consequence, for fundamental group purposes, one can restrict their attention to the study of complements of curves in the projective plane, as stated in the foundational paper by O. Zariski [26].

The richness of coverings for a space depends on its fundamental group. This is why we are mostly interested in curves $C \subset \mathbb{P}^2$ whose $\pi_1(\mathbb{P}^2 \setminus C)$ is non-abelian. The first known example is probably the curve formed by three lines $C := L_1 \cup L_2 \cup L_3$ intersecting at one point P. There is an easy way to compute this fundamental group; the pencil of lines through P is parametrized by $\mathbb{P}^1$; this pencil induces an epimorphism of $\mathbb{P}^2 \setminus C$ onto $\mathbb{P}^1 \setminus \{p_1, p_2, p_3\}$ (the punctures corresponding to the three lines). Moreover, this map is a locally trivial fibration (with fiber isomorphic to $\mathbb{C}$) and hence $\pi_1(\mathbb{P}^2 \setminus C) \cong \pi_1(\mathbb{P}^1 \setminus \{p_1, p_2, p_3\})$, which is a free group of two generators.

The first known examples of irreducible curves whose fundamental groups are known to be non-abelian appeared in [26]. The first one corresponds to a hexacuspidal sextic, with its six cusps on a conic; the equation of such a curve is of the form $f_2^3 + f_3^2 = 0$, where f_j is a homogeneous polynomial of degree j. Its fundamental group is $\mathbb{Z}/2 * \mathbb{Z}/3$; in §2 we will see the relation between this group and the pencil generated by $f_2^3 = 0$ and $f_3^2 = 0$. This kind of examples have been generalized by various authors replacing $(2, 3)$ by (p, q). In the same paper, Zariski found the irreducible curve with smallest possible degree having a non-abelian fundamental group: the tricuspidal quartic. This example and many others appearing in the literature are also connected with pencils.

The precise connection with pencils can be stated as follows: a pencil defines a dominant morphism to a quasi-projective curve, inducing an epimorphism at the level of fundamental groups. The multiplicities of the fibers of the pencil induce an orbifold structure on the quasi-projective group, and the map defines an epimorphism onto the orbifold fundamental group. When such an orbifold fundamental group is non-abelian, then the original fundamental group has a surjection onto a non-abelian group. Such surjections coming from dominant maps will be referred to as *geometric* surjections.

The tricuspidal quartic is the only irreducible curve of degree 4 with a non-abelian fundamental group. The degree-five case was studied by A. Degtyarev [9]; he found exactly two irreducible

Partially supported by MTM2007-67908-C02-01.

quintics with non-abelian fundamental groups. One of them, also studied by the first author [2], has an infinite fundamental group. In §2, we will study its relationship with a pencil. The question whether or not all non-abelian fundamental groups have a *geometric* surjection onto an orbifold group naturally arises. A positive result in this direction is given in [8] for certain roots of the Alexander polynomial. In addition, all the examples studied, up to now, supported an affirmative answer to this question.

In this paper we will show an explicit example of a non-abelian fundamental group whose complement admits no geometric surjections. This curve is one of the quintics referred to in the previous paragraph, which will be called the *projective Degtyarev curve* throughout this text. As a brief description, the projective Degtyarev curve has exactly three singular points of type $\mathbb{A}_4$; its fundamental group is finite and non-abelian. In Proposition 4.4, we prove that this group admits no geometric surjections. Once the group is computed, the proof is rather straightforward; it depends on the orders of the group and its abelianization and on the properties of orbifold groups.

If we add a tangent line to one of the singular points of the projective Degtyarev curve, the complement of the union in $\mathbb{P}^2$ is the complement of an affine curve, which will be called the *affine Degtyarev curve*. This affine curve has an infinite non-abelian fundamental group and non-trivial characteristic varieties (see §1 for the definition). Extending results of Arapura and others, it is known that irreducible components of positive dimension (for the fundamental group of a quasiprojective variety) are obtained as pull-back of irreducible components of characteristic varieties of orbifolds. A natural question arises: Is it also true for irreducible components of dimension 0 (isolated points)? Plenty of computations supported a positive answer: most quasiprojective groups satisfy the property for irreducible components of any dimension (see §2 for examples). The main Theorem 4.5 of this paper shows that the fundamental group of the complement of the affine Degtyarev curve does not satisfy this property. This is the only known example, up to now.

The paper is organized as follows. In §1, the concepts of orbifold and characteristic varieties are recalled, also some orbifold groups are studied. In §2, we relate non-abelian fundamental groups of the complements of curves (which are known in the literature) with orbifold morphisms (via pencils of curves). In §3, we describe Degtyarev curves and, in order to obtain a presentation for their fundamental groups, we compute a special braid monodromy. The fundamental groups are obtained in §4, where also the main results of the paper are stated and proved. Finally, further properties of the affine Degtyarev curve are sketched in §5.

1. Orbifold groups and characteristic varieties

The fundamental groups of oriented Riemann surfaces have been extensively studied. The fundamental group of a compact Riemann surface of genus g is

$$\pi_g := \left\langle a_i, b_i, 1 \leq i \leq g \,\middle|\, \prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} \right\rangle.$$

If C is a surface with genus g and $k > 0$ punctures then its fundamental group is free of rank $2g + k - 1$. We are going to extend this family by considering *orbifold groups*.

In this paper, we will refer to an *orbifold* X_φ as an orbifold Riemann surface, that is, a quasiprojective Riemann surface X with a function $\varphi : X \to \mathbb{N}$ with value 1 outside a finite number of points. The finite set $M_\varphi = \{x \in X \mid \varphi(x) > 1\}$ will be called the *set of orbifold points* and $\varphi(x)$ is the *orbifold index* of $x \in M_\varphi$.

We may think that a neighborhood of a point $P \in X_\varphi$ such that $\varphi(P) = n$ is the quotient of a disk (centered at P) by a rotation of angle $\frac{2\pi}{n}$. We will consider that a loop around P is trivial if its lifting bounds a disk. Following this idea, we define orbifold fundamental groups.

Definition 1.1. For an orbifold X_φ, let $p_1, \dots, p_n$ the points such that $m_j := \varphi(p_j) > 1$. Then, the *orbifold fundamental group* of X_φ is

$$\pi_1^{\text{orb}}(X_\varphi) := \pi_1(X \setminus \{p_1, \dots, p_n\}) / \langle \mu_j^{m_j} = 1 \rangle$$

where μ_j is a meridian of p_j. We denote X_φ by $X_{m_1,\dots,m_n}$.

Example 1.2. If X is a compact surface of genus g and type $X_{m_1,\dots,m_n}$, then

$$\pi_1^{\text{orb}}(X_\varphi) = \left\langle a_1, \dots, a_g, b_1, \dots, b_g, \mu_1, \dots, \mu_n \;\middle|\; \prod_{i=1}^{g} [a_i, b_i] = \prod_{j=1}^{n} \mu_j,\ \mu_j^{m_j} = 1 \ (j = 1, \dots, n) \right\rangle,$$

where products are supposed to respect the order. If X is not compact and $\pi_1(X)$ is free of rank r, then

$$\pi_1^{\text{orb}}(X_\varphi) = \left\langle a_1, \dots, a_r, \mu_1, \dots, \mu_n \;\middle|\; \mu_j^{m_j} = 1 \ (j = 1, \dots, n) \right\rangle.$$

Definition 1.3. A dominant algebraic morphism $\rho : Y \to X$ between an algebraic manifold Y and a Riemann surface X defines an *orbifold morphism* $Y \to X_\varphi$ if for all $p \in X$, the divisor $\rho^*(p)$ has multiplicity $\varphi(p)$, that is, $\rho^*(p) = \varphi(p) D$, where D is a (possibly non-reduced) divisor in Y.

Proposition 1.4. *Let $\rho : Y \to X$ define an orbifold morphism $Y \to X_\varphi$. Then ρ induces a homomorphism $\rho_* : \pi_1(Y) \to \pi_1^{\text{orb}}(X_\varphi)$. Moreover, if the generic fiber is connected, then ρ_* is surjective.*

Proof. Let $M_\varphi := \{x \in X \mid \varphi(x) > 1\}$; we consider the restriction mapping $\tilde{\rho} := \rho_| : Y \setminus \rho^{-1}(M_\varphi) \to X \setminus M_\varphi$. This map induces a morphism $\tilde{\rho}_* : \pi_1(Y \setminus \rho^{-1}(M_\varphi)) \to \pi_1(X \setminus M_\varphi)$ fitting in the following commutative diagram:

$$\begin{array}{ccc} \pi_1(Y \setminus \rho^{-1}(M_\varphi)) & \xrightarrow{\tilde{\rho}_*} & \pi_1(X \setminus M_\varphi) \\ i_* \downarrow & & \downarrow j_* \\ \pi_1(Y) & \xrightarrow{\rho_*} & \pi_1(X). \end{array}$$

The vertical mappings are induced by the inclusions. They are both surjective; the kernel of j_* is generated by the meridians of the points in M_φ while the kernel of i_* is generated by the meridians of the irreducible components of $\rho^{-1}(M_\varphi)$, i.e., the components of the pull-back divisor $\rho^*(M_\varphi)$.

Let us consider an irreducible component D of $\rho^*(M_\varphi)$ such that $\rho(D) =: x \in M_\varphi$. Let $n := \varphi(x)$; note that the multiplicity m_D of D in $\rho^*(M_\varphi)$ is a multiple of n. We can interpret m_D as follows. If μ_D denotes a meridian of D, then there is a meridian μ_x of x such that $\tilde{\rho}_*(\mu_D) = (\mu_x)^{m_D}$. Following Definition 1.1, it is easily seen that $\tilde{\rho}_*$ factorizes through a morphism (also called ρ_*) $\pi_1(Y) \to \pi_1^{\text{orb}}(X_\varphi)$.

The above argument also works if one replaces M_φ by a finite set $M \supseteq M_\varphi$. In particular, one can choose M to be the bifurcation locus of ρ, i.e., the mapping is a differentiable locally trivial fibration outside M. If the fiber is generically connected, the long exact homotopy sequence of this fibration implies the surjectivity of $\tilde{\rho}_*$ (for M). The result follows. □

Definition 1.5. A fundamental group $G := \pi_1(Y)$ of an algebraic manifold is said to *posses a geometric surjection* if Y possesses an orbifold morphism $Y \to X_\varphi$ whose generic fiber is connected, and such that $\pi_1^{\text{orb}}(X_\varphi)$ is non-abelian.

We recall the notion of characteristic varieties and its relationship with orbifolds. We focus our attention on the characteristic varieties of quasiprojective manifolds, though they can be defined in general and depend only on the fundamental group. Let X be a connected topological space X, having the homotopy type of a finite CW-complex, and let $G := \pi_1(X, x_0)$, $x_0 \in X$ which will be omitted if it is not necessary. Recall that the space of characters of G is

$$H^1(X; \mathbb{C}^*) = \operatorname{Hom}(H_1(X; \mathbb{Z}), \mathbb{C}^*) = \operatorname{Hom}(\pi_1(X), \mathbb{C}^*) =: \mathbb{T}_G. \tag{1.1}$$

Remark 1.6. Since G is finitely generated, then it is also the case for $H_1(X; \mathbb{Z})$. Let $n := \operatorname{rk} H_1(X; \mathbb{Z})$ and Tors_G be the torsion subgroup of $H_1(X; \mathbb{Z})$. Then $\mathbb{T}_G$ is an abelian complex Lie group with $|\operatorname{Tors}_G|$ connected components (each one is isomorphic to $(\mathbb{C}^*)^n$) satisfying the following exact sequence:

$$1 \to \mathbb{T}_G^{\mathbf{1}} \to \mathbb{T}_G \to \operatorname{Tors}_G \to 1,$$

where $\mathbb{T}_G^{\mathbf{1}}$ is the connected component containing the trivial character $\mathbf{1}$.

For a character $\xi \in \mathbb{T}_G$, we can construct a local system of coefficients $\mathbb{C}_\xi$ over X.

Definition 1.7. The k-th *characteristic variety* of X is the subvariety of $\mathbb{T}_G$, defined by:

$$\mathcal{V}_k(X) = \{\xi \in \mathbb{T}_G \mid \dim H^1(X, \mathbb{C}_\xi) \geq k\},$$

where $H^1(X, \mathbb{C}_\xi)$ is the cohomology with coefficients in the local system ξ. In some cases we will use the notation $\mathcal{V}_k(G)$ since it is independent of X as far as $\pi_1(X) \cong G$. The definition also applies to orbifolds replacing π_1 by π_1^{orb}.

The following result is straightforward.

Proposition 1.8. *Let $\varphi : G \to H$ be a group epimorphism. Then φ^* induces injections $\mathbb{T}_H \cong \varphi^*\mathbb{T}_H \hookrightarrow \mathbb{T}_G$ and $\mathcal{V}_j(H) \cong \varphi^*\mathcal{V}_j(H) \hookrightarrow \mathcal{V}_j(G)$.*

Remark 1.9. Let us explain how to compute these invariants. For the sake of simplicity, the twisted homology, instead of the cohomology, will be computed. Let us consider a finite CW-complex homotopy equivalent to X; for the sake of simplicity the CW-complex will be denoted X. Let $\pi : \tilde{X} \to X$ be the maximal abelian covering. Note that $\tilde{X}$ inherits a CW-complex structure. The group of automorphisms of π is $H_1(X; \mathbb{Z})$. The action of this Abelian group endows the chain complex $C_*(\tilde{X}; \mathbb{C})$ with a module structure over the ring $\Lambda := \mathbb{Z}[H_1(X; \mathbb{Z})]$. The differentials of the complex are Λ-homomorphisms. Moreover, $C_*(\tilde{X}; \mathbb{C})$ is a free Λ-module of finite rank. If we fix a character ξ, $\mathbb{C}$ has a natural Λ-module structure which is denoted by $\mathbb{C}_\xi$ (as the local system of coefficients). The twisted homology of X is the homology of the $C_*(X; \mathbb{C})^\xi := C_*(\tilde{X}; \mathbb{C}) \otimes_\Lambda \mathbb{C}_\xi$. Following this interpretation, it is not difficult to prove that the characteristic varieties are algebraic subvarieties of $\mathbb{T}_G$, defined with integer equations.

This i-th jumping loci of $C_*(\tilde{X}; \mathbb{C})$ with respect to $\_ \otimes_\Lambda \mathbb{C}_\xi$ can also be viewed as the zero locus of the i-th Fitting ideal of $H_1(\tilde{X}; \mathbb{C})$ or, analogously, the support of the module $\wedge^i H_1(\tilde{X}; \mathbb{C})$ over the ring Λ (see [17]).

Following the theory developed by various authors (Beauville [6], Arapura [1], Simpson [22], Budur [7], Delzant [11], Dimca [13]), the structure of characteristic varieties for quasiprojective manifolds can be stated as follows.

Theorem 1.10 ([4])**.** *Let Σ be an irreducible component of $\mathcal{V}_k(G)$, $k \geq 1$. Then one of the two following statements holds:*

- *There exists a surjective orbifold morphism $\rho : X \to C_\varphi$ and an irreducible component Σ_1 of $\mathcal{V}_k(\pi_1^{orb}(C_\varphi))$ such that $\Sigma = \rho^*(\Sigma_1)$.*
- *Σ is an isolated torsion point.*

Remark 1.11. In general, both G and its characteristic varieties are difficult to compute. For the complement of hypersurfaces in a projective space, Libgober [17] gave an alternative way of computing most components of the characteristic varieties from algebraic properties of the hypersurface without computing G.

Remark 1.12. Characteristic varieties can also be understood from Alexander-invariant point of view. Following Theorem 1.10, characteristic varieties are determined by finite-degree abelian coverings.

We compute the invariants for some orbifold groups.

Proposition 1.13. *Let G be the orbifold group of $\mathbb{P}^1_{2,5,10}$. Then G is a semidirect product of the fundamental group of a compact surface of genus 2 and $\mathbb{Z}/10\mathbb{Z}$. The torus $\mathbb{T}_G$ is μ_{10}, the group of 10-th roots of unity, $\mathcal{V}_1(G)$ consists of the primitive 10-th roots of unity and $\mathcal{V}_2(G)=\emptyset$.*

Proof. Let us consider the short exact sequence associated with the abelianization map ($\mathbb{Z}/10 := \langle t \mid t^{10}=1\rangle$ is G/G'). This sequence corresponds to an orbifold morphism, which is a ramified cyclic covering of degree 10 of $\mathbb{P}^1$. The ramification points correspond with the orbifold points whose ramification indices equal the orbifold index. Using Riemann-Hurwitz one checks that the covering space is a compact Riemann surface of genus 2. Since the meridian of an orbifold point of index 10 is of order 10 in G, then the exact sequence splits and we have a semidirect-group structure.

In order to compute $\mathcal{V}_1(G)$ we follow the construction outlined in Remark 1.9, applied to the CW-complex associated with the presentation of G given by $\langle x,y \mid x^2=y^5=(xy)^{10}=1\rangle$. Let us denote p the unique 0-cell, x,y the 1-cells and A,B,C the 2-cells (corresponding to the relations in the given order). Let us fix a character $\xi\in\mathbb{T}_G$. It is clear that $\mathbf{1}\notin\mathcal{V}_1(G)$. We can assume that $\zeta:=\xi(t)\neq 1$. The complex $C_*(X;\mathbb{C})^\xi$ is given by

$$0 \longrightarrow \mathbb{C}^3 \xrightarrow{\partial_2} \mathbb{C}^2 \xrightarrow{\partial_1} \mathbb{C} \longrightarrow 0.$$

The matrix for ∂_1 is $\begin{pmatrix}\zeta^5-1 & \zeta^2-1\end{pmatrix}$. In particular, $\dim\ker\partial_1=1$. The matrix for ∂_2 equals

$$\begin{pmatrix} \zeta^5+1 & 0 & \dfrac{\zeta^{10}-1}{\zeta-1} \\ 0 & \zeta^8-\zeta^6+\zeta^4-\zeta^2+1 & \zeta^5\dfrac{\zeta^{10}-1}{\zeta-1} \end{pmatrix}$$

In order to have non-trivial homology, this matrix must vanish and this happens only when ζ is a primitive 10-th root of unity. □

Proposition 1.14. *Let G be the orbifold group of $\mathbb{P}^1_{2,2,5,5}$. Then G is an extension of $\mathbb{Z}/10\mathbb{Z}$ by the fundamental group of a compact surface of genus 4. The torus $\mathbb{T}_G$ is μ_{10}, the group of 10-th roots of unity, and both $\mathcal{V}_1(G)$ and $\mathcal{V}_2(G)$ consist of the primitive 10-th roots of unity.*

Proof. The short exact sequence associated with the abelianization map ($G/G'=\mathbb{Z}/10$) corresponds to a covering of the orbifold as in the proof of Proposition 1.13, and using Riemann-Hurwitz one obtains that the covering space is a compact Riemann surface of genus 4.

We compute the characteristic varieties as in the proof of Proposition 1.13 for the presentation of G given by $\langle x,y,z \mid x^5=y^5=z^2=(xyz)^2=1\rangle$. Let us denote p the unique 0-cell, x,y,z the 1-cells and A,B,C,D the 2-cells (corresponding to the relations in the given order). Let us fix a character $\xi\in\mathbb{T}_G$. It is clear that $\mathbf{1}\notin\mathcal{V}_1(G)$. We can assume that $\zeta:=\xi(t)\neq 1$. The complex $C_*(X;\mathbb{C})^\xi$ is given by

$$0 \longrightarrow \mathbb{C}^4 \xrightarrow{\partial_2} \mathbb{C}^3 \xrightarrow{\partial_1} \mathbb{C} \longrightarrow 0.$$

The matrix for ∂_1 is $\begin{pmatrix}\zeta^2-1 & \zeta^2-1 & \zeta^5-1\end{pmatrix}$. In particular, $\dim\ker\partial_1=2$. The matrix for ∂_2 equals

$$\begin{pmatrix}\zeta^8-\zeta^6+\zeta^4-\zeta^2+1 & 0 & 0 & \zeta^5+1\\ 0 & \bar\zeta^8-\bar\zeta^6+\bar\zeta^4-\bar\zeta^2+1 & 0 & \zeta(\zeta^5+1)\\ 0 & 0 & \zeta^5+1 & \zeta^5+1\end{pmatrix}$$

In order to have non-trivial homology, this matrix must have rank less than 2 and this happens only when ζ is a primitive 10-th root of unity. Moreover, in that case, the matrix vanishes. □

2. Examples

In this section, we will present a collection of examples of curves with non-abelian fundamental groups and geometric surjections and its relationship with characteristic varieties.

Remark 2.1. If $Y:=\mathbb{P}^2\setminus\mathcal{C}$ admits an orbifold morphism $Y\to X_\varphi$, then the non-singular compactification $\bar X$ of X is $\mathbb{P}^1$.

Remark 2.2. The easiest examples of curves with non-abelian fundamental groups and geometric surjections come from hyperplane (or line) arrangements. If a line arrangement $\mathcal{A}$ has a point P of multiplicity $k\geq 3$, then the pencil of lines through P defines a morphism $\rho:\mathbb{P}^2\setminus\mathcal{A}\to X$, where X is a k-punctured projective line. We have an epimorphism $\rho_*:\pi_1(\mathbb{P}^2\setminus\mathcal{A})\to\pi_1(X)$ and the latter is a free group of rank $k-1$ (hence non abelian).

The following result is well known for specialists.

Proposition 2.3. *The following three assertions are equivalent:*

1. *The group $\pi_1(\mathbb{P}^2\setminus\mathcal{A})$ is non abelian,*
2. *The arrangement $\mathcal{A}$ has a point of multiplicity at least* 3*,*
3. *The group $\pi_1(\mathbb{P}^2\setminus\mathcal{A})$ has a geometric surjection.*

Proof. By the remark above, it is obvious that (2) implies (1) and (3). Also, by definition, (3) implies (1). Hence it is enough to prove that (1) implies (2). Note that, if (2) does not hold, then $\mathcal{A}$ is an arrangement in general position. Either we choose a particular example (e.g. a real arrangement) and a braid monodromy argument implies immediately the abelianity or we use Hattori's topological description of arrangements of hyperplanes in general position [16]. It is also the starting point of *Zariski's proof* of *Zariski's conjecture* in [26] (we thank the referee for pointing this out to us). □

The argument used in Remark 2.2 can be easily generalized when, instead of considering three (or more) incident lines, one considers three (or more) fibers of any pencil of curves in $\mathbb{P}^2$. Of course, any such example corresponds to curves with at least three irreducible components. The notion of orbifold allows for wider generalizations of this concept to curves with any number of irreducible components (for example to irreducible curves).

As stated in the Introduction, the first example of this kind is rather old, see [26]. Let us consider a conic C_2 of equation $f_2=0$ and a cubic C_3 of equation $f_3=0$. Let us assume that they do not have common components and they are not *multiple* lines. Let C be a curve of equation $f_2^3-f_3^2$. Note that the mapping $\rho:\mathbb{P}^2\setminus C\to\mathbb{P}^1\setminus\{[1:1]\}$ given by $[x:y:z]\mapsto[f_2(x,y,z)^3:f_3(x,y,z)^2]$ is well defined (all the base points of the pencil are in C) and surjective. This mapping induces an orbifold map onto a 1-punctured Riemann sphere with two *orbifold* points of multiplicities 2 and 3 (at $[0:1]$ and $[1:0]$ respectively). Thus according to Proposition 1.4, one obtains an epimorphism $\pi_1(\mathbb{P}^2\setminus C)$ onto $\mathbb{Z}/2*\mathbb{Z}/3$.

Proposition 2.4. *Let G be the orbifold fundamental group of $\mathbb{C}_{2,3}$. Then, $\mathbb{T}_G = \mu_6$, $\mathcal{V}_1(G)$ consists of the 6-th primitive roots of unity and $\mathcal{V}_2(G) = \emptyset$. In particular, any curve with equation $f_2^3 - f_3^2 = 0$ has non-trivial characteristic varieties.*

The proof of this Proposition follows easily from the above arguments.

Remark 2.5. For generic choices of f_2 and f_3 this epimorphism is in fact an isomorphism (this is actually the case originally considered by Zariski in [26]). However, this is not the case, for instance, when C is reducible (since $b_1(\mathbb{P}^2 \setminus C) > 1$). Even if C is irreducible one may not necessarily have an isomorphism for several reasons: either there are few non-generic fibers in the pencil (e.g., a sextic with six cusps and four ordinary nodes) or there are several pencils (a sextic with nine cusps).

These examples can be generalized if we replace $(2, 3)$ by any coprimes (p, q), see Oka [21], Némethi [20] and Dimca [12]. In such cases, the fundamental group of a generic curve with equation $f_p^q + f_q^p = 0$ is $\mathbb{Z}/p * \mathbb{Z}/q$. Also Zariski [26] considered another interesting example where the target orbifold is compact.

Let us consider the tricuspidal quartic C_4 with equation $f_4 = 0$. It is not hard to prove that we can choose

$$f_4 := x^2y^2 + y^2z^2 + x^2z^2 - 2xyz(x + y + z). \tag{2.1}$$

and $\operatorname{Sing}(C_4) = \{[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]\}$. The curve C_4 is parametrized by

$$[t : s] \mapsto [t^2s^2 : (t - s)^2s^2 : t^2(t - s)^2] \tag{2.2}$$

and its singular points correspond to $[t : s] = [0 : 1], [1 : 1]$, and $[1 : 0]$. Let $P \in C_4$ be a smooth point with parameter $\alpha \equiv [\alpha : 1]$ and let L_t be the tangent line to C_4 at P, with equation $f_1 = 0$, where

$$f_1 := (\alpha - 1)^3x - \alpha^3y + z. \tag{2.3}$$

Let C_2 be the conic passing through the singular points of C_4 and tangent to C_4 at P. Since five (non-degenerate) conditions are imposed, such a conic is unique. As before, let $f_2 = 0$ be the equation of C_2, where

$$f_2 := \alpha(\alpha - 1)xy - (\alpha - 1)xz + \alpha yz. \tag{2.4}$$

We consider now a cubic C_3 having a nodal point at P (one of the branches tangent to C_4 at P) and tangent to C_4 at the three cuspidal points. Counting the conditions it is easy to prove that only one such cubic exists, with equation $f_3 = 0$, where

(2.5)

$$f_3 := -(\alpha - 2)(2\alpha - 1)(\alpha + 1)xyz - \alpha^3xy^2 - xz^2 - (\alpha - 1)^3x^2y + yz^2 + (\alpha - 1)^3x^2z + \alpha^3y^2z.$$

Lemma 2.6. $f_4f_1^2 = f_3^2 - 4f_2^3$.

A straightforward computation provides a proof of this Lemma, which easily results in the following:

Proposition 2.7. *The fundamental group of $\mathbb{P}^2 \setminus C_4$ possesses a geometric surjection onto $\mathbb{P}^1_{2,2,3}$.*

Remark 2.8. Zariski proved in [26] that $\pi_1(\mathbb{P}^2 \setminus C_4)$ is a non-abelian group of order 12. The above mapping induces a central extension of $\mathbb{D}_6$ (dihedral group of order 6) whose kernel is cyclic of order 2. Note that there is an epimorphism from $\pi_1(\mathbb{P}^2 \setminus (C_4 \cup L_t))$ onto the orbifold group of a 1-punctured Riemann sphere with two multiple points $(2, 3)$. For a generic P it is possible to prove that $\pi_1(\mathbb{P}^2 \setminus (C_4 \cup L))$ equals $\mathbb{B}_3$. There is a particular case corresponding to the bitangent line L_b. In this case there are two such mappings and $\pi_1(\mathbb{P}^2 \setminus (C_4 \cup L_b))$ is the Tits-Artin group of a triangle.

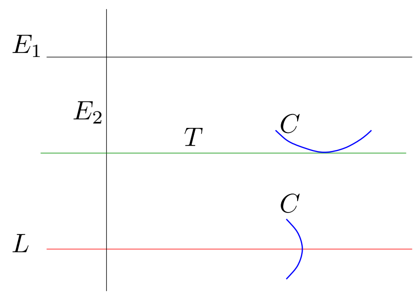

FIGURE 1. Cremona transformation

In [9], Degtyarev proved that only two irreducible curves of degree 5 have non-abelian fundamental groups. One of them is extensively studied in §3. The other one was also studied by the first author in [2]. It is a rigid curve with one point of type $\mathbb{A}_6$ and three cuspidal points (it is the dual curve of the quartic with one $\mathbb{A}_6$). Let C_5 be this curve (with equation $f_5 = 0$). In [2] this group was shown to be non-abelian by finding an epimorphism from an actual presentation of $\pi_1(\mathbb{P}^2 \setminus C_5)$ onto the triangle group of type $2, 3, 7$, which is the orbifold group of $\mathbb{P}^1_\varphi$ with three multiple points of these orders. In fact, one has the following:

Proposition 2.9. *The fundamental group of* $\mathbb{P}^2 \setminus C_5$ *possesses a geometric surjection onto* $\mathbb{P}^1_{2,3,7}$.

Proof. The three summands of Lemma 2.6, which are polynomials of degree six, obviously belong to a pencil of sextics and, hence, they define a map outside the base points. For a particular parameter (a primitive 6-th root of unity, $L_t = L_b$ is the bitangent of C_4. We are going to consider the Cremona transformation $\rho : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ associated with the net of conics having three infinitely near points in common with C_5 at P, the singular point of type $\mathbb{A}_6$. Let us describe this transformation. After blowing up these three infinitely near points one obtains a rational surface X with a morphism $\sigma_1 : X \to \mathbb{P}^2$. Let us denote the three exceptional components (in order of appearance) by E_1, E_2, and T, and finally the tangent line of C at P by L (see Figure 1).

Convention 2.10. For birational morphisms, we keep the notation of a curve for its strict transform unless otherwise stated.

In X one has $E_1 \cdot E_1 = -2$ and $E_1 \cdot E_1 = T \cdot T = L \cdot L = -1$. Since L and T are *combinatorially equivalent*, one can consider the birational morphism $\sigma_2 : X \to \mathbb{P}^2$ obtained as the composition of the contractions of L, E_2 and E_1. The resulting surface is rational with Euler characteristic 3 and hence it is a projective plane. It is not hard to prove that $\rho = \sigma_2^{-1} \circ \sigma_1$. Let us denote $\tilde{C} := \rho(C)$. Note that $\tilde{C}$ is a tricuspidal quartic and T is its unique bitangent line L_b, one point $\tilde{P}$ comes from the infinitely near point of C at P and the other one Q comes from the other intersection point of C and L.

We consider the pencil defined by the orbifold map of Proposition 2.7, where the base point is $\tilde{P}$. Let C_3 be the cubic of equation (2.4) such that $2C_3$ is in the pencil. Following C_3 by σ_2 and σ_1, $C_6 := \rho^*(C_3)$ is a sextic with only one singular point at P (with two branches, one of type $\mathbb{A}_6$ and a smooth branch with maximal contact with the singular branch). With the same ideas, if C_2 is the conic of equation (2.5) such that $3C_2$ is in the pencil, then $C_4 := \rho^*(C_2)$ is a quartic with an $\mathbb{A}_6$ singular point at P.

Finally $\rho^*(\tilde{C} + 2T) = C + 7L$. We have a pencil of degree 12 containing the fibers $2C_6$, $3C_4$ and $C + 7L$. This pencil produces the desired morphism. $\square$

One can find more examples in the literature: Degtyarev [9], Flenner-Zaĭdenberg [14], and Tono [24]. In what follows, the last two families will be described. We start with some definitions.

Definition 2.11. A *Hirzebruch* surface is a rational surface X with a morphism $\pi : X \to \mathbb{P}^1$ which is a holomorphic (or algebraic) fibration with fiber $\mathbb{P}^1$. Such a surface is either $\Sigma_0 := \mathbb{P}^1 \times \mathbb{P}^1$ or it has a unique section S_n with negative self-intersection $-n$, $n > 0$; in that case π is unique and X is denoted by Σ_n (Σ_1 is the blowing-up of one point in $\mathbb{P}^2$).

For any Hirzebruch surface X there is a family of birational maps which are called *elementary Nagata transformations*. They are obtained as follows. Let us consider $\pi : X \to \mathbb{P}^1$, $P \in X$ and $F := \pi^{-1}(\pi(P))$; we consider the blowing-up $\sigma : \hat{X} \to X$ of P, with exceptional component $\tilde{F}$. Since $(F \cdot F)_X = 0$, we have that $(F \cdot F)_{\hat{X}} = -1$. By Castelnuovo criterion, we can blow down F and we obtain a new Hirzebruch surface $\tilde{X}$ where $\tilde{F}$ is a fiber.

Definition 2.12. An elementary Nagata transformation is said to be *positive* (resp. *negative*) if P belongs (resp. does not belong) to a section with non-positive self-intersection. For a positive one, one goes from Σ_n to Σ_{n+1}; for a negative one, from Σ_n to Σ_{n-1}.

In [3], the first author computed the fundamental group of Flenner-Zaĭdenberg curves and showed when it is non-abelian using orbifold groups. We show here that this can also be geometrically proved. In order to construct these curves, we start with a smooth conic C with two tangent lines L_1 and L_2, intersecting at some point P. After blowing up P one obtains $\pi : \Sigma_1 \to \mathbb{P}^1$ with exceptional component E. Let L_3 be another line in the pencil through P which intersects C at two points Q_1 and Q_2. Let us fix two positive integers a, b. After performing a positive elementary Nagata transformations at the point corresponding to the fiber of L_1 and b at the point corresponding to the fiber of L_2 one obtains a Hirzebruch surface Σ_{a+b+1}. One can then perform $a + b$ negative elementary Nagata transformations on the fiber corresponding to L_3 and based at a point in C (say Q_2 for the first one). After this process, E can be blown down which turns our surface into $\mathbb{P}^2$. The curve $C_{a,b}$ obtained has degree $d := a + b + 2$ and three singular points of type $\mathbb{A}_{2a}$, $\mathbb{A}_{2b}$, and a third one with local equation $u^{d-2} = v^{d-1}$.

Proposition 2.13. *The fundamental group of* $\mathbb{P}^2 \setminus C_{a,b}$ *possesses a geometric surjection onto* $\mathbb{P}^1_{2,a+b,c}$*, where* $c := \gcd(2a + 1, 2b + 1)$.

Proof. It is enough to follow the pencil of conics generated by $L_1 + L_2$ and C through the above transformations. We obtain a pencil of curves of degree $2(d - 1)$, where one fiber is $(2a + 1)\tilde{L}_1 + (2b + 1)\tilde{L}_2$ (they are the lines corresponding to the fibers of L_1 and L_2). The fiber containing $C_{a,b}$ is of the form $C_{a,b} + (d - 2)\tilde{L}_3$. Finally the double line in the pencil becomes a double curve of degree $d - 1$. □

In [24], K. Tono describes all rational unicuspidal curves such that its complement in $\mathbb{P}^2$ has logarithmic Kodaira dimension 1. The construction given in [24, Theorem 1] shows that the complement of these curves have non-abelian fundamental group. Any other known rational unicuspidal curve has abelian fundamental group (for the complement).

Proposition 2.14. *For any Tono's curve* C *their fundamental group possesses a geometric surjection onto* $\mathbb{P}^1_{\mu_A,\mu_G,n(C)}$*, where* $\mu_A, \mu_G \geq 2$ *and the number* $n(C)$ *is the opposite of the self-intersection of the strict transform of* C *after the minimal embedded resolution of its unique singular point. This number is at least* 2.

Proof. It is enough to consider the construction of [24, Theorem 1] where a pencil is obtained with two multiple fibers $\mu_A A$ and $\mu_G G$ and a reducible fiber of the form $C + n(C)B$, where B is either a line (type I) or a smooth conic (type II). □

Example 2.15. The curves of type I are parametrized by two integers $n, s \geq 2$. The curve C has degree $(n+1)^2(s-1)+1$, where $n(C) = n$, $\mu_A = n+1$ and $\mu_G = (n+1)(s-1)+1$. For $n = s = 2$, we obtain the multiplicities $2, 3, 4$; in fact, one can compute that this group is finite.

3. Degtyarev curves

Let us consider a projective Degtyarev curve, i.e., a plane projective curve of degree 5 such that $\operatorname{Sing}(C)$ consists of three points, and for each point $P \in \operatorname{Sing}(C)$ the germ (C, P) is topologically equivalent to an $\mathbb{A}_4$-singularity, i.e. with local equation $v^2 - u^5 = 0$; note that in this case, the germs are also analytically equivalent.

Most of the following properties appear in [9] and [19], but we include for the sake of completeness.

Properties 3.1. *Let $C \subset \mathbb{P}^2$ be a projective Degtyarev curve. Then:*

(D1) *The curve C is irreducible.*
(D2) *The tangent line L of C at a singular point P satisfies $(L \cdot C)_P = 4$.*
(D3) *Two Degtyarev projective curves are projectively equivalent.*
(D4) *The subgroup of projective transformations preserving C is cyclic of order 3.*
(D5) *The curve C is autodual.*

Proof. Since the three singular points are locally irreducible, (D1) is true. For (D2), note that $4 \leq (L \cdot C)_P \leq 5$. Let us assume that $(L \cdot C)_P = 5$; considering L as the line at infinity, $C \setminus L$ is an affine curve homeomorphic to $\mathbb{C}$. This case is discarded using Zaĭdenberg-Lin Theorem [25] and (D2) results.

In order to prove (D3), there are two approaches. The direct approach consists of computing the equations of the curve C fixing the position of the singular points and some of their tangent lines. The second method is quite simple and worth describing here: Let C_1, C_2 be two projective Degtyarev curves. By Bézout's Theorem, the singular points are not aligned; and hence, after a projective transformation, one may assume that $\operatorname{Sing}(C_1) = \operatorname{Sing}(C_2) =: S$. Assuming that $S := \{[1:0:0], [0:1:0], [0:0:1]\}$, one can perform a standard Cremona transformation $\psi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ based on the three singular points and defined by $\psi([x:y:z]) = [yz : xz : xy]$.

Geometrically, this rational map is obtained by blowing-up the three vertices of S (obtaining a rational surface X_ℓ) and then blowing down the strict transforms of the lines joining the points of S (which have self-intersection -1 in X_ℓ). One can easily compute that $\tilde{C}_i := \psi(C_i)$ is a tricuspidal quartic. It is well known that there is only one tricuspidal quartic, up to projective transformation, therefore, after a suitable change of coordinates, one may assume $\tilde{C}_1 = \tilde{C}_2 =: \tilde{C}$, where $\tilde{C}$ is the curve with equation given in (2.1). The tricuspidal quartic satisfies the following properties. Let $\operatorname{Sing}(\tilde{C}) = \{P_1, P_2, P_3\}$; there are three points $Q_1^\ell, Q_2^\ell, Q_3^\ell \in \tilde{C}$, $\ell = 1, 2$ such that P_i, Q_j^ℓ, Q_k^ℓ are aligned for all the possibilities with $\#\{i, j, k\} = 3$. Let $\mathcal{A}_\ell$ be the arrangements of curves given by $\tilde{C}$ and the lines joining Q_i^ℓ and Q_j^ℓ.

The curve $\tilde{C}$ is parametrized as in (2.2) and the singular points $P_1 = [0:1:0]$, $P_2 = [1:0:0]$, and $P_3 = [0:0:1]$ correspond to $[t:s] = [0:1], [1:1]$, and $[1:0]$. It is not hard to check that $A_\ell := (\alpha_\ell, 2+\alpha_\ell, -\alpha_\ell)$ are affine parameters of $(Q_1^\ell, Q_2^\ell, Q_3^\ell)$. The last condition implies that $\alpha_\ell^2 + \alpha_\ell - 1 = 0$. If $\alpha_1 = \alpha_2$ then $\mathcal{A}_1 = \mathcal{A}_2$.

The group of projective transformations fixing $\tilde{C}$ is the group of the permutation of the coordinates. The mapping $[x:y:z] \overset{\sigma}{\mapsto} [x:z:y]$ induces $[t:s] \mapsto [s:t]$ in the parametrization, and $[x:y:z] \overset{\tau}{\mapsto} [y:z:x]$ induces $[t:s] \mapsto [s:s-t]$.

Let us assume that $\alpha_1 \neq \alpha_2$ Applying the projective transformation σ, results in two operations on A_ℓ: the permutation $(1,3)$, and the change of parameters. Thus, $\sigma(A_1) = (-\alpha_1^{-1}, (\alpha_1 + 2)^{-1}, \alpha_1^{-1}) = A_2$, which implies $\sigma(\mathcal{A}_1) = \mathcal{A}_2$.

Note that any projective transformation sending $\mathcal{A}_1$ to $\mathcal{A}_2$ lifts to an isomorphism $X_1 \to X_2$ and this isomorphism induces a projective transformation of the source $\mathbb{P}^2$, hence (D3) results.

In order to prove (D4) one can use a similar argument on the projective transformations fixing C (this last property was communicated to the authors by C.T.C. Wall).

The property (D5) follows from (D2) and Plücker generalized formulæ, see [19]. More precisely, given a curve D and a point $P \in D$, the order of the curve is the degree of its dual curve of $\check{D}$:

$$\deg(\check{D}) = \deg(D)(\deg(D)-1) - \sum_{P\in D}(\mu(C,P) - 1 + m(C,P)).$$

This formula implies that $\deg(\check{D}) = 5$. The dual of a singular point of type $\mathbb{A}_4$ is either of the same type or of type $\mathbb{E}_8$ (in case the tangent line has multiplicity of intersection 5 with the curve at the singular point). Thus (D5) holds. $\square$

Remark 3.2. Note that any two projective Degtyarev curves are isotopic. Using the direct approach, we can give a symmetric equation:

$$\begin{aligned}\left(7+3\sqrt{5}\right)(x^3z^2+x^2y^3+y^2z^3) + \left(2\sqrt{5}+6\right)(x^3yz+xy^3z+xyz^3) +\\ +2(x^3y^2+x^2z^3+y^3z^2) + \left(33+11\sqrt{5}\right)(x^2yz^2+x^2y^2z+xy^2z^2) = 0.\end{aligned}$$

Note that the permutation of two variables comes from the Galois transformation in $\mathbb{Q}(\sqrt{5})$. The curve also admits an equation with rational coefficients; in that case one of the singular points has rational coordinates but the other two are conjugate in $\mathbb{Q}(\sqrt{5})$:

(3.1)

$$z^2y^3 - z(33xz+2x^2+8z^2)y^2 + (21z^2+21xz-x^2)(z^2+11xz-x^2)y + (x-18z)(z^2+11xz-x^2)^2 = 0$$

Properties 3.1 imply that the affine Degtyarev curve is also *rigid*, i.e. any two affine Degtyarev curves are projectively equivalent, and in particular, they are isotopic. In order to study its complement, it is convenient to assume that the line corresponds to the line at infinity and hence it is enough to consider the complement of the affine curve whose equation is obtained from (3.1) by taking $z = 1$.

The fundamental group of the projective Degtyarev curve was computed in [9]. Here we will compute the fundamental group of the affine curve and also show how to recover the group of the projective curve. In order to compute the group we will use the braid monodromy associated with the projection $(x, y) \mapsto x$. Note that the discriminant of the equation (3.1) (with $z = 1$) is (up to a constant) $x(x^2 - 11x - 1)^5$. Since the three roots are real and the projection is $3:1$ with *enough* real roots, the real picture in Figure 2 contains all the required information to obtain the braid monodromy (the dotted lines represent the real part of the complex conjugate roots).

The braid monodromy is defined as a representation $\nabla_0 : \pi_1(\mathbb{C} \setminus \{0, a_+, a_-\}; x_0) \to \mathbb{B}_3$. The source is a free group of rank three generated by:

$$\mu_+ := \alpha_+ \cdot \beta_+ \cdot \gamma_+ \cdot \alpha_+^{-1}, \quad \mu_0 := \alpha_+ \cdot \beta_+ \cdot \alpha_0 \cdot \beta_0 \cdot \gamma_0 \cdot \alpha_0^{-1} \cdot \beta_+^{-1} \cdot \alpha_+^{-1} \text{ and}$$

$$\mu_- := \alpha_+ \cdot \beta_+ \cdot \alpha_0 \cdot \beta_0 \cdot \alpha_- \cdot \beta_- \cdot \gamma_- \cdot \alpha_-^{-1} \cdot \beta_0^{-1} \cdot \alpha_0^{-1} \cdot \beta_+^{-1} \cdot \alpha_+^{-1}.$$

Figure 3 shows a geometric basis of $\pi_1(\mathbb{C} \setminus \{0, a_+, a_-\}; x_0)$. The braids are obtained by considering the way the roots with respect to y move when the parameters move along x. We follow these conventions:

- ($\mathbb{B}$1) In order to draw the braids we consider the projection onto the real axis.
- ($\mathbb{B}$2) When two points have the same real part, we perturb the projection such that positive imaginary parts go to the right and negative imaginary parts go to the left.
- ($\mathbb{B}$3) Roots will be numbered from right to left.

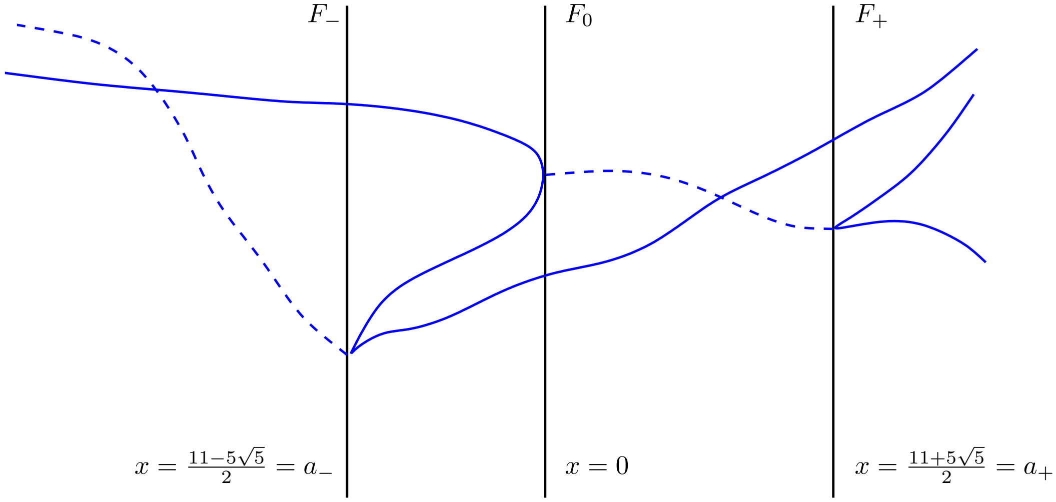

FIGURE 2. Real picture of the affine Degtyarev curve

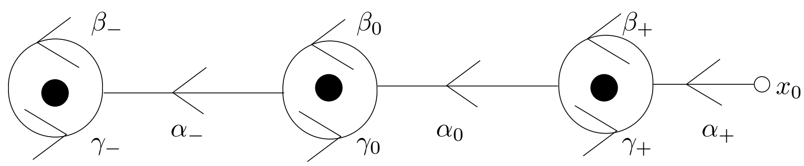

FIGURE 3. Paths in $\mathbb{C} \setminus \{0, a_+, a_-\}$

Paths	**Braids**
α_+	1
β_+	σ_2^2
γ_+	σ_2^3
α_0	$\sigma_1^{-1}\sigma_2$
β_0	1
γ_0	σ_1
α_-	1
β_-	σ_2^2
γ_-	σ_2^3

TABLE 1. Braids

($\mathbb{B}$4) The above conventions give a canonical way to identify open braids with closed braids.

Using the standard Artin generators of the braid groups, the braids obtained from following the paths in $\mathbb{C} \setminus \{0, a_+, a_-\}$ shown in Figure 3 are presented in Table 1.

Proposition 3.3. *The braid monodromy for the chosen projection of the affine Degtyarev curve is given by:*

$$\nabla_0(\mu_+) = \sigma_2^5, \quad \nabla_0(\mu_0) = (\sigma_2^2\sigma_1^{-1}\sigma_2) * \sigma_1, \quad \nabla_0(\mu_-) = (\sigma_2^2\sigma_1^{-1}\sigma_2\sigma_1) * \sigma_2^5 = \sigma_2^2 * \sigma_1^5,$$

where $a * b := aba^{-1}$.

4. Groups of Degtyarev curves

In order to compute the fundamental groups we apply the Zariski-van Kampen method. Let us consider the vertical line F of equation $x = x_0$. The set $F \setminus C$ is of the form $\{x_0\} \times \mathbb{C} \setminus \{y_1, y_2, y_3\}$, where $y_1, y_2, y_3 \in \mathbb{R}$. We choose a *big* real number y_0 in order to fix $(x_0, y_0) =: p_0$ as the base point. The free group $\pi_1(F \setminus C; p_0)$ has a free basis g_1, g_2, g_3 constructed as in Figure 3. The natural action of $\mathbb{B}_3$ on the free group $\mathbb{F}_3$ is expressed in this case as

$$g_i^{\sigma_j} := \begin{cases} g_{i+1} & \text{if } i = j, \\ g_{i+1} * g_i & \text{if } i = j+1, \\ g_i & \text{if } i \neq j, j+1. \end{cases} \tag{4.1}$$

Proposition 4.1. *The fundamental group of the affine Degtyarev curve has a presentation*

$$\left\langle g_1, g_2, g_3 \;\middle|\; g_i^{\nabla_0(\mu_j)} = g_i, i = 1,2,3, j = -,0,+ \right\rangle. \tag{4.2}$$

In this presentation, a meridian of the line at infinity is (up to conjugation) $\left(g_3(g_2 g_1)^2\right)^{-1}$. In particular, a presentation for the projective Degtyarev curve is

$$\left\langle g_1, g_2, g_3 \;\middle|\; (4.2), \quad g_3 = (g_2 g_1)^{-2} \right\rangle. \tag{4.3}$$

Proof. The first presentation is a consequence of the Zariski-van Kampen method by means of the braid monodromy. In order to prove the second one may consider a small deformation of the vertical line F. It will intersect the curve at five points. Three of them are close to (x_0, y_i), $i = 1,2,3$, and the other two ones lie in the *real* branches which go *faster* to infinity. The boundary of a big disk in this line is the inverse of a meridian of the line at infinity. □

Remark 4.2. Proposition 4.1 provides right presentations of the group, but they may be quite cumbersome to work with by hand. Even if one wants to work with them with computer programs, like `GAP`[15], the presentations could be intractable. There are several ways around this problem

(P1) The presentation (4.2) works if we replace the braid monodromy ∇_0 for a conjugate. For example, conjugating the braids in Proposition 3.3 by σ_2^2 produces simpler braids and hence a simpler presentation of the group.

(P2) Instead of finding a good braid to perform the conjugation in (P1) by inspection, one can try to interpret this conjugation in a geometric way. Changing the base point in $\mathbb{C} \setminus \{0, a_+, a_-\}$ might produce simpler braids. For example choosing a real number $\tilde{y}_0 \in (a_-, 0)$ as a base point, one obtains the following as braid monodromy (for the new generators of the group):

$$\tilde{\mu}_+ \mapsto (\sigma_2^{-1}\sigma_1) * \sigma_2^5, \quad \tilde{\mu}_0 \mapsto \sigma_1, \quad \tilde{\mu}_- \mapsto \sigma_2^5. \tag{4.4}$$

These braids have been obtained by conjugation of the ones in Proposition 3.3 by $\sigma_2^2\sigma_1\sigma_2^{-1}$.

(P3) If g is a meridian of the line at infinity obtained using a braid monodromy ∇_0, then, for a braid monodromy $(\nabla_0)^\tau := \tau^{-1}\nabla_0\tau = (\tau^{-1}) * \nabla_0$, a meridian of the line at infinity is g^τ.

(P4) There is another geometric way to reduce the presentation. Note that among the relations $(g_j)^{\sigma_2^5} = g_j$, $j = 1,2,3$, one only needs to keep the relation given by $j = 2$. First of all, the relation for $j = 1$ is trivial; secondly $(g_3 g_2)^\tau = g_3 g_2$ and hence one of them is redundant. In the general case, this can be summarized as follows:

- Let us consider the action (4.1) (replacing 3 by n) of $\mathbb{B}_n$ on the free group with basis $g_1, \dots, g_n$; let us consider a braid $\tau \in \mathbb{B}_n$ which can be decomposed as $\tau = \tau_1 \cdot \dots \cdot \tau_r$, where τ_j involves only a set of n_j consecutive strings and $n = \sum_{j=1}^r n_j$. Then, among the relations $g_j^\tau = g_j$, we only need to consider $s := \sum_{j=1}^r (n_j - 1) = n - r$, disregarding one for each block of strings. Let J_τ be the chosen subset of indices.
- If $\beta = (\tau)^\sigma$, and τ can be decomposed as above, then the set of relations $g_j^\beta = g_j$, $j = 1, \dots, n$, is equivalent to $(g_j^\sigma)^\tau = g_j^\sigma$, $j \in J$.

For example, in our case the presentation (4.2) can be reduced to have 3 relators.

Proposition 4.3. *The group G of the affine Degtyarev curve has a presentation:*

$$\left\langle x, y \,\middle|\, xyxyx = yxyxy, [x, yxy^{-1}xyxy^{-1}xy] = 1 \right\rangle \tag{4.5}$$

A presentation of the group $G_\mathbb{P}$ of the projective Degtyarev curve is obtained from (4.5) *by adding $x^5 = 1$. It turns out that $G_\mathbb{P}$ is a group of order* 320 *with the following properties:*

($G_\mathbb{P}1$) *$G_\mathbb{P}/G'_\mathbb{P}$ is cyclic of order* 5.
($G_\mathbb{P}2$) *The center $Z(G_\mathbb{P})$ is the Klein group of order* 4.
($G_\mathbb{P}3$) *The group $G/Z(G_\mathbb{P})$ is a semidirect product of $(\mathbb{Z}/2)^4$ by $\mathbb{Z}^5$, where the action of a generator of $\mathbb{Z}^5$ cyclically permutes a generator system $h_1, \dots, h_5$ of order* 2 *elements of $(\mathbb{Z}/2)^4$ satisfying $\sum h_i \equiv 0$.*

Proof. The presentation of G is obtained using the braid monodromy 4.4 and Remark 4.2(P4), where $x = g_1, g_2$ and $y = g_3$; note that x and y are conjugate. In order to obtain the presentation of $G_\mathbb{P}$ the relation of the line at infinity needs to be added. This is a *complicated* product of five conjugates of x. If one types this presentation in `GAP`, the output is that $G_\mathbb{P}$ has order 320 and that x is an element of order 5. Also according to `GAP`, the order of the quotient of G obtained by adding the relation $x^5 = 1$ is 320. These facts give the presentation of the statement. The properties of $G_\mathbb{P}$ are either trivial or easily computed using `GAP`. □

Proposition 4.4. *The group $G_\mathbb{P}$ possesses no geometric surjections.*

Proof. The only properties needed for this are the size of both the group $G_\mathbb{P}$ and its abelianization. Let us assume that $G_\mathbb{P}$ possesses a geometric surjection. Since it is finite, the orbifold group must be finite. The only orbifolds having a finite non-abelian fundamental group are those of type $\mathbb{P}^1_{a,b,c}$, with $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} > 1$ (the so-called *spherical* orbifolds): either $\mathbb{P}^1_{2,2,n}$, $n \geq 3$, or $\mathbb{P}^1_{2,3,m}$, $m = 3, 4, 5$. Since the order of the orbifold group must divide 320, the only possibilities are $(2, 2, n)$, where $n|160$. The group is dihedral and its abelianization is either $\mathbb{Z}/2$ or $(\mathbb{Z}/2)^2$. Since the abelianization of $G_\mathbb{P}$ is of order 5, the result follows. □

We finish this section with the main result of this paper. We are going to compute the characteristic varieties of the complement of the affine Degtyarev curve and we will prove that these components cannot come from the characteristic varieties of an orbifold.

Theorem 4.5. *Let $\mathbb{T}_G = \mathbb{C}^*$ be the character torus of G. Then $\mathcal{V}_1(G)$ is the set containing* 1 *and the* 10*-th primitive roots of unity, whereas $\mathcal{V}_2(G) = \emptyset$. Therefore there is no geometric surjection of G onto an infinite orbifold group.*

Since finite group orbifolds do not have characteristic varieties, the following Corollary holds.

Corollary 4.6. *No irreducible component of $\mathcal{V}_1(G)$ is obtained as the pull-back of an irreducible component of the $\mathcal{V}_1(\Gamma)$ where Γ is an orbifold group.*

Proof of Theorem 4.5. We are going to change the presentation (4.5), by taking a new generator t satisfying $y = xt$:

$$\left\langle x, t \,\middle|\, xtx^2tx = tx^2tx^2t, [x, txt^{-1}xtxt^{-1}xt] = 1 \right\rangle \tag{4.6}$$

It is clear that $1 \in \mathcal{V}_1(G) \setminus \mathcal{V}_2(G)$ since the non-twisted homology has rank 1. Let us consider a non-trivial character $\xi \in \mathbb{T}_G$, which is identified by the image $1 \neq \zeta$ of a positive generator of $\mathbb{Z}$. One can associate a CW-complex with the presentation (4.6) with one 0-cell p, two 1-cells x, t and two 2-cells A, B (corresponding to the relations). Then, the complex $C_*(X; \mathbb{C})^\xi$ with which to compute the twisted homology is

$$0 \longrightarrow \mathbb{C}^2 \xrightarrow{\partial_2} \mathbb{C}^2 \xrightarrow{\partial_1} \mathbb{C} \longrightarrow 0.$$

The matrix for ∂_1 is $\begin{pmatrix} \zeta - 1 & 0 \end{pmatrix}$. In particular, $\dim \ker \partial_1 = 1$ and hence $\mathcal{V}_2(G) = 0$. The matrix for ∂_2 equals

$$\begin{pmatrix} 0 & 0 \\ 1 - \zeta + \zeta^2 - \zeta^3 + \zeta^4 & (1 - \zeta + \zeta^2 - \zeta^3 + \zeta^4)(\zeta - 1) \end{pmatrix}.$$

The homology is non trivial if and only if the matrix vanishes and hence $\mathcal{V}_1(G)$ is as in the statement.

Since we are working with the complement of an affine (hence projective) curve, if G admits a geometric surjection onto an infinite orbifold group, the orbifold must be over a rational curve. Since the abelianization has rank 1, the rational curve must be either $\mathbb{C}$ or $\mathbb{P}^1$. Any dominant morphism with target $\mathbb{C}$ can be considered as dominant on $\mathbb{P}^1$ and we treat only this case.

One needs to consider only orbifolds over $\mathbb{P}^1$ whose fundamental groups are infinite, have cyclic abelianizations and admit the 10-th primitive roots of unity in their characteristic varieties. In particular, the abelianization must be of the type $\mathbb{Z}/n\mathbb{Z}$, where 10 divides n.

Let us prove that any such orbifolds admit dominant morphisms in $\mathbb{P}^1_{2,5,10}$ and $\mathbb{P}^1_{2,2,5,5}$. It is not hard to prove (see, e.g., [4] for details) that for a prime p, the abelianization of G has non-trivial p-factors if at least two orbifold points have indices divisible by p. Using the identity mapping, we obtain dominant morphisms in either the above orbifolds or $\mathbb{P}^1_{10,10}$. We need to exclude the case where only a dominant morphism in $\mathbb{P}^1_{10,10}$ exists. In this case, $\mathbb{P}^1_{10n_1,10n_2,n_3,\ldots,n_r}$, $\gcd(n_j, 10) = 1$, $j = 1, \ldots, r$. We proved in [4] that no element of order 10 is in the characteristic varieties of this orbifold, and hence, these orbifolds do not satisfy the claim of the statement.

The properties of $\mathcal{V}_2$ allow us to discard $\mathbb{P}^1_{2,2,5,5}$, see Proposition 1.14. Let us assume that there is a geometric surjection onto the orbifold $\mathbb{P}^1_{2,5,10}$. Proposition 1.13 does not provide a direct obstruction in terms of $\mathcal{V}_1$. Moreover, the kernel of the abelianization map is the fundamental group K_2 of a compact Riemann surface of genus 2, see Proposition 1.13.

Note that $(xy)^5 = (x^2t)^5$ is a central element and the group K generated by this element defines an injection in G/G'. Following [10], if $G_0 := G/K$, the groups G'_0 and G' are isomorphic and hence G' is finitely presented. Using the Reidemeister-Schreier method, we find the following presentation:

$$G' = \langle t_0, t_1, t_2, t_3, t_4 \mid t_{n+1}t_{n+3} = t_n t_{n+2} t_{n+4}, B_n = B_{n+1} \rangle, \tag{4.7}$$

where $B_n := t_n t_{n+1}^{-1} t_{n+2} t_{n+3}^{-1} t_{n+4}$ and $x * t_n = t_{n+1}$. Note that $x^{10} * t_n = t_{n+10} = A * t_n$, where $A := t_n t_{n+2} t_{n+4} t_{n+6} t_{n+8}$ for any n. This guarantees that the above presentation is finite. Summarizing, one can deduce that the kernel K_1 of the epimorphism onto $\mathbb{Z}/10$ equals $\mathbb{Z} \times G'$. Note that the rank of K_1 equals 5 and the rank of K_2 equals 4, so no contradiction arises.

According to `GAP` the next quotients of the lower central series have ranks 5 and 16 for K_2, and 2 and 0 (order 5) for K_1 and hence such an epimorphism cannot exist. ☐

5. Further properties of the affine Degtyarev curve

The affine Degtyarev curve is related with elliptic fibrations as follows. In order to work in a projective setting, one can first consider the projective Degtyarev curve, and fix a singular point P. We will denote by L the tangent line of C at P, and the remaining singular points by $P_\pm$. Let $\sigma : \Sigma_1 \to \mathbb{P}^2$ be the blow-up of P where E denotes the exceptional component. Strict transforms will follow Convention 2.10.

Each generic fiber of Σ intersects C at three points. There are four exceptions; three of them can be seen in Figure 2 and they are denoted by F_+, F_0, and F_-. The fourth one is L, which intersects C at two points: one is smooth and transversal and the other one is the infinitely near point of P in E, which is of type $\mathbb{A}_2$. In order to separate C and E we perform a positive elementary Nagata transformation $\rho : \Sigma_1 \dashrightarrow \Sigma_2$ on the fiber corresponding to L. The fiber which replaces L is denoted by F_∞. Note that F_∞ intersects C at two points: one of them corresponds to the blow-down of L and the other one is a point with a generic tangency. In particular, the combinatorics of the intersections at F_0 and F_∞ coincides.

Remark 5.1. Properties 3.1 imply the rigidity of this arrangement of curves in Σ_2. In particular, once the four fibers are ordered the cross-ratio of their images in $\mathbb{P}^1$ provides an invariant of the arrangement. The existence of an automorphism of Σ_2 preserving C and exchanging the two fibers containing the singular points can be easily checked. As a consequence of the cross-ratio argument, the two tangent fibers must also be exchanged. This automorphism defines a birational map of $\mathbb{P}^2$ which is related to the two solutions in $\mathbb{Q}(\sqrt{5})$ exhibited in the proof of Property 3.1(D3).

Let us consider the minimal resolution Z of the double covering of Σ_2 ramified at $C + E$. The ruling of Σ_2 induces a morphism $\rho : Z \to \mathbb{P}^1$ such that the generic fiber is elliptic. The only singular fibers are the preimages of F_+, F_- (of type I_5 in Kodaira notation), F_0, and F_∞ (of type I_1). These elliptic fibrations have been extensively studied in [18]. Once a section is fixed (e.g. the preimage of E), the set of sections has an abelian group structure (inherited by the structure on the fibers) which is called the Mordell-Weil group. Note that the involution associated with the double covering is defined by taking the opposite. It is known that the Mordell-Weil group of Z is cyclic of order 5.

Let us consider a conic C_1 tangent to C both at P and at another singular point and transversal to the third singular point. The preimage of C_1 by the double covering has two irreducible components which are denoted by E_1 and $-E_1$: they are opposite sections in the Mordell-Weil group. Interchanging the two singular points, one obtains the remaining two sections E_2 and $-E_2$ of Z.

Let us recall that G denotes the fundamental group of the complement of the affine Degtyarev curve, i.e. $\mathbb{P}^2 \setminus (C \cup L) = \Sigma_2 \setminus (C \cup E \cup L_\infty)$.

Remark 5.2. Despite Proposition 4.4, note that its affine version, $G = \pi_1(\mathbb{P}^2 \setminus (C \cup L))$ *does* posses a geometric surjection onto the orbifold over $\mathbb{P}^1_{2,2,5}$, since G admits an epimorphism onto the dihedral group of order 10, see for instance [5].

In order to construct this morphism, we may use the ideas in [23]. The mapping is obtained by a pencil of rational curves of degree 10, with the following non-reduced fibers:

- A smooth conic C_2 of multiplicity 5 such that $(C \cdot C_2)_{P_+} = 2$, $(C \cdot C_2)_{P_-} = 4$ and $(C \cdot C_2)_P = 4$.
- A quintic C_5 of multiplicity 2 such that $(C \cdot C_5)_{P_+} = 5$ (P_+ is a smooth point of C_5), $(C \cdot C_2)_{P_-} = 10$ (P_- is a singular point of C_5 of type $\mathbb{A}_4$), and $(C \cdot C_2)_P = 10$ (P is a singular point of C_5 of type $\mathbb{D}_6$).

- The curve $C+L+2D_2$ where D_2 is a smooth conic such that $(C\cdot D_2)_{P_+}=0$, $(C\cdot D_2)_{P_-}=5$, and $(C\cdot D_2)_P=4$.

We finish this section by describing some properties of the group G. For a point $Q\in C$, the local fundamental group $\pi_1^{\text{loc}}(C,Q)$ of C at Q is $\pi_1(\mathbb{B}_Q\setminus C)$, where $\mathbb{B}_Q$ is a Milnor ball. The inclusion $\mathbb{B}_Q\setminus C\hookrightarrow\mathbb{C}^2\setminus C$ induces a conjugacy class of subgroups (since the base point is not fixed) which will be called the *image of the local fundamental group*.

Proposition 5.3. *Let $P_\pm$ be the two singular points of the affine Degtyarev curve.*

(a) *The images of the local fundamental groups at P_+ and P_- are the whole group G.*
(b) *The center of G contains an abelian free subgroup of rank 2.*

Proof. The property about the image of the local fundamental group at P_- is obvious from the presentation (4.5). For P_+ it can be deduced using `GAP`. As a consequence we obtain two central elements (the images of the central elements of the local fundamental groups). The last property can be deduced by studying some quotients of subgroups of G. □

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Journal *of* Singularities
Volume 2 (2010), 19-50

Proc. of Singularities in Aarhus
received 23 November 2009
DOI: 10.5427/jsing.2010.2b

$\mathcal{A}_0$-SUFFICIENCY OF JETS FROM $\mathbb{R}^2$ TO $\mathbb{R}^2$

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Dedicated to professor Andrew du Plessis on his 60th birthday

Abstract. An r-jet $z \in J^r(2,2)$ is $\mathcal{A}_0$-sufficient in $\mathcal{E}_{[r]}(2,2)$ if every C^r realization of z is topologically right-left equivalent to z. We give sufficient conditions for $\mathcal{A}_0$-sufficiency in $\mathcal{E}_{[r]}(2,2)$. For a certain class of jets, we prove that our sufficient conditions are also necessary. Finally, we use the techniques developed in the course of the proofs of these results to give sufficient conditions for a 1-parameter family of C^r plane-to-plane map-germs to be topologically trivial.

1. Introduction

Let $\mathcal{E}_{[r]}(n,p)$ denote the set of C^r-map-germs $(\mathbb{R}^n,0) \to (\mathbb{R}^p,0)$. Let $\omega : (\mathbb{R}^n,0) \to (\mathbb{R}^p,0)$ be an r-jet. We say that ω is $\mathcal{A}_0$-sufficient in $\mathcal{E}_{[r]}(n,p)$ if, for any C^r-germ $f : (\mathbb{R}^n,0) \to (\mathbb{R}^p,0)$ with $j^r f(0) = \omega$, there exist germs of homeomorphisms $h : (\mathbb{R}^n,0) \to (\mathbb{R}^n,0)$ and $k : (\mathbb{R}^p,0) \to (\mathbb{R}^p,0)$ such that $f = k \circ \omega \circ h$.

The study of sufficiency of jets started with the classical papers of Kuiper [7], Kuo [8], [9] and Bochnak and Łojasiewicz [3]. In these papers the sufficiency of r-jets in $\mathcal{E}_{[r]}(n,1) = \mathcal{E}_{[r]}$ and $\mathcal{E}_{[r+1]}$ with respect to $\mathcal{R}_0$-equivalence and the sufficiency of r-jets in $\mathcal{E}_{[r+1]}(n,p)$ with respect to $\mathcal{V}$-equivalence were studied, and necessary and sufficient conditions for sufficiency were given. (Two map-germs f, g are $\mathcal{R}_0$-equivalent if there exists a germ of homeomorphism h such that $f = g \circ h$, and they are $\mathcal{V}$-equivalent if $f^{-1}(0)$ and $g^{-1}(0)$ are homeomorphic.) In these cases the necessary and sufficient condition was formulated in terms of a Łojasiewicz inequality. This Łojasiewicz inequality implies that every representative of the jet is, in some sense, non-singular outside 0.

In this article we will study $\mathcal{A}_0$-sufficiency of jets, and we will only consider jets from $\mathbb{R}^2$ to $\mathbb{R}^2$. The nice geometric conditions we expect for representatives of such jets are that they only have fold singularities outside the origin and that they do not have singular double points. We must therefore put up Łojasiewicz inequalities avoiding such singularities outside 0, and hopefully such Łojasiewicz inequalities will be necessary and sufficient conditions for $\mathcal{A}_0$-sufficiency of plane-to-plane jets. We have however not been able to prove this in general. Let $\omega : (\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$ be a singular r-jet (identified with a polynomial map of degree $\leq r$) with singular set $\Sigma(\omega)$. Assume that ω is not the zero jet and that 0 is not isolated in $\Sigma(\omega)$. Then $\Sigma(\omega)$ is a 1-dimensional algebraic set. It follows that for small balls $B(0,\rho)$ around 0, 0 is in the closure of all components of $(\Sigma(\omega) - \{0\}) \cap B(0,\rho)$ and the number of such components are independent of the radius ρ. Let $C_1, \dots, C_N$ be these components. By the curve selection lemma, we can find analytic curves $\gamma_i : [0,\epsilon) \to \mathbb{R}^2$ for $i = 1, \dots, N$ with $\gamma_i(0) = 0$ and $\gamma_i(0,\epsilon) \subset C_i$. Let $\mathbf{n}_i = \lim_{t\to 0^+} \gamma_i'(t)/\|\gamma_i'(t)\|$. If all the $\mathbf{n}_i$ are distinct, we say that $C_1, \dots, C_N$ have different tangent directions at 0. Assume that $C_1, \dots, C_N$ have different tangent directions at 0. For such jets, we prove that there exist two Łojasiewicz inequalities which together are necessary and sufficient

2000 *Mathematics Subject Classification.* 14B05, 58A20 (primary), 58A20 (secondary).

Key words and phrases. Sufficiency of jets, finite determinacy, jets from the plane to the plane.

conditions for sufficiency. This result is Theorem 2.2 in Section 2. If we drop the hypothesis about the tangent directions, we can prove that our inequalities are sufficient conditions, but we have not suceeded in proving the necessity of both of these inequalities in the general case. For jets ω such that two components C_i and C_j of $\Sigma(\omega) - \{0\}$ have the same tangent direction at 0, the distance between points in C_i and C_j may be small compared to the distance to 0. This makes perturbation arguments complicated.

If we consider jets ω where 0 is isolated in $\Sigma(\omega)$, we can discard the second Łojasiewicz inequality, and the first Łojasiewicz inequality will be a necessary and sufficient condition for $\mathcal{A}_0$-sufficiency. In fact, it turns out that this inequality is a necessary and sufficient condition for $\mathcal{R}_0$-sufficiency in $\mathcal{E}_{[r]}(2,2)$ for such jets.

The statement of Theorem 2.2 in Section 2 below is a generalized and improved version of a theorem announced without proof in the article [5]. Also Theorem 2.3 is announced without proof in [5].

The article is organized in the following way: In Section 2 we introduce some notation and formulate the main results of the article. In Section 3 we write down the equations in the jet space for certain sets of singular 1- and 2-jets, and we find expressions for distance functions from jets to these singular sets. These distance functions will be used throughout the article. We also discuss the smoothness of one of these distance functions and we prove Propostion 2.1 and Theorem 2.3 of Section 2. In Section 4 we prove that the two Łojasiewicz inequalities formulated in Theorem 2.2 in Section 2 are stable in the sense that all C^r-representatives of a jet satisfying the inequalities also satisfy similar inequalities. We also derive a number of geometrical consequences of our Łojasiewicz inequalities.

In Section 5, we prove that the two Łojasiewicz inequalities of Theorem 2.2 imply sufficiency of the jet. In Section 6, we point out that every jet has a nice realization which has at most only fold singularities outside 0, and avoids singular double points. We then prove that if some of the inequalities of Theorem 2.2 are not satisfied, then we can find another bad realization of the jet having singularities which are topologically different from the singularities of the nice representative. (When we here consider the failure of the second Łojasiewicz inequality, we consider only jets ω such that the tangent directions at 0 of the components $C_1, \ldots, C_N$ are distinct.) This will prove that the Łojasiewicz inequalities are necessary for sufficiency of the jet and therefore complete the proof of Theorem 2.2.

In Section 7 we give examples of sufficient and non-sufficient jets.

Finally, in Section 8, we look at germs of one-parameter families of C^r-maps and state sufficient conditions for such families to be topologically trivial. The conditions are analogous to those satisfied by one-parameter families of C^r-realizations of sufficient jets.

2. The Main Theorem

Let $J^1(2,2)$ be the set of 1-jets $(\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$. An element $z \in J^1(2,2)$ can be identified with a linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$ and thus with a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ or (when we find it convenient) a vector $(a, b, c, d) \in \mathbb{R}^4$. Let $J^2(2,2)$ be the set of 2-jets $(\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$. An element $z \in J^2(2,2)$ can be identified with a polynomial map

$$z(x,y) = (ax + by + ex^2 + 2fxy + gy^2, cx + dy + hx^2 + 2ixy + jy^2).$$

Now $J^2(2,2)$ can be identified with $\mathbb{R}^{10}$ by identifying z with the tuple $(a, b, \ldots, j)$ and we can therefore consider the splitting

$$(L, H) = (L_z, H_z) = ((a, b, c, d), (e, f, g, h, i, j)) \in \mathbb{R}^4 \times \mathbb{R}^6.$$

Consider the set $\mathcal{E}_{[r]}(2,2)$ of C^r-germs $f : (\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$. Let $r \geq 2$ and let $f : U \to \mathbb{R}^2$ be a representative of a germ in $\mathcal{E}_{[r]}(2,2)$. For $p \in U$ we can define $j^1f(p) \in J^1(2,2)$ and $j^2f(p) \in J^2(2,2)$ as the 1- and 2-jet, respectively, of $f\big((x,y)+p\big) - f(p)$ at $(x,y) = 0$. For any $f \in \mathcal{E}_{[r]}(2,2)$ we can consequently define germs $j^1f : (\mathbb{R}^2,0) \to J^1(2,2)$ and $j^2f : (\mathbb{R}^2,0) \to J^2(2,2)$ and thus define the germ (L_f, H_f) by $(L_f,H_f)(p) = (L_{j^2f(p)}, H_{j^2f(p)})$. Let $\Gamma \subset J^2(2,2)$ be defined by

$$\Gamma = \{\, (a,\ldots,j) \,|\, ad - bc = 0, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} aj - bi - cg + df \\ -ai + bh + cf - de \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \},$$

via our identifications. We will see in Section 3 that Γ is the set of singular 2-jets which are not folds.

Let $\omega \in J^r(2,2)$ be a singular jet which we identify with a polynomial map $\omega : \mathbb{R}^2 \to \mathbb{R}^2$ of degree $\leq r$. Assume that 0 is not isolated in $\Sigma(\omega)$ and that ω is not the zero jet. Since $\Sigma(\omega)$ is algebraic, it follows that there exists $\rho_0 > 0$ such that when $0 < \rho < \rho_0$ then $(\Sigma(\omega) - \{0\}) \cap B(0,\rho)$ (where $B(0,\rho) \subset \mathbb{R}^2$ is the open ball with center 0 and radius ρ) is non-singular, has finitely many topological components, 0 is in the closure of each component, and the number of such components is independent of ρ (this follows for example from the results of chapter 2 of [11]). We denote these components by $C_1,\ldots,C_N$ with no reference to the ball $B(0,\rho)$. As explained in the introduction, these curves have a well defined tangent direction at the origin.

For each $\epsilon > 0$, define

$$H_\epsilon = \left\{\, p \,|\, d(j^1\omega(p), \Sigma) \leq \epsilon \,\|p\|^{r-1} \right\}.$$

Here $\Sigma \subset J^1(2,2)$ is the set of singular 1-jets, $d(j^1\omega(p),\Sigma)$ denotes the distance $\inf\{\left\|j^1\omega(p) - z\right\| \mid z \in \Sigma\}$, where $\|\cdot\|$ is the usual Euclidean norm when 1-jets are identified with vectors in $\mathbb{R}^4$ (when points in some finite dimensional linear spaces are identified with vectors in Euclidean spaces $\|\cdot\|$ will always (unless otherwise stated) denote the Euclidean norm via the identification). For every $\epsilon > 0$, H_ϵ is a closed semialgebraic set with $\Sigma(\omega) \subset H_\epsilon$ (this is a consequence of Proposition 2.2.8 of [1] and the Tarski-Seidenberg Theorem).

We now have the following proposition:

Proposition 2.1. *Let $r \geq 2$ and $\omega \in J^r(2,2)$ be a singular, non-zero jet such that 0 is not isolated in $\Sigma(\omega)$. Let Γ, ρ_0, $C_1,\ldots,C_N$ and H_ϵ be as explained above. Consider the following condition:*

(I) *There is a neighbourhood U of 0 and constants $C > 0$ such that if $p \in U$ and $(L,H) \in \Gamma$, then*

$$\|L_\omega(p) - L\| + \|H_\omega(p) - H\| \,\|p\| \geq C\,\|p\|^{r-1}.$$

Assume that condition (I) is satisfied. Then there exists $\epsilon_0 > 0$ such that if ρ_0 above is sufficiently small, then the following is satisfied: For each open ball $B(0,\rho) \subset \mathbb{R}^2$ with center 0 and radius $\rho < \rho_0$, and for each ϵ, $0 < \epsilon < \epsilon_0$, $(H_\epsilon - \{0\}) \cap B(0,\rho)$ has exactly N connected components, and we can label the components of $(H_\epsilon - \{0\}) \cap B(0,\rho)$ by $H_1,\ldots,H_N$, such that $C_i \subset H_i$.

Now we have:

Theorem 2.2 (Main Theorem)**.** *Let $r > 2$ and let $\omega \in J^r(2,2)$ be a jet as described in Proposition 2.1 . Let Γ, $C_1,\ldots,C_N$ and H_ϵ be as defined above and assume that condition (I) of Proposition 2.1 is satisfied. Let ρ_0 and ϵ_0 be as in the conclusion of 2.1. Consider the following* *condition :*

(II) *There exist $\rho > 0$ with $\rho < \rho_0$ and $\epsilon > 0$ with $\epsilon < \epsilon_0$ and a constant $C > 0$ such that if H_i and H_j, $i \neq j$ are components of $(H_\epsilon - \{0\}) \cap B(0, \rho)$ and $p \in H_i \cup \{0\}$ and $q \in H_j \cup \{0\}$ then*

$$\|\omega(p) - \omega(q)\| \geq C(\|p\|^{r-1} + \|q\|^{r-1}) \|p - q\| .$$

Assume also that the condition (II) *above is satisfied, then ω is $\mathcal{A}_0$-sufficient in $\mathcal{E}_{[r]}(2, 2)$.*

Moreover, the condition (I) *of Proposition 2.1 is a necessary condition for $\mathcal{A}_0$-sufficiency in $\mathcal{E}_{[r]}(2, 2)$ for all jets in $J^r(2, 2)$ with $r > 2$, and if we consider singular, non-zero jets ω where 0 is not isolated in $\Sigma(\omega)$, and where all the components $C_1, \ldots, C_N$ of $\Sigma(\omega) - \{0\}$ have different tangent directions at 0, then condition* (II) *above is also a necessary condition for $\mathcal{A}_0$-sufficiency in $\mathcal{E}_{[r]}(2, 2)$.*

Remark 1. One may conjecture that (I) together with (II) is equivalent to $\mathcal{A}_0$-sufficiency for all jets with non isolated critical point at 0. In fact one may sharpen this, and restrict (II) to $\Sigma(\omega)$ and conjecture that (I) together with this restricted version of (II) is equivalent to $\mathcal{A}_0$-sufficiency for all such jets. In two preprints [12] and [13], the second author has verified this conjecture for jets where all the components $C_1, \ldots, C_N$ of $\Sigma(\omega) - \{0\}$ have different tangent directions at 0, for jets of rank 1 and for weighted homogeneous jets. In fact for homogeneous jets, $\mathcal{A}_0$-sufficiency is equivalent to the geometrical condition that the jets only have fold singularities outside 0 and have no singular double points. The proofs of these results given in [12] and [13] depend however heavily on the results and techniques given in this article.

For jets ω where 0 is isolated in $\Sigma(\omega)$ we have the following sufficiency theorem:

Theorem 2.3. *Let $\omega \in J^r(2, 2)$ with $r \geq 2$ be a singular jet and assume that there exists a neighborhood U of 0 such that $\Sigma(\omega) \cap U = \{0\}$. Then ω is $\mathcal{R}_0$-sufficient in $\mathcal{E}_{[r]}(2, 2)$ if and only if ω satisfies the condition* (I) *in Proposition 2.1.*

Remark 2. Let $\omega = (f, g)$. Note that $\mathcal{R}_0$-sufficiency is by [2] (or [14]) equivalent to an inequality $d(\nabla f(p), \nabla g(p)) \geq C||p||^{r-1}$, in fact in [2] it is proven that this inequality also is equivalent to $\mathcal{A}_0$-sufficiency for jets with an isolated critical point at 0. We will see in Subsection 4.1 below that this inequality is trivially equivalent to the inequality $d(j^1\omega(p), \Sigma) \geq C||p||^{r-1}$. The left hand side of the inequality (I) in Proposition 2.1 is a sort of measure of the distance from the jet $j^2\omega(p)$ to the set of singular 2-jets which are not folds. So a priori, this is a much weaker inequality than the inequality $d(j^1\omega(p), \Sigma) \geq C||p||^{r-1}$, but we will show in Subsection 4.1 that these two inequalities actually are equivalent for jets ω with $\Sigma(\omega) = \{0\}$, proving Theorem 2.3. Together with the conclusion of Theorem 2.2 we thus get that $\mathcal{R}_0$-sufficiency, $\mathcal{A}_0$-sufficiency and (I) are equivalent conditions for jets in $J^r(2, 2)$ with an isolated critical point at 0.

3. Folds

As remarked above, the left hand side of the inequality (I) of Proposition 2.1 somehow measures the distance from the 2-jet $j^2\omega(p)$ to the set of singular jets which are not folds. To see this we first have to study fold points and make some estimates in both $J^1(2, 2)$ and $J^2(2, 2)$.

By definition, a mapping $F : \mathbb{R}^2 \to \mathbb{R}^2$ has a fold singularity at a point p if $j^1F(p) \in \Sigma^1$, where Σ^1 is the set of jets of rank 1, $j^1F \pitchfork \Sigma^1$ at p and $\ker DF(p) + T_p\Sigma(F) = \mathbb{R}^2$. We say that a jet $z = (a, \ldots, j) \in J^2(2, 2)$ is a fold if the associated polynomial mapping $z(x, y) = (f(x, y), g(x, y)) = (ax + \cdots + gy^2, cx + \cdots + jy^2)$ has a fold singularity at 0.

We want to describe the set of folds in $J^2(2,2)$ explicitly. Since the Jacobian matrix of z at 0 is $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $ad-bc=0$ is the equation of the singular jets Σ in $J^1(2,2)$. Consider the mapping $(a,b,c,d)\mapsto ad-bc$. When $(a,b,c,d)\in\Sigma^1$ the gradient of this mapping, $(d,-c,-b,a)$, will be a normal vector of Σ^1 at (a,b,c,d). Then $j^1z \pitchfork \Sigma^1$ if and only if at least one of $(\frac{\partial}{\partial x}j^1z)(0)$, $(\frac{\partial}{\partial y}j^1z)(0)$ is not perpendicular to $(d,-c,-b,a)$, that is $\left(\begin{smallmatrix} ai-bh-cf+de \\ aj-bi-cg+df \end{smallmatrix}\right) \neq \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$. On the other hand, we have $Jz(x,y) = (\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x})(x,y)$, and a direct computation gives us that

$$\nabla Jz(0) = 2\begin{pmatrix} ai-bh-cf+de \\ aj-bi-cg+df \end{pmatrix}.$$

For $j^1z \pitchfork \Sigma^1$, the vector $\left(\begin{smallmatrix} ai-bh-cf+de \\ aj-bi-cg+df \end{smallmatrix}\right)$ is therefore a normal vector to $\Sigma(z)$ at 0. The vector $\left(\begin{smallmatrix} aj-bi-cg+df \\ -ai+bh+cf-de \end{smallmatrix}\right)$ will consequently span $T_0\Sigma(z)$, and the condition $\ker Dz(0) + T_0\Sigma(z) = \mathbb{R}^2$ is obviously equivalent to

$$Dz(0)\begin{pmatrix} aj-bi-cg+df \\ -ai+bh+cf-de \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} aj-bi-cg+df \\ -ai+bh+cf-de \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus we see that the set

$$\Gamma = \{\, (a,\dots,j) \mid ad-bc=0, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} aj-bi-cg+df \\ -ai+bh+cf-de \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \}$$

is the set of singular 2-jets which are not folds.

3.1. **Distance from a jet to Σ in $J^1(2,2)$.** Let F,G be nonnegative functions. We will use the notation $F\sim G$ if there are constants $s,t>0$ such that $sF\le G\le tF$. Consider a jet $z\in J^1(2,2)$ identified with a matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. By $\|M\|$, we mean the standard Euclidean norm $\|M\| = (a^2+b^2+c^2+d^2)^{\frac{1}{2}}$. Our first task will be to estimate the distance $d(z,\Sigma)$ from a z to $\Sigma\subset J^1(2,2)$.

Suppose $X = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is a singular jet realizing the distance R from $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ to Σ. It is clear that X is an element of Σ^1. A normal vector to Σ^1 at $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is $\left(\begin{smallmatrix} D & -C \\ -B & A \end{smallmatrix}\right)$, so there is a t with

$$M - X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} A & B \\ C & D \end{pmatrix} = t\begin{pmatrix} D & -C \\ -B & A \end{pmatrix}$$

giving

$$\det M = t\left\|\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right\|^2, \quad R = |t|\cdot\left\|\begin{pmatrix} D & -C \\ -B & A \end{pmatrix}\right\| = \frac{|ad-bc|}{\left\|\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right\|}.$$

Now, suppose $\|\left(\begin{smallmatrix} a \\ c \end{smallmatrix}\right)\| \ge \|\left(\begin{smallmatrix} b \\ d \end{smallmatrix}\right)\|$. Since $\left(\begin{smallmatrix} a & 0 \\ c & 0 \end{smallmatrix}\right)\in\Sigma$,

$$R \le \left\|\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}\right\| = \left\|\begin{pmatrix} b \\ d \end{pmatrix}\right\| \le \frac{1}{\sqrt{2}}\left\|\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right\|.$$

The same argument can be applied if $\|\left(\begin{smallmatrix} a \\ c \end{smallmatrix}\right)\| \le \|\left(\begin{smallmatrix} b \\ d \end{smallmatrix}\right)\|$, so in any case,

$$R \le \frac{1}{\sqrt{2}}\left\|\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right\|.$$

By the triangle inequality,

$$(1-\frac{1}{\sqrt{2}})\left\|\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right\| \le \left\|\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right\| \le (1+\frac{1}{\sqrt{2}})\left\|\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right\|. \tag{3.1}$$

So from this and from the expression for R above, we get that

$$(2-\sqrt{2})\frac{|Jz(x,y)|}{\|Dz(x,y)\|} \leq R = d\big(j^1 z(x,y),\Sigma\big) \leq (2+\sqrt{2})\frac{|Jz(x,y)|}{\|Dz(x,y)\|}, \tag{3.2}$$

and hence,

$$\frac{|Jz(x,y)|}{\|Dz(x,y)\|} \sim d\big(j^1 z(x,y),\Sigma\big) \tag{3.3}$$

for every non-zero jet $z \in J^r(2,2)$.

3.2. **Distance from a singular jet to Γ in $J^2(2,2)$.** Let $z = (a,b,\ldots,j) \in J^2(2,2)$ with $ad - bc = 0$. Let

$$E = E_z = \{\omega \in \Gamma \,|\, L_\omega = (a,b,c,d)\}.$$

We want to estimate distance $d(z,E)$, i.e. the distance from a singular 2-jet z to the set of singular 2-jets with the same linear part as z satisfying the equation

$$\begin{pmatrix} L_1 \\ L_2 \end{pmatrix} := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} aj - bi - cg + df \\ -ai + bh + cf - de \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.4}$$

If $a = b = c = d = 0$, then the distance is 0 of course. Suppose $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is singular and non-zero. E is the linear subspace $\mathbb{R}^6$ with coordinates $(e,\ldots,j)$ satisfying

$$\begin{aligned} &\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} aj - bi - cg + df \\ -ai + bh + cf - de \end{pmatrix} \\ &= \begin{pmatrix} (e,f,g,h,i,j)\cdot(-bd, ad+bc, -ac, b^2, -2ab, a^2) \\ (e,f,g,h,i,j)\cdot(-d^2, 2cd, -c^2, bd, -ad-bc, ac) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

So $E = \mathrm{sp}\{v_1, v_2\}^\perp$, where

$$\begin{aligned} v_1 &= (-bd, ad+bc, -ac, b^2, -2ab, a^2) \\ v_2 &= (-d^2, 2cd, -c^2, bd, -ad-bc, ac). \end{aligned}$$

If $H_z = (e,f,g,h,i,j)$, then the distance we are seeking is the length of the projection $\mathbf{p}_E$ of (e,f,g,h,i,j) onto $E^\perp$. We notice that since $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is singular, v_1 and v_2 are linearly dependent, and assuming that none of them are zero (otherwise, the expressions simplify),

$$\mathbf{p}_E = \frac{1}{2}\left(\frac{L_1}{\|v_1\|^2}v_1 + \frac{L_2}{\|v_2\|^2}v_2\right),$$

and so the distance R is

$$R = \|\mathbf{p}_E\| = \frac{1}{2}\left(\frac{|L_1|}{\|v_1\|} + \frac{|L_2|}{\|v_2\|}\right).$$

Suppose $\sup\{a^2,b^2,c^2,d^2\} \in \{a^2,b^2\}$ and put $N = \left\|\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right\|^2$. It is easily seen that

$$\frac{1}{16}N^2 \leq (\sup\{a^2,b^2,c^2,d^2\})^2 \leq \|v_1\|^2 \leq 12(\sup\{a^2,b^2,c^2,d^2\})^2 \leq 12N^2,$$

and we get

$$\frac{1}{4}N \leq \|v_1\| \leq 2\sqrt{3}N. \tag{3.5}$$

In this case, $R = \frac{|L_1|}{\|v_1\|}$ and

$$\frac{|L_1|}{2\sqrt{3}N} \leq R \leq \frac{4|L_1|}{N}. \tag{3.6}$$

Similarly, if $\sup\{a^2, b^2, c^2, d^2\} \in \{c^2, d^2\}$,

$$\frac{|L_2|}{2\sqrt{3}N} \leq R \leq \frac{4|L_2|}{N}. \tag{3.7}$$

Notice that the left inequalities in (3.6) and (3.7) hold without the assumptions regarding which elements are realizing $\sup\{a^2, b^2, c^2, d^2\}$. By adding the left sides of the inequalities (3.6) and (3.7) we get $(|L_1| + |L_2|)/(2\sqrt{3}N) \leq 2R$. Also, one of the inequalities on the right side of either (3.6) or (3.7) must hold, so certainly $2R \leq 8(|L_1| + |L_2|)/N$. We get

$$\frac{\left\|\begin{pmatrix} L_1 \\ L_2 \end{pmatrix}\right\|}{2\sqrt{3}N} \leq \frac{|L_1| + |L_2|}{2\sqrt{3}N} \leq 2R \leq 8\frac{|L_1| + |L_2|}{N} \leq 16\frac{\left\|\begin{pmatrix} L_1 \\ L_2 \end{pmatrix}\right\|}{N}. \tag{3.8}$$

From this we see that

$$\frac{\left\|\begin{pmatrix} L_1 \\ L_2 \end{pmatrix}\right\|}{N} = \frac{\left\|\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} aj - bi - cg + df \\ -ai + bh + cf - de \end{pmatrix}\right\|}{\left\|\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right\|^2} \sim R = d(z, E_z). \tag{3.9}$$

In the language of partial derivatives and differentials of a C^r mapping f with $p \in \Sigma(f)$, inequality (3.9) reads

$$d(H_f(p), E_{j^2 f(p)}) \sim \frac{\left\|Df(p)\begin{pmatrix} \frac{\partial}{\partial y} Jf(p) \\ -\frac{\partial}{\partial x} Jf(p) \end{pmatrix}\right\|}{\|Df(p)\|^2}. \tag{3.10}$$

3.3. **Smoothness of the distance function and proofs of Proposition 2.1 and Theorem 2.3.** Let $\omega \in J^r(2,2)$. Before we can prove Proposition 2.1, we have to investigate the smoothness properties of the distance map we are about to define. Let $d : \mathbb{R}^2 \to \mathbb{R}$ be the map $p \mapsto d(p) = d(j^1\omega(p), \Sigma)$. We want information about where d is smooth. To this end, let $d' : J^1(2,2) \to \mathbb{R}$ be the map $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto d(A, \Sigma) = \inf\{\|A - X\| \mid X \in \Sigma\}$. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in J^1(2,2) \setminus \Sigma$. Consider $B = \{Y \mid A - Y \in \Sigma\}$. Then $Y \in B$ if and only if there exists $\mathbf{w} \in \mathbb{R}^2$ with $\|\mathbf{w}\| = 1$ such that $A\mathbf{w} = Y\mathbf{w}$. Since $\|Y\mathbf{w}\| \leq \|Y\|$, we get that

$$\inf\{\|A\mathbf{w}\| \mid \|\mathbf{w}\| = 1\} \leq d'(A).$$

On the other hand, let $\lambda = \inf\{\|A\mathbf{w}\| \mid \|\mathbf{w}\| = 1\}$ and let $\mathbf{w} = \left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)$ be a unit vector such that $\lambda = \|A\mathbf{w}\|$. Let Y be the matrix given by $Y\mathbf{w} = A\mathbf{w}$ and $Y\left(\begin{smallmatrix} -v \\ u \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$. Then $\|Y\| = \|A\mathbf{w}\|$ and it follows that

$$d'(A) = \inf\{\|A\mathbf{w}\| \mid \|\mathbf{w}\| = 1\}. \tag{3.11}$$

From this we see that

$$d'(A) = (\inf\{|\mathbf{w}^T A^T A\mathbf{w}|\,;\, \|\mathbf{w}\| = 1\})^{\frac{1}{2}} = (\inf\{|\beta|\,;\, \beta \text{ eigenvalue of } A^T A\})^{\frac{1}{2}}.$$

Calculating the eigenvalues of the symmetric matrix $A^T A$, we find that

$$d'(A) = \frac{1}{\sqrt{2}}\sqrt{\|A\|^2 - \sqrt{\|A\|^4 - 4(\det A)^2}}.$$

If we want to find an explicit expression for $X = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \Sigma$ such that $d'(A) = \|A - X\|$, we can use the method of Lagrange multipliers. The coordinates of X have to satisfy the following equations:

$$x - a = \lambda w \tag{3.12}$$
$$y - b = -\lambda z \tag{3.13}$$
$$z - c = -\lambda y \tag{3.14}$$
$$w - d = \lambda x \tag{3.15}$$
$$xw - yz = 0. \tag{3.16}$$

Analyzing this system, we find that if $|\det(A)| < \frac{1}{2}\|A\|^2$ (note that the inequality $|\det(A)| \leq \frac{1}{2}\|A\|^2$ holds for any A), then $\lambda \neq \pm 1$ and then the solution of the above system is given by

$$x = \frac{a + \lambda d}{1 - \lambda^2}, \quad y = \frac{b - \lambda c}{1 - \lambda^2}, \quad z = \frac{c - \lambda b}{1 - \lambda^2}, \quad w = \frac{d + \lambda a}{1 - \lambda^2}. \tag{3.17}$$

where λ is given by

$$\lambda_1 = \frac{-\|A\|^2 + \sqrt{\|A\|^4 - 4(\det A)^2}}{2\det A} \quad \text{or} \quad \lambda_2 = \frac{-\|A\|^2 - \sqrt{\|A\|^4 - 4(\det A)^2}}{2\det A},$$

and X is given by (3.17) with $\lambda = \lambda_1$. From the expression of d' above we see that d' is smooth when $\det A \neq 0$ and $|\det A| \neq \frac{1}{2}\|A\|^2$. d is consequently smooth on the complement of the set $\Sigma(\omega) \cup \{p \mid |J\omega(p)| = \frac{1}{2}\|D\omega(p)\|^2\}$. Denote this complement by V. Let

$$S = \{p = (x, y) \in V \mid \nabla d(p) \cdot (y, -x) = 0\}.$$

Then

$$S = \{p \in V \mid d|_{\{q \in V \mid \|q\| = \|p\|\}} \text{ has a stationary point at } p\}.$$

From the definition of V and the expression of d' given above it follows that S is a semialgebraic set. Now we have the following lemma:

Lemma 3.1. *Assume ω satisfies condition (I) of 2.1, then there is a neighborhood U of 0 and a $C > 0$ such that*

$$d(p) = d(j^1\omega(p), \Sigma) \geq C\|p\|^{r-1}$$

when $p \in S \cap U$.

Proof. Consider the set

$$\begin{aligned} D = \{(p, A) \subset S \times \Sigma \mid \ & \left\|j^1\omega(p) - A\right\| \leq \left\|j^1\omega(q) - B\right\| \\ & \text{for all } q \in S \text{ with } \|p\| = \|q\| \neq 0 \text{ and } B \in \Sigma\}. \end{aligned}$$

An application of the Tarski-Seidenberg Theorem shows that D is semialgebraic. Assume that the inequality of the lemma is not satisfied. Then $(0, j^1\omega(0)) \in \overline{D}$ and the curve selection lemma implies that we can find an analytic curve $\tilde{\gamma} : [0, \delta) \to \mathbb{R}^2 \times \Sigma$ with $\tilde{\gamma}((0, \delta)) \subset D$ and $\tilde{\gamma}(0) = (0, j^1\omega(0))$. Let $\tilde{\gamma}(t) = (\gamma(t), A(t))$. We must have that $\left\|j^1\omega(\gamma(t)) - A(t)\right\| = o(\|\gamma(t)\|^{r-1})$.

Let

$$A(t) = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}.$$

Then

$$j^1\omega(\gamma(t)) - A(t) = s(t)\frac{\begin{pmatrix} d(t) & -c(t) \\ -b(t) & a(t) \end{pmatrix}}{\|A(t)\|}$$

where $|s(t)| = \left\|j^1\omega(t) - A(t)\right\|$. For each t let $\beta_t(u)$ be a curve such that $\beta_t(0) = \gamma(t)$, $\|\beta_t'(u)\| = 1$ and $\|\beta_t(u)\| = \|\gamma(t)\|$ for each u. Let $A_t(u) \in \Sigma$ be such that $d(j^1\omega(\beta_t(u)), \Sigma) = A_t(u)$. It is clear that $A_t(u) \in \Sigma^1$, and since $A_t(u)$ is given by equation (3.17) with $\lambda = \lambda_1$, it is clear that $A_t(u)$ is unique and smooth in u for small u. Moreover, $A_t(0) = A(t)$. By construction, $\left\|j^1\omega(\beta_t(u)) - A_t(u)\right\|^2$ must have a stationary point for $u = 0$. So

$$\begin{aligned} &\frac{d}{du}\left\|j^1\omega(\beta_t(u)) - A_t(u)\right\|^2|_{u=0} \\ &= 2\left(\frac{d}{du}j^1\omega(\beta_t(u))|_{u=0} - \frac{d}{du}A_t(u)|_{u=0}\right)\cdot(j^1\omega(\gamma(t)) - A(t)) = 0. \end{aligned}$$

(Here "$\cdot$"denotes the standard Euclidean inner product in $J^1(2,2)$ identified with $\mathbb{R}^4$ via the coordinates (a,b,c,d).)

Now, $\frac{d}{du}A_t(u)|_{u=0} \in T_{A(t)}\Sigma^1$, and since $j^1\omega(\gamma(t)) - A(t)$ is a normal vector to $T_{A(t)}\Sigma^1$, we get that

$$\left(\frac{d}{du}j^1\omega(\beta_t(u))|_{u=0}\right)\cdot(j^1\omega(\gamma(t)) - A(t)) = 0.$$

So

$$\left(\frac{d}{du}j^1\omega(\beta_t(u))|_{u=0}\right)\cdot\frac{\begin{pmatrix} d(t) & -c(t) \\ -b(t) & a(t) \end{pmatrix}}{\|A(t)\|} = \left(Dj^1\omega(\gamma(t))w(t)\right)\cdot\frac{\begin{pmatrix} d(t) & -c(t) \\ -b(t) & a(t) \end{pmatrix}}{\|A(t)\|} = 0,$$

where $w(t)$ is the unit vector $\frac{d}{du}\beta_t(u)|_{u=0}$.

Let $\|\gamma(t)\| \sim t^l$ and $|s(t)| = \left\|j^1\omega(\gamma(t)) - A(t)\right\| \sim t^q$. Then $q > l(r-1)$. Since we have that

$$\frac{d}{dt}\left\|j^1\omega(\gamma(t)) - A(t)\right\|^2 \sim t^{2q-1},$$

we get that

$$\left(\frac{d}{dt}(j^1\omega(\gamma(t)) - A(t))\right)\cdot(j^1\omega(\gamma(t)) - A(t)) \sim t^{2q-1}$$

and consequently that

$$\frac{d}{dt}(j^1\omega(\gamma(t)) - A(t))\cdot\frac{\begin{pmatrix} d(t) & -c(t) \\ -b(t) & a(t) \end{pmatrix}}{\|A(t)\|} \sim t^{q-1}.$$

Since $\frac{d}{dt}A(t) \in T_{A(t)}\Sigma^1$, and $\begin{pmatrix} d(t) & -c(t) \\ -b(t) & a(t) \end{pmatrix}$ is a a normal vector to $T_{A(t)}\Sigma^1$, we must have

$$\frac{d}{dt}(j^1\omega(\gamma(t)))\cdot\frac{\begin{pmatrix} d(t) & -c(t) \\ -b(t) & a(t) \end{pmatrix}}{\|A(t)\|} \sim t^{q-1}.$$

Now $\frac{d}{dt}(j^1\omega(\gamma(t))) = Dj^1\omega(\gamma(t))\gamma'(t)$. Let $v(t) = \frac{\gamma'(t)}{\|\gamma'(t)\|}$. Since $\|\gamma'(t)\| \sim t^{l-1}$, we get that

$$t^{q-l} \sim (Dj^1\omega(\gamma(t))v(t))\cdot\frac{\begin{pmatrix} d(t) & -c(t) \\ -b(t) & a(t) \end{pmatrix}}{\|A(t)\|} = o(t^{l(r-1)-l}) = o(\|\gamma(t)\|^{r-2}).$$

Let us consider $v(t)$ and $w(t)$ above as two unit vectors in $T_{\gamma(t)}\mathbb{R}^2$. Since $\gamma(t)$ is analytic, $v(t) = \frac{\gamma'(t)}{\|\gamma(t)\|}$ and $w(t)\cdot\gamma(t) = 0$, we must have $v(t)\cdot w(t) \to 0$ as $t \to 0$. Let $e_1(t) = \frac{\partial}{\partial x}\circ\gamma(t)$ and $e_2(t) = \frac{\partial}{\partial y}\circ\gamma(t)$, we must then have $e_1(t) = s_1(t)v(t)+p_1(t)w(t)$ and $e_2(t) = s_2(t)v(t)+p_2(t)w(t)$, where $|s_i(t)| < 2$ and $|p_i(t)| < 2$ for small t. From this and from above we get that

$$(Dj^1\omega(\gamma(t))e_1(t)) \cdot \frac{\begin{pmatrix} d(t) & -c(t) \\ -b(t) & a(t) \end{pmatrix}}{\|A(t)\|} = o(\|\gamma(t)\|^{r-2})$$

and

$$(Dj^1\omega(\gamma(t))e_2(t)) \cdot \frac{\begin{pmatrix} d(t) & -c(t) \\ -b(t) & a(t) \end{pmatrix}}{\|A(t)\|} = o(\|\gamma(t)\|^{r-2}).$$

For fixed t, write $j^2\omega(\gamma(t))$ as in Section 2 in the form

$$j^2\omega(\gamma(t)) = \left(\tilde{a}(t)x + \tilde{b}(t)y + \cdots + \tilde{g}(t)y^2, \tilde{c}(t)x + \tilde{d}(t)y + \cdots + \tilde{j}(t)y^2\right).$$

Then

$$Dj^1\omega(\gamma(t))e_1(t) = 2\begin{pmatrix} \tilde{e}(t) & \tilde{f}(t) \\ \tilde{h}(t) & \tilde{i}(t) \end{pmatrix}.$$

We thus get that

$$\begin{aligned} &2\begin{pmatrix} \tilde{e}(t) & \tilde{f}(t) \\ \tilde{h}(t) & \tilde{i}(t) \end{pmatrix} \cdot \frac{\begin{pmatrix} d(t) & -c(t) \\ -b(t) & a(t) \end{pmatrix}}{\|A(t)\|} \\ =&2\frac{a(t)\tilde{i}(t) - b(t)\tilde{h}(t) - c(t)\tilde{f}(t) + d(t)\tilde{e}(t)}{\|A(t)\|} \\ =&o(\|\gamma(t)\|^{r-2}). \end{aligned}$$

In a similar way we get that

$$2\frac{a(t)\tilde{j}(t) - b(t)\tilde{i}(t) - c(t)\tilde{g}(t) + d(t)\tilde{f}(t)}{\|A(t)\|} = o(\|\gamma(t)\|^{r-2}).$$

Let $\tilde{z}(t)$ be the singular 2- jet with $L_{\tilde{z}(t)} = (a(t), b(t), c(t), d(t))$ and $H_{\tilde{z}(t)} = H_\omega(\gamma(t)) = \left(\tilde{e}(t), \tilde{f}(t), \tilde{g}(t), \tilde{h}(t), \tilde{i}(t), \tilde{j}(t)\right)$. From above it is clear that

$$\frac{\left\|\begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}\begin{pmatrix} a(t)\tilde{j}(t) - b(t)\tilde{i}(t) - c(t)\tilde{g}(t) + d(t)\tilde{f}(t) \\ -a(t)\tilde{i}(t) + b(t)\tilde{h}(t) + c(t)\tilde{f}(t) - d(t)\tilde{e}(t) \end{pmatrix}\right\|}{\left\|\begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}\right\|^2} = o(\|\gamma(t)\|^{r-2}).$$

From (3.9) it is then clear that there exists a jet $z(t) = (L_{z(t)}, H_{z(t)}) \in \Gamma$ with $L_{z(t)} = (a(t), b(t), c(t), d(t))$ such that $\left\|H_\omega(\gamma(t)) - H_{z(t)}\right\| = o(\|\gamma(t)\|^{r-2})$. It follows that

$$\left\|L_\omega(\gamma(t)) - L_{z(t)}\right\| + \left\|H_\omega(\gamma(t)) - H_{z(t)}\right\| \|\gamma(t)\| = o(\|\gamma(t)\|^{r-1})$$

contradicting (I). ◻

Lemma 3.2. *Assume ω satisfies condition (I) of 2.1 with neighbourhood U and constant $C > 0$, then*

$$\|D\omega(p)\| \geq C\,\|p\|^{r-1}.$$

when $p \in U$.

Proof. It is clear that $(0, H_\omega(p)) \in \Gamma$ (where 0 is the zero-jet in $J^1(2,2)$) for each p, so

$$\|D\omega(p)\| = \|L_\omega(p) - 0\| + \|H_\omega(p) - H_\omega(p)\| \|p\| \geq C \|p\|^{r-1}.$$

and the lemma follows. □

Proof of Proposition 2.1. Let ω be as in Proposition 2.1 satisfying condition (I). As pointed out above, the function d is smooth at points p which are not singular and satisfy $|J\omega(p)| \neq \frac{1}{2}\|D\omega(p)\|^2$. Let the radius ρ_0 in the statement of Proposition 2.1 also be chosen so small that the conclusions of Lemma 3.1 and Lemma 3.2 hold when $U = B(0,\rho)$ and $0 < \rho < \rho_0$. From Lemma 3.2 it then follows that if d is not smooth at p and p is a regular point, then $|J\omega(p)| = \frac{1}{2}\|D\omega(p)\|^2 \geq \frac{C}{2}\|D\omega(p)\| \|p\|^{r-1}$, where C is given in Lemma 3.2. So, if $\epsilon < \frac{2-\sqrt{2}}{2}C$, it follows from inequality (3.2) that d is smooth in $(H_\epsilon - \Sigma(\omega)) \cap B(0,\rho)$ when $\rho < \rho_0$. Also assume that $\epsilon < C$ where this time C is the constant of Lemma 3.1. It follows that $(H_\epsilon - \{0\}) \cap B(0,\rho)$ contains no points in S when $\rho < \rho_0$.

The set $(H_\epsilon - \{0\}) \cap B(0,\rho)$ is semialgebraic and has consequently finitely many connected components, and each component C_i is contained in one such component. If ρ_0 is chosen small enough, we may apply Theorem 9.3.6 of [1], and conclude that $H_\epsilon \cap \overline{B(0,\rho)}$ is homeomorphic to the cone with vertex 0 and basis $H_\epsilon \cap \{p \mid \|p\| = \rho\}$. Since this basis is semialgebraic, and hence a finite union of closed segments and isolated points, it follows that each component of $(H_\epsilon - \{0\}) \cap \overline{B(0,\rho)}$ is a cone with the vertex 0 removed and with basis either a closed segment or a point of the circle $\{p \mid \|p\| = \rho\}$. Consider such a component H_k of $(H_\epsilon - \{0\}) \cap B(0,\rho)$ and a point $p \in H_k$. Assume H_k contains none of the components C_i. If p is an isolated point in $H_k \cap \{q \mid \|q\| = \|p\|\}$, then p is a local minimum of the function $d|_{\{q \mid \|q\|=\|p\|\}}$. If p is not isolated, then p is a point in $H_k \cap \{q \mid \|q\| = \|p\|\}$ and this set is a 1-dimensional compact curve which also must contain a local minimum of the function $d|_{\{q \mid \|q\|=\|p\|\}}$ in its interior. Since H_k does not contain any of the curves C_i, $d|_{\{q \mid \|q\|=\|p\|\}}$ is smooth at this local minimum so this minimum must be a point in S. From above we have that this is impossible.

If H_k contains two components C_i, C_j of $\Sigma(\omega) - \{0\}$, then $H_k \cap \{q \mid \|q\| = \|p\|\}$ contains a 1-dimensional compact curve such that the end-points of this curve are singular points and the interior points are non-singular. Then the function $d|_{\{q \mid \|q\|=\|p\|\}}$ must have a local maximum at an interior point of this curve. Again, this point must be a point in S which is impossible. We therefore conclude that it is impossible that a component of $(H_\epsilon - \{0\}) \cap B(0,\rho)$ contains several or no components of $\Sigma(\omega) - \{0\}$. This completes the proof of Proposition 2.1. □

Proof of Theorem 2.3. We will need the following lemma.

Lemma 3.3. *Let ω be a jet with $\Sigma(\omega) = \{0\}$ (as a set germ at 0). Consider the following inequality:*
There exist a constant C and a neighbourhood U of 0 such that

$$(\mathrm{I}') \qquad d(p) = d(j^1\omega(p), \Sigma) \geq C \|p\|^{r-1}$$

for $p \in U$. Then (I') is equivalent with the inequality (I) of Proposition 2.1.

Proof of Lemma 3.3. Assume that the inequality (I′) is not satisfied. Then we can find a sequence $p_n \to 0$ such that $d(j^1\omega(p_n), \Sigma) = o(\|p_n\|^{r-1})$. If p is a point such that $|J\omega(p)| =$

$\frac{1}{2}\|D\omega(p)\|^2$, then it follows from the estimates in (3.2) and Lemma 3.2 that

$$d(j^1\omega(p),\Sigma) \geq (2-\sqrt{2})\frac{|J\omega(p)|}{\|D\omega(p)\|} =$$
$$\frac{2-\sqrt{2}}{2}\|D\omega(p)\| \geq \frac{(2-\sqrt{2})}{2}C\|p\|^{r-1},$$

where C is given in Lemma 3.2. It follows from this and the existence of the sequence p_n that the function $d|_{\{p\,|\,\|p\|=\rho\}}$ must have an absolute minimum at points p where d is smooth, hence in the set S, when ρ is sufficiently small. Let p be such a point. From Lemma 3.1 it follows however that if $\|p\|$ is small then $d(p) \geq C\|p\|^{r-1}$ for some C independent of p, and since $d|_{\{p\,|\,\|p\|=\rho\}}$ attains an absolute minimum at p this contradicts the existence of the sequence p_n. So (I$'$) must be satisfied.
Let $\pi_1^2 : J^2(2,2) \to J^1(2,2)$ be the canonical projection. Then $\Gamma \subset (\pi_1^2)^{-1}(\Sigma)$ and from this, the implication (I$'$)$\Rightarrow$(I) is obvious. □

Let f and g be the components of ω. As pointed out in Remark 2, it follows from Lemma 3.3 that we only need to prove the equivalence of the inequality $d(j^1\omega(p),\Sigma) \geq C\|p\|^{r-1}$ and the inequality $d(\nabla f(p),\nabla g(p)) \geq C\|p\|^{r-1}$ of [2] (or [14]). From Subsection 3.2, we have $d(j^1\omega(p),\Sigma) \sim \frac{|J\omega(p)|}{\|D\omega(p)\|}$. From the definition in [2], we get that

$$d(\nabla f(p),\nabla g(p)) = \min\{||\nabla f(p) - \frac{\nabla f(p)\cdot\nabla g(p)}{||\nabla g(p)||^2}\nabla g(p)||, ||\nabla g(p) - \frac{\nabla g(p)\cdot\nabla f(p)}{||\nabla f(p)||^2}\nabla f(p)||\}.$$

If say, $\|\nabla f(p)\| \geq \|\nabla g(p)\|$, then a straightforward calculation shows that

$$d(\nabla f(p),\nabla g(p)) = \frac{|J\omega(p)|}{\|\nabla f(p)\|} \geq \frac{|J\omega(p)|}{\|D\omega(p)\|} \geq \frac{1}{\sqrt{2}}d(\nabla f(p),\nabla g(p)),$$

hence

$$d(\nabla f(p),\nabla g(p)) \sim \frac{|J\omega(p)|}{\|D\omega(p)\|}$$

and consequently

$$d(\nabla f(p),\nabla g(p)) \sim d(j^1\omega(p),\Sigma).$$

The conclusion of Theorem 2.3 follows from this. □

4. Stability of the Lojasiewicz inequalities

In this section we prove that the Lojasiewicz inequalities (I) of 2.1 and (II) of 2.2 are in some sense stable under perturbations of the jet by C^r- mappings with r-jet vanishing to r-th order at 0, and we derive some important geometrical consequences of the two Lojasiewicz inequalities.

4.1. **Lojasiewicz inequality (I)..** From now on, let $\omega = (f,g) \in J^r(2,2)$ for some $r \geq 2$ and with 0 not isolated in $\Sigma(\omega)$. Let $\tilde{\omega} = (\tilde{f},\tilde{g})$ be a C^r map with $j^r\tilde{\omega}(0) = 0$. For $t \in \mathbb{R}$, put $\omega_t(p) = \omega(p) + t\tilde{\omega}(p) = (f_t, g_t)$. Also, let $\epsilon > 0$ and let U be a neighbourhood of $0 \in \mathbb{R}^2$.

Lemma 4.1. *Assume that ω satisfies the condition (I) of Proposition 2.1 for some neighbourhood U of 0 and some constant $C > 0$. Then there are constants $0 < C' < C$ and $\epsilon > 0$ and a neighbourhood U' of 0 such that if $t \in (-\epsilon, 1+\epsilon)$, then condition (I) with constant C' holds for ω_t in U'.*

Proof. Let $(L, H) \in \Gamma$. By the triangle inequality,

$$\|L_{\omega_t}(p) - L\| \geq \|L_\omega(p) - L\| - |t| \, \|L_{\tilde{\omega}}(p)\| \geq \|L_\omega(p) - L\| - (1+\epsilon) \, \|L_{\tilde{\omega}}(p)\| \, ,$$

and similarly,

$$\|H_{\omega_t}(p) - H\| \geq \|H_\omega(p) - H\| - (1+\epsilon) \, \|H_{\tilde{\omega}}(p)\| \, .$$

Hence,

$$\begin{aligned} &\|L_{\omega_t}(p) - L\| + \|H_{\omega_t}(p) - H\| \, \|p\| \\ \geq &\|L_\omega(p) - L\| - (1+\epsilon) \, \|L_{\tilde{\omega}}(p)\| + \|H_\omega(p) - H\| - (1+\epsilon) \, \|H_{\tilde{\omega}}(p)\|) \, \|p\| \\ \geq &\frac{C}{2} \, \|p\|^{r-1} \end{aligned}$$

when U' is so small that $\frac{\|L_{\tilde{\omega}}(p)\|}{\|p\|^{r-1}} \leq \frac{C}{4(1+\epsilon)}$ and $\frac{\|H_{\tilde{\omega}}(p)\|}{\|p\|^{r-2}} \leq \frac{C}{4(1+\epsilon)}$. Such a neighbourhood U' exists for any $\epsilon > 0$ since $j^r \tilde{\omega}(0) = 0$ implies that $\|L_{\tilde{\omega}}(p)\| = o(\|p\|^{r-1})$ and that $\|H_{\tilde{\omega}}(p)\| = o(\|p\|^{r-2})$. Putting $C' = \frac{C}{2}$ completes the proof. $\square$

4.2. **Stability of Lojasiewicz inequality (II).** Let ω and ω_t be as in Subsection 4.1, but assume that $r > 2$. We assume that ω satisfies condition (I) of Proposition 2.1 and that U, C and ϵ are so small that by Lemma 4.1, (I) also is satisfied for ω_t, $t \in (-\epsilon, 1+\epsilon)$. Let $F : U \times (-\epsilon, 1+\epsilon) \to \mathbb{R}^3$ be the 1-parameter unfolding of ω given by $(p, t) \mapsto (\omega_t(p), t)$.

Lemma 4.2. *There are constants $C', \epsilon > 0$ such that if $t \in (-\epsilon, 1+\epsilon)$ and $p \in U \cap (\Sigma(\omega_t) \setminus \{0\})$, then*

$$\frac{\|\nabla J\omega_t(p)\|}{\|D\omega_t(p)\|} \geq C' \, \|p\|^{r-2} \, . \tag{4.1}$$

Proof. For $p \in U \cap (\Sigma(\omega_t) \setminus \{0\})$ we can choose H such that $(L_{\omega_t}(p), H) \in \Gamma$. Inequality (I) implies that for $t \in I = (-\epsilon, 1+\epsilon)$,

$$\|H_{\omega_t}(p) - H\| \, \|p\| \geq C \, \|p\|^{r-1} \, . \tag{4.2}$$

Choose H of this type, minimizing the distance $\|H_{\omega_t}(p) - H\|$. It follows from Schwartz inequality and (3.8) that

$$\frac{\|\nabla J\omega_t(p)\|}{\|D\omega_t(p)\|} \geq \frac{\left\| D\omega_t(p) \begin{pmatrix} \frac{\partial}{\partial y} J\omega_t(p) \\ -\frac{\partial}{\partial x} J\omega_t(p) \end{pmatrix} \right\|}{\|D\omega_t(p)\|^2} \geq \frac{1}{8} \, \|H_{\omega_t}(p) - H\| \geq \frac{C}{8} \, \|p\|^{r-2} \, . \tag{4.3}$$

The lemma follows by choosing $C' \leq \frac{C}{8}$. $\square$

Let $F_0 = F|_{(U \setminus \{0\}) \times (-\epsilon, 1+\epsilon)}$. It is easily seen that $JF(p, t) = J\omega_t(p)$. Thus, Lemma 4.2 implies that 0 is a regular value of JF_0 and we can conclude that $\Sigma(F_0)$ is a 2-dimensional C^{r-1} submanifold of $\mathbb{R}^3$. Define a vector field $\mathbf{v}$ on $\Sigma(F)$ by

$$\mathbf{v}(p, t) = \begin{cases} (0, 0, 1), & \text{if } p = 0 \\ \frac{\mathbf{p}_T(p,t)}{[\mathbf{p}_T(p,t)]_t}, & \text{otherwise,} \end{cases}$$

where $\mathbf{p}_T$ means the projection of $\mathbf{k} = (0, 0, 1)$ into the tangent plane of the manifold $\Sigma(F_0)$ and v_t denotes the t-component of any vector v. Notice that $\mathbf{v}_t \equiv 1$ on $\Sigma(F)$.

Lemma 4.3. $\|\mathbf{v}(p, t) - (0, 0, 1)\| = o\big(\|p\| \big)$.

Proof. In block-form the matrix of DF reads

$$DF = \begin{pmatrix} D\omega_t & \tilde{\omega} \\ 0 & 1 \end{pmatrix}.$$

As mentioned above, we see that $JF = 0 \Leftrightarrow J\omega_t = 0$. Put $h(p,t) = J\omega_t(p)$. Then $\Sigma(F) = h^{-1}(0)$, and hence, $\nabla h(p,t) \perp T_{(p,t)}\Sigma(F_0)$.

Let $\mathbf{p}_N(p,t)$ be the projection of $\mathbf{k} = (0,0,1)$ onto $\mathrm{sp}\{\nabla h(p,t)\}$. The projection $\mathbf{p}_T(p,t)$ of $\mathbf{k}$ into $T_{(p,t)}\Sigma(F_0)$ is

$$\mathbf{p}_T = \mathbf{k} - \mathbf{p}_N = \mathbf{k} - \frac{\frac{\partial h}{\partial t}}{\|\nabla h\|^2}\nabla h.$$

The t-component of $\mathbf{p}_T$ equals $\frac{\|\nabla J\omega_t\|^2}{\|\nabla h\|^2}$. Thus,

$$\mathbf{v} = \frac{\|\nabla h\|^2}{\|\nabla J\omega_t\|^2}\mathbf{k} - \frac{\frac{\partial h}{\partial t}}{\|\nabla J\omega_t\|^2}\nabla h.$$

Using that $\mathbf{v}_t = 1$, we get

$$\|\mathbf{v} - \mathbf{k}\| = \frac{\left|\frac{\partial h}{\partial t}\right|}{\|\nabla J\omega_t\|}.$$

Now, $\frac{\partial h}{\partial t} = \frac{\partial}{\partial t} J\omega_t$, where

$$J\omega_t = J\omega + t\left(\frac{\partial f}{\partial x}\frac{\partial \tilde{g}}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial \tilde{g}}{\partial x} + \frac{\partial \tilde{f}}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial \tilde{f}}{\partial y}\frac{\partial g}{\partial x}\right) + t^2 J\tilde{\omega}.$$

From Lemma 3.2 we have that $\|D\omega(p)\| \geq C\|p\|^{r-1}$. Since

$$J\tilde{\omega}(p) = o(\|p\|^{r-1} \cdot \|p\|^{r-1})$$

and

$$\left(\frac{\partial f}{\partial x}\frac{\partial \tilde{g}}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial \tilde{g}}{\partial x} + \frac{\partial \tilde{f}}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial \tilde{f}}{\partial y}\frac{\partial g}{\partial x}\right)(p) = o(\|p\|^{r-1})\,\|D\omega(p)\|\,,$$

we can conclude that $\frac{\partial h}{\partial t}(p,t) = o(\|p\|^{r-1})\,\|D\omega(p)\|$. By rearranging the terms of (4.1) we obtain

$$\frac{1}{\|\nabla J\omega_t(p)\|} \leq \frac{\|p\|^{2-r}}{C'\,\|D\omega_t(p)\|}.$$

Combining all this and the fact that $\|D\omega_t(p)\| = \|D\omega(p)\| + o(\|p\|^{r-1})$, we get

$$\begin{aligned}\|\mathbf{v}(p,t) - (0,0,1)\| &= \frac{\left|\frac{\partial h}{\partial t}(p,t)\right|}{\|\nabla J\omega_t(p)\|} \\ &= o(\|p\|^{r-1}) \cdot \|D\omega(p)\| \cdot \frac{\|p\|^{2-r}}{C'\,\|D\omega_t(p)\|} \\ &= o(\|p\|).\end{aligned}$$

□

We are now going to extend the vector field $\mathbf{v}$ to a vector field ξ defined and continuous on all of $U \times (-\epsilon, 1+\epsilon)$. For simplicity, let $I = (-\epsilon, 1+\epsilon)$ and $U_0 = U - \{0\}$. Recall that $F_0 = F|_{U_0 \times I}$.

Let $q \in U_0 \times I$, If $q \in \Sigma(F_0)$, we can find an open neighbourhood V of q in $\mathbb{R}^3$ and a C^{r-1}-diffeomorphism $\Phi : V \to W$ of V onto an open neighbourhood W of the origin in $\mathbb{R}^3$ such that $\Phi(V \cap \Sigma(F_0)) = W \cap (\mathbb{R}^2 \times \{0\})$. In W, define a vector field $\mathbf{v}_\Phi$ by

$$\mathbf{v}_\Phi(x,y,z) = D\Phi(\Phi^{-1}(x,y,0))\mathbf{v}(\Phi^{-1}(x,y,0)).$$

Now, put $V_q = V$ and define

$$\mathbf{w}_q(p,t) = D\Phi^{-1}(\Phi(p,t))\mathbf{v}_\Phi(\Phi(p,t))$$

for $(p,t) \in V_q$. When $q \in U_0 \times I - \Sigma(F)$, put $V_q = U_0 \times I - \Sigma(F)$ and define $w_q = (0,0,1)$ on V_q. Gluing these locally defined vector fields together by a partition of unity argument and scaling the resulting vector field such that the t-component becomes identically 1, we get a vector field ξ defined on $U_0 \times I$ extending $\mathbf{v}$. If the V_q's corresponding to points $q \in \Sigma(F_0)$ are chosen small enough, we obtain

$$\|\xi(p,t) - (0,0,1)\| = o(\|p\|). \tag{4.4}$$

We can extend ξ to all of $U \times I$ by defining $\xi(0,0,t) = (0,0,1)$.

This new vector field ξ is continuous, and by construction, ξ is C^{r-2} on $U_0 \times I$. We have assumed that $r > 2$, so ξ is at least C^1. Thus for every $p \in U_0 \times I$ there is a local flow line through p. Of course, the curve $\gamma : I \to U \times I$, $t \mapsto (0,0,t)$, is a flow line through every point of $\{0\} \times I$. Thus we have local solutions of ξ through every point of $U \times I$. Although ξ itself is not differentiable on the t-axis we will see that 4.4 is sufficient for ξ to have a continuous flow near the t-axis. In fact we have:

Lemma 4.4. *There is an open neighbourhood $U' \subset U$ of the origin in $\mathbb{R}^2$ and an injective continuous map $\phi : U' \times I \to \mathbb{R}^3$ such that $\forall\,(p,t) \in U' \times I$,*

$$\phi(p,t) = (h_t(p),t), \qquad \phi(p,0) = (p,0) \qquad \textit{and} \qquad \frac{\partial}{\partial t}\phi(p,t) = \xi(\phi(p,t)).$$

Proof. Equation 4.4 and the differentiability of ξ in $U_0 \times I$ imply that the Lipschitz condition of Theorem 2 in [8] is satisfied by ξ. Thus we can find the flow φ of Theorem 2 in [8]. From 4.4 and the fact that the t-component of ξ is 1, it clear that if U' is small enough the flow line through each $(p,0)$, $p \in U'$ must reach every t-level in I before it reaches the boundary of $U \times I$. Putting $\phi(p,t) = \varphi(t,(p,0))$ we get the desired map ϕ. Since ξ has t-component equal 1, ϕ can be written as $\phi(p,t) = (h_t(p),t)$ for some level-map h_t. □

Since ξ is tangent to $\Sigma(F)$, we get a map $\Sigma(\omega) \times \{0\} \to \Sigma(\omega_t) \times \{t\}$ given by $(p,0) \mapsto \phi(p,t) = (h_t(p),t)$. This is a homeomorphism of $\Sigma(\omega) \times \{0\}$ onto its image. The map ϕ therefore induces homeomorphisms $h_t|_{\Sigma(\omega)} : \Sigma(\omega) \to \Sigma(\omega_t)$.

Lemma 4.5. *Let $0 < \delta < \epsilon$, then* $\sup\limits_{t \in [-\delta,1+\delta]} \|h_t(p) - p\| = o(\|p\|)$.

Proof. Suppose there is a constant $K > 0$ and a sequence $\{p_n\}$ such that $\|p_n\| \to 0$ and

$$\sup_{t \in [-\delta,1+\delta]} \|h_t(p_n) - p_n\| > K\,\|p_n\|\,.$$

Write $\phi(p,t) = \phi_p(t) = \big(\phi_p^1(t), \phi_p^2(t), t\big)$, $\xi(v,t) = \big(\xi^1(v,t), \xi^2(v,t), 1\big)$ and $p_n = (p_n^1, p_n^2)$. Applying the Mean Value Theorem and equation (4.4), we get that

$$\begin{aligned}
K\|p_n\| &< \sup_{t\in[-\delta,1+\delta]} \|h_t(p_n) - p_n\| \\
&= \sup_{t\in[-\delta,1+\delta]} \big\|\big(\phi_{p_n}^1(t), \phi_{p_n}^2(t)\big) - (p_n^1, p_n^2)\big\| \\
&\le 2(1+2\delta) \sup_{t\in[-\delta,1+\delta]} \left\|\big(\frac{\partial}{\partial t}\phi_{p_n}^1(t), \frac{\partial}{\partial t}\phi_{p_n}^2(t)\big)\right\| \\
&= 2(1+2\delta) \sup_{t\in[-\delta,1+\delta]} \big\|\big(\xi^1(\phi_{p_n}(t)), \xi^2(\phi_{p_n}(t))\big)\big\| \\
&= 2(1+2\delta) \big\|\big(\xi^1(v_n,t_n), \xi^2(v_n,t_n)\big)\big\| = o(\|v_n\|)
\end{aligned}$$

for (v_n, t_n) on the curve ϕ_{p_n} with

$$\big\|\big(\xi^1(v_n,t_n), \xi^2(v_n,t_n)\big)\big\| = \sup_{-\delta\le s\le 1+\delta} \big\|\big(\xi^1(\phi_{p_n}(s)), \xi^2(\phi_{p_n}(s))\big)\big\|.$$

Suppose $\|v_n\| < 2\|p_n\|$. Then we get the contradiction $K\|p_n\| < o(\|p_n\|)$. If this assumption is wrong, we can find a subsequence of $\{v_n\}$ with $\|v_n\| \ge 2\|p_n\|$. Let C be the trace of $\pi \circ \phi_{p_n}$, where $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ is the projection onto the first two coordinates. We consider the arc length of C, and see that

$$\begin{aligned}
\frac{1}{2}\|v_n\| \le \|v_n\| - \|p_n\| &\le \int_C \big\|\big(\xi^1(\phi_{p_n}(s)), \xi^2(\phi_{p_n}(s))\big)\big\|\, ds \\
&\le (1+2\delta) \big\|\big(\xi^1(v_n,t_n), \xi^2(v_n,t_n)\big)\big\| = o(\|v_n\|)
\end{aligned}$$

which is a new contradiction. The lemma follows. □

Lemma 4.6. *For small $\epsilon > 0, t \in I$, $\Sigma(\omega_t) \subset H_\epsilon$ in a neighbourhood of the origin in $\mathbb{R}^2$.*

Proof. From the proof of Lemma 4.3, we get

$$J\omega(p) = J\omega_t(p) + o(\|p\|^{r-1})\,\|D\omega(p)\|.$$

If $p \in \Sigma(\omega_t)$, $\frac{|J\omega(p)|}{\|D\omega(p)\|} = o(\|p\|^{r-1})$, and the lemma follows from (3.3) of Subsection 3.1. □

Remark 3. Let $\hat\omega$ be a C^r-realization of ω, then we can define a family $\omega_t = \omega + t(\hat\omega - \omega)$ of C^r-realizations such that $\omega_1 = \hat\omega$. Let $C_1, \dots, C_N$ be the connected components of $\Sigma(\omega)\setminus\{0\}$. Since $\Sigma(\omega)\setminus\{0\}$ and $\Sigma(\omega_t)\setminus\{0\}$ are homeomorphic, $\Sigma(\omega_t)\setminus\{0\}$ consists of N connected components for each t. Let h_t, $t \in I$ be the family of homeomorphisms constructed above. Since $h_0(C_i) = C_i$ and the set $\{h_t(p)\,|\,p \in C_i,\, t \in I\}$ is connected, it follows from the Lemma 4.6 and Proposition 2.1 that each $\Sigma(\omega_t)\setminus\{0\}$ has exactly one connected component in each H_i. So if $\epsilon > 0$ is chosen so small that the conclusion of 2.1 holds, each such realization $\hat\omega$ of ω has exactly one connected component of $\Sigma(\hat\omega)\setminus\{0\}$ in each connected component of $H_\epsilon - \{0\}$.

The corollary below gives a sort of stability property of inequality (II) under perturbation of the jet by C^r-mappings with r-jet vanishing at 0.

Corollary 4.7. *Let the hypothesis be as in Theorem 2.2, and assume that inequality (II) holds for ω with a constant $C > 0$. Let ω_t be as above. Then there exists a neighbourhood U of $0 \in \mathbb{R}^2$ such that if $t \in [0,1]$ and $p, q \in \Sigma(\omega_t) \cap U$ are points belonging to different components of $\Sigma(\omega_t)\setminus\{0\}$, then*

$$\|\omega_t(p) - \omega_t(q)\| \ge \frac{C}{2}(\|p\|^{r-1} + \|q\|^{r-1})\,\|p - q\|.$$

The inequality also holds if either p or q is equal 0.

Proof. Write $\omega_t = \omega + t\tilde{\omega}$. Since $j^r\tilde{\omega}(0) = 0$ we have $\|D\tilde{\omega}(p)\| = o(\|p\|^{r-1})$ where this time $\|D\tilde{\omega}(p)\|$ denotes the operator norm. From this follows that

$$\begin{aligned}\|\tilde{\omega}(p) - \tilde{\omega}(q)\| &= \left\| \int_0^1 D\tilde{\omega}(sp + (1-s)q)(p-q)\, ds \right\| \\ &\leq \sup_{s\in[0,1]} \|D\tilde{\omega}(sp + (1-s)q)\| \, \|p-q\| \\ &= o(\|p\|^{r-1} + \|q\|^{r-1}) \, \|p-q\| \,.\end{aligned}$$

From Remark (3) we get that, if U is sufficiently small, then there exists i, j, $i \neq j$ such that $p \in H_i$ and $q \in H_j$. From inequality (II) and above it follows that

$$\begin{aligned}\|\omega_t(p) - \omega_t(q)\| &= \|(\omega(p) - \omega(q)) + t(\tilde{\omega}(p) - \tilde{\omega}(q))\| \\ &\geq \|\omega(p) - \omega(q)\| - |t| \, \|\tilde{\omega}(p) - \tilde{\omega}(q)\| \\ &\geq C(\|p\|^{r-1} + \|q\|^{r-1}) \, \|p-q\| - o(\|p\|^{r-1} + \|q\|^{r-1}) \, \|p-q\| \\ &\geq \frac{C}{2}(\|p\|^{r-1} + \|q\|^{r-1}) \, \|p-q\| \,.\end{aligned}$$

Since $\|\tilde{\omega}(p)\| = o(\|p\|^r)$, the last statement of the corollary follows easily from (II) if say, $q = 0$. ☐

4.3. **Consequences of (I) and (II).** The inequalities (I) and (II) from Proposition 2.1 and Theorem 2.2 have several implications which will be important to us.

Lemma 4.8. *If $\omega \in J^r(2,2)$ satisfies (I) and (II) in a neighbourhood U of the origin, then there is a constant $K > 0$ such that $\|\omega(p)\| \geq K \|p\|^r$ for all p in a neighbourhood of the origin.*

Proof. Let

$$A = \left\{ p \,|\; \|\omega(p)\| = \min_{\|q\|=\|p\|} \|\omega(q)\| \,,\, p, q \in U_0 \right\}.$$

An application of the Tarski-Seidenberg Theorem shows that A is a semi-algebraic set. Hence, we can apply the curve selection lemma to find an analytic curve $\beta : [0,\epsilon) \to \mathbb{R}^2$ with $\beta(0) = 0$ and $\beta(0,\epsilon) \subset A$. Let s be chosen such that $\|\beta(t)\| \sim t^s$ as $t \to 0$. Assume that the lemma is false. Then $\|\omega(\beta(t))\| = o(\|\beta(t)\|^r) = o(t^{rs})$, and differentiation with respect to t gives $\|D\omega(\beta(t))\beta'(t)\| = o(t^{rs-1})$, and since we have that $\|\beta'(t)\| \sim t^{s-1}$ we obtain

$$\left\| D\omega(\beta(t)) \frac{\beta'(t)}{\|\beta'(t)\|} \right\| = o(t^{rs-1-s+1}) = o(\|\beta(t)\|^{r-1}).$$

Since $\beta'(t)/\|\beta'(t)\|$ is a unit vector, it follows from (3.11) of Subsection 3.3 that $d(j^1\omega(\beta(t)), \Sigma) = o(\|\beta(t)\|^{r-1})$ and we get that $\beta(t) \in H_\epsilon$. From (II) with $p = \beta(t)$ and $q = 0$, we get that $\|\omega(\beta(t))\| \geq C \|\beta(t)\|^r$, which is a contradiction. ☐

Corollary 4.9. *Suppose (I) and (II) hold. Then there is a neighbourhood U of the origin and a constant $K > 0$ such that*

$$\|\omega_t(p)\| \geq K \|p\|^r$$

for all $t \in I$ and $p \in U$.

Proof. This follows easily from Lemma 4.8 since $\|\omega_t(p)\| = \|\omega(p)\| + o(\|p\|^r)$. ☐

Remark 4. The hypothesis of Lemma 4.8 can be weakened. In fact, the lemma follows from inequality (I) alone. This can be seen as follows: If there is a sequence $p_n \to 0$ such that $\|\omega(p_n)\| = o(\|p_n\|^r)$, then we may apply a variant of the technique in the proof of Lemma 4.10 below to show that ω has a C^r- representative which is identically equal 0 along some non-constant curve starting at 0. Such a representative has singular points different from folds along this curve, and hence cannot satisfy (I). This will however contradict the conclusion of Lemma 4.1.

Lemma 4.10. *Let $r > 2$. Let $\omega = (f, g) \in J^r(2, 2)$ be as in the hypothesis of Proposition 2.1 and assume ω satisfies (I) of Proposition 2.1, then there is a neighbourhood U of 0 and a constant $C > 0$ such that for each i either*

$$\forall p \in H_i,\ \|\nabla f(p)\| \geq C \|p\|^{r-1}$$

or

$$\forall p \in H_i,\ \|\nabla g(p)\| \geq C \|p\|^{r-1}.$$

Proof. Assume the lemma is false. Then, by the technique employed in the proof of Lemma 4.8, there exist analytic curves $\beta(t)$ and $\gamma(t)$, $t \in [0, \delta)$ with $\beta(0) = \gamma(0) = (0, 0)$, $\beta(0, \delta), \gamma(0, \delta) \subset H_i$ for sufficiently small $\delta > 0$ such that

$$\|\nabla f(\beta(t))\| = o(\|\beta(t)\|^{r-1}) \tag{4.5}$$

and

$$\|\nabla g(\gamma(t))\| = o(\|\gamma(t)\|^{r-1}), \tag{4.6}$$

for $t > 0$. We claim that

$$f(\beta(t)) = o(\|\beta(t)\|^r). \tag{4.7}$$

To see this, assume $\|\beta(t)\| \sim t^s$ and let u be such that $|f(\beta(t))| \sim t^u$. Then $|\frac{d}{dt} f(\beta(t))| \sim t^{u-1}$, and also

$$|\frac{d}{dt} f(\beta(t))| = |\nabla f(\beta(t)) \cdot \beta'(t)| \leq \|\nabla f(\beta(t))\| \cdot \|\beta'(t)\| = o(t^{sr-1}).$$

It follows that $u - 1 > sr - 1$ and the claim follows from this. In the same manner we get

$$g(\gamma(t)) = o(\|\gamma(t)\|^r). \tag{4.8}$$

We consider the curve β, and follow an argument of Kuo's article [9]. By a suitable rotation of $\mathbb{R}^2$ we can make β tangent to the x-axis at 0. Assume this is the case. By a change of parameter if necessary, $\beta_1(t) = t^s$ and $|\beta_2(t)| = o(t^s)$. We make a C^1 change of coordinates: $X = x$, $Y = y - \beta_2(|x|^{\frac{1}{s}})$. In these coordinates, β is the positive X-axis.

Using the Taylor expansion of f about 0, we can write f as a polynomial in Y as follows:

$$f(x, y) = f(X, Y + \beta_2(|X|^{\frac{1}{s}})) = \tilde{f}_0(X) + \tilde{f}_1(X)Y + \tilde{f}_2(X)Y^2 + \cdots \tag{4.9}$$

Putting $Y = 0$, we get that $\tilde{f}_0(X) = f(X, \beta_2(|X|^{\frac{1}{s}}))$ and we see from (4.7) that the function $f_0(x) = \tilde{f}_0(|x|)$, is a C^r map with $j^r f_0(0) = 0$. Differentiating (4.9) with respect to Y and putting $Y = 0$, we see that $\tilde{f}_1(X) = \frac{\partial f}{\partial y}(X, \beta_2(|X|^{\frac{1}{s}}))$, and it follows from (4.5) that the function $f_1(x) = \tilde{f}_1(|x|)$ is a C^{r-1} function with $j^{r-1} f_1(0) = 0$.

Let $K = \{(x, y) \mid |y| \leq |x|, x \geq 0\} \cap \overline{B_r(0)}$ where $B_r(0)$ is some small open ball around 0. Define $\tilde{F}(x, y) = f_1(x)(y - \beta_2(|x|^{\frac{1}{s}}))$. $\tilde{F}$ is analytic at points (x, y) with $x \neq 0$. From (4.5) it follows that $\frac{d^m}{dx^m} f_1(x) = o(|x|^{r-1-m})$ for $m \geq 0$. Furthermore since $\beta_2(t) = o(t^s)$, we get that $\frac{\partial^m}{\partial x^m}(y - \beta_2(|x|^{\frac{1}{s}})) = o(|x|^{1-m})$ when $m > 0$. Also, $|y - \beta_2(|x|^{\frac{1}{s}})| \leq 2|x|$ when

$(x,y) \in K$. Altogether this implies that $\frac{\partial^{|m|}}{\partial x^{m_1} \partial y^{m_2}}(\tilde{F})(p) = o(|(x,y)|^{r-|m|})$ with $m = (m_1, m_2)$ when $p = (x,y) \in K - \{0\}$.

Now let Q be the r-th order Taylor field on K with values in $\mathbb{R}$ defined by $Q^m(0) = 0$ for all m and $Q^m(p) = \frac{\partial^{|m|}}{\partial x^{m_1} \partial y^{m_2}}(\tilde{F})(p)$ for all $m = (m_1, m_2)$ and all $p \in K \setminus \{0\}$. It follows from Lemma 4.11 below that Q is a C^r-Whitney field. Thus, by Whitney's Extension Theorem Q has a C^r-extension F defined on a neighbourhood of $0 \in \mathbb{R}^2$ such that $j^r F(0) = 0$ (see [10] for a statement and proof of Whitney's Extension Theorem).

Apply the same construction to g along γ to obtain g_0 and G as C^r-functions both with r-jet equal 0 at (0). Then define

$$\hat{\omega} = (\hat{f}, \hat{g}) = (f - f_0 - F, g - g_0 - G).$$

Then $\hat{\omega}$ is a C^r-realization of ω, and by construction, $\nabla \hat{f} = 0$ along $\beta(t)$ and $\nabla \hat{g} = 0$ along $\gamma(t)$. If the traces of β and γ are the same, then obviously $\hat{\omega}$ has singularities which are not folds along this curve, which contradicts Lemma 4.1. If the traces of β and γ are not intersecting in a neighbourhood of 0, then we have found a C^r-realization $\hat{\omega}$ of ω such that $\Sigma(\hat{\omega}) \setminus \{0\}$ has at least two connected components in H_i. This will however contradict Remark 3. □

Lemma 4.11. *Let $U \subset \mathbb{R}^n$ be an open set with $0 \in \overline{U}$. Let F be a C^r-function defined on U. Assume that $\frac{\partial^{|\alpha|} F}{\partial x^\alpha}(p) \to 0$ when $p \to 0$ for each multiindex α with $|\alpha| \leq r$. Let $K \subset \{0\} \cup U$ be a compact, convex set with $0 \in K$. Let Q be the r-th order Taylor field on K defined by $Q^\alpha(p) = \frac{\partial^{|\alpha|} F}{\partial x^\alpha}(p)$ if $p \neq 0$, and $Q^\alpha(p) = 0$ if $p = 0$, $|\alpha| \leq r$. Then Q is a C^r-Whitney field.*

Proof. Let $p, q \in K$. Let $m = (m_1, \dots, m_n)$ be a multiindex with $|m| \leq r$. Let

$$R_q Q^m(p) = Q^m(p) - \frac{\partial^{|m|}}{\partial x^m}\Big(\sum_{|\alpha| \leq r} \frac{1}{\alpha_1! \dots \alpha_n!} Q^\alpha(q)(x-q)^\alpha\Big) \Bigg|_{x=p}.$$

We must show that $R_q Q^m(p) = o(\|p-q\|^{r-|m|})$ for each such multiindex m. We will only show this when $m = \mathbf{0} = (0, \dots, 0)$ since the proof is similar when $|m| > 0$. Extend F to $\{0\} \cup U$ by putting $F(0) = 0$. Let $p, q \in K$ and define $g(t) = F(tp + (1-t)q)$. Then g can be extended to a C^r function on some open interval containing $[0,1]$ (if q or p is 0 extend g to the zero-function on $(-\epsilon, 0)$ or $(1, 1+\epsilon)$ respectively). Note that

$$g^{(k)}(t) = \sum_{|\alpha|=k} \frac{k!}{\alpha_1! \dots \alpha_n!} Q^\alpha(tp + (1-t)q)(p-q)^\alpha,$$

for $t \in [0,1]$. So, by applying an integral version of Taylor's formula with remainder, we get

$$\begin{aligned}
R_q Q^{\mathbf{0}}(p) = g(1) - \sum_{k=0}^{r} \frac{1}{k!} g^{(k)}(0) &= \frac{1}{(r-1)!} \int_0^1 g^{(r)}(t)(1-t)^{r-1} dt - \frac{1}{r!} g^{(r)}(0) \\
&= \frac{1}{(r-1)!} \int_0^1 (g^{(r)}(t) - g^{(r)}(0))(1-t)^{r-1} dt \\
&= \frac{1}{(r-1)!} \sum_{|\alpha|=r} \Big(\int_0^1 \frac{r!}{\alpha_1! \dots \alpha_n!} (Q^\alpha(tp + (1-t)q) - Q^\alpha(q))(1-t)^{r-1} dt\Big)(p-q)^\alpha.
\end{aligned}$$

Now, each Q^α is continuous on the compact, convex set K and therefore uniformly continuous, and from this it follows easily that $\int_0^1 (Q^\alpha(tp + (1-t)q) - Q^\alpha(q))(1-t)^{r-1} dt \to 0$ when $\|p-q\| \to 0$. Since $|(p-q)^\alpha| \leq \|p-q\|^r$ when $|\alpha| = r$, we thus get that $R_q Q^{\mathbf{0}}(p) = o(\|p-q\|^r)$. □

Lemma 4.12. *If $\omega = (f, g) \in J^r(2,2)$ satisfies the inequalities (I) and (II) in a neighbourhood U of 0, then there is a smaller neighbourhood U' of 0 such that $F|_{\Sigma(F)\cap(U'\times I)}$ is injective.*

Proof. It is enough to show that ω_t is injective when restricted to $\Sigma(\omega_t)$. Consider the component H_i for some i. By Lemma 4.10 we may assume that $\|\nabla f(p)\| \geq C\|p\|^{r-1}$ for all p in H_i. Then there is a smaller neighbourhood V of 0 such that $\|\nabla f_t(p)\| \geq \frac{C}{2}\|p\|^{r-1}$ for all t and $p \in H_i$. As before, $j^2\omega_t(p)$ is identified with the 10-tuple $(a, \ldots, j)$. $\Sigma(\omega_t)$ is given by the equation $ad - bc = 0$. Suppose $f_t|_{\Sigma(\omega_t)}$ has an extremum at $p \in H_i \cap \Sigma(\omega_t)$. By the method of Lagrange multipliers, at p,

$$a = \lambda \cdot \frac{\partial J\omega_t}{\partial x} = \lambda(ai - bh - cf + de)$$
$$b = \lambda \cdot \frac{\partial J\omega_t}{\partial y} = \lambda(aj - bi - cg + df).$$

We have $(a, b) = \|\nabla f_t(p)\| \geq \frac{C}{2}\|p\|^{r-1} \neq 0$ which implies that $\lambda \neq 0$ and hence,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} aj - bi - cg + df \\ -ai + bh + cf - de \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This means that $(L_{\omega_t}(p), H_{\omega_t}(p)) \in \Gamma$. The conclusion must be that every such p lies outside some open neighbourhood of the origin, since ω_t by assumption satisfies (I). Hence f_t and consequently ω_t is injective when restricted to the component of $\Sigma(\omega_t)\setminus\{0\}$ lying in H_i. Together with Corollary 4.7, this proves the lemma. □

Recall the definition of F_0 given above Lemma 4.3. Let $M = \Sigma(F_0)$ and $\Omega = F(M)$.

Lemma 4.13. *Let U be chosen so small that the conclusions of Lemma 4.1 and Lemma 4.12 hold. Then Ω is a two-dimensional C^{r-1} submanifold of the target.*

Proof. $F|_{\overline{M}}$ is an injective continuous map from a compact space to a Hausdorff space, so it must be a homeomorphism onto its image. So $F|_M$ is a topological embedding and by Lemma 4.1, $F|_M$ is a C^{r-1} immersion, hence a C^{r-1} embedding. Thus Ω is a C^{r-1} manifold. □

5. Construction of trivializing vector fields in source and target

Let $\omega \in J^r(2,2)$ be as in the hypothesis of Proposition 2.1, assume that $r > 2$ and that ω satisfies the inequalities (I) and (II) in a neighbourhood U of 0. Let F, M and Ω be as in Section 4, and assume that the neighborhood U in the definition of M and Ω also is chosen so small that the conclusion of Corollary 4.9 holds. Clearly, Corollary 4.9 implies that $F((\overline{U}-U)\times I)\cap\{0\}\times I = \varnothing$. It follows that we can find a neighborhood V of 0 in $\mathbb{R}^2$ such that $(\Omega\cup(\{0\}\times I))\cap(V\times I)$ is closed in $V\times I$. Let us change notation and denote $\Omega\cap(V\times I)$ by Ω. Let $(q,t) = (\omega_t(p), t) = F(p,t) \in \Omega$ for $(p,t) \in M$. Since $F|_M$ has rank 2 everywhere, $DF(p,t)\mathbf{v} \in T_{F(p,t)}\Omega$ for all $\mathbf{v} \in \mathbb{R}^3$. With this in mind, we can define a tangent vector field $\mathbf{u}$ on Ω by

$$\mathbf{u}(q,t) = \mathbf{u}(F(p,t)) = DF((p,t))(0,0,1) = (\tilde{\omega}(p), 1).$$

Since $\omega_t(p) = \omega(p) + t\tilde{\omega}(p)$ and $\|\tilde{\omega}(p)\| = o(\|p\|^r)$, it follows from Corollary 4.9 that

$$\|\mathbf{u}(F(p,t)) - (0,0,1)\| = o(\|\omega_t(p)\|) = o(\|q\|).$$

This equation is similar to the conclusion of Lemma 4.3. Put $\mathbf{u}|_{\{0\}\times I} = (0,0,1)$. Since Ω is a C^{r-1} manifold in the target, $\mathbf{u}$ can be extended to a neighborhood $V \times I$ of $\{0\} \times I$ in a way completely analogous to the way the vector field $\mathbf{v}$, defined in Subsection 4.2, was extended to all of source. We scale this extended vector field such that the component in the t-direction becomes 1 and denote this vector field by η. By this construction, η becomes C^{r-2} outside the t-axis, and we get the following lemma which is similar to (4.4).

Lemma 5.1. $\|\eta(q,t)-(0,0,1)\| = o(\|q\|)$.

This lemma implies that η like the vector field ξ constructed in Subsection 4.2 satisfies the hypothesis of Kuo's Theorem 3 in [8]. Therefore η has a continuous flow ψ in $V \times I$. Moreover, since the component of η in the t-direction equals 1, each flow line will live until it reaches either $(\overline{V}-V)\times I$ or $V\times\{-\epsilon, 1+\epsilon\}$. An easy estimate using 5.1 shows that if $V_1 \subset V$ is sufficiently small and $(q,t) \in V_1 \times I$ then $\psi_{(q,t)}$ will stay close to $\{0\}\times I$ and therefore reach $V\times\{-\epsilon,1+\epsilon\}$ and therefore cannot have any closure points in $(\overline{V}-V)\times I$. So when $(q,t)\in V_1\times I$ we can define the flow $\psi_{(q,t)}(s)$ for $s\in(-\epsilon-t, 1+\epsilon-t)$, especially each flow line through points in $V_1\times\{0\}$ can be defined on I, and we will get a map $k : V_1\times I \to \mathbb{R}^3$ defined by $k(q,t) = k_t(q) = \psi_{(q,0)}(t)$. Each k_t is a homeomorphism which maps the 0-level of Ω to the t-level of Ω. Let us choose such a neighborhood V_1 and let $U_1 \subset U$ be a neighborhood of 0 in $\mathbb{R}^2$ such that $F(U_1\times I)\subset V_1\times I$. Define a tangent vector field $\mathbf{w}$ on $M\cap(U_1\times I)$ by

$$DF((p,t))\mathbf{w}(p,t) = \mathbf{u}(F(p,t)).$$

This definition is unambiguous because we have required $\mathbf{w}$ to be tangential and $F|_{M\cap(U_1\times I)} : M\cap(U_1\times I)\to\Omega$ is an immersion. Put $\mathbf{w}|_{\{0\}\times I} = (0,0,1)$. Outside $M\cup\{0\}\times I$, DF is invertible so we can define an extension ζ of $\mathbf{w}$ to all of source by the equation

$$DF_{(p,t)}\zeta(p,t) = \eta(F(p,t)).$$

We are now going to show that ζ has a continuous flow. To this end, we will need the lemma below.

Lemma 5.2. *If $p\in M$, then there is a neighbourhood W of p such that for all $q\in W$, $F(q)\in\Omega \Rightarrow q\in M$.*

Proof. Let $p\in M$. Then p is a fold point and if $r\geq 4$, this will follow from the standard normal form of a fold. When $r>2$, there are (for example following the arguments in [15] Section 15), C^{r-1}-coordinates (x,y,t) around p, (u,v,t) around $F(p)$ in which $p=(0,0,0)=F(p)$ and such that in these coordinates F has the form $F(x,y,t) = (x, h(x,y,t), t)$ where $h(x,0,t) = \frac{\partial h}{\partial y}(x,0,t) = 0 \neq \frac{\partial^2 h}{\partial y^2}(0,0,0)$. In these coordinates, $\Sigma(F) = \{y=0\}$ and $F(\Sigma(F)) = \{v=0\}$. The lemma now follows by an easy argument using Taylor's formula. □

The existence and continuity of the flow of η is given in the following lemma.

Lemma 5.3. *Let $0<\delta<\epsilon$. Then there exists a neighborhood $U_2\subset U_1$ such that ζ has a continuous flow $\vartheta(p,t,s) = \vartheta_{(p,t)}(s)$ in the set $\{(p,t,s)\,|\,(p,t)\in U_2\times(-\delta,1+\delta),\ s\in(-\delta-t,1+\delta-t)\}$.*

Proof. Again, change notation and put $M := M\cap(U_1\times I)$. Consider $\{\{0\}\times I, M, U_1\times I\setminus\Sigma(F)\}$ as a stratification of $U_1\times I$. We can think of ζ as a stratified vector field whose restriction to each stratum is a C^{r-2}-vector field. These restrictions have each a C^{r-2}- flow defined on each stratum. For each $p=(x,y,t)\in U_1\times I$, let $\vartheta(p,s)=\vartheta_p(s)$ denote the flow through p of the restriction of ζ to the stratum of p. Let ϑ_p be defined on its maximal interval of existence. Now we will prove that this flow is continuous, by using the continuous flow in the target to control the flow in the source.

To this end, consider the vector field η in the target which also can be considered as a stratified vector field with respect to the stratification $\{\{0\}\times I, \Omega, (V\setminus\{0\})\times I\setminus\Omega\}$. Since η has a continuous flow on $V\times I$ and each flow line lives until it reaches the boundary of $V\times I$, each flow line stays in its respective stratum and no flow line can have closure points belonging to lower dimensional strata. From the equation $DF((p,t))\zeta(p,t) = \eta(F(p,t))$, we get that the flow of ζ is mapped to the flow ψ of η. Let $p\in(U_1-\{0\})\times I$. Then the flow line ϑ_p is mapped to the flow line $\psi_{F(p)}$ which is a flow line either in Ω or in $(V\setminus\{0\})\times I\setminus\Omega$, and therefore cannot have a closure point in $\{0\}\times I$. It follows that ϑ_p cannot have a closure point in $\{0\}\times I$ either. By the same sort of arguments it follows that if $F(p)\in(V\setminus\{0\})\times I\setminus\Omega$ then ϑ_p cannot have a closure point in M either. When $p\in U_1\times I\setminus\Sigma(F)$ and $F(p)\in\Omega$, $F(\vartheta_p)$ is a flow line of η in Ω. It then follows from Lemma 5.2 that ϑ_p cannot have a closure point in M either. So, for each $p\in U\times I$, each flow line ϑ_p does not have closure points in lower dimensional strata and since the component of ζ in the t-direction equals 1, each flow line ϑ_p can be continued until it meets the boundary of $U_1\times I$.

Let U' be a neighborhood of $0\in\mathbb{R}^2$ such that $\overline{U}'\subset U_1$, and let $0<\delta<\epsilon$. We will prove that there exists another neighborhood $\tilde U\subset\overline{\tilde U}\subset U'$ such that flow lines of ζ through points in $\overline{\tilde U}\times[-\delta,1+\delta]$ cannot have closure points in $(\overline{U'}-U')\times[-\delta,1+\delta]$.

Corollary 4.9 implies that there exists $\rho>0$ such that $B(0,\rho)\times[-\delta,1+\delta]\subset V_1\times I$ and $F((\overline{U'}-U')\times[-\delta,1+\delta])\subset\mathbb{R}^3\setminus(B(0,\rho)\times[-\delta,1+\delta])$, where $B(0,\rho)$ is the ball around $0\in\mathbb{R}^2$ with radius ρ. Since the flow ψ of η is continuous, we can find $\rho_1<\rho$ such that when $(q,t)\in B(0,\rho_1)\times[-\delta,1+\delta]$ the flow line $\psi_{(q,t)}(s)$ stays in $B(0,\rho)\times[-\delta,1+\delta]$ for $s\in[-\delta-t,1+\delta-t]$. By continuity of F, let $\tilde U\subset U'$ be such that $F(\overline{\tilde U}\times[-\delta,1+\delta])\subset B(0,\rho_1)\times[-\delta,1+\delta]$. Let $(p,t)\in\overline{\tilde U}\times[-\delta,1+\delta]$. Then the flow $\vartheta_{(p,t)}(s)$ is mapped to $\psi_{F(p,t)}(s)$ and since the latter flow stays in $B(0,\rho)\times[-\delta,1+\delta]$ for $s\in[-\delta-t,1+\delta-t]$ and $(\overline{U'}-U')\times[-\delta,1+\delta]$ is mapped to the complement of $B(0,\rho)\times[-\delta,1+\delta]$, the flow $\vartheta_{(p,t)}(s)$ can never meet $(\overline{U'}-U')\times[-\delta,1+\delta]$ but must stay in $U'\times[-\delta,1+\delta]$ when $s\in[-\delta-t,1+\delta-t]$. Putting $U_2=\tilde U$ the above argument shows that ζ has a flow $\vartheta(x,y,t,s)$ in

$$\hat U=\{(p,t,s)\,|\,(p,t)\in U_2\times[-\delta,1+\delta],\ s\in[-\delta-t,1+\delta-t]\}.$$

Since U' can be chosen arbitrarily small, the argument also shows that this flow is continuous in $\{0\}\times(-\delta,1+\delta)$.

Since ζ is a C^{r-2} vector field in the open set $U_1\times I-\Sigma(F)$ and we have seen that the flow stays in this set until it meets the closure of $U_1\times I$ the flow is continuous in this set. Especially the flow $\vartheta(p,t,s)$ is continuous when $(x,y,t)\in U_2\times(-\delta,1+\delta)-\Sigma(F)$ and $s\in(-\delta-t,1+\delta-t)$.

We will show that by replacing U_2 with a smaller neighbourhood U_3, we will get a continuous flow at all points. To this end, let $U_3\subset\overline{U_3}\subset U_2$ be a neighborhhood of 0 of $\mathbb{R}^2$ such that when $(p,t)\in\overline{U_3}\times[-\delta,1+\delta]$ then $\vartheta_{(p,t)}(s)\in U_2\times[-\delta,1+\delta]$ for $s\in[-\delta-t,1+\delta-t]$. (Such a neighbourhood exists since we have seen that the flow ϑ is continuous in $\{0\}\times(-\delta,1+\delta)$.) It remains to see that ϑ is continuous at points (p,t,s) when $(p,t)\in U_3\times(-\delta,1+\delta)\cap M$, and $s\in(-\delta-t,1+\delta-t)$. Assume this is not the case. Then there exist such (p,t,s) and a sequence $(p_n,t_n,s_n)\to(p,t,s)$ such that $\vartheta_{(p_n,t_n)}(s_n)\nrightarrow\vartheta_{(p,t)}(s)$. Since the restriction of ζ is C^{r-2} on M and the restriction of the flow therefore is continuous there, we must have $(p_n,t_n)\in U_3\times(-\delta,1+\delta)\setminus M$. Since the flow lines in $U_3\times[-\delta,1+\delta]$ stay in $U_2\times[-\delta,1+\delta]$, the sequence $\vartheta_{(p_n,t_n)}(s_n)$ is contained in the compact subset $\overline{U_2}\times[-\delta,1+\delta]$ and we may therefore assume that it converges to some point $(\tilde p,t+s)\in\overline{U_2}\times[-\delta,1+\delta]$. Since the flow in the source is mapped to the flow in the target and the flow in the target is continuous, we get that $F(\tilde p,t+s)=F(\vartheta_{(p,t)}(s))$. Since the flow line $\vartheta_{(p,t)}(s)$ is in M and $F|\Sigma(F)$ is 1-1, $(\tilde p,t+s)\in\overline{U_2}\times(-\delta,1+\delta)\setminus\Sigma(F)$. Since the flow $\vartheta_{(p',t)}(s)$ through points (p',t) in $\overline{U_2}\times[-\delta,1+\delta]$ stays in $U_1\times[-\delta,1+\delta]$ and can be defined for $s\in[-\delta-t,1+\delta-t]$, $\vartheta_{(\tilde p,t+s)}(-s)$ is defined. Since the flow ϑ is continuous on $U_1\times I\setminus\Sigma(F)$, $(p_n,t_n)=\vartheta_{\vartheta_{(p_n,t_n)}(s_n)}(-s_n)\to\vartheta_{(\tilde p,t+s)}(-s)$. This implies that $\vartheta_{(\tilde p,t+s)}(-s)=(p,t)$ which is impossible since flow lines in $U_1\times I\setminus\Sigma(F)$ never meet M. Putting $U_2:=U_3$ we thus get continuity of the flow ϑ in $\{(p,t,s)\,|\,(p,t)\in U_2\times(-\delta,1+\delta),\ s\in(-\delta-t,1+\delta-t)\}$. $\square$

When $r > 4$, we only need to check continuity of the flow ϑ at points in the t-axis, the remaining cases we treat above will follow automatically from the lemma below.

Lemma 5.4. *If $r > 4$, then $\zeta|_{U_0 \times I}$ is C^{r-4}.*

Proof. Let $p = (x_p, y_p, t_p) \in M$. Then p is a fold point of ω_{t_p}, and by Theorem 15A of [15], there are suitable centered coordinates H around p and K around $F(p)$ such that $(u, v, t) = K \circ F \circ H(x, y, z) = (x, y^2, t)$. If we look closely into the proof of this theorem we find that K can be chosen to be C^{r-1} and H to be C^{r-3}. We know that both $\zeta = (\zeta^1, \zeta^2, \zeta^3)$ and $\eta = (\eta^1, \eta^2, \eta^3)$ are tangential on M and Ω respectively, and hence, $\zeta^2(x, 0, t) = \eta^2(u, 0, t) = 0$. Thus, since K is C^{r-1} and η is C^{r-2}, we can, in the new coordinates, write $\eta^2(u, v, t) = v\eta'(u, v, t)$ for some C^{r-3} function η'. For $y \neq 0$, we get from the definition of ζ that

$$DF_{(x,y,t)}\zeta(x, y, t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \zeta^1(x, y, t) \\ \zeta^2(x, y, t) \\ \zeta^3(x, y, t) \end{pmatrix} = \begin{pmatrix} \eta^1(x, y^2, t) \\ y^2\eta'(x, y^2, t) \\ \eta^3(x, y^2, t) \end{pmatrix}.$$

From this relation we see that $\zeta^2(x, y, t) = \frac{1}{2}y\eta'(x, y^2, t)$. Because $\zeta^2(x, 0, t) = 0$, we see that the same equation must hold also for $y = 0$. Hence we can conclude that ζ is C^{r-3} in our new coordinates around p, and since DH is C^{r-4}, ζ is C^{r-4} in $U_0 \times I$ where U has been shrinked as to be contained in $F^{-1}(V)$. □

Proof of the sufficiency part of Theorem 2.2. Consider the neighborhood V_1 of $0 \in \mathbb{R}^2$ and the homeomorphisms k_t defined on V_1, in the beginning of this section. Let $0 < \delta < \epsilon$ and let U_2 be the neighborhood of 0 given in Lemma 5.3. Since the flow $\vartheta(p, t, s)$ of the vector field ζ can be defined and is continuous for $p \in U_2$, $t \in (-\delta, 1 + \delta)$ and $s \in (-\delta - t, 1 + \delta - t)$, we can define $h_t : U_2 \to \mathbb{R}^2$ by the equation $(h_t(p), t), = \vartheta(p, 0)(t)$. Since the flow is continuous it is clear that each h_t is a homeomorphism onto its image. Since the flow ϑ is mapped by $F = (\omega_t, t)$ to the flow ξ in the target, it follows from the definition of k_t that $\omega_t(h_t(p)) = k_t(\omega(p))$. For $t = 1$, this means precisely that ω and $\omega_1 = \omega + \tilde{\omega}$ are $\mathcal{A}_0$-equivalent. Since $\tilde{\omega}$ was arbitrarily chosen, this means that ω is $\mathcal{A}_0$-sufficient. □

6. Realizations of r-jets

Every r-jet has a quite well behaved realization in the sense to be made precise below. If an r-jet fails to satisfy (I) or (II), then it has another realization with different topological properties. We use this to prove the necessity part of Theorem 2.2.

6.1. **A nice realization of an r-jet.**

In this section we show that every r-jet has a realization which has no singular double points and only fold points and regular points outside the origin. Let $\omega : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be an r-jet.

Lemma 6.1. *There is some finite determined smooth germ f with $j^r f = \omega$.*

Proof. This is true because for smooth germs $(\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$, finite determinacy holds in general. See [6] for details. □

Since f is finitely determined, we can assume that f is a polynomial map. Also, the germ f is stable outside the origin. From the classification of stable germs we conclude that every singular point of $f|_{U_0}$ is either a fold or a cusp. Moreover the only singular double points occuring are double points of folds in general positions, which are isolated.

Lemma 6.2. *f has no singular double points and only fold points and regular points outside the origin.*

Proof. Let

$$C_1 = \{p \in \mathbb{R}^2 \,|\, p \text{ is a cusp point}\}$$

and let

$$C_2 = \{p \in \Sigma(f) \,|\, \exists\, q \in \Sigma(f),\, q \neq p \text{ with } f(p) = f(q)\}.$$

From the Tarski-Seidenberg Theorem it will follow that both C_1 and C_2 are semialgebraic sets. Since C_1 and C_2 also consists of isolated points, they cannot have 0 in their closure. □

6.2. **Bad realizations.**

Lemma 6.3. *If (I) fails for an r-jet $\omega \in J^r(2,2)$, then there is a a C^r-germ g with $j^r g(0) = \omega$ having a sequence of distinct cusp points converging to the origin.*

Proof. If (I) fails for ω, then there is a sequence (x_n) converging to 0 and a sequence $(L_n, H_n) \in \Gamma$ such that

$$\|L_n - L_\omega(x_n)\| + \|H_n - H_\omega(x_n)\|\, \|x_n\| = o(\|x_n\|^{r-1}).$$

Define a Taylor field Q on $S = \{0\} \cup (\bigcup_n \{x_n\})$ with values in $\mathbb{R}^2$ by $Q^m(0) = 0$ for all m and

$$Q^m(x_n) = \begin{cases} 0, & m = 0, \\ L_n^m - L_\omega^m(x_n), & |m| = 1, \\ H_n^m - H_\omega^m(x_n), & |m| = 2, \\ 0, & |m| \geq 3. \end{cases}$$

This notation requires some explanation. For instance, let $\omega = (f, g)$ and

$$(L_n, H_n) = (a_n, b_n, \ldots, j_n).$$

Then

$$Q^{(1,0)}(x_n) = \big(a_n - \frac{\partial f}{\partial x}(x_n), c_n - \frac{\partial g}{\partial x}(x_n)\big)$$
$$Q^{(0,1)}(x_n) = \big(b_n - \frac{\partial f}{\partial y}(x_n), d_n - \frac{\partial g}{\partial y}(x_n)\big)$$
$$Q^{(2,0)}(x_n) = \big(2e_n - \frac{\partial^2 f}{\partial x^2}(x_n), 2h_n - \frac{\partial^2 g}{\partial x^2}(x_n)\big)$$

etc. Assuming $\|x_{n+1}\| < \frac{1}{2}\|x_n\|$, it is straight forward to verify that Q is a C^r-Whitney field. Therefore we can find a C^r map extending Q. Let h be one such map. Then $j^r h(0) = 0$ and also

$$(L_{\omega+h}(x_n), H_{\omega+h}(x_n)) = (L_n, H_n). \tag{6.1}$$

By construction, $j^2(\omega + h)(x_n) \in \Gamma$.

Now it is not hard to see that the set of 2-jets in Γ which are transverse to Σ^1 is a dense subset. Recall that whether or not a point is a cusp point is determined by the 3-jet at that point. It is not hard to see that in the set of 3-jets with a given 2-jet in Γ transverse to Σ^1 the subset of 3-jets which are cusps is a dense subset. Therefore we can always suppose that $j^2(\omega + h)(x_n) \in \Gamma$ is transverse to Σ^1, and by perturbing the 3-jet if necessary, we can suppose the $j^3(\omega + h)(x_n)$ are cusps for all n.

(If $\omega + h$ has singularities appearing along (x_n) besides simple cusps, then one can define a new Whitney field providing a C^r perturbation h' (in fact h' can be taken to be smooth) with $j^r h'(0) = 0$ and $j^2 h'(x_n) = 0$ such that $\omega + h + h'$ has only cusps along (x_n). Then $g = \omega + h + h'$ is the desired realization of ω.) □

Lemma 6.4. *Assume $\omega \in J^r(2,2)$ satisfies condition (I), but assume condition (II) fails. Also assume that $\Sigma(\omega) - \{0\}$ has N connected components $C_1, \ldots, C_N$ with 0 in their closure all with distinct oriented tangent directions at 0. Then there is a C^r-germ g with $j^r g(0) = \omega$ having a sequence of singular double points converging to the origin.*

Proof. Assume that (II) fails for ω and let the sets H_i and H_ϵ be defined as before. Then we can find a sequence $\epsilon_n \to 0$ and sequences of points (x_n) and (y_n), both converging to $0 \in \mathbb{R}^2$, with each $x_n \in H_i \cup \{0\}$ and $y_n \in H_j \cup \{0\}$ $i \neq j$ where H_i and H_j are components of $H_{\epsilon_n} - \{0\}$such that (II) fails for $\{x_n, y_n\}$. Assume first that $x_n \neq 0$ and $y_n \neq 0$. Then

$$d(j^1\omega(w_n), \Sigma) = o(\|w_n\|^{r-1}), \tag{6.2}$$

for $w_n = x_n, y_n$ and

$$\|\omega(x_n) - \omega(y_n)\| = o\big((\|x_n\|^{r-1} + \|y_n\|^{r-1})\,\|x_n - y_n\|\,\big). \tag{6.3}$$

Then, since $d(j^1\omega(x_n), \Sigma) = o(\|x_n\|^{r-1})$, using an argument with The Whitney Extension Theorem (similar to the one given in the proof of Lemma 6.3), we can find a representative $\hat{\omega}$ such that $\hat{\omega}$ has singular points along the sequence $\{x_n\}$. By the results of Subsection 4.2, we can find a homeomorphism h mapping $\Sigma(\omega)$ to $\Sigma(\hat{\omega})$ and therefore points $p_n \in C_i$ such that $h(p_n) = x_n$, and by Lemma 4.5, we get that $\|p_n - x_n\| = o(\|p_n\|)$. By the same sort of argument, there is a point $q_n \in C_j$ with $\|q_n - y_n\| = o(\|q_n\|)$. We may also assume that $\|x_n\| \geq \|y_n\|$ and $\|x_{n+1}\| < \frac{1}{2}\|y_n\|$. Notice that because our assumption of the tangent directions of the components $C_1, \ldots C_N$ and the estimates above, there exists $\delta > 0$ such that for all n, $\|x_n - y_n\| > \delta\|x_n\|$.

Let $K = \{0\} \bigcup_n \{x_n, y_n\}$. For each $p \in K$, let $S(p)$ be the singular matrix closest to $D\omega(p)$ in $J^1(2,2)$ and let $M(p) = S(p) - D\omega(p)$. It follows from equation (6.2) that $\|M(w_n)\| = o(\|w_n\|^{r-1})$ for $w_n = x_n, y_n$. Define a r-th order Taylor field Q on K with values in $\mathbb{R}^2$ by

$$Q^m(p) = \begin{cases} 0, & p = 0 \\ 0, & p = y_n, m = 0 \\ \omega(y_n) - \omega(x_n), & p = x_n, m = 0 \\ M^m(p), & |m| = 1 \\ 0, & |m| \geq 2 \end{cases}.$$

Arguments similar to the arguments in [4] show that Q is a Whitney field on K.

Let h be a C^r extension of Q to $\mathbb{R}^2$ and let $g = \omega + h$. Since $j^r h(0) = 0$, g is a realization of ω. For $p \in K$, $Dg(p) = D\omega(p) + Dh(p) = S(p)$, so all points of K are singular points. Also $g(y_n) = \omega(y_n) + h(y_n) = \omega(y_n)$ and $g(x_n) = \omega(x_n) + h(x_n) = \omega(x_n) + \omega(y_n) - \omega(x_n) = \omega(y_n) = g(y_n)$, so (x_n) and (y_n) are sequences of singular double points converging to zero. If say $y_n = 0$ for all n we can use the same Whitney field and we obtain a representative of ω with singular zero-points all along the sequence $\{x_n\}$.

□

Lemma 6.5. *If f and g has only regular points and folds outside the origin and there are homeomorhpisms H and K such that $g = K \circ f \circ H$ then $\Sigma(g) = H^{-1}(\Sigma(f))$.*

Proof. This is clear since regular points and fold points are topologically distinct. □

Lemma 6.6. *If $r > 2$ and $f \in \mathcal{E}_{[r]}(2,2)$ is a cusp, then f is topologically different from regular germs and fold germs.*

Proof. Since f has fold singularities close to the origin, f is clearly topologically different from regular germs. To see that f is topologically different from fold germs, notice that the normal form of a fold implies that the image of a neighbourhood of a fold is not a neighbourhood of its target point. We prove that f maps every neighbourhood of 0 to a neighbourhood of 0. This is easily seen from the normal form of a cusp, but to be able to write f in this form, f has to have a considerable degree of differentiability (see [15]). Consider $j^3 f(0)$ as a polynomial map $P(x,y)$. Then $f(x,y) = P(x,y) + o(\|(x,y)\|^3)$. Since $P(x,y)$ is a cusp, we may change smooth coordinates and write $f(x,y) = (x, xy + y^3) + o(\|(x,y)\|^3)$. Example 7.2 in Section 7 below shows that $(x, xy+y^3)$ is $\mathcal{A}_0$-sufficient in $\mathcal{E}_{[3]}(2,2)$, and hence, f is topologically equivalent to $(x, xy+y^3)$. The conclusion follows. $\square$

Proof of the necessity part of Theorem 2.2. Assume that $\omega \in J^r(2,2)$ does not satisfy (I). Let (x_n) be the sequence in the proof of Lemma 6.3. Let f be a nice realization of ω in the sense of Section 6.1 , and let g be the bad realization of Lemma 6.3.

Suppose the germs at 0 of f and g are $\mathcal{A}_0$-equivalent germs. Then we can find germs at 0 of homeomorphisms H and K such that $g = K \circ f \circ H$. Let U be a neighbourhood of 0 in which g and $K \circ f \circ H$ coincide and choose U so small that f has only fold points and regular points in U_0. Choose N so large that $x_N \in U$. Then the germ of g at x_N and the germ of f at $H(x_N)$ are topologically equivalent. This will however contradict the conclusion of Lemma 6.6, since x_N is a cusp point of g and $H(x_N)$ is either a fold point of f or a regular point of f.

Next, assume that (I) holds and (II) fails for ω, and assume that the oriented tangent directions of the components $C_1, \dots C_N$ of $\Sigma(\omega) \setminus \{0\}$ are all distinct. Let f be as above, but let g be the realization of Lemma 6.4. Suppose there exist germs of homeomorphisms $H : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ and $K : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ such that $f \circ H = K \circ g$ and let U be a neighbourhood of 0 where representatives of the germs are equal. If necessary, choose a smaller U such that f has no singular double points in U. We can find n large enough to ensure that both x_n and y_n are contained in U. According to Lemma 6.5, H maps $\Sigma(g)$ into $\Sigma(f)$. We have that $K \circ g(x_n) = K \circ g(y_n)$ but $f \circ H(x_n) \neq f \circ H(y_n)$ because otherwise $H(x_n)$ and $H(y_n)$ would be singular double points. This contradiction finishes the proof. $\square$

7. Examples

Before we give examples of the use of Theorem 2.2, we prove a proposition which is helpful when trying to verify that a jet is sufficient. To understand where the inequality in the next proposition comes from, recall the expression from Section 3.2 measuring the distance from $(L,H) \in J^2(2,2)$ with L singular to the set $\{\, (J,K) \in J^2(2,2) \mid J = L, (J,K) \in \Gamma \,\}$.

Proposition 7.1. *Let $\omega \in J^r(2,2)$. Then the Lojasiewicz inequality (I) is implied by the following Lojasiewicz inequality:*

There is a neighbourhood U of 0 in $\mathbb{R}^2$ and a real number $C > 0$ such that for all $p \in U$,

$$\left\| D\omega(p) \begin{pmatrix} \frac{\partial}{\partial y} J\omega(p) \\ -\frac{\partial}{\partial x} J\omega(p) \end{pmatrix} \right\| \geq C \, \|p\|^{r-2} . \tag{III}$$

Proof. Assume that (III) holds for an r-jet ω and that there is a sequence $(L_n, H_n) \in \Gamma$ and a sequence (p_n) of points converging to zero in $\mathbb{R}^2$ such that (I) does not hold, that is

$$\|L_\omega(p_n) - L_n\| + \|H_\omega(p_n) - H_n\| \, \|p_n\| = o(\|p_n\|^{r-1}). \tag{7.1}$$

Let us introduce some notation. Let

$$(L_\omega(p_n), H_\omega(p_n)) = (a(p_n), b(p_n), \dots, j(p_n))$$

and let

$$(L_n, H_n) = (a_n, b_n, \dots, j_n).$$

Finally define $(a'_n, \dots, j'_n)$ by $a'_n = a(p_n) - a_n$, $b'_n = b(p_n) - p_n$, etc. It is easily seen from (7.1) that

$$\|(a'_n, \dots, j'_n)\| = o(\|p_n\|^{r-2}).$$

Now, because $(L_n, H_n) \in \Gamma$,

$$\begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \begin{pmatrix} a_n j_n - b_n i_n - c_n g_n + f_n d_n \\ -a_n i_n + b_n h_n + c_n f_n - d_n e_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

By writing $a(p_n) = a_n + a'_n$ etc., it is clear that

$$\begin{aligned} &\left\| \begin{pmatrix} a(p_n) & b(p_n) \\ c(p_n) & d(p_n) \end{pmatrix} \begin{pmatrix} a(p_n)j(p_n) - b(p_n)i(p_n) - c(p_n)g(p_n) + f(p_n)d(p_n) \\ -a(p_n)i(p_n) + b(p_n)h(p_n) + c(p_n)f(p_n) - d(p_n)e(p_n) \end{pmatrix} \right\| \\ &= \left\| \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \begin{pmatrix} a_n j_n - b_n i_n - c_n g_n + f_n d_n \\ -a_n i_n + b_n h_n + c_n f_n - d_n e_n \end{pmatrix} + \begin{pmatrix} A \\ B \end{pmatrix} \right\| = o(\|p_n\|^{r-2}) \end{aligned}$$

because each term of A and B contains at least one primed factor. But (III) implies that

$$\begin{aligned} &\left\| \begin{pmatrix} a(p_n) & b(p_n) \\ c(p_n) & d(p_n) \end{pmatrix} \begin{pmatrix} a(p_n)j(p_n) - b(p_n)i(p_n) - c(p_n)g(p_n) + f(p_n)d(p_n) \\ -a(p_n)i(p_n) + b(p_n)h(p_n) + c(p_n)f(p_n) - d(p_n)e(p_n) \end{pmatrix} \right\| \\ &\quad \geq C \|p_n\|^{r-2} \end{aligned}$$

so we arrive at a contradiction. Thus (I) must hold and the proof is finished. □

Example 7.2. Let $\omega(x, y) = (\ x\ ,\ xy + y^k\ )$ for some integer $k > 2$. For $k = 3$ this is the normal form of a cusp. We want to show that ω is $\mathcal{A}_0$-sufficient in $\mathcal{E}_{[k]}(2, 2)$. A computation gives the following.

$$D\omega(x, y) = \begin{pmatrix} 1 & 0 \\ y & x + ky^{k-1} \end{pmatrix}$$

and

$$J\omega(x, y) = x + ky^{k-1}.$$

It is clear that the singular set is a single curve tangent to the y-axis at the origin. So, $C_1 = \{x + ky^{k-1} = 0 \,|\, y > 0\}$ and $C_2 = \{x + ky^{k-1} = 0 \,|\, y < 0\}$ are the two components of $\Sigma(\omega) - \{0\}$. After some computation, we get that close to the origin we have

$$\left\| D\omega(x, y) \begin{pmatrix} \frac{\partial}{\partial y} J\omega(x, y) \\ -\frac{\partial}{\partial x} J\omega(x, y) \end{pmatrix} \right\| = \left\| \begin{pmatrix} k(k-1)y^{k-2} \\ -x + (k^2 - 2k)y^{k-1} \end{pmatrix} \right\| \geq \|(x, y)\|^{k-2}.$$

Hence, ω satisfies (III). By proposition 7.1, ω satisfies (I).

It is more cumbersome to verify (II). Notice that if (x, y) is close enough to the origin and $\epsilon > 0$ is sufficiently small, then by (3.2) of Subsection 3.1

$$\begin{aligned} H_\epsilon &= \{\ (x, y) \mid d(j^1\omega(x, y), \Sigma) \leq \epsilon \|(x, y)\|^{k-1}\ \} \\ &\subset \{\ (x, y) \mid |J\omega(x, y)| \leq \frac{(2 + \sqrt{2})\epsilon}{2} \|D\omega(x, y)\| \, \|(x, y)\|^{k-1}\ \} \\ &\subset \{\ (x, y) \mid |x + ky^{k-1}| \leq (2 + \sqrt{2})\epsilon |y|^{k-1}\ \} =: H^*_\epsilon. \end{aligned}$$

Let

$$H_\pm = H^*_\epsilon \cap \{\ (x, y) \mid y \gtrless 0\ \}$$

be the two components of $H_\epsilon^* \setminus \{0\}$. It is enough to verify (II) for pairs of points in $H_\pm \cup \{0\}$. It is clear from above that if (x_n, y_n) is a sequence in H_ϵ^* converging to 0, then $2k|y_n|^{k-1} > |x_n| > \frac{k}{2}|y_n|^{k-1}$ provided $\epsilon > 0$ is sufficiently small.

If k is an even number and $p = (x, y) \in H_+$ and $q = (x', y') \in H_-$, then x and x' have opposite signs and the first component of ω becomes dominating and

$$\begin{aligned}
\|\omega(p) - \omega(q)\| &= \big\|(x - x' , \, xy + y^k - x'y' - y'^k)\big\| \\
&\geq |x - x'| \\
&= |x| + |x'| \\
&\geq \frac{k}{2}(|y|^{k-1} + |y'|^{k-1}) \\
&\geq (\|p\|^{k-1} + \|q\|^{k-1}) \\
&\geq (\|p\|^{k-1} + \|q\|^{k-1}) \, \|p - q\|
\end{aligned}$$

as long as $\|p\|$, $\|q\|$ and ϵ are chosen small enough. The same estimate is valid if either $p = (0, 0)$ or $q = (0, 0)$.

If k is odd, then $p = (x, y) \in H_+$ and $q = (x', y') \in H_-$ may have nearly equal first components, but in this case ω separates these points in the second component if the first components are getting very close. We have

$$\begin{aligned}
\|\omega(p) - \omega(q)\| &= \big\|(x - x' , \, xy + y^k - x'y' - y'^k)\big\| \\
&\geq |xy - x'y'| - |y^k - y'^k| = |xy| + |x'y'| - |y|^k - |y'|^k \\
&\geq (\frac{k}{2} - 1)(|y|^k + |y'|^k) \\
&\geq C(\|p\|^k + \|q\|^k)
\end{aligned}$$

for some $C > 0$ as long as $\|p\|$, $\|q\|$ and ϵ are chosen small enough. If $\|p\| \geq \|q\| \geq 1/2 \, \|p\|$, then

$$\|p\|^k + \|q\|^k \geq (\|p\|^{k-1} + \|q\|^{k-1}) \, \|q\| \geq \frac{1}{4}(\|p\|^{k-1} + \|q\|^{k-1}) \, \|p - q\| \, ,$$

and if $\|q\| \leq 1/2 \, \|p\|$, then

$$\|p\|^k + \|q\|^k \geq \|p\|^k \geq \frac{1}{2} \, \|p\|^{k-1} \, \|p - q\| \geq \frac{1}{4}(\|p\|^{k-1} + \|q\|^{k-1}) \, \|p - q\| \, .$$

This shows that ω is $\mathcal{A}_0$-sufficient in $\mathcal{E}_{[k]}(2, 2)$ for every integer $k \geq 3$.

Example 7.3. Let $\omega(x, y) = (xy^2 - \frac{1}{3}x^3 \, , \, y^2 \,)$. We find

$$D\omega(x, y) = \begin{pmatrix} y^2 - x^2 & 2xy \\ 0 & 2y \end{pmatrix},$$

and

$$J\omega(x, y) = 2y(y^2 - x^2).$$

Since $\Sigma(\omega)$ consists of the lines $y = 0$, $y = x$ and $y = -x$, the 6 components of $\Sigma(\omega) - \{0\}$ has different tangent directions. But since $\omega(x, x) = \omega(x, -x)$, (II) of Theorem 2.2 does not hold for any r. So $\omega \in J^r(2, 2)$ is not sufficient in $\mathcal{E}_{[r]}(2, 2)$ for any r.

Example 7.4. Let $\omega(x, y) = (xy^2 - \frac{1}{3}x^3 \, , \, y^2 + y^3 \,)$. We find

$$D\omega(x, y) = \begin{pmatrix} y^2 - x^2 & 2xy \\ 0 & 2y + 3y^2 \end{pmatrix},$$

and

$$J\omega(x,y) = y(2+3y)(y^2-x^2).$$

Since the germ of $\Sigma(\omega)$ at 0 consists of the lines $y=0$, $y=x$ and $y=-x$, $\Sigma(\omega)-\{0\}$ has 6 components which have different tangent directions at the origin. Consider $p=p(t)=(t,t)$ and $q=q(t)=(t+t^2,-t-t^2)$. p and q are singular points from different components of $\Sigma(\omega)-\{0\}$. We find $\|\omega(p(t))-\omega(q(t))\| \sim |t|^4 = o(|t|^3) = o(\|p(t)-q(t)\|\,(\|p(t)\|^2+\|q(t)\|^2))$. This shows that (II) of Theorem 2.2 does not hold when $r=3$, so ω is not sufficient in $\mathcal{E}_{[3]}(2,2)$. However, regarding ω as a jet in $J^4(2,2)$, we will show that (I) of Proposition 2.1 and (II) of Theorem 2.2 will hold when $r=4$, so ω will be sufficient as a 4-jet among C^4-realizations.

Let $p_n=(x_n,y_n)$ be a sequence converging to $(0,0)$, and assume that $p_n \in H_\epsilon$ for any $\epsilon>0$ when n is large. Then it follows from (3.3) of Subsection 3.1 that $\frac{|J\omega(p_n)|}{\|D\omega(p_n)\|} = o(\|p_n\|^3)$. It is enough to consider the following two cases.

Case 1; $y_n = o(|x_n|)$. Then $\|p_n\| \sim |x_n|$, $|J\omega(p_n)| \sim |y_n| x_n^2$, so $\frac{|y_n|}{\|D\omega(p_n)\|} = o(|x_n|)$. Since $\|D\omega(p_n)\| \sim \max\{x_n^2, |y_n|\}$, we must have $\|D\omega(p_n)\| \sim x_n^2$ and therefore $y_n = o(\,|x_n|^3)$.

Case 2; There exists ϵ such that $|y_n| \geq \epsilon |x_n|$ for all n. Then we get that $\|D\omega(p_n)\| \sim |y_n|$ and therefore

$$\frac{|J\omega(p_n)|}{\|D\omega(p_n)\|} \sim \frac{|y_n|\,|y_n^2-x_n^2|}{|y_n|} = |y_n^2-x_n^2| = o(\|p_n\|^3).$$

This will imply that $|y_n| \sim |x_n|$, $\|p_n\| \sim |x_n|$ and $y_n = \pm\, x_n + o(|x_n|^2)$.

We will now prove that (I) of Proposition 2.1 will hold when $r=4$. Assume this is not the case. Then there exist a sequence (p_n) in $\mathbb{R}^2$, $p_n \to 0$, and a sequence $(L_n,H_n) \in \Gamma$ such that $\|L_\omega(p_n)-L_n\| + \|H_\omega(p_n)-H_n\|\,\|p_n\| = o(\|p_n\|^3)$. Since

$$\|L_\omega(p_n)-L_n\| \geq d(D\omega(p_n),\Sigma) \sim \frac{J\omega(p_n)}{\|D\omega(p_n)\|},$$

we must have $\frac{|J\omega(p_n)|}{\|D\omega(p_n)\|} = o(\|p_n\|^3)$, and from above it follows that we can assume that either $y_n = o(|x_n|^3)$ or $y_n = \pm\, x_n + o(|x_n|^2)$. Let $L_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$, and put $\tilde{L}_n = L_n - L_\omega(p_n) = L_n - D\omega(p_n)$. Then $\left\|\tilde{L}_n\right\| = o(\|p_n\|^3)$. Write

$$\begin{aligned} H_\omega(p_n) &= (e(p_n), f(p_n), g(p_n), h(p_n), i(p_n), j(p_n)) \\ &= (-x_n, y_n, x_n, 0, 0, 1+3y_n). \end{aligned}$$

Moreover, let $C_n = \frac{1}{2}\begin{pmatrix} \frac{\partial}{\partial y} J\omega(p_n) \\ -\frac{\partial}{\partial x} J\omega(p_n) \end{pmatrix}$ and let

$$\bar{C}_n = \begin{pmatrix} a_n\, j(p_n) - b_n\, i(p_n) - c_n\, g(p_n) + d_n\, f(p_n) \\ -a_n\, i(p_n) + b_n\, h(p_n) + c_n\, f(p_n) - d_n\, e(p_n) \end{pmatrix}.$$

Let $\tilde{C}_n = \bar{C}_n - C_n$. Let $z_n = (L_n, H_\omega(p_n)) \in J^2(2,2)$ and let $E_n = E_{z_n}$ be the linear subspace of $J^2(2,2)$ defined in Subsection 3.2. By the estimate (3.9) in Subsection (3.2) we get

$$\begin{aligned} o(\|p_n\|^2) &= \|H_\omega(p_n)-H_n\| = \|(L_n,H_\omega(p_n)) - (L_n,H_n)\| \\ &\geq d((L_n,H_\omega(p_n)),E_n) \sim \frac{\left\|L_n(\bar{C}_n)\right\|}{\|L_n\|^2}. \end{aligned} \tag{7.2}$$

We can write

$$L_n(\bar{C}_n) = D\omega(p_n)(C_n) + \tilde{L}_n(C_n) + D\omega(p_n)(\tilde{C}_n) + \tilde{L}_n(\tilde{C}_n).$$

Assume first that $y_n = o(|x_n|^3)$. Then $\|D\omega(p_n)(C_n)\| \sim x_n^4$ and $\left\|\tilde{L}_n(C_n)\right\| = o(|x_n|^5)$. Moreover $\tilde{C}_n = \begin{pmatrix} o(|x_n|^3) \\ o(x_n^4) \end{pmatrix}$ and this implies that $\left\|D_\omega(p_n)(\tilde{C}_n)\right\| = o(|x_n|^5)$ and $\left\|\tilde{L}_n(\tilde{C}_n)\right\| = o(x_n^6)$. Altogether this implies that $\left\|L_n(\bar{C}_n)\right\| \sim x_n^4$. Moreover $\|L_n\| \sim \|D_\omega(p_n)\| \sim x_n^2$, so we get that $\frac{\|L_n(\bar{C}_n)\|}{\|L_n\|^2} \sim |x_n|^0$ which contradicts (7.2).

Assume now that $y_n = \pm x_n + o(x_n^2)$. In this case $\|D\omega(p_n)(C_n)\| \sim |x_n|^3$, $\left\|\tilde{L}_n(C_n)\right\| = o(|x_n|^5)$, $\left\|D_\omega(p_n)(\tilde{C}_n)\right\| = o(|x_n|^5)$ and $\left\|\tilde{L}_n(\tilde{C}_n)\right\| = o(x_n^6)$. From this we get $\left\|L_n(\bar{C}_n)\right\| \sim |x_n|^3$. Furthermore $\|L_n\| \sim \|D_\omega(p_n)\| \sim |x_n|$ so we get that $\frac{\|L_n(\bar{C}_n)\|}{\|L_n\|^2} \sim |x_n|$. Since $\|p_n\| \sim |x_n|$ this again contradicts (7.2). Therefore we cannot find a sequence (L_n, H_n) contradicting inequality (I), and (I) must therefore hold when $r = 4$.

Now let us assume that inequality (II) does not hold . Then there must exist sequences $p_n = (x_n, y_n)$ and $q_n = (u_n, v_n)$ such that $p_n,\ q_n \in H_\epsilon$ for any $\epsilon > 0$ when n is large, $p_n \in H_i$ and $q_n \in H_j$ with $i \neq j$ and $\|\omega(p_n) - \omega(q_n)\| = o(\|p_n - q_n\|\,(\|p_n\|^3 + \|q_n\|^3)) = o(\|p_n\|^4 + \|q_n\|^4)$. (Note that it will follow from what we have shown above that $\|p_n - q_n\| \sim (\|p_n\| + \|q_n\|)$ when $p_n \in H_i$, $q_n \in H_j$ and $i \neq j$. This also follows from Lemma 4.5 and the proof of Lemma 6.4). Since we may assume that p_n and q_n satisfy Case 1 or Case 2 above, we have to consider several subcases. Assume first $y_n = \pm x_n + o(|x_n|^2)$ and $v_n = \pm u_n + o(|u_n|^2)$ and x_n and u_n have different signs. Then

$$|(x_n y_n^2 - \frac{1}{3}\,x_n^3) - (u_n v_n^2 - \frac{1}{3}\,u_n^3)| = \frac{2}{3}\,|x_n|^3 + \frac{2}{3}\,|u_n|^3 + o(|x_n|^4) + o(|u_n|^4) \sim \|p_n\|^3 + \|q_n\|^3\,.$$

So we cannot have such a pair of sequences violating (II). The case $y_n = \pm x_n + o(|x_n|^2)$ and $v_n = o(|u_n|^3)$ where x_n and u_n have the same sign, can be treated in a similar manner and we get the same conclusion. Consider the case $y_n = \pm x_n + o(|x_n|^2)$ and $v_n = o(|u_n|^3)$ where x_n and u_n have opposite signs. Then

$$|(x_n y_n^2 - \frac{1}{3}\,x_n^3) - (u_n v_n^2 - \frac{1}{3}\,u_n^3)| = |\frac{2}{3}\,x_n^3 + \frac{1}{3}\,u_n^3| + o(|x_n|^4) + o(|u_n|^7).$$

If $|u_n| > 2|x_n|$ the right hand side of the equation above is dominated by the term $|\frac{2}{3}\,x_n^3 + \frac{1}{3}\,u_n^3| \sim \frac{1}{3}\,|u_n|^3 \sim \|p_n\|^3 + \|q_n\|^3$. If $|u_n| \leq 2|x_n|$ then $v_n = o(|x_n|^3) = o(|y_n|^3)$. This implies that.

$$|(y_n^2 + y_n^3) - (v_n^2 + v_n^3)| \sim\ |y_n|^2 \sim \|p_n\|^2 + \|q_n\|^2\,.$$

Therefore we cannot find sequences contradicting (II) in this case either.

The next case is $|y_n| = o(|x_n|^3)$ and $|v_n| = o(|u_n|^3)$. Then it is clear that such p_n, q_n must belong to components H_i and H_j containing the positive and negative part of the x-axis respectively, and x_n and u_n must consequently have different signs. Then

$$|(x_n y_n^2 - \frac{1}{3}\,x_n^3) - (u_n v_n^2 - \frac{1}{3}\,u_n^3)| = \frac{1}{3}|x_n|^3 + \frac{1}{3}|u_n|^3 + o(|x_n|^7) + o(|u_n|^7) \sim \|p_n\|^3 + \|q_n\|^3\,.$$

Thus, (II) cannot fail along such sequences.

The only case left is when $y_n = \pm x_n + o(|x_n|^2)$ and $v_n = \pm u_n + o(|u_n|^2)$ and x_n and u_n have the same sign. Since p_n and q_n belong to different H_i's, y_n and v_n must have opposite signs. We may assume that $x_n,\ u_n,\ y_n > 0$ and $v_n < 0$. So $x_n = y_n + o(|y_n|^2)$ and $u_n = -v_n + o(|v_n|^2)$. Assume that

$$\|\omega(p_n) - \omega(q_n)\| = o(\|p_n\|^4 + \|q_n\|^4) = o(|y_n|^4 + |v_n|^4).$$

Let $\tilde{p}_n = (y_n, y_n)$ and $\tilde{q}_n = (-v_n, v_n)$. Then

$$\|\omega(p_n) - \omega(\tilde{p}_n)\| = o(|y_n|^4) \text{ and } \ \|\omega(q_n) - \omega(\tilde{q}_n)\| = o(|v_n|^4).$$

This implies that,

$$\|\omega(\tilde{p}_n) - \omega(\tilde{q}_n)\| \sim |\frac{2}{3}y_n^3 + \frac{2}{3}v_n^3| + |y_n^2 + y_n^3 - v_n^2 - v_n^3| = o(|y_n|^4 + |v_n|^4).$$

So we get that

$$|y_n^2 + y_n^3 - v_n^2 - v_n^3| = |y_n - v_n|\, |y_n + v_n + y_n^2 + y_n\, v_n + v_n^2| = o(|y_n|^4 + |v_n|^4).$$

Then since y_n and v_n have opposite signs, $|y_n - v_n| \sim |y_n| + |v_n|$, and we get

$$|y_n + v_n + y_n^2 + y_n\, v_n + v_n^2| = o(|y_n|^3 + |v_n|^3).$$

But since $|y_n^2 + y_n\, v_n + v_n^2| \sim |y_n|^2 + |v_n|^2$ we must then have that $|y_n + v_n| \sim |y_n|^2 + |v_n|^2$. This will imply that

$$|\frac{2}{3}y_n^3 + \frac{2}{3}v_n^3| = \frac{2}{3}\,|y_n + v_n|\, |y_n^2 - y_n v_n + v_n^2| \sim (|y_n|^4 + |v_n|^4)$$

which gives a contradiction. This proves that in any case, we cannot find a pair of sequences (p_n) and (q_n) violating (II). So (II) must hold when $r = 4$. We conclude that ω satisfies the hypothesis of Theorem 2.2 and hence is sufficient for $r = 4$.

8. Topological trivialization of 1-parameter families of germs

So far we have studied the perturbation of an r-jet z by an arbitrary C^r mapping h with $j^r h(0) = 0$. In particular, we have studied the 1-parameter family of C^r map-germs $z + th$. In this section we deal with a somewhat different problem. We are going to consider C^r 1-parameter families $\alpha_t = (f_t, g_t)$ of C^r map-germs $\alpha_t : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$. By this we mean that there exists a C^r map $F : U \times I \to \mathbb{R}^3$ given by $F(p,t) = (\beta_t(p), t)$ such that each β_t is a representative of the germ α_t. (We call such F a representative of the family.) The techniques we have developed in the earlier sections can be used to give some sufficient conditions to decide that such a 1-parameter family of germs can be topological trivialized, i.e. there are 1-parameter families of homeomorphisms H_t and K_t such that $\alpha_t \circ H_t = K_t \circ \alpha_0$.

Proposition 8.1. *Let $r > 2$ and let $\alpha_t = (f_t, g_t) : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a C^r 1-parameter family of C^r germs from the $\mathbb{R}^2$ to $\mathbb{R}^2$. The following conditions are sufficient for α_t to be topologically trivializable:*

There exists a representative $F : U \times I \to \mathbb{R}^3$, $F(p,t) = (\beta_t(p), t)$ having the following properties.

(1) Each $\beta_t|_{U_0}$ has only fold singularities (recall that $U_0 = U - \{\,0\,\}$.)
(2) $F|_{\Sigma(F)}$ is 1-1.
(3) $\|\beta_t(p)\| > 0$ for $(p,t) \in U_0 \times I$.
(4) $\left\|\frac{\partial F}{\partial t}(p,t) - (0,0,1)\right\| = o(\|\beta_t(p)\|)$ as $p \to 0$.

Proof. The proof of this proposition is very similar to the proof of the sufficiency part of Theorem 2.2. We will therefore only sketch this proof refering to the relevant details of that proof. Property 1 above ensures that $M = \Sigma(F_0)$ is a C^{r-1} submanifold of $\mathbb{R}^3$ and that $F|_{\Sigma(F_0)}$ is an immersion. Together with property 2 this also makes $N = F(M)$ a C^{r-1} submanifold, completely analogous to Lemma 4.13. Now we can define a vector field $\mathbf{w}$ on N by

$$\mathbf{w}(F(p,t)) = DF_{(p,t)} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{\partial \alpha_t}{\partial t} \\ 1 \end{pmatrix}$$

which will be tangent to N because F has rank 2 at every point of M. Also define $\mathbf{w}(0,0,t) = (0,0,1)$. This gives a vector field on all of $F(\Sigma(F))$. Property 4 guarantees us that $\mathbf{w}$ satisfies Kuo's condition. Indeed, the situation is exactly the same as for the vector field $\mathbf{u}$ on Ω in

Section 5. Recall the technique we used to extend $\mathbf{u}$ to all of the target. We can use the same technique to extend $\mathbf{w}$ to a vector field μ defined on some open neighbourhood $V \times I$ of $\{0\} \times I$ in the target, and as in Lemma 5.1 we get

$$\|\mu(q,t) - \mathbf{k}\| = o(\|q\|).$$

Thus we can integrate μ and get a continuous flow $\theta(q,t)$ defined on $V \times I$. The vector field μ is C^{r-2} outside the t-axis, just like the vector fields η and ζ of earlier sections.

The next step is to define a corresponding vector field ν on the source. This is defined to be the unique vector field whose restriction to M is a tangent vector field and which is mapped onto μ under DF. We can now use the same arguments as in the proof of Lemma 5.3 to see that ν has a continuous flow. Let U' be a neighborhood of 0 in the source such that $U' \subset \overline{U}' \subset U$. Then property 4 give us that if J is a compact interval with $J \subset I$, then $F(\overline{U}' - U') \times J$ is bounded away from $\{0\} \times I$. Using this we can use the continuous flow $\theta(q,t)$ in the target to control the flow in the source, and we can argue exactly as in Lemma 5.3 to obtain the existence and continuity of the flow of ν. The flows of μ and ν induce the required homeomorphisms, and the proof is finished. □

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Journal *of* Singularities
Volume 2 (2010), 51-55

Proc. of Singularities in Aarhus
received 25 August 2009
DOI: 10.5427/jsing.2010.2c

CLASSICAL ZARISKI PAIRS

ALEX DEGTYAREV

Abstract. We compute the fundamental groups of all irreducible plane sextics constituting classical Zariski pairs.

1. Introduction

A *classical Zariski pair* is a pair of irreducible plane sextics that share the same combinatorial type of singularities but differ by the Alexander polynomial [10]. The first example of such a pair was constructed by O. Zariski [13]. Then, it was shown in [4] that the curves constituting a classical Zariski pair have simple singularities only and, within each pair, the Alexander polynomial of one of the curves is $t^2 - t + 1$, whereas the polynomial of the other curve is trivial. The former curve is called *abundant*, and the latter *non-abundant*. The abundant curve is necessarily of *torus type*, *i.e.*, its equation can be represented in the form $f_2^3 + f_3^2 = 0$, where f_2 and f_3 are homogeneous polynomials of degree 2 and 3, respectively.

A complete classification of classical Zariski pairs up to equisingular deformation was recently obtained by A. Özgüner [1]. Altogether, there are 51 pairs, one of them being in fact a triple (assuming that the complex orientations of both $\mathbb{P}^2$ and of complex curves are taken into account): the non-abundant curves with the set of singularities $\mathbf{E}_6 \oplus \mathbf{A}_{11} \oplus \mathbf{A}_1$ form two distinct complex conjugate deformation families. The purpose of this note is to compute the fundamental groups of (the complements of) the curves constituting classical Zariski pairs. We prove the following theorem.

1.0.1. **Theorem.** *Within each classical Zariski pair, the fundamental group of the abundant* (*respectively, non-abundant*) *curve is* $\mathbb{B}_3/(\sigma_1\sigma_2)^3$ (*respectively,* $\mathbb{Z}_6$).

This theorem is proved in Section 4, using the list of [1] and a case by case analysis. In fact, most groups are already known, see [2], [5], [8], [3], and [9], and the few missing curves can be obtained by perturbing the set of singularities $\mathbf{A}_{17} \oplus 2\mathbf{A}_1$. The construction and the computation of the fundamental group are found in Sections 2 (the non-abundant curves) and 3 (the abundant curves).

2. The curve not of torus type

2.1. Up to projective transformation, there is a unique curve $C \subset \mathbb{P}^2$ with the set of singularities $\mathbf{A}_{17} \oplus 2\mathbf{A}_1$ and not of torus type, see [11]; its transcendental lattice is $\begin{bmatrix} 4 & 2 \\ 2 & 10 \end{bmatrix}$. (In the case under consideration, the transcendental lattice can be defined as the orthogonal complement $NS(\tilde{Y})^\perp \subset H_2(\tilde{Y})$, where $\tilde{Y}$ is the minimal resolution of singularities of the double plane ramified at C. Recall that $\tilde{Y}$ is a $K3$-surface.) After nine blow-ups, the curve transforms to the union of two of the three type $\tilde{\mathbf{A}}_0^*$ fibers in a Jacobian rational elliptic surface with the combinatorial

1991 *Mathematics Subject Classification.* Primary: 14H45; Secondary: 14H30, 14H50.
Key words and phrases. Plane sextic, Zariski pair, torus type, fundamental group, elliptic surface.

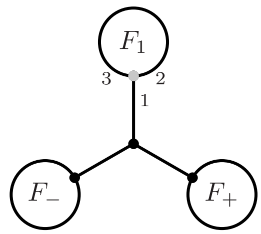

FIGURE 1. The skeleton Sk of $\bar{B}$

type of singular fibers $\tilde{\mathbf{A}}_8 \oplus 3\tilde{\mathbf{A}}_0^*$ (in Kodaira's notation, one fiber of type I_9 and three fibers of type I_1). For the equation, consider the pencil of cubics given by

$$f_b(x, y) := b(-x^2 - xy^2 + y) + (x^3 - xy + y^3) = 0, \qquad b \in \mathbb{P}^1,$$

and take two fibers corresponding to two distinct roots of $b^3 = 1/27$. (All three roots give rise to nodal cubics, which are the three type $\tilde{\mathbf{A}}_0^*$ fibers in the elliptic pencil above. The curve corresponding to $b = \epsilon/3$, $\epsilon^3 = 1$, has a node at $x = (2/5)\epsilon^{-1}$, $y = (1/5)\epsilon$. The type $\tilde{\mathbf{A}}_8$ fiber blows down to the nodal cubic $\{f_0 = 0\}$.)

2.1.1. **Lemma.** *For the curve C as in 2.1, one has*

$$\pi_1(\mathbb{P}^2 \smallsetminus C) = \langle p, \gamma_+ \,|\, p^9 = 1,\ \gamma_+^{-1} p \gamma_+ = p^4 \rangle.$$

Proof. Consider the trigonal curve $\bar{B} \subset \Sigma_2$ with a type $\mathbf{A}_8$ singular point. Its skeleton Sk, see [7], is shown in Figure 1.

Let F_1, $F_\pm$ be the type $\tilde{\mathbf{A}}_0^*$ singular fibers of $\bar{B}$ (vertical tangents), and let F_∞ be the type $\tilde{\mathbf{A}}_8$ fiber. (Recall that F_1, $F_\pm$ are located inside the small loops in Figure 1, whereas F_∞ is inside the outer region.) Consider the minimal resolution of the double covering $\tilde{X} \to \Sigma_2$ ramified at $\bar{B}$ and the exceptional section $E \subset \Sigma_2$, and denote by tildes the pull-backs of the fibers in $\tilde{X}$.

Consider the nonsingular fiber F over the $\bullet$-vertex v of Sk next to F_1 (shown in grey in Figure 1), denote $\pi_F := \pi_1(F \smallsetminus (\bar{B} \cup E))$, and pick a canonical basis $\{\alpha_1, \alpha_2, \alpha_3\}$ for π_F defined by the marking of Sk at v shown in Figure 1, see [7]. Then the fundamental group $\tilde{\pi}_F := \pi_1(\tilde{F} \smallsetminus E)$ of the punctured torus $\tilde{F} \smallsetminus E$ is obtained from π_F by adding the relations $\alpha_1^2 = \alpha_2^2 = \alpha_3^2 = 1$ and passing to the kernel of the homomorphism $\pi_F \to \mathbb{Z}_2$, $\alpha_1, \alpha_2, \alpha_3 \mapsto 1$. Hence, $\tilde{\pi}_F$ is the free group generated by

$$p := \alpha_1\alpha_2 = (\alpha_2\alpha_1)^{-1} \quad \text{and} \quad q := (\alpha_3\alpha_2) = (\alpha_2\alpha_3)^{-1}.$$

Start with the group

$$G_1 = \pi_1(\tilde{X} \smallsetminus (E \cup \tilde{F}_+ \cup \tilde{F}_- \cup \tilde{F}_\infty))$$

and compute it applying Zariski–van Kampen's approach [12] to the elliptic pencil on $\tilde{X}$. Let γ_1, $\gamma_\pm$ be the generators of the free group

$$\pi_1(\mathbb{P}^1 \smallsetminus (F_1 \cup F_+ \cup F_- \cup F_\infty), F)$$

represented by the shortest loops in Sk starting at v and circumventing the corresponding fibers in the counterclockwise direction. (We identify fibers of the ruling and their projections to the base.) Fix a closed disk Δ in the base and consider a *proper section* over Δ, *i.e.*, a topological section of the ruling disjoint from the fiberwise convex hull of $\bar{B}$, see [7]. Using this proper section, one can lift these generators to $\Sigma_2 \smallsetminus (\bar{B} \cup E)$ and to $\tilde{X} \smallsetminus E$. Using the same proper section, define the braid monodromies $m_1, m_\pm \in \operatorname{Aut}\pi_F$ and their lifts $\tilde m_1, \tilde m_\pm \in \operatorname{Aut}\tilde\pi_F$. In this notation, the group G_1 has the following presentation, *cf.* [12]:

$$G_1 = \bigl\langle p, q, \gamma_+, \gamma_- \bigm| p = \tilde m_1(p),\ q = \tilde m_1(q),\ \gamma_\pm^{-1} p \gamma_\pm = \tilde m_\pm(p),\ \gamma_\pm^{-1} q \gamma_\pm = \tilde m_\pm(q) \bigr\rangle.$$

The braid monodromy is computed as explained in [7]; for $\bar B$ it is

$$m_1 = \sigma_2, \quad m_+ = \sigma_1^{-3}\sigma_2\sigma_1^3, \quad m_- = \sigma_1^{-1}\sigma_2^2\sigma_1\sigma_2^{-2}\sigma_1,$$

where σ_1, σ_2 are the Artin generators of $\mathbb{B}_3$ (we assume that the braid group $\mathbb{B}_3$ acts on π_F from the left), and in terms of p and q it takes the form

$$\begin{gathered}\tilde m_1 \colon p \mapsto pq, \quad q \mapsto q;\\ \tilde m_+ \colon p \mapsto pqp^3, \quad q \mapsto p^{-4}q^{-1}p^{-4}q^{-1}p^{-1};\\ \tilde m_- \colon p \mapsto (pq)^2(p^2q)^2p, \quad q \mapsto p^{-1}q^{-1}(p^{-2}q^{-1})^3p^{-1}q^{-1}p^{-1}.\end{gathered}$$

The very first relation $p = pq$ implies $q = 1$. Hence also $\tilde m_\pm(q) = 1$ and $p^9 = 1$. Thus, one has

$$G_1 = \bigl\langle p, \gamma_+, \gamma_- \bigm| p^9 = 1,\ \gamma_+^{-1}p\gamma_+ = p^4,\ \gamma_-^{-1}p\gamma_- = p^7 \bigr\rangle. \qquad \textbf{(2.1.2)}$$

In order to pass to the group $\pi_1(\mathbb{P}^2 \smallsetminus B)$, we need to patch back in one of the nine irreducible components of the type $\tilde{\mathbf{A}}_8$ fiber F_∞. (The component to be patched in is the proper transform of the nodal curve $\{f_0(x, y) = 0\}$.) This operation adds to (**2.1.2**) an additional relation $[\partial\tilde\Gamma] = 1$, where $\tilde\Gamma$ is a small holomorphic disk in $\tilde X$ transversal to the component in question. Using a proper section again, one can see that in G_1 there is a relation $[\partial\tilde\Gamma]^{-1}p^{?} = \gamma_-\gamma_+$, where $p^{?}$ is merely an element of the group $\tilde\pi_F$ of the fiber (modulo the relations in G_1), which we do not bother to compute. Adding the extra relation $[\partial\tilde\Gamma] = 1$ to (**2.1.2**) and eliminating γ_-, one arrives at the presentation announced in the statement. (Note that $7 = 4^{-1} \bmod 9$, hence the order of p remains 9.) ☐

2.1.3. **Corollary.** *The commutant of the group $\pi_1(\mathbb{P}^2 \smallsetminus C)$ as in Lemma 2.1.1 is a central subgroup of order* 3.

Proof. The commutant is normally generated by the commutator $p^{-1}\gamma_+^{-1}p\gamma_+ = p^3$; it is a central element of order 3. ☐

2.1.4. **Corollary.** *For any irreducible perturbation C' of the curve C as in 2.1, one has $\pi_1(\mathbb{P}^2 \smallsetminus C') = \mathbb{Z}_6$.*

Proof. Let $G = \pi_1(\mathbb{P}^2 \smallsetminus C')$. Due to Corollary 2.1.3, the commutant $[G, G]$ is a quotient of $\mathbb{Z}_3$, hence either $\mathbb{Z}_3$ or $\{1\}$. Furthermore, $[G, G] \subset G$ is a central subgroup. On the other hand, since C is irreducible, $G/[G, G] = \mathbb{Z}_6$, and any central extension

$$\{1\} \to \mathbb{Z}_3 \to G \to \mathbb{Z}_6 \to \{1\}$$

of the cyclic group $\mathbb{Z}_6$ would be abelian. ☐

3. The curve of torus type

3.1. Up to projective transformation, there is a unique torus type curve $C \subset \mathbb{P}^2$ with the set of singularities $\mathbf{A}_{17} \oplus 2\mathbf{A}_1$, see [11]; its transcendental lattice is $\left[\begin{smallmatrix}2&0\\0&2\end{smallmatrix}\right]$. Similar to 2.1, this curve blows up to the union of the two type $\tilde{\mathbf{A}}_0^*$ fibers in a Jacobian rational elliptic surface with the combinatorial type of singular fibers $\tilde{\mathbf{E}}_8 \oplus 2\tilde{\mathbf{A}}_0^*$ (in Kodaira's notation, one fiber of type II^* and two fibers of type I_1). The curve can be given by the equation

$$f(x, y) := (y^3 + y^2 + x^2)\Bigl(y^3 + y^2 + x^2 - \frac{4}{27}\Bigr) = 0,$$

and its torus structure is

$$f(x,y) = \Big(y^3 + y^2 + x^2 - \frac{2}{27}\Big)^2 + \Big(\frac{\sqrt[3]{4}}{9}\Big)^3.$$

3.1.1. **Lemma.** *Let C be a curve as in 3.1, and let U be a Milnor ball about the type $\mathbf{A}_{17}$ singular point of C. Then the homomorphism $\pi_1(U \smallsetminus C) \to \pi_1(\mathbb{P}^2 \smallsetminus C)$ induced by the inclusion $U \hookrightarrow \mathbb{P}^2$ is surjective.*

Proof. In the coordinates $\tilde{y} = y/x$, $\tilde{z} = 1/x$, the curve is given by the equation

$$(\tilde{y}^3 + \tilde{y}^2\tilde{z} + \tilde{z})\Big(\tilde{y}^3 + \tilde{y}^2\tilde{z} + \tilde{z} - \frac{4}{27}\tilde{z}^3\Big) = 0,$$

the type $\mathbf{A}_{17}$ singular point is at the origin, and each component is inflection tangent to the line $\{\tilde{z} = 0\}$ at this point. To compute the group, apply Zariski–van Kampen theorem [12] to the vertical pencil $\{\tilde{z} = \text{const}\}$, choosing for the reference a generic fiber $F = \{\tilde{z} = \epsilon\}$ close to the origin. On the one hand, one has an epimorphism $\pi_1(F \smallsetminus C) \twoheadrightarrow \pi_1(\mathbb{P}^2 \smallsetminus C)$. On the other hand, the intersection $C \cap \{\tilde{z} = 0\}$ consists of a single 6-fold point; hence, if ϵ is small enough, all six points of the intersection $C \cap F$ belong to U and the generators of $\pi_1(F \smallsetminus C)$ can be chosen inside U. □

3.1.2. **Corollary.** *Let C' be a perturbation of the curve C as in 3.1 with the set of singularities $\mathbf{A}_{14} \oplus \mathbf{A}_2 \oplus 2\mathbf{A}_1$. Then $\pi_1(\mathbb{P}^2 \smallsetminus C') = \mathbb{B}_3/(\sigma_1\sigma_2)^3$.*

Proof. Let U be as in Lemma 3.1.1. Then $\pi_1(U \smallsetminus C') = \mathbb{B}_3$ and, due to the lemma, there is an epimorphism $\mathbb{B}_3 \twoheadrightarrow \pi_1(\mathbb{P}^2 \smallsetminus C')$. Since C' is necessarily irreducible and of torus type (so that the abelianization of $\pi_1(\mathbb{P}^2 \smallsetminus C')$ is $\mathbb{Z}_6$ and $\pi_1(\mathbb{P}^2 \smallsetminus C')$ factors to $\mathbb{B}_3/(\sigma_1\sigma_2)^3$), the latter epimorphism factors through an isomorphism $\mathbb{B}_3/(\sigma_1\sigma_2)^3 \cong \pi_1(\mathbb{P}^2 \smallsetminus C')$. □

3.1.3. **Remark.** The other irreducible perturbations of C that are of torus type are considered elsewhere, see [5]. Their groups are also $\mathbb{B}_3/(\sigma_1\sigma_2)^3$.

4. Proof of Theorem 1.0.1

4.1. The groups of all but one sextics of torus type occurring in classical Zariski pairs are known, see [5] for a 'map' and further references; all groups are $\mathbb{B}_3/(\sigma_1\sigma_2)^3$. The only missing curve has the set of singularities $\mathbf{A}_{14} \oplus \mathbf{A}_2 \oplus 2\mathbf{A}_1$. Such a curve can be obtained by a perturbation from a reducible sextic of torus type with the set of singularities $\mathbf{A}_{17} \oplus 2\mathbf{A}_1$ (see Proposition 5.1.1 in [6]), and its group is given by Corollary 3.1.2.

4.2. The fundamental groups of most non-abundant sextics appearing in classical Zariski pairs are computed in [5], [8], [3], with a considerable contribution from [9]. According to [3], unknown are the groups of the curves with the sets of singularities

$$\mathbf{A}_{17} \oplus \mathbf{A}_1, \quad \mathbf{A}_{14} \oplus \mathbf{A}_2 \oplus 2\mathbf{A}_1, \quad 2\mathbf{A}_8 \oplus 2\mathbf{A}_1, \quad 2\mathbf{A}_8 \oplus \mathbf{A}_1.$$

The first curve can be obtained by a perturbation from a sextic with a single type $\mathbf{A}_{19}$ singular point. According to [2], its group is abelian. The three other curves are perturbations of the curve C constructed in 2.1, and their groups are abelian due to Corollary 2.1.4. (Note that the perturbations exist due to Proposition 5.1.1 in [6], and the resulting curves are unique up to equisingular deformation due to [1].) □

4.2.1. **Remark.** A curve C as in 2.1 can also be perturbed to a sextic with the set of singularities $\mathbf{A}_{17} \oplus \mathbf{A}_1$, but the result is reducible.

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Journal *of* Singularities
Volume 2 (2010), 56-66

Proc. of Singularities in Aarhus
received 13 December 2009
DOI: 10.5427/jsing.2010.2d

SHEAVES ON SINGULAR VARIETIES

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Abstract. We prove existence of reflexive sheaves on singular surfaces and threefolds with prescribed numerical invariants and study their moduli.

1. Motivation

Sheaves on singular varieties have become very popular recently because of their appearance in Physics, String Theory and Mirror Symmetry. In particular, many open questions about sheaves on singular varieties have come to light. The corresponding mathematical tools, however, are waiting to be developed. Our aim in this paper is to entice singularists to develop some basic techniques needed to approach such questions.

It is extremely common for a physicist or string theorist to start up a lecture by giving a partition function for a theory, and now even algebraic geometers are quite often doing the same. It is not just a fashion, but the fact is that this is an extremely efficient way to present results. The general format of such partition functions is of an infinite sum whose terms contain integrals over moduli spaces. Here are some examples. We will not need details from these expressions, just the observation that they all contain integrals over moduli spaces.

Example 1.1. (*String Theory*) The Nekrasov partition function for $N = 2$ supersymmetric $SU(r)$ pure gauge theory on a complex surface X is given by an expression of the form

$$Z_X := \Lambda^{(1-r)d\cdot d} \sum_{n\geq 0} \Lambda^{2rn} \int_{\mathfrak{M}_{r,d,n}(X)} 1 \ ,$$

where $\mathfrak{M}_{r,d,n}(X)$ is the moduli space of framed torsion-free sheaves or rank r, and Chern classes $c_1 = d$ and $c_2 = n$. For the case of gauge theories with matter, one writes a similar expression but with more interesting integrands, see [GL].

Example 1.2. (*Donaldson–Thomas Theory*) For a Calabi–Yau threefold X, the partition function for Donaldson–Thomas theory is given by:

$$Z_X := \sum_{\beta\in H_2(X,\mathbb{Z})} \sum_{n\in\mathbb{Z}} Q^n v^\beta \int_{[I_n(X,\beta)]^{\mathrm{vir}}} 1 \ ,$$

where $I_n(X, \beta)$ is the moduli space of ideal sheaves $\mathcal{I}$ fitting into an exact sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

and satisfying

$$\chi(\mathcal{O}_Y) = n$$

and $[Y] = \beta \in H_2(X, \mathbb{Z})$, where χ is the holomorphic Euler characteristic, see [MNOP].

Key words and phrases. Reflexive sheaves, local holomorphic Euler characteristic, moduli spaces.

Example 1.3. (*Gromov–Witten Theory*) For a Calabi–Yau threefold X, the partition function for degree-β Gromov–Witten invariants is given by

$$Z_X := \exp \sum_{\beta \neq 0} \sum_{g \geq 0} u^{2g-2} v^{\beta} \int_{[\overline{M_g}(X,\beta)]^{\mathrm{vir}}} 1 \ ,$$

where $\overline{M_g}(X, \beta)$ is the moduli space of genus-g curves representing the class $\beta \in H_2(X, \mathbb{Z})$. There is a precise sense in which this partition function is equivalent to the one in Example 1.2, see [MNOP].

These examples illustrate the appearance of integrals over moduli spaces of sheaves. Even in the case of moduli spaces of maps of Example 1.3 the theory is still related to a theory given by integration over moduli of sheaves. Observe that the definition of moduli spaces itself requires a choice of numerical invariants: in Example 1.1 the Chern classes and in Example 1.2 the Euler characteristic. So, we now agree that we are interested in moduli spaces of sheaves on surfaces and threefolds. Of course, the physics motivation is just a bonus, and we could have been interested in such moduli spaces for purely geometric reasons, as they are part of classical algebraic geometry. Now physics dictates that we should consider theories defined over singular varieties. In fact, some of the most popular categories considered currently by physicists and string theorists turn out empty in the absence of singularities; such is the case of the Fukaya–Seidel category and the Orlov category of singularities. Thus we arrive at the conclusion that we need to understand moduli of sheaves on singular varieties. Both the case of global moduli of sheaves on projective varieties and the case of local moduli on a small neighborhood of a singularity are of interest. For the local case there is an added difficulty: What are the correct numerical invariants to be considered? In this paper we will show that the local holomorphic Euler characteristic provides a satisfactory invariant for sheaves on local surfaces. For the case of local threefolds however, the study of numerical invariants is work in progress, and much remains to be done. The goal of this paper is to describe partial progress in the understanding of these questions. We define new numerical invariants for the threefold case, and give existence of sheaves with given local numerical data.

Acknowledgements

We are very happy to contribute to this volume in honour of Andrew DuPlessis, and thankful to Christophe Eyral for giving us this opportunity. The final version of this article was completed while the first author visited UNICAMP under FAPESP project number 2009/08587-5; their hospitality and generous support is hereby gladly acknowledged.

2. Main Results

In this paper we consider rational surface singularities obtained by contracting a line $\ell \cong \mathbb{P}^1$ inside a smooth surface or threefold. Numerous approaches using numerical invariants or characteristic classes of some sort have been proposed in the past, see e.g. [Br], and in Section 3 we define numerical invariants, some of which are new, that we need for the present situation. To set the stage, in Section 4 we recall some of our earlier results for sheaves on singular surfaces. The results for threefolds presented in Section 5 are new and will appear in more detail in [Kö].

In Section 3 we define the local holomorphic Euler characteristic $\chi(\ell, \mathcal{F})$ of a reflexive sheaf $\mathcal{F}$. We will present the following results.

Theorem 4.4. *Let $\mathfrak{M}_n(X_k)$ be the moduli of reflexive sheaves on $\mathbb{C}^2/\mathbb{Z}_k$ with local holomorphic Euler characteristic equal to n. Then for all $n \geq 0$, $\mathfrak{M}_n(X_k)$ is non-empty.*

Theorem 5.1. *For every rank-2 bundle E on $W_1 := \mathrm{Tot}\big(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)\big)$ with $c_1(E) = 0$ and $E|_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(-j) \oplus \mathcal{O}_{\mathbb{P}^1}(j)$, the following bounds are sharp:*

$$j - 1 \le \chi(\ell, \pi_* E) = \mathbf{h}(E) \le (j^2 + j)(j-1)/6 \ .$$

Here $\pi\colon W_1 \to X$ is the contraction of the zero section ℓ and X is the singular threefold $xy - zw = 0$ in $\mathbb{C}^4$.

Theorem 5.7. *Let X be the singular threefold $xy - zw = 0$ in $\mathbb{C}^4$. For each $j \ge 2$ there exists a $(4j-5)$-dimensional family of rank-2 reflexive sheaves on X with local holomorphic Euler characteristic $j-1$.*

For each of the cases $j = 0$ or 1, our methods produce only the direct images of the split sheaves; both have local holomorphic Euler characteristic 0.

3. Numerical Invariants

In this section we define numerical invariants for sheaves on a neighborhood of a singularity. Our first invariant is defined for any dimension, and is particularly adapted to study reflexive sheaves that are direct images of vector bundles on a resolution. Let $\pi\colon (Z, \ell) \to (X, x)$ be a resolution of an isolated quotient singularity, $\mathcal{F}$ a reflexive sheaf on Z and $n := \dim X$. The following definition is due to Blache, [Bl, Def. 3.9].

Definition 3.1. The *local holomorphic Euler characteristic* of $\pi_*\mathcal{F}$ at x is

$$\chi\big(x, \pi_*\mathcal{F}\big) := \chi\big(\ell, \mathcal{F}\big) := h^0\big(X;\ (\pi_*\mathcal{F})^{\vee\vee}/\pi_*\mathcal{F}\big) + \sum_{i=1}^{n-1} (-1)^{i-1} h^0\big(X;\ R^i\pi_*\mathcal{F}\big) \ . \tag{3.1}$$

For the case when X is a compact orbifold, Blache [Bl] shows that the global Euler characteristics of X and its resolution are related by

$$\chi\big(X, (\pi_*\mathcal{F})^{\vee\vee}\big) = \chi\big(Z, \mathcal{F}\big) + \sum_{x \in \mathrm{Sing}\, X} \chi\big(x, \pi_*\mathcal{F}\big) \ . \tag{3.2}$$

Example 3.2. (*Homological dimension 1*) Consider the case when Z is itself the total space of a bundle on $\mathbb{P}^1$. Then Z has homological dimension one, and the expression on the right-hand side of (3.1) reduces to two terms, which we call the *width* and *height* of $\mathcal{F}$, respectively:

$$\mathbf{w}(\mathcal{F}) := h^0\big(X;\ (\pi_*\mathcal{F})^{\vee\vee}/\pi_*\mathcal{F}\big) \qquad \text{and} \qquad \mathbf{h}(\mathcal{F}) := h^0\big(X;\ R^1\pi_*\mathcal{F}\big) \ . \tag{3.3}$$

Hence $\chi = \mathbf{w} + \mathbf{h}$.

The case when Z is the total space of a negative line bundle on $\mathbb{P}^1$ was studied in [BGK1] and [GKM]. Unfortunately, the width vanishes in higher dimensions.

Lemma 3.3. [BGK1, Lemma 5.2] *Let C be a curve of codimension $n \ge 2$ in Z and $\pi\colon Z \to X$ the contraction of C to a point. Then for any reflexive sheaf $\mathcal{F}$ on Z we have*

$$h^0\big(X;\ (\pi_*\mathcal{F})^{\vee\vee}/\pi_*\mathcal{F}\big) = 0 \ .$$

Example 3.4. (*Flop*) When $W_1 = \mathrm{Tot}\big(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)\big)$, Lemma 3.3 shows that $\mathbf{w} = 0$. The height is still a non-trivial invariant, but less powerful than on surfaces.

However, we can define new invariants by restricting to sub-surfaces. We have two divisors $D_0 := \mathrm{Tot}\big(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \{0\}\big)$ and $D_\infty := \mathrm{Tot}\big(\{0\} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)\big)$, which are both isomorphic to Z_1, and they span the linear system

$$|D| := \big\{ \lambda_0 D_0 + \lambda_\infty D_\infty : [\lambda_0 : \lambda_\infty] \in \mathbb{P}^1 \big\} \ .$$

Then each $D_\lambda \in |D|$ is isomorphic to Z_1, and by restriction to D_λ we can define an entire family of pairs $(\mathbf{w}, \mathbf{h})$.

We now return to the case when Z is the total space of a vector bundle over $\ell = \mathbb{P}^1$ and there is a contraction $\pi\colon Z \to X$. We will construct sheaves on X as direct images of bundles on Z, which we now describe. For simplicity, we consider rank-2 bundles with vanishing c_1. The general case is no more difficult, but more unwieldy to present. When $E|_\ell \cong \mathcal{O}_{\mathbb{P}^1}(-j) \oplus \mathcal{O}_{\mathbb{P}^1}(j)$, we call the integer $j \geq 0$ the *splitting type* of E. It turns out that the ampleness of the conormal bundle of ℓ implies that E is an algebraic extension of line bundles,

$$0 \longrightarrow \mathcal{O}(-j) \longrightarrow E \longrightarrow \mathcal{O}(j) \longrightarrow 0 \ . \tag{3.4}$$

A line bundle $\mathcal{O}(n)$ is uniquely determined as the pullback of $\mathcal{O}_{\mathbb{P}^1}(n)$ from $\mathbb{P}^1$, since $\operatorname{Pic} Z \cong \operatorname{Pic} \mathbb{P}^1$. For every $j \geq 0$, there is the trivial extension $\mathcal{O}(-j) \oplus \mathcal{O}(j)$, which we call the *split bundle* of splitting type j. For convenience, we sometimes write E_{split} for the split bundle of the same splitting type as a given bundle E.

The first cohomology of $\mathcal{E}nd\, E$ is finite-dimensional and furnishes us with our next invariant:

$$h^1\big(Z;\ \mathcal{E}nd\, E\big)$$

Naturally, we wish to consider the zeroth cohomology as well. Sadly, this is infinite-dimensional, so extra effort is required. We consider the m^{th} infinitesimal neighbourhood of ℓ, denoted $\ell^{(m)}$, which is a projective scheme. The restriction $E^{(m)} := E|_{\ell^{(m)}}$ is coherent. For $i = 0, 1$, we set

$$\psi^i_m(E) := h^i\big(\ell^{(m)};\ E^{(m)}\big) \ ,$$

thus ψ^i_m takes finite values. We find that the difference $\psi^i_m(E_{\text{split}}) - \psi^i_m(E)$ is eventually constant.

Definition 3.5. For $i = 0, 1$ and $m \gg 0$, set

$$\Delta_i(E) := \psi^i_m(E_{\text{split}}) - \psi^i_m(E) \ .$$

For h^1, of course, this step is needlessly complicated, as the first cohomology is actually finite-dimensional, but this way the method may be applied to spaces in which the conormal bundle of ℓ is not ample.

The two numbers Δ_0 and Δ_1 are related via the Hilbert polynomial. Recall that for any coherent sheaf $\mathcal{A}$ on a projective scheme S, the Hilbert series

$$\phi(\mathcal{A}, n) := \chi\big(\mathcal{A}(n)\big) := \sum_{i \geq 0} (-1)^i h^i\big(S;\ \mathcal{A}(n)\big)$$

is a polynomial of degree $\dim S$. We have

$$\Delta_0(E) - \Delta_1(E) = \phi\big(E^{(m)}, 0\big) - \phi\big(E^{(m)}_{\text{split}}, 0\big) \ .$$

But the Hilbert polynomials of $E^{(m)}$ and $E^{(m)}_{\text{split}}$ are the same, as we will show momentarily, and so we have $\Delta_0 = \Delta_1$, and for computational ease we just stick with $h^1(\mathcal{E}nd\, E)$. The equality of the Hilbert polynomials, and consequently the fact that the Hilbert polynomial does not see the extension (3.4), is a consequence of the following result.

Lemma 3.6. *Let E be an extension of type* (3.4) *with splitting type j on either $Z_k := \operatorname{Tot}\big(\mathcal{O}_{\mathbb{P}^1}(-k)\big)$ or $W_1 := \operatorname{Tot}\big(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)\big)$. Then the Hilbert polynomial of $E|_{\ell^m}$,*

$$\begin{aligned} \phi\big(E^{(m)}, n\big) &= \chi\big(E^{(m)}(n)\big) \\ &:= \sum_i (-1)^i h^i\big(\ell^{(m)};\ E(n)|_{\ell^{(m)}}\big) = \begin{cases} (m+1)(km+2+2n) & \text{on } Z_k, \\ \tfrac{1}{3}(m+2)(m+1)(2m+3n+3) & \text{on } W_1, \end{cases} \end{aligned}$$

is independent of the extension class, and independent of the splitting type j. Similarly, the Hilbert polynomial of the endomorphism bundle $\mathcal{E}nd\, E|_{\ell^{(m)}}$ is $2\phi\big(E^{(m)}, n\big)$.

	$w(E)$	$h(E)$	$h^1(\mathcal{E}nd\, E)$	$w(G)$	$h(G)$	$h^1(\mathcal{E}nd\, G)$
Z_1	6	3	15	1	2	9
Z_2	2	2	9	0	2	7
Z_3	1	2	7	0	2	6
W_1	0	4	35	0	2	17

TABLE 1. The invariants width, height and $h^1(\mathcal{E}nd)$ for the split bundle E and a generic bundle G of splitting type $j = 3$ on the spaces Z_1, Z_2, Z_3 and W_1.

Proof. By the additivity of the Hilbert polynomial on short exact sequences, the Hilbert polynomials in question are determined by the Hilbert polynomial of the line bundles $\mathcal{O}_{\ell^{(m)}}(p)$ for all p. Since $\mathcal{O}_{\ell^{(m)}}(1)$ is ample, the higher cohomology of $\mathcal{O}_{\ell^{(m)}}(p)$ vanishes for sufficiently large p. (We can verify this by direct computation.)

Being a polynomial, the Hilbert polynomial is determined by finitely many values, so it suffices to compute $\phi(\mathcal{E}nd\, E^{(m)}, n) = h^0\big(\ell^{(m)};\ \mathcal{O}_{\ell^{(m)}}(p)\big)$ for large p. Since E and $\mathcal{E}nd\, E$ have filtrations by line bundles, which restrict to filtrations on every infinitesimal neighbourhood $\ell^{(m)}$, we compute:

$$\begin{aligned} \phi\big(E^{(m)}, n\big) &= \phi\big(\mathcal{O}_{\ell^{(m)}}(-j), n\big) + \phi\big(\mathcal{O}_{\ell^{(m)}}(j), n\big)\ , \text{ and} \\ \phi\big(\mathcal{E}nd\, E^{(m)}, n\big) &= \phi\big(\mathcal{O}_{\ell^{(m)}}(-2j), n\big) + 2\phi\big(\mathcal{O}_{\ell^{(m)}}, n\big) + \phi\big(\mathcal{O}_{\ell^{(m)}}(2j), n\big). \end{aligned}$$

We conclude this proof by computing $H^0\big(\ell^{(m)};\ \mathcal{O}(p)\big)$. Now we have to consider the spaces Z_k and W_1 separately. We pick a chart U with local coordinates (z, u) on Z_k and (z, u, v) on W_1, respectively, which transform to $(z^{-1}, z^k u)$ and (z^{-1}, zu, zv).

On $\ell^{(m)} \subset Z_k$, a section $a \in \mathcal{O}(p)(U)$ is a function $a(z, u) = \sum_{r=0}^{m} \sum_{s=0}^{\infty} a_{rs} z^s u^r$ such that $\sum_{r,s} a_{rs} z^{s-p} u^r$ is holomorphic in $(z^{-1}, z^k u)$, i.e. $s - p \le kr$. Thus

$$a(z, u) = \sum_{r=0}^{m} \sum_{s=0}^{kr+p} a_{rs} z^s u^r\ ,$$

which has $\frac{1}{2}(m + 1)(km + 2 + 2p) =: \phi_{\mathcal{O}}(p)$ coefficients.

On $\ell^{(m)} \subset W_1$, a section $a \in \mathcal{O}(p)(U)$ is $a(z, u, v) = \sum_{t=0}^{m} \sum_{r=0}^{m-t} \sum_{s=0}^{\infty} a_{trs} z^s u^r v^t$ such that $\sum_{t,r,s} a_{trs} z^{s-p} u^r v^t$ is holomorphic in (z^{-1}, zu, zv), i.e. $s - p \le r + t$. Thus

$$a(z, u, v) = \sum_{t=0}^{m} \sum_{r=0}^{m-t} \sum_{s=0}^{r+t+p} a_{trs} z^s u^r v^t\ ,$$

which has $\frac{1}{6}(m + 2)(m + 1)(2m + 3p + 3) =: \phi\big(\mathcal{O}, p\big)$ coefficients.

Putting it all together, we have

$$\begin{aligned} \phi\big(E^{(m)}, n\big) &= \phi\big(\mathcal{O}, -j + n\big) + \phi\big(\mathcal{O}, j + n\big)\ , \\ \phi\big(\mathcal{E}nd\, E^{(m)}, n\big) &= \phi\big(\mathcal{O}, -2j + n\big) + 2\phi\big(\mathcal{O}, n\big) + \phi\big(\mathcal{O}, 2j + n\big)\ , \end{aligned}$$

which gives the desired functions. □

3.1. **Examples of invariants.** To make the notion of the numbers we defined above more concrete, we tabulate examples for the two bundles $E = \mathcal{O}(-3) \oplus \mathcal{O}(3)$ (the split bundle of splitting type 3) and G, the "most generic" bundle of splitting type 3 (which has the lowest invariants among all bundles of splitting type 3), on the spaces Z_1, Z_2, Z_3 and W_1; see Table 1.

4. Surfaces

Let $Z_k := \mathrm{Tot}\big(\mathcal{O}_{\mathbb{P}^1}(-k)\big)$ and let E be a rank-2 bundle on Z_k with $c_1(E) = 0$ and splitting type j. Then E is determined by an element $p \in \mathrm{Ext}^1\big(\mathcal{O}(j), \mathcal{O}(-j)\big)$ as in (3.4). The direct image $\pi_*(E)$ is a reflexive sheaf on X_k, and there are bounds for its local holomorphic Euler characteristic around the singular point $x \in X_k$ in terms of j. An efficient algorithm to compute $\mathbf{w}, \mathbf{h}$ and χ is given in `http://www.maths.ed.ac.uk/~s0571100/Instanton/`, hence we can explicitly calculate the values of these numerical invariants for any such bundle E. We present here a useful existence result.

Lemma 4.1. *Let E be a rank-2 bundle over Z_k, $k > 1$, with $c_1(E) = 0$ and splitting type $j < k$. Then*

$$\chi(x, \pi_* E) = j - 1 \ .$$

Proof. By [G, Theorem 3.3] it follows that if $j < k$ then $E \cong \mathcal{O}_{Z_k}(j) \oplus \mathcal{O}_{Z_k}(-j)$. By definition, $\chi(x, \pi_* E) = \mathbf{w}(E) + \mathbf{h}(E)$. Direct computation (see [BGK1]) then shows that $\mathbf{w}(E) = 0$ and $\mathbf{h}(E) = j - 1$. □

In fact, we can say a lot more.

Lemma 4.2. [BGK1, Corollary 2.18] *Let E be a rank-2 bundle over Z_k, $k > 1$, with splitting type $j > 0$. Set $j = qk + r$ with $0 \le r < q$. The following bounds are sharp:*

$$j - 1 \le \chi(x, \pi_* E) \le \begin{cases} q^2 k + (2q+1)r - 1 & \text{if } 1 \le r < k \ , \\ q^2 k & \text{if } r = 0 \ . \end{cases}$$

Remark 4.3. Note that every bundle that satisfies the conditions of Lemma 4.1 is split, whereas in general there are many distinct isomorphism classes of bundles, which attain a whole range of numerical invariants. The lower bound in Lemma 4.2 is attained by a class of generic bundles, while the upper bound is obtained by the split bundle of splitting type j, and moreover, the split bundle is the only bundle to attain the bound when $r = 0$.

These two lemmas directly imply the following existence result.

Theorem 4.4. *Let $\mathfrak{M}_n(X_k)$ be the moduli of reflexive sheaves on X_k with local holomorphic Euler characteristic equal to n. Then for all $n \ge 0$, $\mathfrak{M}_n(X_k)$ is non-empty.*

4.1. **Applications to physics.**

To illustrate applications to physics, we mention some results on the existence of instantons. We stress that this particular instance of gaps on instantons charges presented below was completely new to physicists. In fact, there was a folklore belief that 1-instantons are always the most common, and that higher instantons of charge k should decay to k instantons of charge 1 over time. Our results showed that over the spaces Z_k with $k \ge 3$ there do not exist any 1-instantons, nevertheless higher charge instantons do exist (of course we mean mathematical existence proofs).

In [GKM, Proposition 54] we studied the Kobayashi–Hitchin correspondence for the spaces Z_k: We showed that an $SU(2)$-instanton on Z_k of charge n corresponds to a holomorphic $SL(2)$-bundle E on Z_k with $\chi(\ell, E) = n$ together with a trivialization of $E|_{Z_k^\circ}$, where $Z_k^\circ := Z_k - \ell$. A simple observation [GKM, Proposition 4.1] shows that there exists a trivialization of $E|_{Z_k^\circ}$ if and only if $n = 0 \mod k$. This restricts the splitting type of an instanton bundle over Z_k to be of the form nk and lead us to the following existence/non-existence result:

Proposition 4.5. [GKM, Theorem 6.8] *The minimal local charge of a non-trivial $SU(2)$-instanton on Z_k is $\chi_k^{\min} = k - 1$. The local moduli space of (unframed) instantons on Z_k with fixed local charge $\chi_k^{\min}$ has dimension $k - 2$.*

This result shows a straightforward passage from the algebraic geometry of bundles on surfaces to meaningful mathematical physics. Similar results for Calabi–Yau threefolds promise to have exciting interpretations in string theory and physics, whenever the mathematical background is constructed.

Remark 4.6. (*Gaps of instanton charges*) The non-existence of instantons with certain local charges on the spaces Z_k for $k > 2$ is in stark contrast with what happens in the case $k = 1$, where there is no gap [BG, Theorem 0.2].

Open Question 4.7. Theorem 4.2 gives sharp bounds for χ – are the intermediate values achieved? Given an integer α such that $j - 1 < \alpha < q^2 k$, does there exist an instanton bundle on Z_k with splitting type j and $\chi = \alpha$? We have a positive answer for analogous question when $k = 1$, all other cases are open.

We illustrate also an application to topology:

Theorem 4.8. [BGK1, Theorem 4.15] *If $j = qk$ for some $q \in \mathbb{N}$, then the pair $(\mathbf{w}, \mathbf{h})$ stratifies instanton moduli stacks $\mathfrak{M}_{j,k}$ into Hausdorff components.*

Open Question 4.9. Find invariants that stratify the moduli stacks $\mathfrak{M}_{j,k}$ in the case $j = nk + r$ with $r \neq 0 \mod k$. We know that the pair $(\mathbf{w}, \mathbf{h})$ does not provide a fine enough invariant to stratify the moduli stacks in these cases. Thus, some extra numerical invariant is needed. At the moment the authors are completely unaware of any suitable candidate.

We find it completely surprising that the case $r = 0$, whose physics interpretation is known, turned out to be much simpler to solve. From a topological point of view one should of course have Hausdorff stratifications for the moduli stacks in all cases.

5. Threefolds

Consider a smooth threefold W containing a line $\ell \cong \mathbb{P}^1$. We will focus on the Calabi-Yau cases

$$W_i := \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-i) \oplus \mathcal{O}_{\mathbb{P}^1}(i-2) \text{ for } i = 1, 2, 3.$$

The existence of a contraction of ℓ imposes heavy restrictions on the normal bundle [Jim], namely $N_{\ell/W}$ must be isomorphic to one of

$$\text{(a)}\ \ \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)\ , \qquad \text{(b)}\ \ \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(0)\ , \ \text{ or } \qquad \text{(c)}\ \ \mathcal{O}_{\mathbb{P}^1}(-3) \oplus \mathcal{O}_{\mathbb{P}^1}(+1)\ .$$

Conversely, Jiménez states that if $\mathbb{P}^1 \cong \ell \subset W$ is any subspace of a smooth threefold W such that $N_{\ell/W}$ is isomorphic to one of the above, then:

- in (a) ℓ always contracts,
- in (b) either ℓ contracts or it moves, and
- in case (c) there exists an example in which ℓ does not contract nor does any multiple (i.e. any scheme supported on ℓ) move.

W_1 is the space appearing in the basic flop. Let X be the cone over the ordinary double point defined by the equation $xy - zw = 0$ on $\mathbb{C}^4$. The basic flop is described by the diagram:

$$\begin{tikzcd} & W \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[dd] & \\ W_1^- \arrow[rr, dashed] \arrow[dr, "\pi_1"'] & & W_1^+ \arrow[dl, "\pi_2"] \\ & X & \end{tikzcd} \tag{5.1}$$

Here $W := W_{x,y,z,w}$ is the blow-up of X at the vertex $x = y = z = w = 0$, $W_1^- := Z_{x,z}$ is the small blow-up of X along $x = z = 0$ and $W_1^+ := Z_{y,w}$ is the small blow-up of X along $y = w = 0$. The *basic flop* is the rational map from W^- to W^+.

In $W_2 \cong Z_2 \times \mathbb{C}$ the zero section does not contract to a point (so it must be able to move), but it is possible to contract it partially and obtain a singular family $X_2 \times \mathbb{C}$, where X_2 is the surface containing an ordinary double-point singularity defined by $xy - z^2 = 0$ in $\mathbb{C}^3$. Holomorphic bundles on W_2 have infinite local holomorphic Euler characteristic, but the restriction $E|_{Z_2 \times \{0\}}$ has well-defined and finite width and height. Note that in contrast to W_1, there are strictly holomorphic (non-algebraic) bundles on W_2, although every rank-2 bundle on W_2 is still an extension of line bundles.

In W_3 not even a partial contraction of the zero section is possible. Nevertheless we can still calculate the width and height of the restriction $E|_{Z_3}$ of a bundle E to a subsurface $Z_3 \hookrightarrow W_3$. Again, on W_3 there are strictly holomorphic (non-algebraic) bundles, and moreover, there are (many) rank-2 bundles which are not extensions of line bundles.

5.1. **Bounds and generating functions.** We can compute the invariants $\mathbf{w}(E)$, $\mathbf{h}(E)$ and $h^1(\mathcal{E}nd\, E)$ directly and algorithmically. We have an implementation of each of the algorithms for the commutative algebra software *Macaulay 2*, which led us to discover several formulae for the bounds of these invariants. Bounds for the local holomorphic Euler characteristic $\chi = \mathbf{w} + \mathbf{h}$ on surfaces were presented in Section 4; now we turn to the flop space W_1, were by Lemma 3.3, we have $\chi = \mathbf{h}$.

Theorem 5.1. *For every rank-2 bundle E on W_1 with $c_1(E) = 0$ and splitting type j, the following bounds are sharp:*

$$j - 1 \leq \chi(\ell, E) = \mathbf{h}(E) \leq (j^2 + j)(j - 1)/6 \ .$$

Proof. The lower bound is attained by a class of generic bundles, and the upper bound by the split bundle $\mathcal{O}(-j) \oplus \mathcal{O}(j)$. This can be seen by direct computation as explained in [BGK1] and [Kö]. □

We also have a concise expression for the numbers $h^1(\mathcal{E}nd)$ of the extremal cases, that is generic and the split bundles of splitting type j.

Definition 5.2. A power series of the form $g(z) = \sum_{j=0}^{\infty} a_j z^j$ is called a *generating function* for the sequence $(a_j)_{j=0}^{\infty}$. Hence, $a_j = \dfrac{1}{j!} \dfrac{d^j g}{dz^j}\Big|_{z=0}$.

Set $a_j^{X,E} := h^1(X;\ \mathcal{E}nd\, E)$. Then if the base space is $X = Z_k$ or W_1 and the bundle E over X is either split or generic of splitting type j, we have generating functions for $a_j^{X,E}$, as shown in Table 2. Since the generating function of a sum of two sequences is the sum of the generating functions, we can easily deduce from this the generating functions for Δ_0 and Δ_1. We spell out the inequalities.

Theorem 5.3. *For every rank-2 bundle E on W_1 with $c_1(E) = 0$ and splitting type j, the following bounds are sharp:*

$$(j^3 + 3j^2 - j)/3 \leq h^1(W_1;\ \mathcal{E}nd\, E) \leq (4j^3 - j)/3$$

Proof. The lower bound is attained by a generic bundle and the upper bound by the split bundle, the values are found by direct computation. □

Space	Split bundle E_j	Generic bundle G_j
$Z_k,\ k=2n$	$\dfrac{-z(z^{n+1}+z^n+z+1)}{(z-1)^2(z^k-1)}$	$\dfrac{z^{k+2}-z^3-z^2-z}{(z-1)^2(z^k-1)}$
$Z_k,\ k=2n+1$	$\dfrac{-z(2z^{n+1}+z+1)}{(z-1)^2(z^k-1)}$	
W_1	$\dfrac{z(z^2+6z+1)}{(z-1)^4}$	$\dfrac{z(-z^2+2z+1)}{(z-1)^4}$

TABLE 2. Generating functions for $a_j^{X,E} := h^1(X;\ \mathcal{E}nd\,E)$ on various spaces for the split and the generic bundle of splitting type j (data for G_j only valid for $j \geq k$); the value $a_j^{X,E}$ is the j^{th} coefficient in the Taylor series.

5.2. **Moduli of sheaves.** We consider sheaves on singular varieties obtained as direct images of bundles on W_i. First we study such bundles and their moduli. The topological structure of these moduli is not yet well understood. Most numerical invariants defined in Section 3 can be computed over any W_i; however, the invariants Δ_0 and Δ_1 in (3.5) are infinite on W_2 and W_3, so more refined counterparts are required.

Open Question 5.4. Construct a Hausdorff stratification of the moduli stacks $\mathfrak{M}_n(W_i)$ of bundles on W_i with $c_1 = 0$ and $\chi(\ell, E) = n$.

We obtain a partial understanding of these moduli by looking at first-order deformations, and this will provide enough bundles for an existence theorem of reflexive sheaves on the corresponding singular varieties.

Proposition 5.5. (*First-order deformations*) *Set* $F := \mathcal{O}_\ell(-j) \oplus \mathcal{O}_\ell(j)$ *with* $\ell \subset W_i$.

(1) *For any bundle* E *on* W_i *with* $E|_\ell \cong F$*, the space of first-order deformations of* G *is isomorphic to* $\mathbb{C}^{\gamma_1}$*, where*
$$\gamma_1 := h^1\big(\ell;\ \mathcal{E}nd(E|_\ell) \otimes \mathcal{I}_\ell/\mathcal{I}_\ell^2\big) < \infty\ .$$
(2) *If* $\mathcal{I}_\ell/\mathcal{I}_\ell^2$ *is ample (i.e. if* $i = 1$*), then there exists a vector bundle* A *on* W_1 *such that* $A|_\ell \cong F$.

Proof. The dimension count is standard deformation theory. Existence of extensions to formal and small analytic neighbourhoods of ℓ are given by Peternell's Existence Theorem [Pet]. The fact that we actually get existence on the entire space W_1 rather than just a small neighbourhood of ℓ is due to the fact that every bundle on W_1 is determined by its restriction to a finite infinitesimal neighborhood of ℓ. □

Corollary 5.6. (*Dimension of moduli*) *The moduli space of first order deformations of* $\mathcal{O}(j) \oplus \mathcal{O}(-j)$ *over* W_i *modulo holomorphic isomorphisms is isomorphic to* $\mathbb{P}^{4j-5}$.

Proof. It is well known that multiplying the extension class by a non-zero constant does not change the holomorphic type of the underlying bundle. It turns out that on the first formal neighborhood this is the only identification. This was proved for surfaces in [BGK1, Theorem 4.9] and for W_i, $i = 1, 2, 3$ in [Kö]. We can then compute γ_1 directly as the dimension of the first cohomology of $\mathcal{E}nd\big(\mathcal{O}_{\mathbb{P}^1}(-j) \oplus \mathcal{O}_{\mathbb{P}^1}(j)\big) \otimes N^*_{\ell/W_i}$ on $\mathbb{P}^1$. The $\mathcal{E}nd$-bundle splits into a direct sum of line bundles, and the computation is straightforward. □

If instead of the first-order deformations we wish to consider all deformations, then the dimension of the deformation space is given by

$$\gamma := \sum_{m=0}^{\infty} h^1\big(\ell;\ \mathcal{E}nd(E|_\ell) \otimes \mathrm{Sym}^m(\mathcal{I}_\ell/\mathcal{I}_\ell^2)\big)\ , \tag{5.2}$$

which is finite when $\mathcal{I}_\ell/\mathcal{I}_\ell^2$ is ample, but infinite in general. Though the space of deformations may be infinite, it turns out that for a fixed j the moduli space $\mathfrak{M}_n(W_i)$ of holomorphic bundles E on W_i with $\chi(\ell, E) = n = j - 1$ has a Zariski-open set of dimension $4j - 5$ consisting of of first-order deformations of $\mathcal{O}(j) \oplus \mathcal{O}(-j)$ (cf. Corollary 5.6). Now, using these moduli for the case of W_1, we obtain sheaves on the singular threefold X appearing on the flop diagram (5.1).

Theorem 5.7. *Let X be the singular threefold $xy - zw = 0$ in $\mathbb{C}^4$. For each $j \geq 2$ there exists a $(4j - 5)$-dimensional family of rank-2 reflexive sheaves on X with local holomorphic Euler characteristic $j - 1$.*

Proof. These reflexive sheaves are obtained as direct images of generic bundles on W_1 with splitting type $(-j, j)$. Combine Corollary 5.6 with the value of χ found for the generic bundle as given in Table 2. □

For the case of $j = 0$ or 1 our methods give only the direct images of the split bundles $\mathcal{O} \oplus \mathcal{O}$ and $\mathcal{O}(1) \oplus \mathcal{O}(-1)$, both have $\chi = 0$.

We stop short of stating a similar theorem for the singular spaces obtained by partial contractions on W_i with $i = 2, 3$ because strictly speaking the definition of local Euler characteristic was given for isolated singularities. We do obtain existence of reflexive sheaves on those spaces, but we do not yet have a good feel for what would be the correct numerical numerical invariants to use.

Open Question 5.8. Describe the full moduli of reflexive rank-2 sheaves on W_1 with $c_1 = 0$ and $\chi = n$, that is, include all sheaves that do not occur as direct images of bundles on W_1.

Open Question 5.9. Describe moduli of sheaves with fixed numerical invariants on germs of singularities.

The latter is of course a very big question, actually infinitely many open questions, starting with the definition of the correct invariants up to their computation and then construction of moduli. It is certainly an entire research project for a whole group of singularists. We hope some singularists get inspired to work on these questions.

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Journal *of* Singularities
Volume 2 (2010), 67-80

Proc. of Singularities in Aarhus
received 28 December 2009
DOI: 10.5427/jsing.2010.2e

GRAPHS OF STABLE MAPS FROM CLOSED ORIENTABLE SURFACES TO THE 2-SPHERE

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Abstract. We prove that any bipartite weighted graph can be associated to some stable map from a closed orientable surface to the sphere and obtain necessary and sufficient conditions on a graph to be attached to a fold map of a given degree.

1. Introduction

The local behaviour of stable maps between surfaces was described by Whitney, who determined the typical singularities that these maps may have, namely fold curves with isolated cusps. More recently, the work of T. Ohmoto and F. Aicardi [17], based on the Vassiliev-type isotopy invariants [21], has thrown new light on the study of stable maps from a non local viewpoint. These invariants are related to the behaviour of the branch sets (or apparent contours) of these maps.

In order to investigate the global classification of stable maps from surfaces to the plane, graphs of stable maps were introduced in [12] to provide a combinatorial description of the topology of the singular set (see §2 for the definition). A natural question is to characterize graphs of stable maps (for example they are necessarily bipartite). In [13] the special case of stable maps from the sphere to the plane was studied, with emphasis on fold maps (i.e. those without cusps). The classification of fold maps between manifolds and possible related homotopy principles has been addressed by various authors ([1], [2], [8], [19], [20]). In [13] it was shown that *any tree with zero weights is the graph of a stable map from the sphere to the plane*. On the other hand, the vertices of any tree may be labelled alternately positive and negative (i.e the tree is bipartite). Graphs of fold maps from the sphere to the plane were then characterized as being trees with an equal number of positive and negative vertices. In [14] it was shown that graphs of stable maps of closed orientable surfaces to the plane are precisely non negatively weighted bipartite graphs. As for fold maps, it was shown that the characterization in the spherical case extends to fold maps all of whose regions are planar (this corresponds to the zero weight condition).

Potential applications of stable maps such as the global study of Gauss maps on closed surfaces, or the determination of linking numbers of closed curves in terms of secant maps lead one to consider stable maps and fold maps from surfaces to the sphere of arbitrary degree (the degree zero case being essentially that of maps into the plane). In the present article we characterize graphs of stable maps in this more general setting. The main results are as follows.

1) Any bipartite graph $\mathcal{G}$ with non negatively weighted vertices is the graph of a stable map of a connected orientable and closed (compact and boundaryless) surface into the 2-sphere of

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2000 *Mathematics Subject Classification.* 57R45, 57M15, 57R65.
Key words and phrases. Stable maps, graphs, branch sets, degree.
The second author is partially supported by FAPEMIG (EDT 214/05) and CAPES (BEX4529/06-5).
The third author is partially supported by DGCYT-FEDER (MTM2009-08933).

arbitrary degree. The Euler characteristic of the surface is $2(\chi(\mathcal{G}) - g)$*, where* g *stands for the sum of all the weights in* $\mathcal{G}$*.*

A bipartite graph is said to be *balanced* provided the difference $V^+ - V^-$ between the numbers of positive and negative vertices equals the difference $g^+ - g^-$ between the sums of the weights of the positive and the negative vertices.

2) A bipartite graph can be the graph of some fold map from a closed orientable surface to the plane if and only if it is balanced. Moreover, any bipartite graph can be the graph of some fold map from a closed orientable surface to the sphere of degree $(V^+ - V^-) - (g^+ - g^-)$ *(in particular, the degree of a fold map may be deduced from its weighted graph).*

The basic techniques used here are surgeries on stable maps (together with the corresponding modification of the graph) and Quine's Theorem relating the degree and the number of cusps (with signs) of stable maps between surfaces ([18]). In §3 *reduction* and *extension* of graphs are defined, based on a suitable interpretation of certain codimension one transitions of stable maps ([17]) and used in §4 and §6 in the characterization of graphs of fold maps.

Finally, we notice that the pair given by the graph and the branch set is not enough to determine the isotopy class of a stable map from a surface to the plane or the sphere. As explained in §3, there are examples of non equivalent stable maps sharing both, their graph and their branch set. In order to distinguish between them we need to add some extra information which can be encoded in the form of Blank's words [5, 4, 9, 10] conveniently associated to the curves of the branch set.

2. Stable maps

We first recall some definitions and basic results. Two smooth maps f and g from a surface M to a surface N are said to be C^∞ *right-left equivalent* (simply, *equivalent*) if there are diffeomorphisms, l and k, such that $g \circ l = k \circ f$. The maps f and g are *isotopic* if both the above diffeomorphisms are isotopic to the identity. A map f is said to be *stable* if all maps sufficiently close to f (in the Whitney C^∞-topology) are equivalent to f.

A point of the source surface M is a *non singular point* of f if the map f is a local diffeomorphism around that point, and *singular* otherwise. The singular set Σf of f is the set of singular points of f, and its image $Bf = f(\Sigma f)$ is called the *branch set* of f. By Whitney's theorem [11], for any stable map $f : M \to N$, its singularities are locally of fold type $(x, y) \mapsto (x^2, y)$, and of cusp type $(x, y) \mapsto (x^3 + yx, y)$; Σf is a union of embedded curves on M and Bf is a union of smooth curves on N with transverse double points and possibly many cusp points. The non-singular set (which is immersed into the surface N by the map) consists of finitely many *regions*. Given orientations of the surfaces M and N, a region is *positive* if the map preserves orientation and *negative* otherwise. The singular set is the frontier of each half (positive or negative) of the surface M, i.e. any singular curve lies in the frontier of a positive and a negative region. We denote by M^+ (resp. M^-) the union of all the positive (resp. negative) regions including their boundaries. Clearly, M^+ and M^- meet in their common boundary, the singular set of f.

Topological information of stable maps f may be conveniently encoded in a weighted graph from which the pair $M, \Sigma f$ may be reconstructed (up to diffeomorphism) ([13], [14]). The edges and vertices of the graph correspond (respectively) to the singular curves and the regions (i.e. the connected components of the non-singular set). An edge is incident to a vertex if and only if the singular curve corresponding to the edge lies in the frontier of the region corresponding to the vertex. In other words, given a stable map $f : M \to N$, its graph $\mathcal{G}(f)$ is the *dual graph* of Σf in M. We attach a label to each vertex of the graph, $+$ (or $-$) for positive (resp. negative) regions. Since each component of Σf is the boundary of a positive and of a negative region, the signs of the vertices are assigned alternatively, that is, the graph $\mathcal{G}(f)$ is bipartite. The weight g_v of a vertex v is defined to be the genus of the corresponding region i.e the genus of the closed surface obtained by adding disk to the region, one for each boundary curve. Figure 1 shows different stable maps of zero degree from the torus and bi-torus to the plane and their weighted graphs.

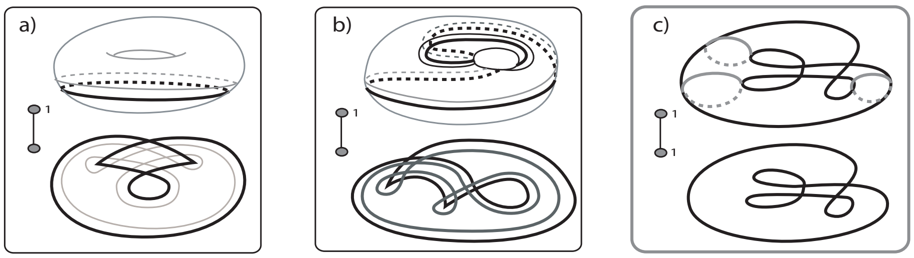

FIGURE 1. Stable maps and their graphs

In the particular case of stable maps from the sphere to the sphere, S. Demoto [7] has studied the isotopy classes corresponding to a graph with a unique edge and 2 vertices. In this case, the branch set is a connected closed curve which may have cusps and/or self-intersections. For $d = deg(f) \geq 2$, Demoto proves that when the branch set has no self-intersections the number of cusps of f is at least $2d$. Example c) in Figure 2 illustrates a map $f : T \to S^2$ with degree 1 whose graph has exactly one edge and the branch set has 4 self-intersections and no cusps. The examples a) and b) in Figure 2 correspond respectively to stable maps from the sphere and the torus to the sphere, whose branch set has no cusps and c) its singular set consists of a unique curve, whereas the second one has degree 1. The corresponding graphs are shown on the left of each picture. As we shall see later, the basic examples displayed in Figures 1 and 2 will take an important role in the proofs of the results of this paper.

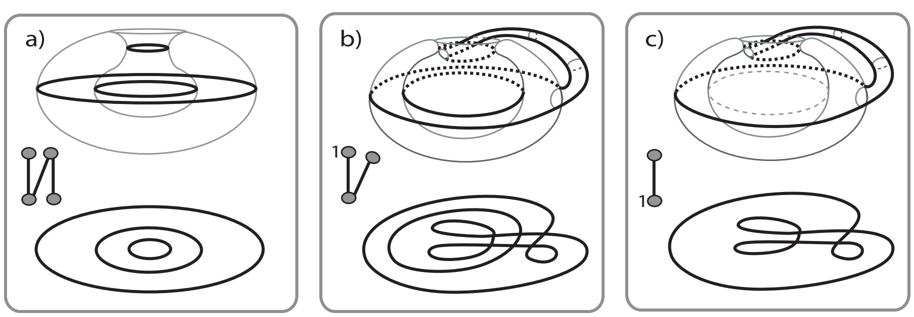

FIGURE 2. Branch sets with 4 self-intersections and no cusps.

We say that the graph $\mathcal{G}(f)$ is of *type* $T(\mathcal{G}) = (m, n, g)$ if it has m edges, n vertices and the total sum of the weights of its vertices is g (called the *total weight* of $\mathcal{G}(f)$). We observe that the following relation holds: $g(M) = \beta_1(\mathcal{G}(f)) + g$, where $g(M)$ denotes the genus of M and $\beta_1(\mathcal{G}(f))$ the $1st$ Betti number of the graph.

A cusp is called positive (resp. negative) if its local mapping degree is $+1$ (resp. -1) with respect to given orientations.

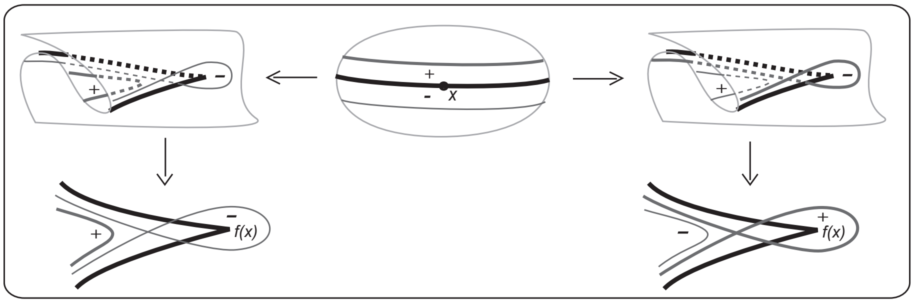

Figure 3. Example of negative and positive cusps.

Let f be a stable map between closed surfaces M and N of degree $deg(f)$. In [18] it was shown that

$$\chi(M) - 2\chi(M^-) + C = deg(f)\chi(N),$$

where χ denotes the Euler characteristic and $C = C^+ - C^-$, the number of positive cusps minus the number of negative cusps.

Lemma 2.1. *For a stable map $f : M \to S^2$ with $C = 0$ one has*

$$deg(f) = (V^+ - V^-) - (g^+ - g^-),$$

where V^+ (resp. V^-) is the number of positive (resp. negative) regions and g^+ (resp. g^-) the genus of M^+ (resp. M^-).

Proof: It follows from Quine's formula that $\chi(M) - 2\chi(M^-) = 2deg(f)$. Now, $\chi(M) = \chi(M^+) + \chi(M^-) - \chi(M^+ \cap M^-) = \chi(M^+) + \chi(M^-)$, and thus

$$\chi(M^+) - \chi(M^-) = 2deg(f). \tag{1}$$

Then the result follows from the relation $\chi(M^\pm) = 2(V^\pm - g^\pm - m)$. □

3. Surgery of stable maps

One way of constructing a stable map is to glue together two stable maps. In particular, in a surgery, a pair of disjoint disks in the surface is removed and replaced by a tube, the map then being extended over the interior of the tube. There are two types of surgery: horizontal and vertical. These were introduced in [14] for stable maps from surfaces to the plane. The extension of these definitions for stable maps between closed surfaces in general is straightforward:

a) Horizontal surgery. Given a stable map h between two surfaces M and N, a *bridge* is an embedded rectangle β in N which meets the branch set Bh in opposite edges (and nowhere else) compatibly with the orientation of the branch set as shown in Figure 4(a) (see [16]). The stable map h_β is constructed as follows. The bridge meets $h(M)$ in two intervals, $h(I)$ and $h(J)$, say. Choose small disks in M one containing I, the other J and replace their interiors by a tube (i.e. an annulus), respecting the orientation of M, so as to obtain an oriented surface. As illustrated in Figure 4(a), the map h may then be extended over the tube to give the required stable map h_β. In particular, if M is the disjoint union of surfaces P and Q and f and g denote the restrictions of h to P and to Q, with I in P and J in Q then we obtain the *horizontal sum* $f +_{hor} g$. In other words $h = f \cup g$ and $(f \cup g)_\beta = f +_{hor} g$.

b) Vertical surgery. In this case we take a connected sum by identifying two small non-singular disks in the domain, one positive and one negative (as in Figure 4 (b)) whose images in

N coincide. The disks are replaced by a tube which is mapped into the plane, with a singular curve running around the middle of the tube. Thus the surgery adds a disjoint embedded curve to the branch set. We denote this sum as $f +_{ver} g$. It is possible also to perform vertical surgery using a bridge, but this will not be needed here. Observe that horizontal (resp. vertical) surgery decreases (resp. increases) the number of edges by one.

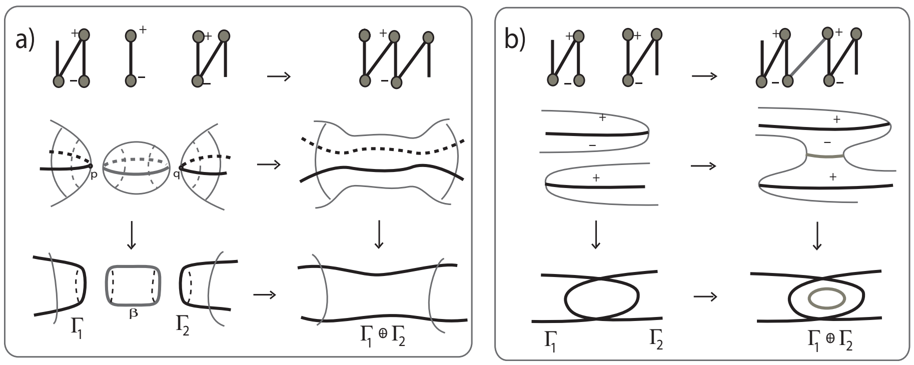

FIGURE 4. Surgeries: (a) horizontal, (b) vertical.

Figure 4 also shows the effects of the surgeries on the graphs. It is easy to see that if $\mathcal{G}_i$ represents the graphs of $f_i, i = 1, 2$ and $\mathcal{G}_1 +_{hor} \mathcal{G}_2$, $\mathcal{G}_1 +_{ver} \mathcal{G}_2$ respectively represent the graphs of $f_1 +_{hor} f_2$ and $f_1 +_{ver} f_2$, then

- $T(\mathcal{G}_1 +_{hor} \mathcal{G}_2) = T(\mathcal{G}_1) + T(\mathcal{G}_2) - (1, 0, 0)$,
- $T(\mathcal{G}_1 +_{ver} \mathcal{G}_2) = T(\mathcal{G}_1) + T(\mathcal{G}_2) + (1, 0, 0)$,

Observe that surgeries do not affect the degree. In particular, the degree of a horizontal or vertical sum of f and g is the sum of the degrees of f and g. In particular, as illustrated in Figure 5, taking the horizontal connected sum of any stable map $f : M \to S^2$ with $g : S^2 \to S^2$ having two cusps depicted below increases the degree of f by one but does not change its graph.

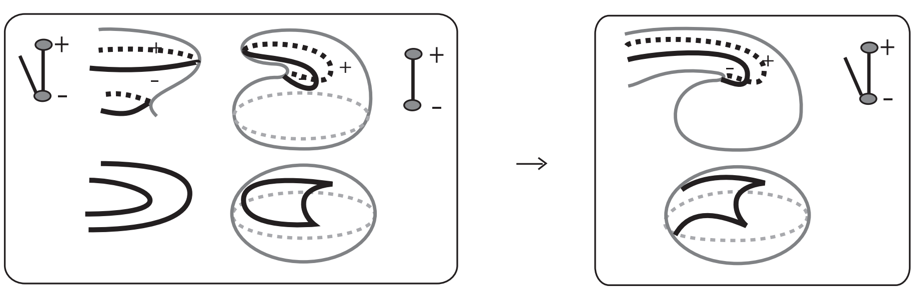

FIGURE 5. Altering the degree and preserving the graph.

c) Transitions. Apart from connected sums we can also use certain transitions in order to alter the graph and/or the branch set of a stable map. A codimension one transition corresponds to a generic homotopy from a given stable map f_0 to another stable map f_1 which is not right-left equivalent to f_0. In other words, this means a path transverse to all the strata of the *the discriminant hypersurface* in the mapping space $C^\infty(M, S^2)$. See [12] or [17] for the description of all the possible transitions. The interesting transitions, from our viewpoint, are those altering the numbers of cusps, or of singular curves, namely the swallowtail, beaks and lips transitions. Figure 6 and 7 show some examples of swallowtail, lips and beaks transitions on a degree one map from the sphere to the sphere. Clearly, the transitions do not alter the degree, for the new map remains in the same pathcomponent of $C^\infty(M, S^2)$.

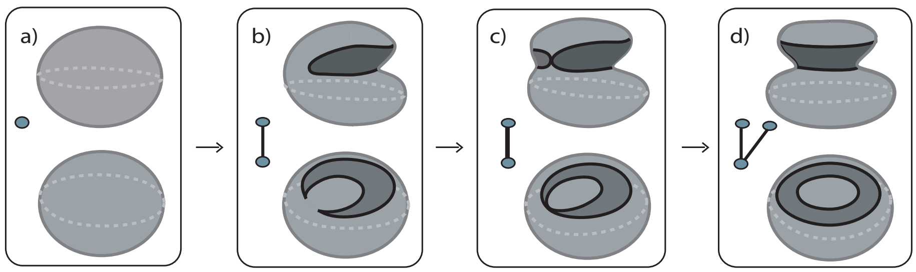

FIGURE 6. Lips ($a \to b$) and beaks ($b \to d$) transitions on maps of the sphere.

We shall focus our attention into a special combination of transitions that will be useful in the last section of this paper: The *double beaks+double inverse swallowtail*. This is obtained by successive application of beaks transitions in two nearby segments of neighbouring singular curves (with opposite orientations), followed by successive annihilations of two pairs of cusps (with opposite signs) trough swallowtail transitions. The effects of this homotopy on the graph and branch set are shown in Figure 7. We observe that the total number of singular curves decreases by two, which corresponds to the identification of three successive edges to form one edge of the new graph (referred to as the *reduced graph*). In particular, by means of successive reductions, any odd number of consecutive edges in a tree may be identified to form a single edge in the reduced tree.

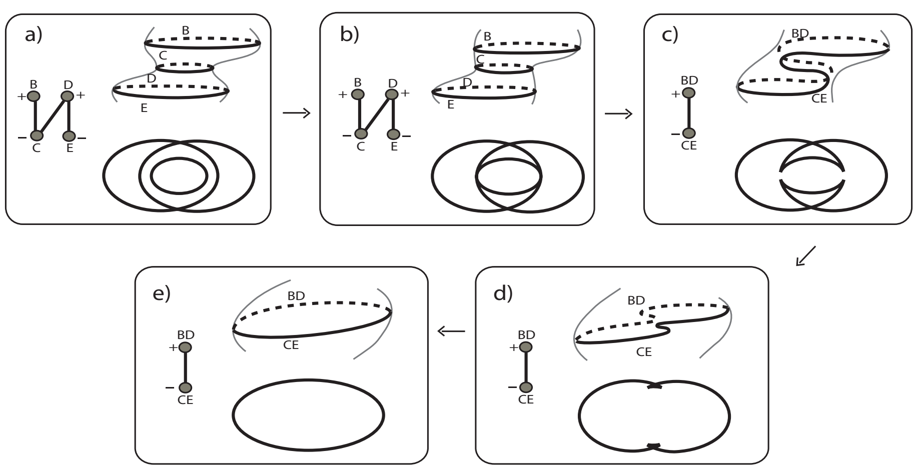

FIGURE 7. Double beaks + double inverse swallowtail: Reduction of a graph .

The inverse homotopy, *double swallowtail+double beaks*, obtained by creating two couples of cusps in a singular curve by means of two swallowtail transitions, followed by a suitable pair of beaks transitions has the effect of replacing an edge by three consecutive edges in a new graph (referred to as the *extended graph*). Note that the extended graph depends on the location where transitions happen. For example, as in Figure 8, given an edge pq, the set $\mathcal{H}$ (resp. $\mathcal{F}$) of edges emanating from p (resp. q) is divided into two subsets $\mathcal{H}_i$ (resp. $\mathcal{F}_i$), $i = 1, 2$, so that these subsets of edges are distributed to created vertices p_1, p_1, p_2, q_2 in the extended graph. Also the weight of p (resp. q) is divided into two weights of p_1, p_2 (resp. q_1, q_2). Conversely, the homotopy in the opposite direction is a reduction of graphs, which gathers edeges and weights. Clearly these homotopies do not affect the degree of a map.

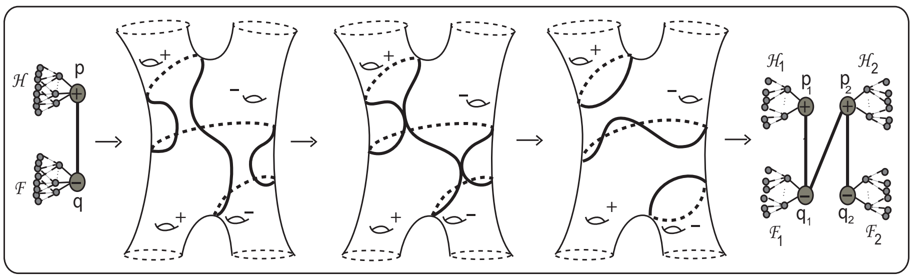

FIGURE 8. Extensions of a graph.

Observe that the graph of any stable map $f : M \to S^2$ is bipartite and that $\chi(M) = 2\chi(\mathcal{G}(f)) - 2g$. In particular, M is the sphere if and only if the graph is a tree with all weights zero. These considerations lead to

Theorem 3.1. *Any bipartite graph with non-negatively weighted vertices is the graph of a stable map of a surface to the sphere of arbitrary degree.*

Proof: It was shown in [14] that any bipartite graph may be realized by a stable map of degree zero from some surface into the sphere. Since the horizontal surgery in Figure 5 does not change the graph the map may be taken to have arbitrary positive degree. To get negative degree compose with the antipodal map of the sphere which does not change the graph. □

Remark 3.2. *We observe that the pair (graph, branch set) is in general not enough to determine the isotopy class of a stable map from a closed surface to the plane or the sphere. A good example of this is obtained from Milnor's example of a plane curve with 6 double points which can be seen as the image of the boundary of a 2-disc by two different immersions. If we define a mapping $f : S^2 \to \mathbb{R}^2$ by putting it equal to one of these immersions on the lower hemisphere and to the other on the upper hemisphere, we obtain a fold map from S^2 to $\mathbb{R}^2$. On the other hand, by choosing the same immersion on both hemispheres we get a new fold map from S^2 to $\mathbb{R}^2$ which can be joined by a smooth family of fold maps to the orthogonal projection of the unit 2-sphere in $\mathbb{R}^3$ on the equatorial plane. These two maps although share both, their graph and their apparent contour, are not equivalent* [8]. *We thus need some extra information which is encoded in the set of Blank's words (*[4], [5], [10]*) associated to the curves of the branch set. Once we specify a bijection between the edges of the graph and the curves in the branch set, we can work separately at each vertex by applying the techniques described in* [10] *in order to recover the class of the immersion of a surface with boundary associated to it. A convenient assemblage of these immersions will lead to a stable map class.*

4. FOLD MAPS

In this section we consider fold maps of surfaces into the plane which, of course, are also fold maps into the sphere of degree zero. We recall that a *fold map* is a stable map without cusps, so that the branch set consists of curves immersed in the plane. In [13] it was shown that a necessary and sufficient condition for a graph with zero weights to be the graph of a fold map (of an orientable surface) is that the number of positive and negative vertices be equal. We generalize this fact to the case of graphs with arbitrary weights. In fact it immediately follows from Lemma 2.1 that the graph of fold maps $f : M \to S^2$ of degree zero is balanced, i.e., $V^+ - V^- = g^+ - g^-$. Furthermore, the converse is also true (Theorem 4.2 below). To show this, we begin with the case of trees.

Proposition 4.1. *Any balanced tree is the graph of a fold map of a surface into the plane and hence of degree zero into the sphere.*

Proof: The proof is by induction on the total weight g of the tree. The case of trees of total weight zero was proven in [13]. Let $\mathcal{T}$ be a balanced tree of total weight $g > 0$. Denote by g^+ (resp.g^-) the sum of the weights of the positive (resp. negative) vertices of $\mathcal{T}$. We may suppose that $g^+ > 0$. There are two cases to consider: a) $g^- > 0$ and b) $g^- = 0$.

a) We may choose a positive vertex v of weight $g_1 > 0$ and a negative vertex w of weight $g_2 > 0$ and join them by a path in the tree (necessarily consisting of an odd number of edges). We may assume that vertices of the path have weight zero (otherwise we could choose a shorter such path). Let $\mathcal{T}'$ be the tree obtained by reducing the path to a single edge vw of $\mathcal{T}'$. The tree $\mathcal{T}'$ also has total weight g. An important observation is that reduction leaves g^+, g^- and $V^+ - V^-$ unchanged (though not V^+ or V^-). Thus $\mathcal{T}'$ is also balanced. Let $\mathcal{T}''$ be the tree $\mathcal{T}'$ with the weights g_1 and g_2 replaced by $g_1 - 1$ and $g_2 - 1$ (Figure 9a)). Thus $\mathcal{T}''$ is also balanced. The total weight of $\mathcal{T}''$ is clearly $g - 2$ so that, by induction, it is the graph of a fold map of a surface to the plane. The connected sum (along the singular curve corresponding to the edge vw) of this fold map with the fold map of the bitorus to the plane illustrated in Figure 1 (c) is a fold map with graph $\mathcal{T}'$. By applying a sequence of double swallowtail + double beaks transitions we create a fold map whose graph is $\mathcal{T}$, as required.

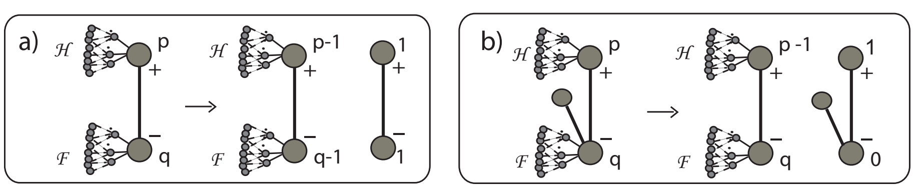

FIGURE 9. Decomposition of trees.

b) In this case, $V^+ - V^- = g^+ > 0$ so that $V^+ > V^-$ and $g^+ < V^+$. Claim: there exists an extreme (i.e. belonging to just one edge) positive vertex of weight zero. Proof of claim: Let L be the number of all positive vertices of weight zero. Then it is easy to see $V^+ - g^+ \le L$ and by the assumption we have $V^- \le L$. Now, suppose that there is no extreme positive vertex of weight zero. Fix a negative vertex n and orient all edges of the tree to be bound for n. Then to each positive vertex p of weight zero we may assign a negative vertex z ($\neq n$) so that zp is an edge pointing toward n. Hence $V^- > L$, that makes a contradiction. This proves the claim. Thus we may choose v a positive extreme vertex of weight zero. Now $g^+ > 0$ so there exists a positive vertex w of weight $g_1 > 0$. There is a path from v to w and we may insist that all vertices of this path between v

and w have weight zero. Since both v and w are positive the length of the path is even so we may reduce $\mathcal{T}$ to a tree $\mathcal{T}'$ in which v and w are connected by a path of length two say vuw. As before, $\mathcal{T}'$ is also balanced. Now let$\mathcal{T}''$ be the tree $\mathcal{T}'$ with the edge uv removed and the weight of w reduced by one to $g_1 - 1$ (recall $g_1 > 0$). $\mathcal{T}''$ is clearly also balanced and of total weight $g - 1$. By hypothesis, $\mathcal{T}''$ is the graph of a fold map. Forming the horizontal connected sum with the fold map from the torus to the plane (illustrated in Figure 2b)) yields a fold map whose graph is $\mathcal{T}'$. Finally, as before, a sequence of double swallowtail + double beaks transitions produces a fold map whose graph is $\mathcal{T}$.

Clearly, in both cases the map f is a fold map from the closed surface M with Euler characteristic $\chi(M) = 2 - 2g$ to the plane. □

Theorem 4.2. *Any bipartite balanced graph is the graph of a fold map from a surface to the plane.*

Proof: As above, it is enough to find some map $f : M \to I\!R^2$ whose graph is the given one. Now observe that given any bipartite graph one may obtain a tree with the same vertices by removal of appropriate edges. Moreover, the graph is balanced if and only if the tree is. We then have from Proposition 4.1 that this tree may be realized by a fold map $f : M \to I\!R^2$, where M is a closed surface with genus equal to the sum of all the weights in the tree. Finally we can apply vertical surgeries on f in order to recover the removed edges, where f may be replaced properly via homotopy of fold maps if necesary. □

We remark that a general result due to Y. Eliashberg (Theorem B, [8]) implies that for any closed non necessarily connected curve C separating a closed orientable surface M into pieces M^+ and M^- with common boundary C, there exists a fold map from M to the plane whose singular set is C if and only if $\chi(M^+) = \chi(M^-)$. We saw in [12] that there is a $1-1$ relation between topological classes of curves in a surface M and weighted graphs satisfying the relation $\chi(M) = 2(\chi(\mathcal{G}) - g)$. Since the condition $\chi(M^+) = \chi(M^-)$ amounts to say that the corresponding graph is balanced, we have that Proposition 4.1 can also be obtained from Eliashberg result. Nevertheless, we emphasize that whereas Eliashberg's techniques guarantee the existence of such a map, those presented here furnish a practical method to construct it.

5. Fold maps with prescribed branching data in the plane

It is a well known fact (see [6] or [15]) that the sum of the winding numbers of the boundary curves of a surface immersed in the plane is equal to the Euler characteristic of the surface. Since we can view a fold map from a surface to the plane as a union of immersed surfaces with boundary, with the boundary curves conveniently identified with the singular set of the map, we can apply this result in order to obtain information on the branch set curves of fold maps from closed surfaces to the plane.

Lemma 5.1. *Any branch curve of a fold map $f : M \to I\!R^2$, whose graph is a (weighted) tree has odd winding number (i.e., an even number of double points).*

Proof: Consider the tree with each edge indexed by one plus the winding number of the corresponding branch curve. At any vertex v, the local sum of the indices must be equal to $\chi(R_v)$, where R_v denotes the region represented by v. Since the graph is a tree there is a vertex v_1 which belongs to just one edge e_1. It follows that the index of e_1 must be equal to $\chi(R_{v_1})+1 = 2-2\omega_1$, where ω_1 is the weight of v_1, and thus even. Removing e_1 we obtain a subtree for which the local sums are also even. By induction on the number of edges of the tree, starting with the case of one edge, the indices of the subtree are all even. In other words, the winding numbers are all odd. □

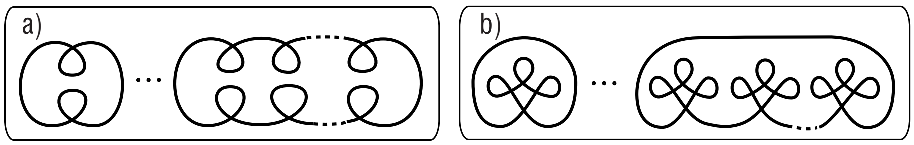

FIGURE 10. Basic plane curves with odd winding numbers.

Figure 10 displays representatives of two different stable isotopy classes (see [3]) with odd winding number. We shall denote them respectively as curves of type $(1,0)$ (10 a)) and $(0,1)$ (10 b)). By a curve of type (a,b) we shall understand a connected sum of a curves of type $(1,0)$ and b curves of type $(0,1)$. We shall refer to these curves as *basic curves*. By a curve of type $(0,0)$ we understand an embedded circle.

Let $\mathcal{T}$ be a weighted tree with vertices $\{v_k\}_{k=1}^n$ and corresponding weights $\{\omega_k\}_{k=1}^n$. We can order the vertices in such a way that $\{v_k\}_{k=1}^r$ are the positive ones and $\{v_k\}_{k=r+1}^n$ the negative. To each edge $v_iv_j, i=1,\cdots,r,\ j=r+1,\cdots,n$, we associate a variable I_{ij}. We write C_k for the sum of the indices I_{kj} for all the edges v_kvj containing v_k.

Lemma 5.2. *The tree $\mathcal{T}$ is balanced if and only if the compatibility conditions*

$$C_k = 2 - 2\omega_k$$

have a unique solution.

Proof: Since $\mathcal{T}$ is a tree the number of edges is $n-1$. The compatibility condition at any vertex v_k is $C_k = 2-2\omega_k$. We thus have a linear system of n equations in $n-1$ variables. On the other hand, we have the conditions,

$$\sum_{i=1}^{r}(C_i - 2 - 2\omega_i) = \sum I_{ij} - 2n = \sum_{j=1+r}^{n}(C_j - 2 - 2\omega_j),$$

where the middle sum runs over all the edges of $\mathcal{T}$.Thus any equation is a consequence of the rest. Now fix a vertex $\star$. For any vertex v_k define d_k to be the length of the (unique) path in the tree between v_k and $\star$. Thus $d_\star = 0,\ \ d_k = 1$ if $v_k\star$ is an edge, for any edge v_iv_j, d_i and d_j differ by one and, for any vertex $v_k \neq \star$ there is a unique edge v_kv_s such that $d_k = d_s+1$. The equation $C_k = 2-2\omega_k$ determines I_{ks} in terms of the other variables i.e. in terms of the I_{ij} for which $d_i = d_j + 1$. For the largest value of d_k, C_k is just I_{ks}, for which $I_{ks} = 2-2\omega_k$ is, of course, the unique solution. Thus the equations $C_k = 2-2\omega_k$ may be solved uniquely for successively smaller values of d_k up to and including $d_1 = 1$. The remaining equation $C_\star = 2-2\omega_*$ is a consequence of the rest. We observe that the solution consists entirely of even integers, corresponding to the fact (already proved) that the winding numbers must all be odd. □

Proposition 5.3. *Any balanced weighted bipartite graph is the graph of a fold map from a surface M to the plane whose branch set consists of basic curves.*

Proof: It is enough to prove the result for a tree, for, given any balanced graph, we may take a maximal tree which will also be balanced. If the tree is the graph of a fold map then by doing vertical surgeries on the fold map we realize the original graph by a fold map. The extra curves introduced into the branch set are all embedded circles hence basic. For a tree the proof goes by induction on the total weight. For zero weighted trees it was shown in [13] by using curves of type $(a, 0), a \in \mathbb{Z}$. Suppose the assertion is true for any balanced tree of total weight g and let $\mathcal{T}$ be a balanced tree with total weight $g+1$. We proceed as in Proposition 4.1 and consider the two cases a) and b) and the corresponding reduced trees. We observe that in both cases, the decomposition of the reduced tree leads to two fold maps:

- f_1, whose branch set is made of a curve of type $(0, 1)$ in case a) and of two curves, one of type $(0, 1)$ and the other of type $(0, 0)$ in case b), and
- f_2, whose graph has total weight lesser than $g+1$ and thus, by the induction hypothesis can be chosen in such a way that all its branch curves are of Type (a, b).

Now observe that their horizontal sum also gives rise to a fold map whose branch curves are of type (a, b). Moreover, the new branch curves produced in the extension process in order to obtain f from $f_1 +_{hor} f_2$ may also be taken in in the family of curves of type (a, b) as can be seen in Figure 11. □

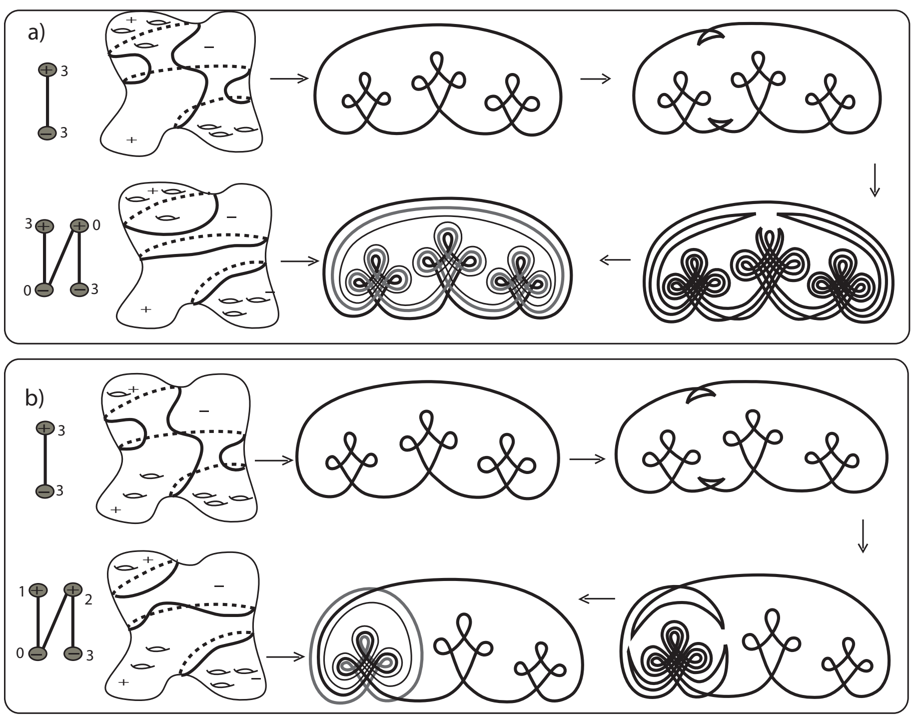

FIGURE 11. Different extensions of a graph.

It can be shown that given a natural number ω and a subset $\{i_1, \cdots, i_k\}$ of odd integers satisfying the relation

$$(i_1 + 1) + \cdots + (i_k + 1) = 2 - 2\omega,$$

we can find an immersion of a surface of genus ω and k boundary components whose respective winding numbers in the plane are $\{i_1, \cdots, i_k\}$. This is proven in a similar way than it was done for discs with holes in [13]. In that case, the family of curves of type $(a, 0)$ was enough to perform all the image curves. Here we must consider all the possible types (a, b), for the curves of type $(0, b)$ contribute to the genus of the considered surface. In fact, for a torus with a unique boundary curve, we can use the curve $(0, 1)$ (as in Figure 2a)) and if the curve has genus ω, then the image curves must be chosen of types $\{(a_1, b_1), \cdots, (a_k, b_k)\}$, with $\omega \leq b_1 + \cdots + b_k$ for different combinations of these curves defining the image of the boundary of surfaces with non zero genus). Figure 12 illustrates an inductive method for constructing the image of the boundary of immersed regions having k boundary components with total winding number i, for all possible compatible integer sets $(i_1 + 1, \cdots, i_k + 1)$, such that $i = i_1 + \cdots + i_k$. This method runs in a similar way to the one used in [13] for fold maps from S^2 to the plane.

In order to construct a fold map corresponding to a given balanced weighted tree we must conveniently assemble different immersed regions whose boundary curves are mapped into a proposed branch set (determined by a given graph).

Proposition 5.4. *Let f be a fold map all whose branch curves are of type (a, b) and suppose that v is an extremal vertex with weight ω. Then the region associated to v has a unique boundary curve whose image by f is of type $(0, \omega)$.*

Proof: Since v is an extreme vertex, there is a unique edge attached to it in the graph. The corresponding branch curve is the image of the boundary of the region R_v represented by v. Supposing that this is a curve of type (a, b), we must have that $a = 0$, for curves wit $a \neq 0$ do not satisfy Blank's criterium in order to be the image of the boundary of immersed regions in the plane [9]. On the other hand, the winding number of this branch curve must coincide with the Euler characteristic of R_v, therefore, $1 - 2\omega = 1 - 2b$ and we have the required result. □

Remark 5.5. *The results of this section can be transported by stereographic projection to fold maps of degree zero from surfaces to S^2.*

6. Biased graphs and Fold maps

Given an integer number d we say that a bipartite weighted graph is *is biased by d* if the following equality holds

$$V^+ - V^- = g^+ - g^- + d,$$

where V^+ and V^- respectively denote the numbers of vértices with positive, and negative labels, and g^+ and g^- the genus of the corresponding regions.

We shall prove now that any bipartite weighted graph $\mathcal{G}$ can be the graph of some fold map whose degree is equal to the bias of $\mathcal{G}$.

Remark 6.1. *We observe that, as illustrated in Figure 13 below, a curve of type $(0, d), d \geq 0$, can be the branch set of a fold map of degree $d'(\geq 0)$ from the surface of genus $2d'' + d'(\geq 0)$ into the sphere, where $d = d' + d''$.*

Any fold map (of a surface into the sphere) has a bipartite graph and degree $(V^+ - V^-) - (g^+ - g^-)$. Conversely,

Theorem 6.2. *Any bipartite weighted graph may be realized by a fold map (of a surface into the sphere).*

Proof: We prove it first for a tree biased by d and then use vertical surgeries, as above, to extend it to any bipartite graph with bias d. Assume that d is positive (resp. negative). Given such a tree $\mathcal{T}$, let v be one of its vertices that we may suppose is a positive (resp. negative) vertex. Consider a new weighted tree, $\mathcal{T}_d$, obtained from $\mathcal{T}$ by adding d to the weight ω of v. Clearly, $\mathcal{T}_d$ is a balanced tree. Then it follows from Propositions 4.1 and 5.3 that there is a zero

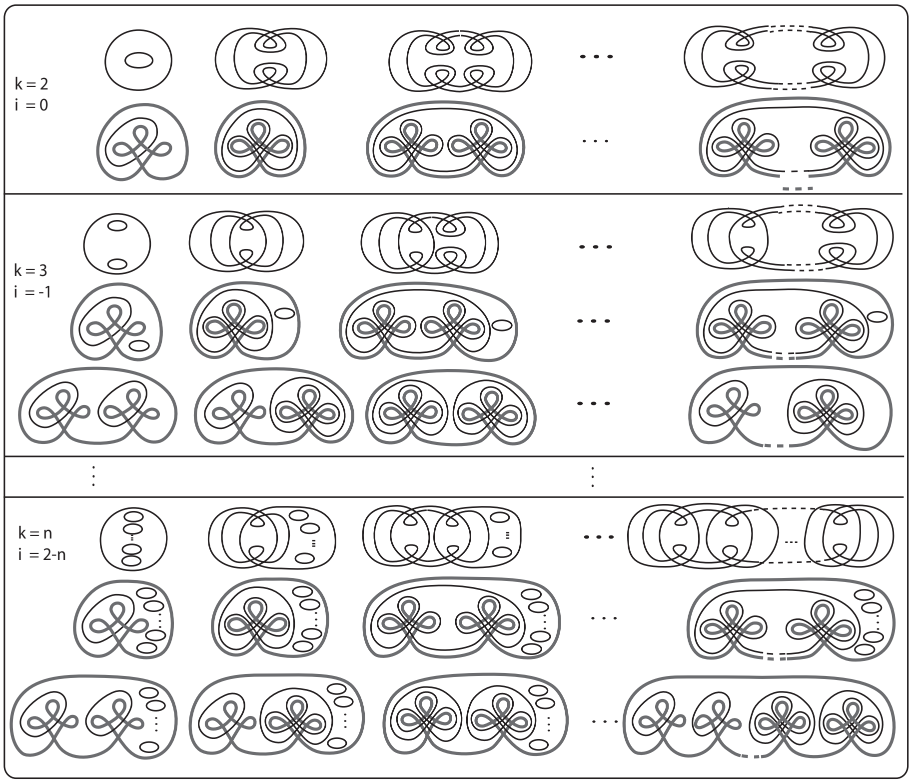

FIGURE 12. Basic curves in the boundary of immersed regions with genus.

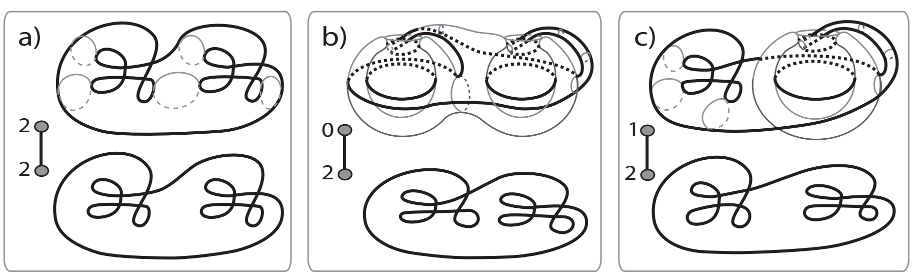

FIGURE 13. genus versus degree.

degree fold map $f : M \to S^2$ whose associated graph is $\mathcal{T}_d$, where $\chi(M) = 2 - 2(g^+ + g^- + d)$ and all the curves in the branch set are of type (a, b). We know from Proposition 5.4 that the branch curve corresponding to the edge e in $\mathcal{T}_d$ must be of type $(0, \omega + d)$. Now, in view of the above remark, we can construct a map $f' : M' \to S^2$ with $\chi(M) = 2 - 2(g^+ + g^-)$, of degree d, without changing the graph and the branch set (see Figure 13. □

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Journal *of* Singularities
Volume 2 (2010), 81-91

Proc. of Singularities in Aarhus
received 2 December 2009
DOI: 10.5427/jsing.2010.2f

CHOW GROUPS AND TUBULAR NEIGHBOURHOODS

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Abstract. We will prove theorems of Zariski-Lefschetz type for the analytic Chow groups of a quasi-projective variety. We will also derive an algebraic analogue, using formal instead of tubular neighbourhoods.

I. In this paper we will look at the algebraic and analytic Chow groups for complex quasi-projective varieties.

First, let X be a scheme over $\mathbb{C}$ of finite type, $k \geq 0$. Then the k-th Chow group $A_k(X)$ is defined as follows: $A_k(X) := C_k(X)/Z_k(X)$. Here $C_k(X)$ is the group of k-cycles in X, i.e. the free abelian group of formal $\mathbb{Z}$-linear combinations of k-dimensional algebraic subvarieties (i.e. closed non-empty reduced and irreducible subschemes) of X, and $Z_k(X)$ is the subspace of $\mathbb{Z}$-linear combinations of elements of the form $div\, f$, where $f \in \mathcal{M}(D)^*$, D a $(k+1)$-dimensional algebraic subvariety of X. Note that $\mathcal{M}(D)$ is the field of rational functions on D and $div\, f$ the divisor of f.

See [Fu] I.1.3, where $A_k(X)$ is called the group of k-cycles modulo rational equivalence. It is reasonable to speak of "Chow groups" because $\oplus_k A_k(X)$ is called "Chow ring" in the non-singular case where we have a ring structure indeed.

If X is everywhere of dimension n we have that $A_{n-1}(X) = Cl(X) :=$ Weil divisor class group = group of Weil divisors modulo principal divisors.

We can define analytic Chow groups, too, for a complex space. However, in the analytic context $C_k(X)$ is defined using locally finite linear combinations instead of finite linear combinations, and $Z_k(X)$ consist of elements $\sum\limits_i div\, f_i$, where $(D_i)_{i\in I}$ is a locally finite set of $(k+1)$-dimensional analytic subvarieties of X and f_i is a non-zero meromorphic function on D_i.

Note that this is not the same definition as in [V] but it is at least reasonable in the following sense: If the complex space X is everywhere of dimension n we have again that $A_{n-1}(X) = Cl(X) :=$ Weil divisor class group.

From now on let X be a closed subscheme of $\mathbb{P}_N(\mathbb{C})$, Y a Zariski-closed subspace of X, and H a hyperplane. The complex space associated to X will be denoted by X^{an}. We assume that X is reduced because this is not an essential restriction.

A Lefschetz type theorem for the Chow groups should compare those of $X \setminus Y$ and $X \cap H \setminus Y$. But looking for such a theorem seems to be very difficult. A considerable simplification is obtained in the analytic context if one replaces the hyperplane section by some neighbourhood ("Zariski-Lefschetz type theorem"). There are two possibilities: first, one can take a fundamental system of neighbourhoods V of $X^{an} \cap H^{an} \setminus Y^{an}$ in $X^{an} \setminus Y^{an}$ and compare $A_k(X^{an} \setminus Y^{an})$ with $A_k(V)$, or one can take a fundamental system of neighbourhoods U of $X^{an} \cap H^{an}$ in X^{an} and compare $A_k(X^{an} \setminus Y^{an})$ with $A_k(U \setminus Y^{an})$. Note that the neighbourhoods $U \setminus Y^{an}$ of

2000 *Mathematics Subject Classification.* 14C15, 32C15.
Key words and phrases. Chow groups, Lefschetz theorem.

$X^{an} \cap H^{an} \setminus Y^{an}$ are big compared with V.

The second alternative has already been studied in [H1] in the special case of the Weil divisor class group: If $\dim X \geq 3$ everywhere we have $Cl(X^{an} \setminus Y^{an}) \simeq Cl(U \setminus Y^{an})$ for some fundamental system of neighbourhoods U of $X^{an} \cap H^{an}$ in X^{an}, see [H1] Theorem 1.2.

II. The analogue of tubular neighbourhoods in the algebraic context is given by formal completion. Let $\hat{X}$ be the formal completion of X along $X \cap H$, see [GD] I §10. Then the formal completion of $X \setminus Y$ along $X \cap H \setminus Y$ is given by $\hat{X} \setminus \hat{Y}$. This is the algebraic analogue of the neighbourhoods V above (in the limit).

This approach in the algebraic context goes back to A.Grothendieck when he studied the Picard group. In fact Grothendieck has proved in [G] a Lefschetz theorem for the Picard group $Pic(X \setminus Y)$ in the case $Y = \emptyset$. This has been generalized in [HL2]. The case where Y is arbitrary has been studied in [HL1] (smooth case) and [HL3] (general case).

Note that $Pic(X \setminus Y) \simeq Cl(X \setminus Y)$ if $X \setminus Y$ is smooth. This could be used in order to derive a Lefschetz theorem for the Weil divisor class group, see [HL1] Theorem 1.5: If $\dim X \geq 4$ everywhere, $\operatorname{codim} Sing\, X \geq 2$ and H is generic we have that $Cl(X) \simeq Cl(X \cap H)$.

When working with formal neighbourhoods we have to make precise what we mean by the dimension: If $\hat{Z}$ is a closed formal subscheme of $\mathbb{P}_N(\mathbb{C}) \setminus Y$, $\dim \hat{Z} \geq k$ everywhere if for all closed points z of $\hat{Z}$ and all associated prime ideals $\mathfrak{p}$ of $\mathcal{O}_{\hat{Z},z}$ we have $\dim \mathcal{O}_{\hat{Z},z}/\mathfrak{p} \geq k$.

Furthermore, a closed formal subscheme $\hat{Z}$ of $\hat{X}$ is called reducible if there are proper formal closed subschemes such that $\hat{Z} = \hat{Z}_1 \cup \hat{Z}_2$, where $\mathcal{J}_1 \cdot \mathcal{J}_2 = 0$ for the ideal sheaves $\mathcal{J}_1, \mathcal{J}_2$ of $\hat{Z}_1$, $\hat{Z}_2$ in $\hat{Z}$. Otherwise, $\hat{Z}$ is called irreducible, of course.

Note that if Z is a subscheme of $\mathbb{P}_N(\mathbb{C}) \setminus Y$ of pure dimension k, $Z \cap H \neq \emptyset$, we have $\dim \hat{Z} = k$, too.

What is the algebraic analogue of neighbourhoods of the form $U \setminus Y^{an}$? It is easier to give a direct definition of the corresponding Chow group than to define an analogue of the space itself. Let us start from a different description of $A_k(X \setminus Y)$ in the algebraic case: we have $A_k(X \setminus Y) \simeq A_k(X, Y) := C_k(X)/(Z_k(X) + C_k(Y))$. The notation might be misleading: obviously we still have an arrow $A_k(X) \to A_k(X, Y)$.

Note that $A_k(X, Y) \simeq C_k(X, Y)/Z_k(X, Y)$ with $C_k(X, Y) := C_k(X)/C_k(Y)$ and $Z_k(X, Y) = Z_k(X)/Z_k(X) \cap C_k(Y) \simeq (Z_k(X) + C_k(Y))/C_k(Y)$, by the isomorphism theorems of group theory.

Then it is natural to define $A_k(\hat{X}, \hat{Y})$ with $\hat{X}, \hat{Y}$ instead of X, Y. Now $A_k(\hat{X}, \hat{Y})$ seems to be the appropriate algebraic analogue of $\lim\limits_{\to} A_k(U \setminus Y^{an})$, as we will see from the results.

We have an analogous notion $A_k(X^{an}, Y^{an})$ in the analytic context which does not, however, coincide necessarily with $A_k(X^{an} \setminus Y^{an})$ in general because analytic subsets of $X^{an} \setminus Y^{an}$ do not necessarily extend to analytic subsets of X^{an}.

III. Now we have all types of Chow groups which we will use at our disposal and can phrase our theorems. As often define $\dim \emptyset := -1$.

In the analytic context we have:

Theorem 1: The mapping $A_k(X^{an} \setminus Y^{an}) \to \lim\limits_{\to} A_k(U \setminus Y^{an})$ is bijective if $k \geq 2$ and injective if $k \geq 1$.

Here U runs through the set of open neighbourhoods of $X^{an} \cap H^{an}$ in X^{an}.

Theorem 1': The mapping $A_k(X^{an}, Y^{an}) \to \varinjlim A_k(U, U \cap Y^{an})$ is bijective if $k \geq 2$ and injective if $k \geq 1$.

Again, U runs through the set of open neighbourhoods of $X^{an} \cap H^{an}$ in X^{an}.

Theorem 2: The mappings $A_k(X^{an} \setminus Y^{an}) \to \varinjlim A_k(U \setminus Y^{an}) \to \varinjlim A_k(V)$ are bijective if $k \geq \dim(Y \cap H) + 3$ and injective if $k \geq \dim(Y \cap H) + 2$.

Here U (resp. V) runs through the set of all open neighbourhoods of $X^{an} \cap H^{an}$ in X^{an} (resp. of $X^{an} \cap H^{an} \setminus Y^{an}$ in $X^{an} \setminus Y^{an}$).

Similarly, in the algebraic context we obtain:

Theorem 3: The mapping $A_k(X \setminus Y) \to A_k(\hat{X}, \hat{Y})$ is bijective if $k \geq 2$ and injective if $k \geq 1$.

Theorem 4: The mappings $A_k(X \setminus Y) \to A_k(\hat{X}, \hat{Y}) \to A_k(\hat{X} \setminus \hat{Y})$ are bijective if $k \geq \dim(Y \cap H) + 3$ and injective if $k \geq \dim(Y \cap H) + 2$.

Remark: In the case $Y = \emptyset$ Theorem 1, 1' and 2 coincide, the same holds for Theorem 3 and 4.

Finally we will compare the algebraic and analytic context, this will make it possible, in particular, to make Theorem 1' more precise. See Remark 3.1 below.

From the literature to be used it is evident that the results in the algebraic context go over to the case of an arbitrary algebraically closed field instead of $\mathbb{C}$.

1. Analytic context: Proof of Theorem 1, 1' and 2

We can identify $\mathbb{P}_N^{an}(\mathbb{C}) \setminus H^{an}$ with $\mathbb{C}^N$. For $R > 0$ let U_R be the complement of $\{z \in \mathbb{C}^N \cap X^{an} \mid \max|z_j| \leq R\}$ in X^{an}. The U_R form a fundamental system of neighbourhoods of $X^{an} \cap H^{an}$ in X^{an}. Fix R.

First let us prove

Lemma 1.1: a) If $k \geq 2$ (resp. $k \geq 1$), for every purely k-dimensional (sc. closed) analytic subset C of $U_R \setminus Y^{an}$ there is exactly (resp. at most) one purely k-dimensional analytic subset C' of $X^{an} \setminus Y^{an}$ such that $C' \cap U_R = C$.

b) The mapping $C_k(X^{an} \setminus Y^{an}) \to C_k(U_R \setminus Y^{an})$ is bijective if $k \geq 2$ and injective if $k \geq 1$.

Proof: a) see Theorem 3.2 in [H1].

b) follows from a).

Lemma 1.2: a) If D is a purely k-dimensional analytic subvariety of $X^{an} \setminus Y^{an}$, $k \geq 2$, every meromorphic function on $D \cap U_R$ extends to a unique meromorphic function on D.

b) $Z_k(X^{an} \setminus Y^{an}) \to Z_k(U_R \setminus Y^{an})$ is bijective if $k \geq 1$.

Proof: a) We modify (and correct) the proof of [H1] Theorem 3.4 which covers the special case where D can be extended to a subvariety of X^{an}:

Let f be a meromorphic function on $U_R \cap D$, and let $p : \tilde{D} \to D$ be the normalization. Let $\tilde{D}_{sing}$ be the singular locus of $\tilde{D}$, $D^* := D \setminus p(\tilde{D}_{sing})$, $\tilde{D}^* := p^{-1}(D^*)$. Let $I_{f\circ p}$ be the set of points of $p^{-1}(U_R \cap D^*)$ where $f \circ p$ is indeterminate. Put $D_R^{**} := U_R \cap D^* \setminus p(I_{f\circ p})$, $\tilde{D}_R^{**} := p^{-1}(D_R^{**})$ and $p_R := p|\tilde{D}_R^{**} : \tilde{D}_R^{**} \to D_R^{**}$. Let $\tilde{W}$ be a sufficiently small neighbourhood of a point in $\tilde{D}_R^{**}$. On $\tilde{W}$, $f\circ p$ can be written in the form g/h where g, h are holomorphic functions on $\tilde{W}$ whose germs are relatively prime. Then (g, h) defines a section of $\mathcal{O}^2_{\tilde{D}}|\tilde{W}$; it generates an invertible $\mathcal{O}_{\tilde{D}}|\tilde{W}$-module which depends only on f. Patching together we obtain an invertible $\mathcal{O}_{\tilde{D}_R^{**}}$-submodule $\mathcal{S}$ of $\mathcal{O}^2_{\tilde{D}_R^{**}}$. Then $(p_R)_*\mathcal{S}$ is an invertible $p_*\mathcal{O}_{\tilde{D}}|D_R^{**}$-submodule of $p_*\mathcal{O}^2_{\tilde{D}}|D_R^{**}$, at the same time we can consider these two sheaves as coherent $\mathcal{O}_{D_R^{**}}$-modules, too.

It is easy to see that $(p_R)_*\mathcal{S}$ coincides with its $(k-1)$-st gap sheaf relative to $p_*\mathcal{O}^2_{\tilde{D}}|D_R^{**}$ (see [S] p. 132): Let W be an open set in D_R^{**} and A an analytic subset of W of dimension $\leq k-1$. Let s be a section of $p_*\mathcal{O}^2_{\tilde{D}}|W$ such that $s|W \setminus A$ is a section of $(p_R)_*\mathcal{S}$. Then s can be considered as an element of $\Gamma(p^{-1}(W), \mathcal{O}^2_{\tilde{D}})$ whose restriction to $p^{-1}(W \setminus A)$ is a section in $\mathcal{S}$. The latter can be uniquely extended to an element of $\Gamma(p^{-1}(W), \mathcal{S})$ which has to coincide with $s \in \Gamma(p^{-1}(W), \mathcal{O}^2_{\tilde{D}})$.

Therefore $(p_R)_*\mathcal{S}$ can be extended to a coherent $\mathcal{O}_{U_R\cap D}$-submodule of $p_*\mathcal{O}^2_{\tilde{D}}|U_R \cap D$ with analogous properties, by the subsheaf extension theorem, see [ST], first part of the proof of Theorem 1b. Note that the resulting sheaf can be considered after trivial extension as a coherent $\mathcal{O}_{U_R\setminus Y}$-module, too.

By Theorem 3.3 of [H1] the subsheaf above can be uniquely extended to a coherent $\mathcal{O}_{X\setminus Y}$-submodule of $p_*\mathcal{O}^2_{\tilde{D}}$ which coincides with its $(k-1)$-st relative gap sheaf; note that $k-1 \geq 1$ because $k \geq 2$. Of course, it must be the trivial extension of a coherent $\mathcal{O}_D$-submodule $\mathcal{T}$ of $p_*\mathcal{O}^2_{\tilde{D}}$.

There is a discrete subset Σ of D such that $\mathcal{T}|D \setminus \Sigma$ is even a $p_*\mathcal{O}_{\tilde{D}}|D \setminus \Sigma$-module: note that we have a multiplication mapping $p_*\mathcal{O}_{\tilde{D}} \otimes_{\mathcal{O}_D} \mathcal{T} \to p_*\mathcal{O}^2_{\tilde{D}}$ whose image is contained in $\mathcal{T}$ if we restrict to $U_R \cap D$. Then use Lemma 3.1 of [H1]. (Note that $X \subset Y$ should be replaced by $X \setminus Y$ there.)

Now $\mathcal{T}|D \setminus \Sigma$ is finite as a $\mathcal{O}_{D\setminus\Sigma}$-module, hence as a $p_*\mathcal{O}_{\tilde{D}}|D \setminus \Sigma$-module. As such it is coherent, and its restriction to $U_R \cap D$ is invertible outside some analytic subset of codimension ≥ 2. Therefore $\mathcal{T}|D \setminus \Sigma$ is an invertible $p_*\mathcal{O}_{\tilde{D}}|D \setminus \Sigma$-module, too, outside some analytic subset of codimension ≥ 2, after enlarging Σ if necessary: Otherwise there would be an irreducible analytic subset of $D \setminus \Sigma$ of dimension $\geq k-1 > 0$ where $\mathcal{T}$ is not invertible. Note that this irreducible subset could be continued to an analytic subset of D, by the theorem of Remmert-Stein ([GR] Theorem V D 5). Then use Lemma 3.1 of [H1] again.

Let D^{**} be the subset of $D^* \setminus \Sigma$ where $\mathcal{T}$ is invertible. If (g, h) is a local generator we obtain using g/h a meromorphic function on D^{**} which can be uniquely continued to a meromorphic function on $D^* \setminus \Sigma$, hence on $D \setminus \Sigma$ and finally on D, by the Kontinuitätssatz [KK] 53.A.9. This gives the desired extension of f.

b) Suppose that f is a meromorphic function on D, where D is a purely $(k+1)$-dimensional analytic subset of $U_R \setminus Y^{an}$. By Lemma 1.1a), there is exactly one purely $(k+1)$-dimensional analytic subset D' of $X^{an} \setminus Y^{an}$ such that $D' \cap U_R = D$. By a) we may extend f to exactly one meromorphic function on D'. The rest is clear.

Proof of Theorem 1: First assume that $k \geq 2$. By Lemma 1.1b), the mapping $C_k(X^{an} \setminus Y^{an}) \to C_k(U_R \setminus Y^{an})$ is bijective. By Lemma 1.2b), $Z_k(X^{an} \setminus Y^{an}) \to Z_k(U_R \setminus Y^{an})$ is bijective. This implies that $A_k(X^{an} \setminus Y^{an}) \to A_k(U_R \setminus Y^{an})$ is bijective, hence $A_k(X^{an} \setminus Y^{an}) \to \varinjlim A_k(U \setminus Y^{an})$, too.

Now assume only $k \geq 1$. Then we know that $C_k(X^{an} \setminus Y^{an}) \to C_k(U_R \setminus Y^{an})$ is injective, whereas $Z_k(X^{an} \setminus Y^{an}) \to Z_k(U_R \setminus Y^{an})$ is bijective. This implies that $A_k(X^{an} \setminus Y^{an}) \to A_k(U_R \setminus Y^{an})$ is injective, hence $A_k(X^{an} \setminus Y^{an}) \to \varinjlim A_k(U \setminus Y^{an})$, too.

Proof of Theorem 1': We apply Lemma 1.1 and Lemma 1.2 in the case $Y = \emptyset$. According to Lemma 1.1 we have that for every purely k-dimensional analytic subset C of U_R there is exactly (resp. at most) one purely k-dimensional analytic subset C' of X^{an} such that $C' \cap U_R = C$. If no irreducible component of C is contained in Y^{an} we know that the same holds for C', too. So we obtain that $C_k(X^{an}, Y^{an}) \to C_k(U_R, Y^{an} \cap U_R)$ is bijective (resp. injective).

Similarly, if $k \geq 1$ and D is an analytic subvariety of X of dimension $k+1$, we can extend D to exactly one analytic subvariety of X^{an} of dimension $k+1$, and if f is meromorphic on D we can extend f to D'. Again, if D is not contained in Y^{an}, D' is not contained in Y^{an}, too. Therefore $Z_k(X^{an}, Y^{an}) \simeq Z_k(U_R, U_R \cap Y^{an})$. Altogether we obtain Theorem 1'.

Now let us turn to the proof of Theorem 2. Suppose that $k \geq \dim(Y \cap H) + 2$, so $k \geq \dim Y + 1$, and that U is an open neighbourhood of $X^{an} \cap H^{an}$ in X^{an}. As we will see in Proposition 3.2, $A_k(X^{an}) \simeq A_k(X^{an} \setminus Y^{an})$; with the same techniques we have $A_k(U) \simeq A_k(U \setminus Y^{an})$.

Therefore we can suppose in the proof of Theorem 2 that $Y \subset H$. Furthermore we can assume $Y \neq \emptyset$ because otherwise Theorem 2 coincides with Theorem 1 and 1'.

Let $\mathcal{A}_k$ be the sheaf of purely k-dimensional analytic subsets on X^{an}: if W is open in X^{an} let $\Gamma(W, \mathcal{A}_k)$ be the set of all closed purely k-dimensional analytic subsets of W. If A is a locally closed subset of X^{an} we have $\Gamma(A, \mathcal{A}_k) = \varinjlim \Gamma(W, \mathcal{A}_k)$ where W runs through the set of all open neighbourhoods of A in X^{an}: this follows from [Go] II 3.3 Corollaire 1.

Lemma 1.3: The mapping $\Gamma(X^{an} \cap H^{an}, \mathcal{A}_k) \to \Gamma(X^{an} \cap H^{an} \setminus Y^{an}, \mathcal{A}_k)$ is bijective if $k \geq \dim Y + 3$ and injective if $k \geq \dim Y + 2$.

Proof: We may suppose $X = \mathbb{P}_N$. It is sufficient to show that the mapping

$$\Gamma(X^{an} \cap H^{an} \setminus (Y')^{an}, \mathcal{A}_k) \to \Gamma(X^{an} \cap H^{an} \setminus (Y'')^{an}, \mathcal{A}_k)$$

is bijective resp. injective if $Y'' \subset Y' \subset Y$ and $Y' \setminus Y''$ is smooth of dimension $l \leq \dim Y$.

Let $j : X^{an} \cap H^{an} \setminus (Y')^{an} \to X^{an} \cap H^{an} \setminus (Y'')^{an}$ be the inclusion. Then it suffices to show that the mapping

$$j_*(\mathcal{A}_k | X^{an} \cap H^{an} \setminus (Y')^{an}) \to \mathcal{A}_k | X^{an} \cap H^{an} \setminus (Y'')^{an})$$

is bijective resp. injective.

We have to show this at every point of $(Y')^{an} \cap H^{an} \setminus (Y'')^{an}$. Choose local coordinates $z_1, \ldots, z_N$ centered at this point such that Y'^{an} is locally described by $z_{l+1} = \ldots = z_N = 0$ and H^{an} by $z_N = 0$. Fix $\epsilon_0 = \delta_0 > 0$ sufficiently small. For $0 < \epsilon, \delta < \epsilon_0$ put $W_{\epsilon,\delta} := \{z \,|\, |z_j| < \epsilon_0, j = 1, \ldots, l, \epsilon < \max(|z_{l+1}|, \ldots, |z_{N-1}|) < \epsilon_0, |z_N| < \delta\}$. Let $(\epsilon_\nu)_{\nu \geq 1}, (\delta_\nu)_{\nu \geq 1}$ be strictly monotonously decreasing sequences of positive real numbers which converge to 0, where $\epsilon_1 \leq \epsilon_0, \delta_1 \leq \delta_0$, and put $W := \bigcup_{\nu=1}^{\infty} W_{\epsilon_\nu, \delta_\nu}$. Note that the closures of the sets W obtained in this way form a fundamental system of neighbourhoods of $\{z\,|\,|z_j| \le \epsilon_0, j = 1,\dots,N, (z_{l+1},\dots,z_{N-1}) \neq 0, z_N = 0\}$ in $\{z\,|\,|z_j| \le \epsilon_0, j = 1,\dots,N, (z_{l+1},\dots,z_N) \neq 0\}$.

Now it is sufficient to show: Every purely k-dimensional closed analytic subset of W admits exactly (resp. at most) one extension to a closed analytic subset of $\{z\,|\,|z_j| < \epsilon, j = 1,\dots,N-1, |z_N| < \delta_1\}$. (*)

Here we proceed similarly as in the proof of Lemma 9 in [H2]. The essential point is the following: Every purely k-dimensional analytic subset of $\{z\,|\,|z_j| < \epsilon_0, j = 1,\dots,l, \epsilon_\nu < \max(|z_{l+1}|,\dots,|z_{N-1}|) < \epsilon_0, \delta_{\nu+2} < |z_N| < \delta_1\}$ admits exactly (resp. at most) one extension to a purely k-dimensional analytic subset of $\{z\,|\,|z_j| < \epsilon_0, j = 1,\dots,l, \max(|z_{l+1}|,\dots,|z_{N-1}|) < \epsilon_0, \delta_{\nu+2} < |z_N| < \delta_1\}$.

But this is just a consequence of [S] Theorem 2.18 resp. Lemma 2.17.

By induction, this makes it possible to extend every purely k-dimensional analytic subset of W to exactly (resp. at most) one purely k-dimensional analytic subset of $W \cup \{z\,|\,|z_j| < \epsilon_0, j = 1,\dots,N-1, \delta_\nu < |z_N| < \delta_1\} \cup \{z\,|\,|z_j| < \epsilon_0, j = 1,\dots,N-1, \max(|z_{j+1}|,\dots,|z_{N-1}| > \epsilon_\nu, |z_N| < \delta_1\}$, hence of $\{z\,|\,|z_j| < \epsilon_0, j = 1,\dots,N-1, |z_N| < \delta_1\} \setminus Y'$, or of $\{z\,|\,|z_j| < \epsilon_0, j = 1,\dots,N-1, |z_N| < \delta_1\}$, by the extension theorem of Remmert-Stein ([GR] Theorem V D 5). This implies (*).

As a consequence we obtain the following Lefschetz type theorem:

Theorem 1.4: The mapping $\Gamma(X^{an} \setminus Y^{an}, \mathcal{A}_k) \to \Gamma(X^{an} \cap H^{an} \setminus Y^{an}, \mathcal{A}_k)$ is bijective if $k \ge \dim Y + 3$ and injective if $k \ge \dim Y + 2$.

Proof: By Lemma 1.1, $\Gamma(X^{an}, \mathcal{A}_k) \simeq \Gamma(X^{an} \cap H^{an}, \mathcal{A}_k)$. Using Lemma 1.3 we conclude that $\Gamma(X^{an}, \mathcal{A}_k) \to \Gamma(X^{an} \cap H^{an} \setminus Y^{an}, \mathcal{A}_k)$ is bijective (resp. injective). By the theorem of Remmert-Stein ([GR] Theorem V D 5), $\Gamma(X^{an}, \mathcal{A}_k) \simeq \Gamma(X^{an} \setminus Y^{an}, \mathcal{A}_k)$.

Now let us look at meromorphic functions:

Lemma 1.5: If $k \ge \dim Y + 3$ and D is a k-dimensional subvariety of X we have that $\Gamma(D^{an} \cap H^{an}, \mathcal{M}_{D^{an}}) \simeq \Gamma(D^{an} \cap H^{an} \setminus Y^{an}, \mathcal{M}_{D^{an}})$.

Proof: Replacing Y by $Y \cap D$ we may assume that $Y \subset D$.

Let us take up the notations of the proof of Lemma 1.3. Then it is sufficient to show:

$$j_*(\mathcal{M}_k | D^{an} \cap H^{an} \setminus (Y')^{an}) \simeq \mathcal{M}_k | D^{an} \cap H^{an} \setminus (Y'')^{an}$$

Again, it suffices to show that every meromorphic function on $W \cap D^{an}$ extends (uniquely) to a meromorphic function on $D^{an} \cap \{z\,|\,|z_j| < \epsilon_0, j = 1,\dots,N-1, |z_N| < \delta_1\}$. The essential point is to show that every meromorphic function on $D^{an} \cap \{z\,|\,|z_j| < \epsilon_0, j = 1,\dots,l, \epsilon_\nu < \max(|z_{l+1}|,\dots,|z_{N-1}|) < \epsilon_0, \delta_{\nu+2} < |z_N| < \delta_1\}$ admits exactly one meromorphic extension on $D^{an} \cap \{z\,|\,|z_j| < \epsilon_0, j = 1,\dots,l, \max(|z_{l+1}|,\dots,|z_{N-1}|) < \epsilon_0, \delta_{\nu+2} < |z_N| < \delta_\nu\}$. (**)

If we have this we proceed as in the proof of Lemma 1.3: Every meromorphic function on $D^{an} \cap W$ admits exactly one meromorphic extension to $D^{an} \cap \{z\,|\,|z_j| < \epsilon_0, j = 1,\dots,N-1, |z_N| < \delta_1\} \setminus Y'$, hence to $D^{an} \cap \{z\,|\,|z_j| < \epsilon_0, j = 1,\dots,N-1, |z_N| < \delta_1\}$, by the Kontinuitätssatz, see [KK] 53.A.9.

In order to prove (**) we proceed as in the proof of Lemma 1.2, case $k \ge 3$. The essential point is to show the following lemma:

Lemma 1.6: Let $\mathcal{G}$ be a coherent analytic sheaf on $\{z \,|\, |z_j| < \epsilon_0, j = 1, \ldots, l, \max(|z_{l+1}|, \ldots, |z_{N-1}|) < \epsilon_0, \delta_{\nu+2} < |z_N| < \delta_1\}$ and $\mathcal{F}$ a coherent analytic subsheaf of $\mathcal{G}|\{z \,|\, |z_j| < \epsilon_0, j = 1, \ldots, l, \epsilon_\nu < \max(|z_{l+1}|, \ldots, |z_{N-1}|) < \epsilon_0, \delta_{\nu+2} < |z_N| < \delta_1\}$. Assume that for all open subsets W of $\{z \,|\, |z_j| < \epsilon_0, j = 1, \ldots, l, \epsilon_\nu < \max(|z_{l+1}|, \ldots, |z_{N-1}|) < \epsilon_0, \delta_{\nu+2} < |z_N| < \delta_1\}$ and all analytic subsets A of W with dim $A \leq l+1$ the following holds:

Every section of $\mathcal{G}|W$ whose restriction to $W \setminus A$ belongs to $\mathcal{F}|W \setminus A$ is a section of $\mathcal{F}|W$.

Then $\mathcal{F}$ extends uniquely to a coherent analytic subsheaf of $\mathcal{G}$ with the analogous property.

Proof: Apply [S] Theorem 4.5, p. 156, with $n = l + 1$.

Therefore we get the following Lefschetz theorem for meromorphic functions:

Theorem 1.7: If $k \geq \dim Y + 3$ and D is a k-dimensional subvariety of X not contained in Y we have that $\Gamma(D^{an} \setminus Y^{an}, \mathcal{M}_{D^{an}}) \simeq \Gamma(D^{an} \cap H^{an} \setminus Y^{an}, \mathcal{M}_{D^{an}})$.

Proof: By the theorem of Remmert-Stein, we have $\Gamma(D^{an}, \mathcal{M}_{D^{an}}) \simeq \Gamma(D^{an} \setminus Y^{an}, \mathcal{M}_{D^{an}})$. The rest follows from Lemma 1.2 and 1.5.

Proof of Theorem 2: By Theorem 1.4, $C_k(X^{an} \setminus Y^{an}) \to \lim\limits_{\to} C_k(V)$ is bijective (resp. injective).

Furthermore, Theorem 1.7 implies that $Z_k(X^{an} \setminus Y^{an}) \simeq \lim\limits_{\to} Z_k(V)$.

This implies that the mapping $A_k(X^{an} \setminus Y^{an}) \to \lim\limits_{\to} A_k(V)$ is bijective (resp. injective). By Theorem 1 we have $A_k(X^{an} \setminus Y^{an}) \simeq \lim\limits_{\to} A_k(U \setminus Y^{an})$. Note that we have assumed $Y \subset H, Y \neq \emptyset$.

2. Algebraic context: Proof of Theorem 3 and 4

Here we need the following two lemmas:

Lemma 2.1: If $k \geq \dim Y + 3$, for every k-dimensional formal subvariety (i.e. non-empty closed irreducible reduced formal subscheme) C of $\hat{X} \setminus \hat{Y}$ there is exactly one subvariety C' of $X \setminus Y$ such that $\hat{C}' = C$.

Proof: Existence: C is also a formal subvariety of $\hat{\mathbb{P}}_N(\mathbb{C}) \setminus \hat{Y}$. Then apply Corollary 6 of [F1] with Y instead of Z: there is an extension of C to a closed subscheme C' of $\mathbb{P}_N(\mathbb{C}) \setminus Y$, "extension" means that $\hat{C}' = C$. Replacing C' by $C' \cap X$ if necessary we may suppose that C' is a closed subscheme of $X \setminus Y$. We may take C' to be reduced. If we take an irreducible component C'_0 with $\hat{C}'_0 \neq \emptyset$ we get that $\hat{C}'_0 = C$, so there is an extension to a subvariety of $X \setminus Y$. The uniqueness is clear.

Lemma 2.2: If D is a $(k+1)$-dimensional subvariety of $X \setminus Y$, $k \geq \dim Y + 2$, every rational function on $\hat{D}$ extends to a (unique) rational function on D.

Proof: This follows from [F1] Corollary 3 with Y instead of Z.

Proof of Theorem 3: Apply Lemma 2.1 and 2.2 with $Y := \emptyset$.

First suppose that $k \geq 1$. If C' is a k-dimensional subvariety of X not contained in Y we have that $C' \cap H \neq \emptyset$, so $\hat{C}' \neq \emptyset$, end $\hat{C}' \not\subset \hat{Y}$ because otherwise $C' \subset Y$. This implies that

$C_k(X\backslash Y) \to C_k(\hat{X}, \hat{Y})$ is injective. By Lemma 2.1 and 2.2 we obtain that $Z_k(X\backslash Y) \to Z_k(\hat{X}, \hat{Y})$ is bijective. So we obtain injectivity.

Now suppose $k \geq 2$. By Lemma 2.1 and 2.2, for every k-dimensional formal subvariety C of $\hat{X}$ not contained in $\hat{Y}$ there is exactly one subvariety C' of X such that $\hat{C}' = C$; in fact, we have $C' \subset X \setminus Y$. Similarly, if f is a rational function on a $(k+1)$-dimensional formal subvariety C of $\hat{X} \setminus \hat{Y}$, we have a unique subvariety D' of $X \setminus Y$ with $\hat{D}' = D$ and a unique rational function on D' which induces f. In total we obtain bijectivity.

Proof of Theorem 4: Using Lemma 2.1 and 2.2 we get that $C_k(X \setminus Y) \to C_k(\hat{X} \setminus \hat{Y})$ is bijective (resp. injective) and that $Z_k(X \setminus Y) \to Z_k(\hat{X} \setminus \hat{Y})$ is bijective. Note that a subvariety C' of $X \setminus Y$ of dimension $\geq \dim Y + 2$ must intersect $H \setminus Y$, so $\hat{C}' \neq \emptyset$ in $\hat{X} \setminus \hat{Y}$. We conclude that $A_k(X \setminus Y) \to A_k(\hat{X} \setminus \hat{Y})$ is bijective (resp. injective).

Furthermore, $A_k(X \setminus Y) \simeq A_k(\hat{X}, \hat{Y})$ by Theorem 3. So we obtain Theorem 4.

3. Remarks on the comparison of the analytic and algebraic context

The comparison is especially simple in the case of $A_k(X, Y)$ and the corresponding analytic object. If we pass to the formal context it seems that the following assertion (*) is considered as a consequence of GAGA theory, see [F2] p. 737 resp. [B] §10, p. 115:

a) For every formal analytic subvariety C of $\hat{X}^{an}$ there is exactly one formal subvariety C' of $\hat{X}$ such that $(C')^{an} = C$.
b) Let D be a formal subvariety of $\hat{X}$. Every formal meromorphic function f on D^{an} is rational, i.e. there is a (unique) formal rational function g on D such that $g^{an} = f$. (*)

Remark 3.1: Adopting (*) we have a commutative diagram

$$\begin{array}{ccccc} A_k(X,Y) & & \longrightarrow & & A_k(\hat{X},\hat{Y}) \\ \downarrow\simeq & & & & \downarrow\simeq \\ A_k(X^{an},Y^{an}) & \to & \lim\limits_{\to} A_k(U, U\cap Y^{an}) & \to & A_k(\hat{X}^{an},\hat{Y}^{an}) \end{array}$$

where all arrows are bijective if $k \geq 2$ resp. injective if $k \geq 1$.

Here U runs through the set of open neighbourhoods of $X^{an} \cap H^{an}$ in X^{an}.

Proof: By Chow's theorem ([GR] Theorem V D 7), analytic subvarieties of X^{an} are algebraic. Therefore it is easy to see that $C_k(X, Y) \simeq C_k(X^{an}, Y^{an})$. Now let D be a subvarity of X. By Hurwitz' theorem, see [Fi] 4.7, every meromorphic function on D^{an} is rational, i.e. comes from a (unique) rational function on D. Therefore $Z_k(X, Y) \simeq Z_k(X^{an}, Y^{an})$. Altogether, the left vertical arrow is bijective.

By (*) it is easy to see that $C_k(\hat{X}, \hat{Y}) \simeq C_k(\hat{X}^{an}, \hat{Y}^{an})$ and $Z_k(\hat{X}, \hat{Y}) \simeq Z_k(\hat{X}^{an}, \hat{Y}^{an})$, so the right vertical is bijective, too.

The upper arrow is bijective (resp. injective) by Theorem 3.

So the composition of the lower horizontal mappings is bijective (resp. injective).

By Theorem 1', the first lower horizontal arrow is bijective (resp. injective). If $k \geq 2$ we obtain our statement. But in order to treat the case $k = 1$ we need that the second lower horizontal arrow is injective in this case, too. This can easily be proved: Let $k \geq 1$. Every purely k-dimensional analytic subvariety C' of U_R is uniquely determined by its completion $\hat{C}'$, so $C_k(U_R, U_R \cap Y^{an}) \to C_k(\hat{X}^{an}, \hat{Y}^{an})$ is injective, and $Z_k(U_R, U_R \cap Y^{an}) \simeq Z_k(\hat{X}^{an}, \hat{Y}^{an})$: the injectivity is clear, the surjectivity comes from that of $Z_k(X^{an}, Y^{an}) \to Z_k(\hat{X}^{an}, \hat{Y}^{an})$: we have

$Z_k(X^{an}, Y^{an}) \simeq Z_k(X,Y) \simeq Z_k(\hat{X},\hat{Y}) \simeq Z_k(\hat{X}^{an},\hat{Y}^{an})$. This makes the proof of Theorem 1' superfluous!

It is plausible that we should have a connection between the algebraic and analytic case with respect to Theorem 2 and 4, too. First notice:

Proposition 3.2: If $k \geq \dim Y + 1$ we have a commutative diagram

$$\begin{array}{ccc} A_k(X) & \simeq & A_k(X \setminus Y) \\ \downarrow\simeq & & \downarrow\simeq \\ A_k(X^{an}) & \simeq & A_k(X^{an} \setminus Y^{an}) \end{array}$$

Proof: By the theorem of Remmert-Stein ([GR] Theorem V D 5), irreducible analytic subsets of $X^{an} \setminus Y^{an}$ of dimension $\geq \dim Y + 1$ extend to X^{an}. By Chow ([GR] V D 7), analytic subsets of X^{an} are algebraic.

Of course, Zariski-closed subsets of $X \setminus Y$ extend to X.

On the other hand, if D is an irreducible subvariety of X of dimension $\geq \dim Y + 2$, every meromorphic function on $D^{an} \setminus Y^{an}$ is meromorphic on D^{an} by the Kontinuitätssatz [KK] 53.A.9. Meromorphic functions on X^{an} are rational by Hurwitz' Theorem, see [Fi] 4.7. Note that rational functions on $X \setminus Y$ coincide wth those on X.

Now let us state the following conjecture:

Conjecture 3.3: The mapping $A_k(\hat{X} \setminus \hat{Y}) \to A_k(\hat{X}^{an} \setminus \hat{Y}^{an})$ is bijective if $k \geq \dim(Y \cap H) + 3$ and injective if $k \geq \dim(Y \cap H) + 2$.

Remark 3.4: Suppose that Conjecture 3.3 holds. Then we have a commutative diagram

$$\begin{array}{ccccc} A_k(X \setminus Y) & & \longrightarrow & & A_k(\hat{X} \setminus \hat{Y}) \\ \downarrow & & & & \downarrow \\ A_k(X^{an} \setminus Y^{an}) & \to & \lim\limits_{\to} A_k(V) & \to & A_k(\hat{X}^{an} \setminus \hat{Y}^{an}) \end{array}$$

where all arrows are bijective if $k \geq \dim(Y \cap H) + 3$ resp. injective if $k \geq \dim(Y \cap H) + 2$.

Here V runs through the set of all open neighbourhoods of $X^{an} \cap H^{an} \setminus Y^{an}$ in $X^{an} \setminus Y^{an}$.

Proof: By Proposition 3.2, the left vertical is bijective. Now Conjecture 3.3 yields that the right vertical is bijective (resp. injective).

The upper horizontal is bijective (resp. injective) because of Theorem 4.

So the composition of the lower horizontal mappings is bijective (resp. injective).

Now suppose $k \geq \dim Y \cap H + 3$. Then the first mapping in the lower horizontal is bijective, by Theorem 2. Altogether this implies that all arrows are bijective.

However we can argue in a simpler way which would lead (if Conjecture 3.3 holds) to a new proof of Theorem 2 and allows to treat the case $k = \dim Y \cap H + 2$, too: It is easy to see that the second arrow in the lower horizontal is injective for $k \geq \dim Y \cap H + 2$.

Every purely k-dimensional analytic subset C of V is uniquely determined by its completion $\hat{C}$, so $\lim\limits_{\to} C_k(V) \to C_k(\hat{X}^{an} \setminus \hat{Y}^{an})$ is injective. Also, $\lim\limits_{\to} Z_k(V) \to Z_k(\hat{X}^{an} \setminus \hat{Y}^{an})$ is surjective: this follows from $Z_k(X^{an} \setminus Y^{an} \simeq Z_k(X \setminus Y) \simeq Z_k(\hat{X} \setminus \hat{Y}) \simeq Z_k(\hat{X}^{an} \setminus \hat{Y}^{an})$. This yields the desired injectivity.

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Journal *of* Singularities
Volume 2 (2010), 92-127

Proc. of Singularities in Aarhus
received 28 September 2009
DOI: 10.5427/jsing.2010.2g

The mandala of Legendrian dualities for pseudo-spheres in Lorentz-Minkowski space and "flat" spacelike surfaces

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Abstract

Using the Legendrian dualities between surfaces in pseudo-spheres in Lorentz-Minkowski 4-space, we study various kind of flat surfaces in pseudo-spheres. We consider a surface in the pseudo-sphere and its dual surface. Flatness of a surface is defined by the degeneracy of the dual surface similar to the case for the Gauss map of a flat surface in the Euclidean space. We study singularities of these flat surfaces and dualities of singularities.

1 Introduction

It has been shown in [25] that a theorem of Legendrian dualities for pseudo-spheres in Lorentz-Minkowski space which gives a commutative diagram between contact manifolds defined by the dual relations. This theorem has been generalized into pseudo-spheres in semi-Euclidean space with general index in [10]. Such a commutative diagram is called a *mandala of Legendrian dualities* now [10, 26]. The mandala of Legendrian dualities is very useful for the study of the differential geometry on submanifolds in pseudo-spheres. Especially, it works well even for spacelike hypersurfaces in the lightcone where the induced metric is degenerate[25].

In this paper we consider various kinds of flatness of surfaces in pseudo-spheres in Lorentz-Minkowski space. In Euclidean space, a flat surface is characterized by the degeneracy of the Gauss map. For example, a surface is a part of a plane if the Gauss map is constant. Moreover, a surface is a developable surface if the image of the Gauss map is a point or a curve (i.e., all points of the surface are singularities of the Gauss map). We remark that the dual surface of a surface plays similar roles to those of the Gauss map of the surface [24, 31]. According to these facts on the Euclidean case, the Legendrian dual of a surface in pseudo-sphere is considered to be a kind of the Gauss map of the surface. In this sense a surface in a pseudo-sphere is "flat"if the Legendrian dual is singular at any point of the surface. Especially, we consider the case when the Legendrian dual is a curve in a pseudo-sphere. In [22] we have studied a surface in Hyperbolic space whose lightcone dual is a curve. In this case the surface is called a *horo-flat surface*. Moreover, such surfaces are one-parameter families of horo-cycles. Therefore, we call it a *horo-flat horocyclic surface*. Horo-flat surfaces are "flat"surfaces in the sense of a new geometry in Hyperbolic space[5, 6, 17, 18, 19, 22] which is called "Horospherical Geometry". In this paper we consider surfaces with similar properties as horo-flat horo-cyclic

2000 Mathematics Subject classification. 53A35, 57R45, 53C42
Keywords and Phrase. Legendrian duality, Horo-spherical geometry, Horo-flat surfaces, Singularities, Lorentz-Minkowski space, Pseudo-spheres

surfaces in other pseudo-spheres. These surfaces can be obtained by the aid of the mandala of Legendrian dualities. One of the main results in this paper is to give classifications of the singularities of these surfaces and show dualities among singularities. Therefore, the mandala of Legendrian dualities still remains on the singularities level. As a consequence, these surfaces are *frontals* which are the projection images of isotropic maps into the total contact manifold of a Legendrian fibration. If the isotropic map is a Legendrian immersion, the frontal is called a *wave front* (or, simply a *front*).

Singularities of wave fronts have been originally investigated by Zakalyukin[34, 35]. See [2] for the detail. He has shown that generic singularities of wave front surfaces are the cuspidal edge and the swallowtail. It is known that generic singularities of frontal surfaces are the cuspidal cross cap in addition to the above two fronts [14, 15].

Here, the *cuspidal edge* is a map germ $((\mathbb{R}^2; u, v), \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ defined by $(u, v) \mapsto (u, v^2, v^3)$ at the origin, the *swallowtail* is a map germ $((\mathbb{R}^2; u, v), \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ defined by $(u, v) \mapsto (u, 3v^4 + u^2v, 4v^3 + 2uv)$ and the *cuspidal cross cap* is a map germ $((\mathbb{R}^2; u, v), \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ defined by $(u, v) \mapsto (u, v^2, uv^3)$ at the origin. Furthermore, the dual surfaces have the more degenerate singularities which called the cuspidal lips or the cuspidal beaks and the cuspidal butterfly. The *cuspidal lips* (resp. *cuspidal beaks*) is a map germ $((\mathbb{R}^2; u, v), \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ defined by $(u, v) \mapsto (u, -2v^3 + u^2v, 3v^4 - u^2v^2)$ (resp. $(u, v) \mapsto (u, -2v^3 - u^2v, 3v^4 - u^2v^2)$). The *cuspidal butterfly* is a map germ $((\mathbb{R}^2; u, v), \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ defined by $(u, v) \mapsto (u, 5v^4 + 2uv, 4v^5 + uv^2 - u^2)$. We can draw the pictures of these singularities here.

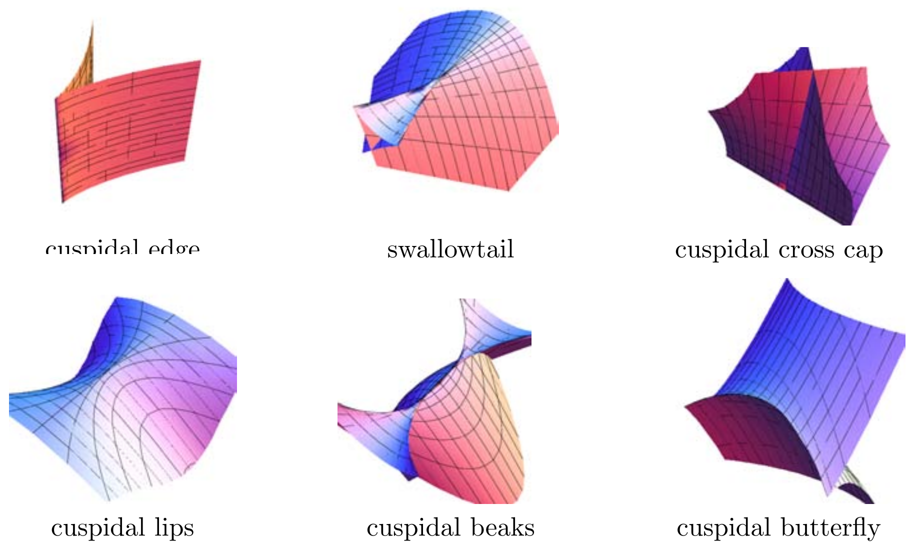


Figure 1.

We study singularities of maps up to $\mathcal{A}$-equivalence among map germs. Here, map germs $f_1, f_2 : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ are $\mathcal{A}$-*equivalent* if there exist diffeomorphism germs $\phi_1 : (\mathbb{R}^2, \mathbf{0}) \to$

$(\mathbb{R}^2, \mathbf{0})$ and $\phi_2 : (\mathbb{R}^3, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ such that $\phi_2 \circ f_1 = f_2 \circ \phi_1$ holds. In Section 8 we give criteria to detect the map-germs in the above list of frontals. In order to give classifications of "flat" surfaces we construct a basic Lorentzian invariant in Section 6. We give characterizations of the above singularities of our surfaces by using such invariants (cf., Theorems 8.6, 8.8, 8.9, 8.10, 8.11, 8.13 and 8.14).

On the other hand, there are many investigations on linear Weingarten surfaces in pseudo-spheres ([1, 8, 11, 12, 27]). The mandala of Legendrian duality is deeply related to linear Weingarten surfaces. By using the mandala of Legendrian dualities, we can unify the notion of linear Weingarten surfaces in different pseudo-spheres. (cf. Theorem 5.2)

We assume throughout the whole paper that all the maps and manifolds are C^∞ unless the contrary is explicitly stated.

2 Basic concepts and notations

In this section we prepare basic notions on Minkowski space. For detailed properties, see [29]. Let $\mathbb{R}^{n+1} = \{(x_0, x_1, \ldots, x_n) | x_i \in \mathbb{R},\ i = 0, 1, \ldots, n\}$ be an $(n+1)$-dimensional vector space. For any vectors $\boldsymbol{x} = (x_0, \ldots, x_n)$, $\boldsymbol{y} = (y_0, \ldots, y_n)$ in $\mathbb{R}^{n+1}$, the *pseudo scalar product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined by $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_0 y_0 + \sum_{i=1}^{n} x_i y_i$. The space $(\mathbb{R}^{n+1}, \langle , \rangle)$ is called *Minkowski* $(n+1)$*-space* and denoted by $\mathbb{R}^{n+1}_1$.

We say that a vector $\boldsymbol{x}$ in $\mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$ is *spacelike*, *lightlike* or *timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0, = 0$ or < 0 respectively. The norm of the vector $\boldsymbol{x} \in \mathbb{R}^{n+1}$ is defined by $\|\boldsymbol{x}\| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle|}$. For a non-zero vector $\boldsymbol{n} \in \mathbb{R}^{n+1}_1$ and a real number c, the hyperplane with pseudo normal $\boldsymbol{n}$ is given by

$$HP(\boldsymbol{n}, c) = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 | \langle \boldsymbol{x}, \boldsymbol{n} \rangle = c\}.$$

We say that $HP(\boldsymbol{n}, c)$ is a *spacelike* , *timelike* or *lightlike hyperplane* if $\boldsymbol{n}$ is timelike, spacelike or lightlike respectively.

We have the following three kinds of pseudo-spheres in $\mathbb{R}^{n+1}_1$: The *hyperbolic n-space* is defined by

$$H^n(-1) = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 |\ \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1\},$$

the *de Sitter n-space* by

$$S^n_1 = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1\ \}$$

and the (*open*) *lightcone* by

$$LC^* = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 \setminus \{\mathbf{0}\} | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0\ \}.$$

For any $\boldsymbol{x}^1, \boldsymbol{x}^2, \ldots, \boldsymbol{x}^n \in \mathbb{R}^{n+1}_1$, we define a vector $\boldsymbol{x}^1 \wedge \boldsymbol{x}^2 \wedge \cdots \wedge \boldsymbol{x}^n$ by

$$\boldsymbol{x}^1 \wedge \boldsymbol{x}^2 \wedge \cdots \wedge \boldsymbol{x}^n = \begin{vmatrix} -\boldsymbol{e}_0 & \boldsymbol{e}_1 & \cdots & \boldsymbol{e}_n \\ x^1_0 & x^1_1 & \cdots & x^1_n \\ x^2_0 & x^2_1 & \cdots & x^2_n \\ \vdots & \vdots & \cdots & \vdots \\ x^n_0 & x^n_1 & \cdots & x^n_n \end{vmatrix}, \tag{2.1}$$

where $\boldsymbol{e}_0, \boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ is the canonical basis of $\mathbb{R}^{n+1}_1$ and $\boldsymbol{x}^i = (x^i_0, x^i_1, \dots, x^i_n)$. We can easily check that

$$\langle \boldsymbol{x}, \boldsymbol{x}^1 \wedge \boldsymbol{x}^2 \wedge \cdots \wedge \boldsymbol{x}^n \rangle = \det(\boldsymbol{x}, \boldsymbol{x}^1, \dots, \boldsymbol{x}^n), \tag{2.2}$$

so that $\boldsymbol{x}^1 \wedge \boldsymbol{x}^2 \wedge \cdots \wedge \boldsymbol{x}^n$ is pseudo orthogonal to any $\boldsymbol{x}^i$ $(i = 1, \dots, n)$.

3 A mandala of Legendrian dualities for pseudo-spheres

We now review some properties of contact manifolds and Legendrian submanifolds. Let N be a $(2n+1)$-dimensional smooth manifold and K be a tangent hyperplane field on N. Locally such a field is defined as the field of zeros of a 1-form α. The tangent hyperplane field K is *non-degenerate* if $\alpha \wedge (d\alpha)^n \neq 0$ at any point of N. We say that (N, K) is a *contact manifold* if K is a non-degenerate hyperplane field. In this case K is called a *contact structure* and α is a *contact form*. Let $\phi : N \longrightarrow N'$ be a diffeomorphism between contact manifolds (N, K) and (N', K'). We say that ϕ is a *contact diffeomorphism* if $d\phi(K) = K'$. Two contact manifolds (N, K) and (N', K') are *contact diffeomorphic* if there exists a contact diffeomorphism $\phi : N \longrightarrow N'$. A submanifold $i : L \subset N$ of a contact manifold (N, K) is said to be *Legendrian* if dim $L = n$ and $di_x(T_xL) \subset K_{i(x)}$ at any $x \in L$. We say that a smooth fiber bundle $\pi : E \longrightarrow M$ is called a *Legendrian fibration* if its total space E is furnished with a contact structure and its fibers are Legendrian submanifolds. Let $\pi : E \longrightarrow M$ be a Legendrian fibration. For a Legendrian submanifold $i : L \subset E$, $\pi \circ i : L \longrightarrow M$ is called a *Legendrian map*. The image of the Legendrian map $\pi \circ i$ is called a *wavefront set* of i which is denoted by $W(L)$. For any $z \in E$, it is known that there is a local coordinate system $(x, p, y) = (x_1, \dots, x_m, p_1, \dots, p_m, y)$ around z such that $\pi(x, p, y) = (x, y)$ and the contact structure is given by the 1-form $\alpha = dy - \sum_{i=1}^m p_i dx_i$ (cf. [2], 20.3).

In [25] we have shown the basic duality theorem which is a fundamental tool for the study of spacelike hypersurfaces in Minkowski pseudo-spheres. We consider the following four double fibrations:

(1) (a) $H^n(-1) \times S^n_1 \supset \Delta_1 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = 0\ \}$,
(b) $\pi_{11} : \Delta_1 \longrightarrow H^n(-1)$,$\pi_{12} : \Delta_1 \longrightarrow S^n_1$,
(c) $\theta_{11} = \langle d\boldsymbol{v}, \boldsymbol{w} \rangle | \Delta_1$, $\theta_{12} = \langle \boldsymbol{v}, d\boldsymbol{w} \rangle | \Delta_1$.

(2) (a) $H^n(-1) \times LC^* \supset \Delta_2 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = -1\ \}$,
(b) $\pi_{21} : \Delta_2 \longrightarrow H^n(-1)$,$\pi_{22} : \Delta_2 \longrightarrow LC^*$,
(c) $\theta_{21} = \langle d\boldsymbol{v}, \boldsymbol{w} \rangle | \Delta_2$, $\theta_{22} = \langle \boldsymbol{v}, d\boldsymbol{w} \rangle | \Delta_2$.

(3) (a) $LC^* \times S^n_1 \supset \Delta_3 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = 1\ \}$,
(b) $\pi_{31} : \Delta_3 \longrightarrow LC^*$,$\pi_{32} : \Delta_3 \longrightarrow S^n_1$,
(c) $\theta_{31} = \langle d\boldsymbol{v}, \boldsymbol{w} \rangle | \Delta_3$, $\theta_{32} = \langle \boldsymbol{v}, d\boldsymbol{w} \rangle | \Delta_3$.

(4) (a) $LC^* \times LC^* \supset \Delta_4 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = -2\ \}$,
(b) $\pi_{41} : \Delta_4 \longrightarrow LC^*$,$\pi_{42} : \Delta_4 \longrightarrow LC^*$,
(c) $\theta_{41} = \langle d\boldsymbol{v}, \boldsymbol{w} \rangle | \Delta_4$, $\theta_{42} = \langle \boldsymbol{v}, d\boldsymbol{w} \rangle | \Delta_4$.

Here, $\pi_{i1}(\boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{v}$, $\pi_{i2}(\boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{w}$, $\langle d\boldsymbol{v}, \boldsymbol{w} \rangle = -w_0 dv_0 + \sum_{i=1}^n w_i dv_i$ and $\langle \boldsymbol{v}, d\boldsymbol{w} \rangle = -v_0 dw_0 + \sum_{i=1}^n v_i dw_i$.

We remark that $\theta_{i1}^{-1}(0)$ and $\theta_{i2}^{-1}(0)$ define the same tangent hyperplane field over Δ_i which is denoted by K_i. The basic duality theorem is the following theorem:

Theorem 3.1. *Under the same notations as the previous paragraph, each* (Δ_i, K_i) $(i = 1,2,3,4)$ *is a contact manifold and both of* π_{ij} $(j = 1,2)$ *are Legendrian fibrations. Moreover those contact manifolds are contact diffeomorphic to each other.*

Since the proof of the theorem was given in [25], we do not give the detailed proof here. We only remark that (Δ_1, K_1) can be canonically identified with the unit tangent bundle $S(TH^n(-1))$ over $H^n(-1)$ with the canonical contact structure ([7, 9]). Moreover, the contact structure K_i $(i = 2,3,4)$ can be canonically induced by the following constructions. We consider smooth mappings $(i \neq j\ ;\ (i,j = 1,2,3,4))$ $\Psi_{ij} : \Delta_i \longrightarrow \Delta_j$ defined by

$$
\begin{aligned}
&\Psi_{12}(\boldsymbol{v},\boldsymbol{w}) = (\boldsymbol{v},\boldsymbol{v}+\boldsymbol{w}),\ \Psi_{21} = (\boldsymbol{v},\boldsymbol{w}-\boldsymbol{v}),\\
&\Psi_{13}(\boldsymbol{v},\boldsymbol{w}) = (\boldsymbol{v}+\boldsymbol{w},\boldsymbol{w}),\ \Psi_{31}(\boldsymbol{v},\boldsymbol{w}) = (\boldsymbol{v}-\boldsymbol{w},\boldsymbol{w})\\
&\Psi_{14}(\boldsymbol{v},\boldsymbol{w}) = (\boldsymbol{v}-\boldsymbol{w},\boldsymbol{v}+\boldsymbol{w}),\ \Psi_{41}(\boldsymbol{v},\boldsymbol{w}) = \left(\frac{\boldsymbol{v}+\boldsymbol{w}}{2},\frac{\boldsymbol{w}-\boldsymbol{v}}{2}\right),\\
&\Psi_{23}(\boldsymbol{v},\boldsymbol{w}) = (\boldsymbol{w},\boldsymbol{w}-\boldsymbol{v}),\ \Psi_{32}(\boldsymbol{v},\boldsymbol{w}) = (\boldsymbol{v}-\boldsymbol{w},\boldsymbol{v}),\\
&\Psi_{24}(\boldsymbol{v},\boldsymbol{w}) = (2\boldsymbol{v}-\boldsymbol{w},\boldsymbol{w}),\ \Psi_{42}(\boldsymbol{v},\boldsymbol{w}) = \left(\frac{\boldsymbol{v}+\boldsymbol{w}}{2},\boldsymbol{w}\right),\\
&\Psi_{34}(\boldsymbol{v},\boldsymbol{w}) = (\boldsymbol{v}-2\boldsymbol{w},\boldsymbol{v}),\ \Psi_{43}(\boldsymbol{v},\boldsymbol{w}) = \left(\boldsymbol{w},-\frac{\boldsymbol{v}-\boldsymbol{w}}{2}\right).
\end{aligned}
$$

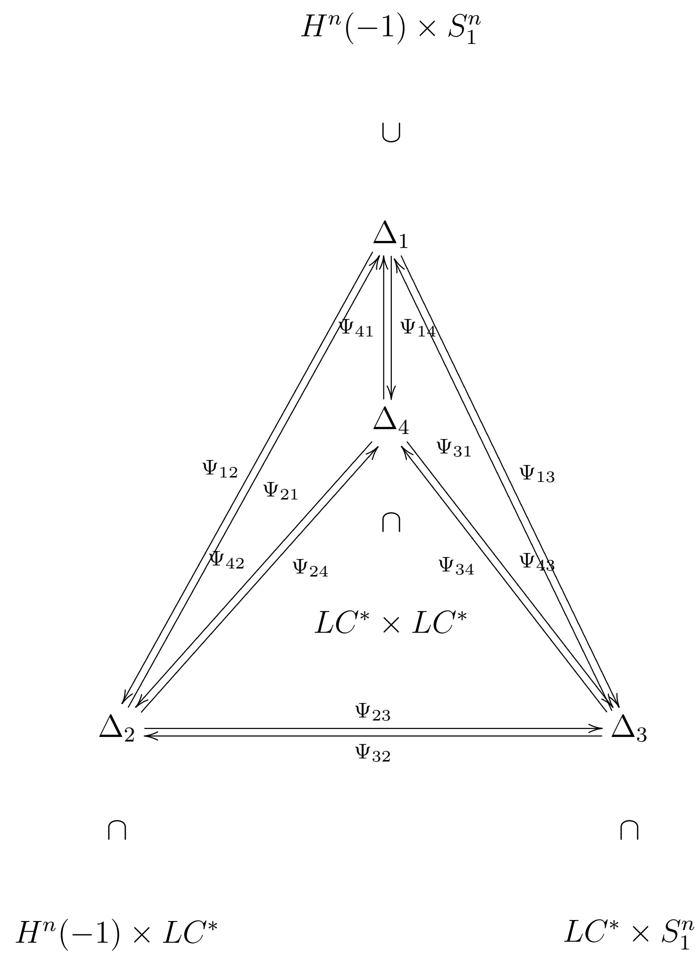
We can easily show that Ψ_{ij} are contact diffeomorphisms such that $\Psi_{ij}^{-1} = \Psi_{ji}$ for any $i,j = 1,2,3,4$. For example, we have

$$\Psi_{12}^*\theta_{21} = \langle d\boldsymbol{v},\boldsymbol{v}+\boldsymbol{w}\rangle|\Delta_1 = (\langle d\boldsymbol{v},\boldsymbol{v}\rangle + \langle d\boldsymbol{v},\boldsymbol{w}\rangle)|\Delta_1 = \langle d\boldsymbol{v},\boldsymbol{w}\rangle|\Delta_1 = \theta_{11}$$

and

$$
\begin{aligned}
\Psi_{41}^*\theta_{11} &= \left\langle d\left(\frac{\boldsymbol{v}+\boldsymbol{w}}{2}\right),\frac{\boldsymbol{v}-\boldsymbol{w}}{2}\right\rangle|\Delta_4\\
&= \frac{1}{4}(\langle d\boldsymbol{v},\boldsymbol{v}\rangle - \langle d\boldsymbol{v},\boldsymbol{w}\rangle + \langle d\boldsymbol{w},\boldsymbol{v}\rangle - \langle d\boldsymbol{w},\boldsymbol{w}\rangle)|\Delta_4\\
&= \frac{1}{4}(-2\langle d\boldsymbol{v},\boldsymbol{w}\rangle)|\Delta_4 = -\frac{1}{2}\langle d\boldsymbol{v},\boldsymbol{w}\rangle|\Delta_4 = -\frac{1}{2}\theta_{41}.
\end{aligned}
$$

Therefore $\Psi_{12} : (\Delta_1, K_1) \longrightarrow (\Delta_2, K_2)$ and $\Psi_{41} : (\Delta_4, K_4) \longrightarrow (\Delta_1, K_1)$ are contact diffeomorphisms. By the similar calculations, we can show that the other Ψ_{ij} are also contact diffeomorphisms. We call these Legendrian dualities a *mandala of Legendrian dualities* (cf.,[10, 26]) because we can explain the situation as the following diagram:



The mandala of Legendrian dualities

The above mandala has the similar structure as the real mandala of Buddhism which is a religious picture of the universe. In the real mandala, the central Buddha is the symbol of the sun (the light). In the above diagram the central contact manifold is corresponding to the light, so that the analogous structure exists. This is the reason why we call the above diagram the mandala of Legendrian dualities. The mandala was generalized into the case for pseudo-spheres in general semi-Euclidean space[10]. Moreover, it can be extended into infinitely many Legendrian dualities[26].

4 Local differential geometry of spacelike hypersurfaces in pseudo-spheres

In this section we consider differential geometry of hypersurfaces in pseudo-spheres as an application of the mandala of Legendrian dualities. We remark that it is deeply related to the previous theory on the differential geometry of submanifold in the hyperbolic space[17]. We now give a quick review on the theory. Let $\boldsymbol{X} : U \longrightarrow H^n(-1)$ be an embedding from an open region $U \subset \mathbb{R}^{n-1}$ and denote that $M = \boldsymbol{X}(U)$. We define the unit normal vector field $\boldsymbol{e} : U \longrightarrow S^n_1$ along M in $H^n(-1)$ by

$$\boldsymbol{e}(u) = \frac{\boldsymbol{X}(u) \wedge \boldsymbol{X}_{u_1}(u) \wedge \cdots \wedge \boldsymbol{X}_{u_{n-1}}(u)}{\|\boldsymbol{X}(u) \wedge \boldsymbol{X}_{u_1}(u) \wedge \cdots \wedge \boldsymbol{X}_{u_{n-1}}(u)\|}.$$

Therefore it satisfies that

$$\langle \boldsymbol{X}(u), \boldsymbol{e}(u)\rangle = \langle \boldsymbol{X}_{u_i}(u), \boldsymbol{e}(u)\rangle = \langle \boldsymbol{X}(u), \boldsymbol{e}_{u_i}(u)\rangle = 0,$$

where $i = 1, \dots, n-1$ and $\boldsymbol{X}_{u_i} = \partial \boldsymbol{X}/\partial u_i$. Since $\langle \boldsymbol{e}(u), \boldsymbol{e}_{u_i}(u)\rangle = 0$, the above relations mean that $\boldsymbol{e}_{u_i}(u)$ is tangent to M at $p = \boldsymbol{X}(u)$. Therefore $d\boldsymbol{e}(u)$ can be considered as a linear transformation on T_pM. We call the linear transformation $A_p = -d\boldsymbol{e}(u) : T_pM \longrightarrow T_pM$ the *de Sitter shape operator* of $M = \boldsymbol{X}(U)$ at $p = \boldsymbol{X}(u)$. Moreover, if we consider $\mathbb{L}^{\pm}(u) = \boldsymbol{X}(u) \pm \boldsymbol{e}(u)$, then $\mathbb{L}^{\pm}(u)$ are lightlike vectors. By the identification of M with U through $\boldsymbol{X}$, $d\boldsymbol{X}(u)$ can be identified with 1_{T_pM}. Therefore we have a linear transformation $d\mathbb{L}^{\pm}(u) : T_pM \longrightarrow T_pM$ with $d\mathbb{L}^{\pm}(u) = 1_{T_pM} \pm d\boldsymbol{e}(u)$. We call the linear transformation $S^{\pm}_p = -d\mathbb{L}^{\pm}(u) : T_pM \longrightarrow T_pM$ the *hyperbolic shape operator* of $M = \boldsymbol{X}(U)$ at $p = \boldsymbol{X}(u)$. The *de Sitter Gauss-Kronecker curvature* of $M = \boldsymbol{X}(U)$ at $p = \boldsymbol{X}(u)$ is defined to be $K_d(u) = \det A_p$ and the *lightcone Gauss-Kronecker curvature* of $M = \boldsymbol{X}(U)$ at $p = \boldsymbol{X}(u)$ is $K^{\pm}_{\ell}(u) = \det S^{\pm}_p$. In [17] we have investigated the geometric meanings of the lightcone Gauss-Kronecker curvature from the contact viewpoint. One of the consequences is that the lightcone Gauss-Kronecker curvature estimates the contact of hypersurfaces with hyperhorospheres. It has been also shown that the Gauss-Bonnet type theorem holds on the normalized lightcone Gauss-Kronecker curvature [18].

On the other hand, we can interpret the above construction by using the Legendrian duality theorem (Theorem 3.1). For any regular hypersurface $\boldsymbol{X} : U \longrightarrow H^n(-1)$, we have $\langle \boldsymbol{X}(u), \mathbb{L}^{\pm}(u)\rangle = -1$. Therefore, we can define embeddings $\mathcal{L}^{\pm}_2 : U \longrightarrow \Delta_2$ by $\mathcal{L}^{\pm}_2(u) = (\boldsymbol{X}(u), \mathbb{L}^{\pm}(u))$. Since $\langle \boldsymbol{X}_{u_i}(u), \mathbb{L}^{\pm}(u)\rangle = 0$, each of $\mathcal{L}^{\pm}_2$ is a Legendrian embedding.

It has been shown that $\pi_{21} : \Delta_2 \longrightarrow H^n(-1)$ is a Legendrian fibration. The fiber is the intersection of LC^* with a spacelike hyperplane (i.e., an elliptic hyperquadric). Therefore the intersection of the fiber with the pseuod-normal plane (i.e., a timelike plane) in $\mathbb{R}^{n+1}_1$ of M consists of two points at each point of M. This is the reason why we have such two Legendrian embeddings. However, one of the results in the theory of Legendrian singularities (cf., the appendix) asserts that the Legendrian submanifold is uniquely determined by the wave front set at least locally. Here, $M = \boldsymbol{X}(U) = \pi_{21} \circ \mathcal{L}^{\pm}_4(U)$ are the wave front sets of $\mathcal{L}^{\pm}_2(U)$ through the Legendrian fibration π_{21}. Therefore each of the Legendrian embeddings $\mathcal{L}^{\pm}_2$ is uniquely determined with respect to $M = \boldsymbol{X}(U)$. It follows that we have a unique pair of lightcone Gauss images $\mathbb{L}^{\pm} = \pi_{22} \circ \mathcal{L}^{\pm}_2$. Moreover, we have a Legendrian embedding $\mathcal{L}_1 : U \longrightarrow \Delta_1$ defined by $\mathcal{L}_1(u) = (\boldsymbol{X}(u), \boldsymbol{e}(u))$. It follows from the mandala of Legendrian dualities that we have

$$\mathcal{L}_3(u) = \Psi_{13} \circ \mathcal{L}_1(u) = (\mathbb{L}^{+}(u), \boldsymbol{e}(u)),\ \mathcal{L}_4(u) = \Psi_{14} \circ \mathcal{L}_1(u) = (\mathbb{L}^{-}(u), \mathbb{L}^{+}(u)).$$

We write $\mathcal{L}_2(u) = \mathcal{L}^{+}_2(u)$. Eventually, we have Legendrian embeddings $\mathcal{L}_i : U \longrightarrow \Delta_i$ $(i = 1, 2, 3, 4)$ such that $\Psi_{ij} \circ \mathcal{L}_i = \mathcal{L}_j$. In this case we started the embedding $\boldsymbol{X} : U \longrightarrow H^n(-1)$. However, we have no reasons why we do not start a spacelike embedding into S^n_1 or LC^*.

According to the above arguments, we consider the following situations. Let $\mathcal{L}_1 : U \longrightarrow \Delta_1$ be a Legendrian embedding and denote that $\mathcal{L}_1(u) = (\boldsymbol{X}^h(u), \boldsymbol{X}^d(u))$. By using the contact diffeomorphism Ψ_{14}, we have a Legendrian embedding $\mathcal{L}_4 : U \longrightarrow \Delta_4$ defined by $\mathcal{L}_4(u) = \Psi_{14} \circ \mathcal{L}_1(u)$. We denote that $\mathcal{L}_4(u) = (\boldsymbol{X}^\ell_-(u), \boldsymbol{X}^\ell_+(u))$, so that we have the following relations:

$$\begin{aligned} \boldsymbol{X}^\ell_-(u) &= \boldsymbol{X}^h(u) - \boldsymbol{X}^d(u), \quad \boldsymbol{X}^\ell_+(u) = \boldsymbol{X}^h(u) + \boldsymbol{X}^d(u), \\ \boldsymbol{X}^h(u) &= \frac{\boldsymbol{X}^\ell_+(u) + \boldsymbol{X}^\ell_-(u)}{2}, \quad \boldsymbol{X}^d(u) = \frac{\boldsymbol{X}^\ell_+(u) - \boldsymbol{X}^\ell_-(u)}{2}. \end{aligned} \tag{4.1}$$

We also denote that $\mathcal{L}_2 = \Psi_{12} \circ \mathcal{L}_1 : U \longrightarrow \Delta_2$ and $\mathcal{L}_3 = \Psi_{13} \circ \mathcal{L}_1 : U \longrightarrow \Delta_3$, so that we have

$$\mathcal{L}_2(u) = (\boldsymbol{X}^h(u), \boldsymbol{X}^\ell_+(u)), \ \mathcal{L}_3(u) = (\boldsymbol{X}^\ell_+(u), \boldsymbol{X}^d(u)). \tag{4.2}$$

Since Ψ_{ij} $(i, j = 1, 2, 3, 4)$ are contact diffeomorphisms, $\mathcal{L}_i(U)$ $(i = 1, 2, 3, 4)$ are Legendrian submanifolds. By definition, $\mathcal{L}_1(U)$ is a Legendrian submanifold in Δ_1 if and only if

$$\langle \boldsymbol{X}^h(u), \boldsymbol{X}^d(u) \rangle = \langle \boldsymbol{X}^h(u), \boldsymbol{X}^d_{u_i}(u) \rangle = \langle \boldsymbol{X}^h_{u_i}(u), \boldsymbol{X}^d(u) \rangle = 0$$

for $i = 1, \dots, n-1$. Therefore if we suppose that $\boldsymbol{X}^h$ is an embedding, then $\boldsymbol{X}^d$ can be considered as the Gauss map of $M^h = \boldsymbol{X}^h(U)$ and $-d\boldsymbol{X}^d(u)$ is the corresponding Weingarten map. If $\boldsymbol{X}^d$ is an embedding, then $\boldsymbol{X}^h$ can be considered as the Gauss map of $M^d = \boldsymbol{X}^d(U)$ and $-d\boldsymbol{X}^h(u)$ is the corresponding Weingarten map. It follows that we can define the corresponding curvatures. The situations are the same as for the other $\mathcal{L}_i(U)$. We now summarize the situations. We denote that $M^H = \boldsymbol{X}^h(U)$ and $M^D = \boldsymbol{X}^d(U)$. If $\boldsymbol{X}^h$ is an embedding, we call $\boldsymbol{X}^d$ the *de Sitter Gauss image* of hypersurface M^H in the hyperbolic space $H^n(-1)$. Moreover, we define $(S^H_d)_p = -d\boldsymbol{X}^d(u) : T_pM^H \longrightarrow T_pM^H$ where $p = \boldsymbol{X}^h(u)$. We also call $(S^H_d)_p$ the *de Sitter Weingarten map* of hypersurface M^H in the hyperbolic space $H^n(-1)$ at $p = \boldsymbol{X}^h(u)$. Then we have *de Sitter principal curvatures* $\kappa^H_{d,i}(u)$ $(i = 1, \dots, n-1)$ defined as the eigenvalues of $(S^H_d)_p$ and the *de Sitter Gauss-Kronecker curvature* $K^H_d(u) = \det(S^H_d)_p$ of M^H at $p = \boldsymbol{X}^h(u)$.

On the other hand, if $\boldsymbol{X}^d$ is an embedding, we call $\boldsymbol{X}^h$ the *hyperbolic Gauss image* of spacelike hypersurface M^D in the d Sitter space S^n_1. Moreover, we define $(S^D_h)_p = -d\boldsymbol{X}^h(u) : T_pM^D \longrightarrow T_pM^D$ where $p = \boldsymbol{X}^d(u)$. We also call $(S^D_h)_p$ the *hyperbolic Weingarten map* of spacelike hypersurface M^D in the de Sitter space S^n_1 at $p = \boldsymbol{X}^d(u)$. Then we have *hyperbolic principal curvatures* $\kappa^D_{h,i}(u)$ $(i = 1, \dots, n-1)$ defined as the eigenvalues of $(S^D_h)_p$ and the *hyperbolic Gauss-Kronecker curvature* $K^D_h(u) = \det(S^D_h)_p$ of M^D at $p = \boldsymbol{X}^d(u)$. If both the mappings $\boldsymbol{X}^h, \boldsymbol{X}^d$ are embeddings, then we define $g^D_{ij}(u) = \langle \boldsymbol{X}^h_i(u), \boldsymbol{X}^h_j(u) \rangle$, $g^H_{ij}(u) = \langle \boldsymbol{X}^d_i(u), \boldsymbol{X}^d_j(u) \rangle$ and $h^{\Delta_1}_{ij}(u) = -\langle \boldsymbol{X}^d_i(u), \boldsymbol{X}^h_j(u) \rangle = \langle \boldsymbol{X}^d_{ij}(u), \boldsymbol{X}^h(u) \rangle = \langle \boldsymbol{X}^d(u), \boldsymbol{X}^h_{ij}(u) \rangle$. We respectively call g^H_{ij}, g^D_{ij} and $h^{\Delta_1}_{ij}$ a *hyperbolic first fundamental invariant* of M^D, a *de Sitter first fundamental invariant* of M^H and a Δ_1-*second fundamental invariant*. In this case we can identify T_pM^H with T'_pM^D for $p = \boldsymbol{X}^h(u)$ and $p' = \boldsymbol{X}^d(u)$. By definition, the principal directions of $(S^H_d)_p$ and $(S^D_h)'_p$ are the same. We have the following Weingarten type formulae.

Proposition 4.1. *Let $\mathcal{L}_1 : U \longrightarrow \Delta_1$ be a Legendrian embedding with $\mathcal{L}_1(u) = (\boldsymbol{X}^h(u), \boldsymbol{X}^d(u))$.*

(1) *Suppose that* $\boldsymbol{X}^h : U \longrightarrow H^n(-1)$ *is an embedding. Then we have*

$$\boldsymbol{X}^d_{u_i} = -\sum_{j=1}^{n-1} \left(h^D\right)_i^j \boldsymbol{X}^h_{u_j}.$$

(2) *Suppose that* $\boldsymbol{X}^d : U \longrightarrow H^n(-1)$ *is an embedding. Then we have*

$$\boldsymbol{X}^h_{u_i} = -\sum_{j=1}^{n-1} \left(h^H\right)_i^j \boldsymbol{X}^d_{u_j}.$$

Here, $\left((h^H)_i^j\right) = (h^{\Delta_1}_{ij})(g^H_{ij})^{-1}$ *and* $\left((h^D)_i^j\right) = (h^{\Delta_1}_{ij})(g^D_{ij})^{-1}$.

The proof of the above formulae is given by the same arguments as those for the Weingarten type formula in [17], so that we omit it. We remark that $\kappa^H_{d,i}(u)$ and $\kappa^D_{h,i}(u)$ are the eigenvalues of $\left((h^H)_i^j\right)$ and $\left((h^D)_i^j\right)$ respectively. We have the following relation between $\kappa^H_{d,i}(u)$ and $\kappa^D_{h,i}(u)$.

Corollary 4.2. *Suppose that both the mappings* $\boldsymbol{X}^h, \boldsymbol{X}^d$ *are embeddings. In this case we have the relation* $\kappa^H_{d,i}(u)\kappa^D_{h,i}(u) = 1$ $(i = 1, \dots n-1)$. *Here* $\kappa^H_{d,i}(u)$ *and* $\kappa^D_{h,i}(u)$ *are principal curvatures corresponding to the same principal direction.*

Proof. Since both the mappings $\boldsymbol{X}^h, \boldsymbol{X}^d$ are embeddings, $K^H_d(u) \neq 0$ and $K^D_h(u) \neq 0$. By the Weingarten type formulae, $\left((h^D)_i^j\right)$ is the inverse matrix of $\left((h^H)_i^j\right)$, so that the eigenvalues have the above relations. □

We say that $\pi_{i1} \circ \mathcal{L}_i$ and $\pi_{i2} \circ \mathcal{L}_i$ are Δ_i-*dual* each other if $\mathcal{L}_i : U \longrightarrow \Delta_i$ is an isotropic mapping with respect to K_i.

5 Linear Weingarten surfaces

Galvez, Martinez and Milan has investigated the linear Weingarten surfaces using the Weierstrass type representation formula [12]. In this section, we discuss linear Weingarten surfaces and their hyperbolic Gauss maps from our point of view. In this section, we identify the Minkowski 4-space with the 2×2 Hermitian matrices. For the detailed description, see [12, Section 2]. A surface $f : U \to H^3_+(-1)$ is called a *linear Weingarten surface* if the mean curvature $H^H_d = (\kappa^H_1 + \kappa^H_2)/2$ and the de Sitter Gauss-Kronecker curvature K^H_d satisfies

$$2a(H^H_d - 1) + b(K^H_d - 1) = 0, \quad a,\ b \in \mathbb{R}, \quad a + b \neq 0.$$

If $a + b \neq 0$ holds, it is called a *linear Weingarten surface of Bryant type.* In [22], we have investigated "horo-flat" horospherical surfaces in $H^3_+(-1)$. It is linear Weingarten surfaces of non-Bryant type, we have considered them as surfaces whose hyperbolic Gauss map degenerates to a curve in the de Sitter space (see [22, Section 4]). This means that a horo-flat horospherical surface is the dual surface of a curve in the de Sitter space. In [12], Galvez, Martinez and Milan showed the following representation formula for linear Weingarten surfaces of Bryant type.

Theorem 5.1. [12, Theorem 2] *Let $V \subset \boldsymbol{C}$ be a simply connected domain. Fix a meromorphic map $A : V \longrightarrow SL(2, \boldsymbol{C})$ satisfying*

$$A^{-1}dA = \begin{pmatrix} 0 & \omega \\ dh & 0 \end{pmatrix},$$

where h is a meromorphic function and ω a holomorphic one-form. If

$$\sigma = (a+b)\left((1+\varepsilon|h|^2)^2|\omega|^2 - \frac{(1-\varepsilon)^2|dh|^2}{(1+\varepsilon|h|^2)^2}\right)$$

is positive definite then $f = A(\Omega_+)A^*$ is a linear Weingarten surface. Moreover, the hyperbolic Gauss map ν of f is given by $\nu = A(\Omega_-)A^*$ where

$$\Omega_\pm = \begin{pmatrix} \dfrac{1 \pm \varepsilon^2|h|^2}{1+\varepsilon|h|^2} & \mp\varepsilon\bar{h} \\ \mp\varepsilon h & \pm(1+\varepsilon|h|^2) \end{pmatrix}, \ \textit{respectively}, \quad \varepsilon = a/(a+b), \quad \textit{and} \quad 1+\varepsilon|h|^2 > 0.$$

By the construction of Legendrian dualities (4.1) and (4.2), we can obtain the dual surfaces in S^3_1 and LC^* by taking $\nu : U \to S^3_1$ and $f \pm \nu : U \to LC^*$:

$$f+\nu = A\begin{pmatrix} 2\dfrac{1}{1+\varepsilon|h|^2} & 0 \\ 0 & 0 \end{pmatrix}A^*, \quad f-\nu = A\begin{pmatrix} 2\dfrac{\varepsilon^2|h|^2}{1+\varepsilon|h|^2} & -2\varepsilon\bar{h} \\ -2\varepsilon h & 2(1+\varepsilon|h|^2) \end{pmatrix}A^*. \tag{5.1}$$

In [3], Aledo and Espinar showed a Weierstrass type representation formula for linear Weingarten surfaces of Bianchi type. A spacelike surface $f : U \to S^3_1$ is a *linear Weingarten surface* if the mean curvature H^D_h and the hyperbolic Gauss-Kronecker curvature K^D_h satisfy

$$2A(H^D_h - 1) + B(K^D_h - 1) = 0, \quad A,\ B \in \mathbb{R}.$$

If $A + B \neq 0$ holds, it is called *Bianchi type*. As a consequence of the duality theorem, we can interpret the relationship between linear Weingarten surfaces in $H^3_+(-1)$ and S^3_1.

Theorem 5.2. *Let $\mathcal{L}_1 : U \to \Delta_1$ be a Legendrian immersion. Suppose that both of $\pi_{11} \circ \mathcal{L}_1 : U \to H^3_+(-1)$ and $\pi_{12} \circ \mathcal{L}_1 : U \to S^3_1$ are immersions. Then $\pi_{11} \circ \mathcal{L}_1 = \boldsymbol{X}^h$ is a linear Weingarten surface of Bryant type if and only if $\pi_{12} \circ \mathcal{L}_1 = \boldsymbol{X}^d$ is a linear Weingarten surface of Bianchi type.*

Proof. Let $\kappa^H_{d,i}$ $(i = 1, 2)$ be the de Sitter principal curvatures of $M^H = \boldsymbol{X}^h(U)$ at $p = \boldsymbol{X}^h(u)$. and $\kappa^D_{h,i}$ $(i = 1, 2)$ the hyperbolic principal curvatures of $M^D = \boldsymbol{X}^d(U)$ at $p' = \boldsymbol{X}^d(u)$. By Corollary 4.2, we have the relations $\kappa^H_{d,i}\kappa^D_{h,i} = 1$. Since $K^H_d(u) = \kappa^H_{d,1}\kappa^H_{d,2}$ and $2H^H_d = \kappa^H_{d,1} + \kappa^H_{d,2}$, we have

$$2a(H^H_d - 1) + b(K^H_d - 1) = a(\kappa^H_{d,1} + \kappa^H_{d,2} - 2) + b(\kappa^H_{d,1}\kappa^H_{d,2} - 1).$$

We also have another relation

$$2A(H^D_h - 1) + B(K^D_d - 1) = A(\kappa^D_{h,1} + \kappa^D_{h,2} - 2) + B(\kappa^D_{h,1}\kappa^D_{h,2} - 1).$$

Since $\kappa^H_{d,i} = 1/\kappa^D_{h,i}$, we have

$$\begin{aligned}
2a(H^H_d - 1) + b(K^H_d - 1) &= a\left(\frac{1}{\kappa^D_{h,1}} + \frac{1}{\kappa^D_{h,2}} - 2\right) + b\left(\frac{1}{\kappa^D_{h,1}\kappa^D_{h,2}} - 1\right) \\
&= \frac{1}{\kappa^D_{h,1}\kappa^D_{h,2}}\left(a(\kappa^D_{h,1} + \kappa^D_{h,2} - 2) + (-2a-b)(\kappa^D_{h,1}\kappa^D_{h,2} - 1\right) \\
&= \frac{1}{\kappa^D_{h,1}\kappa^D_{h,2}}\left(2a(H^D_h - 1) + (-2a-b)(K^D_h - 1)\right).
\end{aligned}$$

If we put $A = a, B = -(2a+b)$, then $2a(H^H_d - 1) + b(K^H_d - 1) = 0$ if and only if $2A(H^D_h - 1) + B(K^D_h - 1) = 0$. Moreover, $A + B = 0$ if and only if $a + b = 0$. This completes the proof. $\square$

This theorem shows that we can bring the representation formula for a surface in $H^3_+(-1)$ to representation formulae for surfaces in S^3_1 and LC^*, and get new surfaces. Remark that we have interesting families of surfaces in the lightcone obtained by taking the dual of linear Weingarten surface of non-Bryant type. In fact, the Gauss map ν of a linear Weingarten surface f given in Theorem 5.1 is a linear Weingarten surface in S^3_1. Furthermore, surfaces $f \pm \nu$ given in (5.1) belong to this class of surfaces. Theorem 5.2 says that Theorem 5.1 also can be considered representation formula for these families of surfaces. Kokubu and Umehara investigated the topological properties of linear Weingarten surfaces giving a variant of this representation formula [27].

6 The Legendrian dualities for "flat"spacelike surfaces

In this section we study general properties of spacelike surfaces in pseudo-spheres which are Δ_i-duals of spacelike curves in pseudo-spheres. Let $\boldsymbol{a}_0 : I \longrightarrow H^3_+(-1)$ be a smooth mapping and $\boldsymbol{a}_i : I \longrightarrow S^3_1$ $(i = 1, 2)$ be smooth mappings from an open interval I with $\langle \boldsymbol{a}_i(t), \boldsymbol{a}_j(t)\rangle = 0$ if $i \neq j$. We define a unit spacelike vector $\boldsymbol{a}_3(t) = \boldsymbol{a}_0(t) \wedge \boldsymbol{a}_1(t) \wedge \boldsymbol{a}_2(t)$, so that we have a pseudo-orthonormal frame $\{\boldsymbol{a}_0, \boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\}$ of $\mathbb{R}^4_1$. We have the following fundamental invariants:

$$\begin{aligned}
c_1(t) &= \langle \boldsymbol{a}_0'(t), \boldsymbol{a}_1(t)\rangle = -\langle \boldsymbol{a}_0(t), \boldsymbol{a}_1'(t)\rangle, & c_4(t) &= \langle \boldsymbol{a}_1'(t), \boldsymbol{a}_2(t)\rangle = -\langle \boldsymbol{a}_1(t), \boldsymbol{a}_2'(t)\rangle, \\
c_2(t) &= \langle \boldsymbol{a}_0'(t), \boldsymbol{a}_2(t)\rangle = -\langle \boldsymbol{a}_0(t), \boldsymbol{a}_2'(t)\rangle, & c_5(t) &= \langle \boldsymbol{a}_1'(t), \boldsymbol{a}_3(t)\rangle = -\langle \boldsymbol{a}_1(t), \boldsymbol{a}_3'(t)\rangle, \\
c_3(t) &= \langle \boldsymbol{a}_0'(t), \boldsymbol{a}_3(t)\rangle = -\langle \boldsymbol{a}_0(t), \boldsymbol{a}_3'(t)\rangle, & c_6(t) &= \langle \boldsymbol{a}_2'(t), \boldsymbol{a}_3(t)\rangle = -\langle \boldsymbol{a}_2(t), \boldsymbol{a}_3'(t)\rangle.
\end{aligned}$$

It can be written in the following form:

$$\begin{pmatrix} \boldsymbol{a}_0'(t) \\ \boldsymbol{a}_1'(t) \\ \boldsymbol{a}_2'(t) \\ \boldsymbol{a}_3'(t) \end{pmatrix} = \begin{pmatrix} 0 & c_1(t) & c_2(t) & c_3(t) \\ c_1(t) & 0 & c_4(t) & c_5(t) \\ c_2(t) & -c_4(t) & 0 & c_6(t) \\ c_3(t) & -c_5(t) & -c_6(t) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{a}_0(t) \\ \boldsymbol{a}_1(t) \\ \boldsymbol{a}_2(t) \\ \boldsymbol{a}_3(t) \end{pmatrix} =: C(t) \begin{pmatrix} \boldsymbol{a}_0(t) \\ \boldsymbol{a}_1(t) \\ \boldsymbol{a}_2(t) \\ \boldsymbol{a}_3(t) \end{pmatrix}.$$

We remark that $C(t)$ is an element of the Lie algebra $\mathfrak{so}(3,1)$ of the Lorentzian group $SO_0(3,1)$. If $\{\boldsymbol{a}_0(t), \boldsymbol{a}_1(t), \boldsymbol{a}_2(t), \boldsymbol{a}_3(t)\}$ is a pseudo-orthonormal frame field as the above, the 4×4-matrix determined by the frame defines a smooth curve $A : I \longrightarrow SO_0(3,1)$. Therefore we have the relation that $A'(t) = C(t)A(t)$. For the converse, let $A : I \longrightarrow SO_0(3,1)$ be a smooth curve, then we can show that $A'(t)A(t)^{-1} \in \mathfrak{so}(3,1)$. Moreover, for any smooth curve $C : I \longrightarrow \mathfrak{so}(3,1)$, we apply the existence theorem on the linear systems of ordinary differential equations, so that there exists a unique curve $A : I \longrightarrow SO_0(3,1)$ such that $C(t) = A'(t)A(t)^{-1}$ with an initial data $A(t_0) \in SO_0(3,1)$. Therefore, a smooth curve $C : I \longrightarrow \mathfrak{so}(3,1)$ might be identified with a pseudo-orthonormal frame in $H^3_+(-1)$. Let $C : I \longrightarrow \mathfrak{so}(3,1)$ be a smooth curve with $C(t) = A'(t)A(t)^{-1}$ and $B \in SO_0(3,1)$, then we have $C(t) = (A(t)B)'(A(t)B)^{-1}$. This means that the curve $C : I \longrightarrow \mathfrak{so}(3,1)$ is a Lorentzian invariant of the pseudo-orthonormal frame $\{\boldsymbol{a}_0(t), \boldsymbol{a}_1(t), \boldsymbol{a}_2(t), \boldsymbol{a}_3(t)\}$. In the followings of this section, we construct dual surfaces of a lightlike curve $\boldsymbol{\ell}$ satisfying $||\boldsymbol{\ell}'|| \neq 0$ by using this frame.

6.1 Δ_2, Δ_3 and Δ_4-dual surfaces of $\boldsymbol{\ell}$

Let $\boldsymbol{\ell}$ be a lightlike curve satisfying $||\boldsymbol{\ell}'|| \neq 0$ and set $\boldsymbol{a}_3 := \boldsymbol{\ell}'/||\boldsymbol{\ell}'||$. Then $\boldsymbol{a}_3$ is spacelike. Since $\boldsymbol{\ell}(t) \in (\boldsymbol{a}_3(t))^{\perp}$, we have curves $\boldsymbol{a}_0$ and $\boldsymbol{a}_2$ satisfying $\langle \boldsymbol{a}_0, \boldsymbol{a}_0 \rangle = -1$, $\langle \boldsymbol{a}_2, \boldsymbol{a}_2 \rangle = 1$, $\boldsymbol{\ell} = \boldsymbol{a}_0 + \boldsymbol{a}_2$ and $\boldsymbol{a}_0, \boldsymbol{a}_2, \boldsymbol{a}_3$ are pseudo-orthonormal each other. If we define $\boldsymbol{a}_1 = \boldsymbol{a}_0 \wedge \boldsymbol{a}_2 \wedge \boldsymbol{a}_3$, then we have a pseudo-orthonormal frame $\{\boldsymbol{a}_0, \boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\}$ satisfying $c_2 \equiv 0$, $c_1 - c_4 \equiv 0$ and $c_{36}(t) \neq 0$ for any t, where $\equiv 0$ means that the function is constantly equal to zero. Thus, we may assume that $\boldsymbol{\ell} = \boldsymbol{a}_0 + \boldsymbol{a}_2$, $c_2 \equiv 0$, $c_1 - c_4 \equiv 0$ and $c_{36}(t) \neq 0$ for any t, this means that $\boldsymbol{\ell}' = c_{36}\boldsymbol{a}_3 \neq 0$.

(1) Δ_2-dual surface of $\boldsymbol{\ell}$: In order to obtain the Δ_2-dual surface of $\boldsymbol{\ell}$, we consider a hight function $F : H^3_+(-1) \times I \longrightarrow \mathbb{R}$ defined by $F(X,t) = \langle X, \boldsymbol{\ell}(t) \rangle + 1$. There exist $x_0, x_1, x_2, x_3 \in \mathbb{R}$ such that $X = x_0\boldsymbol{a}_0 + x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + x_3\boldsymbol{a}_3$. Then the *discriminant set* D_F of F is

$$\begin{aligned} D_F &= \left\{ X \in H^3_+(-1) \;\middle|\; {}^{\exists}t \in I, F(X,t) = \frac{\partial F}{\partial t}(X,t) = 0 \right\} \\ &= \{X \in H^3_+(-1) \mid {}^{\exists}t \in I, -x_0 + x_2 + 1 = 0,\ x_3 = 0\}. \end{aligned}$$

Since $X \in H^3_+(-1)$, we have

$$X = \boldsymbol{a}_0(t) + s\boldsymbol{a}_1(t) + \frac{s^2}{2}\boldsymbol{\ell}(t)$$

for $x_1 = s$, which we write $\boldsymbol{X}^h_\ell(s,t)$.

By the above construction, $(\boldsymbol{X}^h_\ell(s,t), \boldsymbol{\ell}(t)) : I \times \mathbb{R} \to \Delta_2$ is an isotropic map with respect to the contact structure defined in Theorem 3.1, so that $\boldsymbol{X}^h_\ell(s,t)$ and $\boldsymbol{\ell}(t)$ are Δ_2-dual each other.

Since $c_2 \equiv c_1 - c_4 \equiv 0$ hold, the surface $\boldsymbol{X}^h_\ell$ is horo-flat in the sense of [22]. Moreover if we assume $c_3 \equiv 0$, then the singular value of $\boldsymbol{X}^h_\ell$ is $\boldsymbol{a}_0(t)$. We also consider the Δ_1 and Δ_2-dual surfaces of $\boldsymbol{a}_0$. By the same computations as those of the previous paragraph for obtaining the surface $\boldsymbol{X}^h_\ell$, and assumptions $c_2 \equiv c_3 \equiv 0$ instead of $c_2 \equiv c_1 - c_4 \equiv 0$, we have the Δ_1-dual surface $\boldsymbol{X}^d_h$ and the Δ_3-dual surface $\boldsymbol{X}^\ell_h$ of $\boldsymbol{a}_0$ as follows:

$$\boldsymbol{X}^d_h(s,t) := \cos s\, \boldsymbol{a}_2(t) + \sin s\, \boldsymbol{a}_3(t) \quad \text{and} \quad \boldsymbol{X}^\ell_h(s,t) := \boldsymbol{a}_0(t) + \cos s\, \boldsymbol{a}_2(t) + \sin s\, \boldsymbol{a}_3(t).$$

In [22], we introduced these surfaces $\boldsymbol{X}^h_\ell$ and $\boldsymbol{X}^\ell_h$ by the same construction as the above and investigated the geometric properties and singularities of them. It has been shown in [22] that $\boldsymbol{X}^h_\ell$ is a linear Weingarten surface of non-Bryant type.

(2) Δ_3-dual surface of $\boldsymbol{\ell}$: We consider a hight function $F : S^3_1 \times I \longrightarrow \mathbb{R}$ defined by $F(X,t) = \langle X, \boldsymbol{\ell}(t)\rangle - 1$. By the same computations as those for detecting $\boldsymbol{X}^h_\ell(s,t)$, the discriminant set is given by

$$X = \boldsymbol{a}_2(t) + s\boldsymbol{a}_1(t) - \frac{s^2}{2}\boldsymbol{\ell}(t),$$

which we write $\boldsymbol{X}^d_\ell(s,t)$. Like as in the case for $\boldsymbol{X}^h_\ell$, we consider the dual surfaces of $\boldsymbol{a}_2$ here. By exactly the same calculations as those in the previous cases, and assumptions $c_2 \equiv c_6 \equiv 0$ instead of $c_2 \equiv c_1 - c_4 \equiv 0$, the Δ_1-dual surface $\boldsymbol{X}^h_d$ of $\boldsymbol{a}_2(t)$ and the Δ_3-dual surface $\boldsymbol{X}^\ell_d$ of $\boldsymbol{a}_2(t)$ are parameterized by

$$\boldsymbol{X}^h_d(s,t) := \cosh s\, \boldsymbol{a}_0(t) + \sinh s\, \boldsymbol{a}_3(t) \quad \text{and} \quad \boldsymbol{X}^\ell_d(s,t) := \boldsymbol{a}_2(t) + \cosh s\, \boldsymbol{a}_0(t) + \sinh s\, \boldsymbol{a}_3(t).$$

(3) Δ_4-dual surface of $\boldsymbol{\ell}$: We consider a hight function $F : LC^* \times I \longrightarrow \mathbb{R}$ defined by $F(X,t) = \langle X, \boldsymbol{\ell}(t)\rangle + 2$. Putting $x_1 = 2s$ and by exactly the same computations as those of the previous two cases. we have

$$X = \boldsymbol{a}_0(t) - \boldsymbol{a}_2(t) + 2s\boldsymbol{a}_1(t) + s^2\boldsymbol{\ell}(t),$$

which we write $\boldsymbol{X}^\ell_\ell(s,t)$. We study geometric properties of $\boldsymbol{X}^\ell_\ell(s,t)$ in section 7 and investigate the singularities in section 8. Like as in the case of $\boldsymbol{X}^h_\ell$ and $\boldsymbol{X}^d_\ell$, we consider the dual surfaces of $\boldsymbol{\ell}_- := \boldsymbol{a}_0 - \boldsymbol{a}_2$. Under the condition $c_2 \equiv c_1 + c_4 \equiv 0$, the Δ_2-dual surface $\boldsymbol{X}^h_{\ell-}$ of $\boldsymbol{\ell}_-(t)$, the Δ_3-dual surface $\boldsymbol{X}^d_{\ell-}$ of $\boldsymbol{\ell}_-(t)$ and the Δ_4-dual surface $\boldsymbol{X}^\ell_{\ell-}$ of $\boldsymbol{\ell}_-(t)$ are parameterized by

$$\begin{aligned}
\boldsymbol{X}^h_{\ell-}(s,t) &= \boldsymbol{a}_0(t) + s\boldsymbol{a}_1(t) + \frac{s^2}{2}\boldsymbol{\ell}_-(t),\\
\boldsymbol{X}^d_{\ell-}(s,t) &= -\boldsymbol{a}_2(t) + s\boldsymbol{a}_1(t) - \frac{s^2}{2}\boldsymbol{\ell}(t),\\
\boldsymbol{X}^\ell_{\ell-}(s,t) &= \boldsymbol{a}_0(t) + \boldsymbol{a}_2(t) + 2s\boldsymbol{a}_1(t) + s^2\boldsymbol{\ell}_-(t).
\end{aligned}$$

Since we can obtain these surfaces $\boldsymbol{X}^h_{\ell-}$, $\boldsymbol{X}^d_{\ell-}$ and $\boldsymbol{X}^\ell_{\ell-}$ by translating $\boldsymbol{a}_2 \mapsto -\boldsymbol{a}_2$, geometric properties of these surfaces are completely the same as those of $\boldsymbol{X}^h_\ell$, $\boldsymbol{X}^d_\ell$ and $\boldsymbol{X}^\ell_\ell$. Here we explain the meanings of the superscript and the subscript. For example, $\boldsymbol{X}^h_\ell$ means this surface is the dual surface of a curve in the *lightcone* and lies in the *hyperbolic* 3-space. Since surfaces $\boldsymbol{X}^h_\ell$, $\boldsymbol{X}^d_\ell$ and $\boldsymbol{X}^\ell_\ell$ are one-parameter families of parabolas, we call these surfaces *parabollatic surfaces*. If we adopt the word "parabolic" instead of the word "parabollatic", it might be confused with other notions. Now, we summerize the correspondences between these curves and surfaces:

$$
\begin{array}{rlll}
LC^* \supset \boldsymbol{\ell}(t) \longleftrightarrow & \boldsymbol{X}_\ell^h(s,t) & = \boldsymbol{a}_0(t) + s\boldsymbol{a}_1(t) + \dfrac{s^2}{2}\boldsymbol{\ell}(t) & \subset H^3_+(-1) \\
LC^* \supset \boldsymbol{\ell}(t) \longleftrightarrow & \boldsymbol{X}_\ell^d(s,t) & = \boldsymbol{a}_2(t) + s\boldsymbol{a}_1(t) - \dfrac{s^2}{2}\boldsymbol{\ell}(t) & \subset S^3_1 \\
LC^* \supset \boldsymbol{\ell}(t) \longleftrightarrow & \boldsymbol{X}_\ell^l(s,t) & = \boldsymbol{a}_0(t) - \boldsymbol{a}_2(t) + 2s\boldsymbol{a}_1(t) + s^2\boldsymbol{\ell}(t) & \subset LC^* \\
H^3_+(-1) \supset \boldsymbol{a}_0(t) \longleftrightarrow & \boldsymbol{X}_h^d(s,t) & = \cos s\boldsymbol{a}_2(t) + \sin s\boldsymbol{a}_3(t) & \subset S^3_1 \\
H^3_+(-1) \supset \boldsymbol{a}_0(t) \longleftrightarrow & \boldsymbol{X}_h^\ell(s,t) & = \boldsymbol{a}_0(t) + \cos s\boldsymbol{a}_2(t) + \sin s\boldsymbol{a}_3(t) & \subset LC^* \\
S^3_1 \supset \boldsymbol{a}_2(t) \longleftrightarrow & \boldsymbol{X}_d^h(s,t) & = \cosh s\boldsymbol{a}_0(t) + \sinh s\boldsymbol{a}_3(t) & \subset H^3_+(-1) \\
S^3_1 \supset \boldsymbol{a}_2(t) \longleftrightarrow & \boldsymbol{X}_d^\ell(s,t) & = \boldsymbol{a}_2(t) + \cosh s\boldsymbol{a}_0(t) + \sinh s\boldsymbol{a}_3(t) & \subset LC^*.
\end{array}
\tag{6.1}
$$

6.2 Dualities of “flat”surfaces

By using the equations for the pseudo-orthonormal frame, we have

$$
\begin{aligned}
(\boldsymbol{X}_\ell^h)'(s,t) &= sc_1\boldsymbol{a}_0 + c_1\boldsymbol{a}_1 + sc_4\boldsymbol{a}_2 + \left(c_3 + sc_5 + \frac{s^2}{2}c_{36}\right)\boldsymbol{a}_3 \\
(\boldsymbol{X}_\ell^h)_s(s,t) &= s\boldsymbol{a}_0 + \boldsymbol{a}_1 + s\boldsymbol{a}_2,
\end{aligned}
$$

where $(\ )'$ means $\partial/\partial t$ and $(\ )_s$ means $\partial/\partial s$. It follows that we have

$$
\left\langle \boldsymbol{X}_\ell^h(\pm s,t), \boldsymbol{X}_\ell^d(\mp s,t)\right\rangle \equiv 0,
$$

$$
\left\langle (\boldsymbol{X}_\ell^h)'(\pm s,t), \boldsymbol{X}_\ell^d(\mp s,t)\right\rangle \equiv 0 \text{ and } \left\langle (\boldsymbol{X}_\ell^h)_s(\pm s,t), \boldsymbol{X}_\ell^d(\mp s,t)\right\rangle \equiv 0.
$$

This implies that $(\boldsymbol{X}_\ell^h, \boldsymbol{X}_\ell^d) : I \times \mathbb{R} \to \Delta_1$ is an isotropic map with respect to K_1. Therefore $\boldsymbol{X}_\ell^h$ and $\boldsymbol{X}_\ell^d$ are Δ_1-dual each other. Since $\boldsymbol{X}_\ell^h(s,t)$ is a linear Weingarten surface of non-Bryant type, $\boldsymbol{X}_\ell^d(s,t)$ is a linear Weingarten surface of non-Bianchi type by Theorem 5.2. By the same calculation, we can show that the Δ_2-duality between $\boldsymbol{X}_\ell^h(\pm s,t)$ and $\boldsymbol{X}_\ell^\ell(\pm s,t)$, and the Δ_3-duality between $\boldsymbol{X}_\ell^d(\pm s,t)$ and $-\boldsymbol{X}_\ell^\ell(\mp s,t)$ under the assumptions $c_2(t) \equiv 0$, $c_1(t) - c_4(t) \equiv 0$. These assumptions mean that a kind of flatness of $\boldsymbol{X}_\ell^h(s,t), \boldsymbol{X}_\ell^d(s,t)$ and $\boldsymbol{X}_\ell^\ell(s,t)$. For $\boldsymbol{X}_\ell^h(s,t)$, such a flatness is called *horo-flat* in [22].

Furthermore, under the conditions $c_2 \equiv c_3 \equiv 0$ (resp. $c_2 \equiv c_6 \equiv 0$), we have $\left\langle \boldsymbol{X}_h^d, \boldsymbol{X}_h^\ell\right\rangle \equiv 1$ (resp. $\left\langle \boldsymbol{X}_d^h, \boldsymbol{X}_d^\ell\right\rangle \equiv -1$) and $\left\langle \boldsymbol{X}_h^d, d\boldsymbol{X}_h^\ell\right\rangle \equiv 0$ (resp. $\left\langle \boldsymbol{X}_d^h, d\boldsymbol{X}_d^\ell\right\rangle \equiv 0$). Hence $\boldsymbol{X}_h^d$ and $\boldsymbol{X}_h^\ell$ are Δ_3-dual (resp. $\boldsymbol{X}_d^h$ and $\boldsymbol{X}_d^\ell$ are Δ_2-dual) each other. By Theorem 5.2 and the mandala of Legendrian dualities, the surface $\boldsymbol{X}_\ell^\ell(s,t)$ corresponds to the linear Weingarten surfaces of non-Bryant type in $H^3_+(-1)$ and of non-Bianchi type in S^3_1.

Thus we have the following diagram which expresses the duality for flat surfaces in pseudo-spheres:

$$
\begin{tikzcd}[column sep=small]
 & & \big(\boldsymbol{a}_0+s\boldsymbol{a}_1+(s^2/2)\boldsymbol{\ell},\ \boldsymbol{a}_2-s\boldsymbol{a}_1-(s^2/2)\boldsymbol{\ell}\big) \arrow[d, phantom, "\rotatebox{-90}{$\in$}"] & & \\
 & & \Delta_1 \arrow[dddll, shift right, "\Psi_{12}"'] \arrow[d, shift right, "\Psi_{14}"'] \arrow[dddrr, shift right, "\Psi_{13}"'] & & \\
 & & \Delta_4 \arrow[u, shift right, "\Psi_{41}"'] \arrow[ddll, shift right, "\Psi_{42}"'] \arrow[ddrr, shift left, "\Psi_{43}"] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & & \\
 & & \big(\boldsymbol{a}_0-\boldsymbol{a}_2+2s\boldsymbol{a}_1+s^2\boldsymbol{\ell},\ \boldsymbol{\ell}\big) & & \\
\Delta_2 \arrow[uuurr, shift right, "\Psi_{21}"'] \arrow[uurr, shift right, "\Psi_{24}"'] \arrow[rrrr, shift left, "\Psi_{23}"] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & & & & \Delta_3 \arrow[uuull, shift right, "\Psi_{31}"'] \arrow[uull, shift left, "\Psi_{34}"] \arrow[llll, shift left, "\Psi_{32}"] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] \\
\big(\boldsymbol{a}_0+s\boldsymbol{a}_1+(s^2/2)\boldsymbol{\ell},\ \boldsymbol{\ell}\big) & & & & \big(\boldsymbol{\ell},\ -\boldsymbol{a}_2+s\boldsymbol{a}_1+(s^2/2)\boldsymbol{\ell}\big),
\end{tikzcd}
$$

If we start from a curve $\boldsymbol{\ell}$ in the lightcone, we have the following diagram of dualities:

$$
\begin{tikzcd}[column sep=small]
 & & \boldsymbol{a}_0+s\boldsymbol{a}_1+(s^2/2)\boldsymbol{\ell} \arrow[d, phantom, "\rotatebox{-90}{$\in$}"] & & \\
 & & H^3_+(-1) \arrow[dddll, leftrightarrow, "\Delta_1\text{-dual}"'] \arrow[d, leftrightarrow, "\Delta_2\text{-dual}"'] \arrow[dddrr, leftrightarrow, "\Delta_2\text{-dual}"] & & \\
 & & LC^* \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & & \\
 & & \boldsymbol{\ell} \arrow[dll, leftrightarrow, "\Delta_3\text{-dual}"'] \arrow[drr, leftrightarrow, "\Delta_4\text{-dual}"] & & \\
S^3_1 \arrow[rrrr, leftrightarrow, "\Delta_3\text{-dual}"] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & & & & LC^* \arrow[d, phantom, "\rotatebox{90}{$\in$}"] \\
-\boldsymbol{a}_2+s\boldsymbol{a}_1+(s^2/2)\boldsymbol{\ell} & & & & \boldsymbol{a}_0-\boldsymbol{a}_2+2s\boldsymbol{a}_1+s^2\boldsymbol{\ell}.
\end{tikzcd}
$$

Also we can have the diagram on dualities starting from $\boldsymbol{a}_0$ and $\boldsymbol{a}_2$:

$$
\begin{tikzcd}
& \begin{array}{c} H^3_+(-1) \\ \rotatebox{90}{$\in$} \\ \boldsymbol{a}_0 \end{array} \arrow[dl, <->, "\Delta_1\text{-dual}"'] \arrow[dr, <->, "\Delta_2\text{-dual}"] & \\
\begin{array}{c} S^3_1 \\ \rotatebox{90}{$\in$} \\ \cos s\boldsymbol{a}_2 + \sin s\boldsymbol{a}_3 \end{array} \arrow[rr, <->, "\Delta_3\text{-dual}"] & & \begin{array}{c} LC^* \\ \rotatebox{90}{$\in$} \\ \boldsymbol{a}_0 + \cos s\boldsymbol{a}_2 + \sin s\boldsymbol{a}_3 \end{array}
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
& \begin{array}{c} S^3_1 \\ \rotatebox{90}{$\in$} \\ \boldsymbol{a}_2 \end{array} \arrow[dl, <->, "\Delta_1\text{-dual}"'] \arrow[dr, <->, "\Delta_3\text{-dual}"] & \\
\begin{array}{c} H^3_+(-1) \\ \rotatebox{90}{$\in$} \\ \cosh s\boldsymbol{a}_0 + \sinh s\boldsymbol{a}_3 \end{array} \arrow[rr, <->, "\Delta_2\text{-dual}"] & & \begin{array}{c} LC^* \\ \rotatebox{90}{$\in$} \\ \boldsymbol{a}_2 + \cosh s\boldsymbol{a}_0 + \sinh s\boldsymbol{a}_3. \end{array}
\end{tikzcd}
$$

We can also have a diagram starting from the curve $\boldsymbol{\ell}_- = \boldsymbol{a}_0 - \boldsymbol{a}_2$. However, the situation is the same as the case for $\boldsymbol{\ell}$, so that we omit it.

7 Fundamental properties of parabollatic surfaces

In section 6, we construct the dual surfaces of $\boldsymbol{\ell}$ which are called parabollatic surfaces. The analogous notion in Euclidean space is *ruled surfaces* given by one-parameter families of lines in $\mathbb{R}^3$. For the study of singularities and geometric properties of ruled surfaces, the *striction curve* plays a crucial role ([16]). The striction curve is a curve on the ruled surface which contains the singularities of the surface. Similarly, an analogous notion of the striction curve also plays a crucial role for one-parameter families of circles ([23]). Since surfaces $\boldsymbol{X}^h_\ell$, $\boldsymbol{X}^d_\ell$ and $\boldsymbol{X}^\ell_\ell$ are one-parameter families of parabolas, we try to find the analogous notion of striction curves of ruled surfaces. Here, we only consider the surfaces $\boldsymbol{X}^h_\ell$, $\boldsymbol{X}^d_\ell$ and $\boldsymbol{X}^\ell_\ell$. We remark that surfaces $\boldsymbol{X}^d_h, \boldsymbol{X}^\ell_h, \boldsymbol{X}^h_d$ and $\boldsymbol{X}^\ell_d$ have similar properties as the circular surfaces [23]. We shall investigate these surfaces in the forthcoming paper.

7.1 The striction curve of $\boldsymbol{X}^d_\ell$

Let $A = (\boldsymbol{a}_0, \boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3) : I \to SO_0(3,1)$ be a pseudo-orthonormal frame defined in Section 6. The Δ_3-dual surface $\boldsymbol{X}^d_\ell$ of $\boldsymbol{\ell}$ is defined by

$$\boldsymbol{X}^d_{\ell,A}(s,t) := \boldsymbol{a}_2(t) + s\boldsymbol{a}_1(t) - \frac{s^2}{2}\boldsymbol{\ell}(t).$$

For any t, the curve $s \mapsto \boldsymbol{X}^d_{\ell,A}(s,t)$ is a parabola. The each parabola called the *generating parabola*.

On the other hand, for any curve

$$\overline{\boldsymbol{a}_2}(t) = \boldsymbol{a}_2(t) + s(t)\boldsymbol{a}_1(t) - \frac{s(t)^2}{2}\boldsymbol{\ell}(t) \tag{7.1}$$

on the $\boldsymbol{X}^d_\ell$, we define

$$\begin{aligned} \overline{\boldsymbol{a}_0}(t) &= \left(1 + \frac{s(t)^2}{2}\right)\boldsymbol{a}_0(t) - s(t)\boldsymbol{a}_1(t) + \frac{s(t)^2}{2}\boldsymbol{a}_2(t), \\ \overline{\boldsymbol{a}_1}(t) &= -s(t)\boldsymbol{a}_0(t) + \boldsymbol{a}_1(t) - s\boldsymbol{a}_2(t) \text{ and } \overline{\boldsymbol{a}_3}(t) = \boldsymbol{a}_3(t) \end{aligned} \tag{7.2}$$

then $\boldsymbol{X}^d_{\ell,\overline{A}}(s - s(t), t) = \boldsymbol{X}^d_{\ell,A}(s,t)$ holds. Moreover, we define invariants $\overline{C}(t)$ by the formula $\overline{A}'(t) = \overline{C}(t)\overline{A}(t)$, then we have

$$\begin{cases} \overline{c_1} &= \left(\dfrac{1 - s(t)^2}{2}\right) c_1 - s'(t) + \dfrac{s(t)^2}{2} c_4 - s(t)c_2 \\ \overline{c_2} &= s(t)c_1 + c_2 - s(t)c_4 \\ \overline{c_3} &= c_3 - s(t)c_5 + \dfrac{s(t)^2}{2} c_{36} \\ \overline{c_4} &= \dfrac{-s(t)^2}{2} c_1 - s(t)c_2 + \left(1 + \dfrac{s(t)^2}{2}\right) c_4 - s'(t) \\ \overline{c_5} &= c_5 - s(t)c_{36} \\ \overline{c_6} &= c_6 + s(t)c_5 - \dfrac{s(t)^2}{2} c_{36}. \end{cases}$$

It follows that

$$\overline{c_1} - \overline{c_4} = c_1 - c_4$$

and

$$\overline{c_1} - \overline{c_4} = \overline{c_2} = 0 \text{ if and only if } c_1 - c_4 = c_2 = 0.$$

This means that the condition $c_1 - c_4 = c_2 = 0$ is invariant under the adopted coordinate changes. Here, a reparameterization $(s,t) \mapsto (S,T)$ of $\boldsymbol{X}^d_{\ell,A}$ is said to be *adopted* if $S = s - s(t)$ and $T = t$. We have the following proposition.

Proposition 7.1. *Let $\boldsymbol{X}^d_{\ell,A}$ be a parameterization of a parabollatic surfaces of the form*

$$\boldsymbol{X}^d_{\ell,A}(s,t) = \boldsymbol{a}_2 + s\boldsymbol{a}_1 - \frac{s^2}{2}\boldsymbol{\ell}$$

such that $c_1 - c_4$ never vanish. Then Image $\boldsymbol{X}^d_{\ell,A}$ *has an adopted reparameterization of the form*

$$\boldsymbol{X}^d_{\ell,\overline{A}}(s,t) = \overline{\boldsymbol{a}}_2 + s\overline{\boldsymbol{a}}_1 - \frac{s^2}{2}\overline{\boldsymbol{\ell}}$$

satisfying $\langle \overline{\boldsymbol{a}_0}', \overline{\boldsymbol{a}_2} \rangle = 0$ for any t.

Proof. Let us define
$$s(t) = \frac{-c_2(t)}{c_1(t) - c_4(t)}$$
and define curves $\overline{\boldsymbol{a}}_0, \overline{\boldsymbol{a}}_1, \overline{\boldsymbol{a}}_2$ by (7.1) and (7.2). Then $\overline{c_2} \equiv 0$ holds. We do not need to say that $\boldsymbol{X}^d_{\ell,\overline{A}}$ and $\boldsymbol{X}^d_{\ell,A}$ have the same image. Thus the condition of the proposition holds. $\square$

A curve $\boldsymbol{X}^d_{\ell,A}(s(t),t)$ on the surface is called *striction curve* if $\langle \boldsymbol{a}_0'(t), \boldsymbol{a}_2(t)\rangle \equiv 0$ holds. Proposition 7.1 implies that we can take $\boldsymbol{a}_2$ as the striction curve. Singularities of parabollatic surfaces are located on the striction curve. For any parabollatic surfaces satisfying $c_1 - c_4 \neq 0$, there exists a unique striction curve.

Proposition 7.2. *Let $\boldsymbol{X}^d_{\ell,A}$ be a parabollatic surface with the striction curve $\boldsymbol{a}_2$ and $c_1 - c_4 \neq 0$. If (s_0, t_0) is a singular point, then $s_0 = 0$ namely, x_0 is located on the striction curve. Moreover, if $(0, t_0)$ is a singular point, them the generating parabola at t_0 is tangent to the striction curve.*

Proof. Direct calculation and $\boldsymbol{a}_2' = -c_4\boldsymbol{a}_1$ yield the conclusion. $\square$

7.2 The striction curve of $\boldsymbol{X}^\ell_\ell$

In this section, we study general properties of dual surfaces of $\boldsymbol{\ell}$. Let $A = (\boldsymbol{a}_0, \boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3) : I \to SO_0(3,1)$ be a pseudo-orthonormal frame defined in Section 6. The dual surface $\boldsymbol{X}^\ell_\ell$ of $\boldsymbol{\ell}$ is defined by
$$\boldsymbol{X}^\ell_{\ell,A}(s,t) := \boldsymbol{a}_0(t) - \boldsymbol{a}_2(t) + 2s\boldsymbol{a}_1(t) + s^2\boldsymbol{\ell}(t).$$
For any curve, $\overline{\boldsymbol{a}_0} - \overline{\boldsymbol{a}_2}(t) = \boldsymbol{a}_0 - \boldsymbol{a}_2(t) + 2s(t)\boldsymbol{a}_1(t) + s^2\boldsymbol{\ell}(t)$ on $\boldsymbol{X}^\ell_{\ell,\boldsymbol{a}}$, we define
$$\begin{gathered}\overline{\boldsymbol{a}_0}(t) = \boldsymbol{a}_0 + s(t)\boldsymbol{a}_1 + \frac{s(t)^2}{2}\boldsymbol{\ell}, \quad \overline{\boldsymbol{a}_1}(t) = \boldsymbol{a}_1 + s(t)\boldsymbol{\ell}(t), \\ \overline{\boldsymbol{a}_2}(t) = \boldsymbol{a}_2 - s(t)\boldsymbol{a}_1 - \frac{s(t)^2}{2}\boldsymbol{\ell} \text{ and } \overline{\boldsymbol{a}_3}(t) = \boldsymbol{a}_3(t)\end{gathered} \tag{7.3}$$
then $\boldsymbol{X}^\ell_{\ell,\overline{A}}(s - s(t), t) = \boldsymbol{X}^\ell_{\ell,A}(s,t)$ holds. Moreover, we define invariants $\overline{C}(t)$ by the formula $\overline{A}'(t) = \overline{C}(t)\overline{A}(t)$, then we have
$$\begin{cases} \overline{c_1} &= -\dfrac{s(t)^2}{2}(c_1 - c_4) + c_1 + s'(t) + sc_2 \\ \overline{c_2} &= -s(t)c_1 + c_2 + s(t)c_4 \\ \overline{c_3} &= c_3 + s(t)c_5 + \dfrac{s(t)^2}{2}c_{36} \\ \overline{c_4} &= -\dfrac{s(t)^2}{2}c_1 + s(t)c_2 + \dfrac{s(t)^2}{2}c_4 + s'(t) \\ \overline{c_5} &= c_5 - s(t)c_{36} \\ \overline{c_6} &= c_6 - s(t)c_5 - \dfrac{s(t)^2}{2}c_{36} \end{cases}$$

Thus it follows that

$$\overline{c_1} - \overline{c_4} = c_1 - c_4$$

and

$$\overline{c_1} - \overline{c_4} = \overline{c_2} = 0 \text{ if and only if } c_1 - c_4 = c_2 = 0.$$

A reparameterization $(s,t) \mapsto (S,T)$ of $\boldsymbol{X}^{\ell}_{\ell,A}$ is said to be *adopted* if $S = s - s(t)$ and $T = t$. We have the following proposition.

Proposition 7.3. *Let $\boldsymbol{X}^{\ell}_{\ell,A}$ be a parameterization of a parabollatic surfaces of the form*

$$\boldsymbol{X}^{\ell}_{\ell,A}(s,t) = \boldsymbol{a}_0 - \boldsymbol{a}_2 + 2s\boldsymbol{a}_1 + s^2\boldsymbol{\ell}$$

such that $c_1 - c_4$ never vanish. Then Image $\boldsymbol{X}^{\ell}_{\ell,A}$ *has an adopted reparameterization of the form*

$$\boldsymbol{X}^{\ell}_{\ell,\overline{A}}(s,t) = \overline{\boldsymbol{a}_0} - \overline{\boldsymbol{a}}_2 + 2s\overline{\boldsymbol{a}}_1 + s^2\overline{\boldsymbol{\ell}}$$

satisfying $\langle \overline{\boldsymbol{a}_0}', \overline{\boldsymbol{a}_2} \rangle \equiv 0$.

Proof. Let us define

$$s(t) = \frac{c_2(t)}{c_1(t) - c_4(t)}$$

and define curves $\overline{A}$ as (7.1) and (7.2). Then $\overline{c_2} \equiv 0$ holds. We do not need to say that $\boldsymbol{X}^{\ell}_{\ell,\overline{A}}$ and $\boldsymbol{X}^{\ell}_{\ell,A}$ have the same image. Thus the condition of the proposition holds. □

A curve on the surface $\boldsymbol{X}^{\ell}_{\ell,A}(s(t),t)$ is called *striction curve* if $\langle \boldsymbol{a}_0'(t), \boldsymbol{a}_2(t) \rangle \equiv 0$ holds. Proposition 7.3 implies that one can take $\boldsymbol{a}_2$ as the striction curve. Singularities of parabollatic surfaces are located on the striction curve.

Proposition 7.4. *Let $\boldsymbol{X}^{\ell}_{\ell,A}$ be a parabollatic surface with the striction curve $\boldsymbol{a}_2$ and $c_1 - c_4 \neq 0$. If (s_0, t_0) is a singular point, then $s_0 = 0$ namely, x_0 is located on the striction curve. Moreover, if $(0, t_0)$ is a singular point, then the generating parabola at t_0 is tangent to the striction curve.*

Proof. For a parabollatic surface $\boldsymbol{X}^{\ell}_{\ell,A}$, point (s_0, t_0) is a singular point if and only if

$$c_2(t_0) - s_0(c_1(t_0) - c_4(t_0)) = 0 \text{ and } c_3(t_0) - c_6(t_0) + 2s_0c_5(t_0) + s_0^2c_{36}(t_0) = 0.$$

Thus if $\boldsymbol{a}_0 - \boldsymbol{a}_2$ is the striction curve, them $s_0 = 0$ and $c_3(t_0) - c_6(t_0) = 0$ holds. Moreover, if $c_3(t_0) - c_6(t_0) = 0$, then the parabola is tangent to the striction curve at $(0, t_0)$. Because of $\boldsymbol{a}_0' - \boldsymbol{a}_2' = -c_2(\boldsymbol{a}_0 - \boldsymbol{a}_2) + (c_1 + c_4)\boldsymbol{a}_1$. □

Singularities of these surface are studied in Section 8. Although we can construct dual surfaces from $\boldsymbol{\ell}_-$, their geometric properties are the same as those of dual surfaces constructed from $\boldsymbol{\ell}$, so that we omit the study of their striction curves.

8 Singularities of flat parabollatic surfaces

8.1 Criteria for singularities of frontals

All surfaces investigating here have an isotropic lift to some contact manifold. They are called frontals which are originally investigated by Zakalyukin[34, 35]. In order to investigate singularities of concretely parameterized surfaces, the identification problem for singularities are important. Let f_0 be a given map germs. The *identification problem for* f_0 is to find a condition such that a map germ f satisfies the condition if and only if f is $\mathcal{A}$-equivalent to f_0. We call the condition a *criterion* for f_0. Such criteria are given by many people now. Simple criteria for the cuspidal edge and the swallowtail were given by Kokubu, Rossman, Saji, Umehara and Yamada [28]. Other criteria for singularities for frontals are investigated in [13, 32, 22]. Here, we briefly review the criteria for frontals. Let $\pi : E \to M$ be a Legendrian fibration from a five-dimensional contact manifold E to a three-dimensional manifold M. A C^∞-map $f : U \to M$ is called a frontal (resp. front) if there exists an isotropic lift (resp. Legendrian immersion) $L_f : U \to E$, where $U \subset \mathbb{R}^2$ be an open set. Recall that the image of the Legendrian submanifold is called the wavefront set (see Section 3). By the generalized Darboux theorem (cf., [2], 20.3), any Legendrian fibration $E \to M$ is locally equivalent to the standard fibration $PT\mathbb{R}^3 \to \mathbb{R}^3$. Therefore, we assume that $E \to M$ is $PT\mathbb{R}^3 \to \mathbb{R}^3$ and that f is a C^∞ map germ $(U, p) \to (\mathbb{R}^3, f(p))$. Taking the fiber component, let us denote $L_f = (f, [\nu])$. The *discriminant function* of a frontal f is defined by $\lambda(u, v) = \det(f_u,\ f_v,\ \nu)(u, v)$ using the coordinate system (u, v) on U, where $f_u = \partial f/\partial u$, for example. A singular point p of f is *non-degenerate* if $d\lambda(p) \neq \mathbf{0}$ holds. Let p be a non-degenerate singular point of a frontal f. In this case, there exists a smooth parameterization $\gamma(t) : (-\varepsilon, \varepsilon) \to U$, $\gamma(0) = p$ of $S(f)$ near p. Moreover, there exists a smooth vector field $\eta(t)$ along γ satisfying that $\eta(t)$ generates the kernel of $df_{\gamma(t)}$. We call this vector field the *null vector field*. Now we define a function $\phi_f(t)$ on γ by

$$\phi_f(t) = \det\big((f \circ \gamma)', \nu \circ \gamma, d\nu(\eta)\big)(t) \tag{8.1}$$

Using these notations, the following criteria have been obtained.

Theorem 8.1. [28, 13] *Let $f : U \to \mathbb{R}^3$ be a frontal and p a non-degenerate singular point of f and $\gamma : (\varepsilon, \varepsilon) \to U$, $\gamma(0) = p$ be a smooth parameterization of $S(f)$ near p. Then the following assertions hold.*

- *If $\eta\lambda(p) \neq 0$ then f to be a front near p if and only if $\phi_f(0) \neq 0$ holds.*
- *The map germ f at p is $\mathcal{A}$-equivalent to the cuspidal edge if and only if f to be a front near p and $\eta\lambda(p) \neq 0$ hold.*
- *The map germ f at p is $\mathcal{A}$-equivalent to the swallowtail if and only if f to be a front near p, $\eta\lambda(p) = 0$ and $\eta\eta\lambda(p) \neq 0$.*
- *The map germ f at p is $\mathcal{A}$-equivalent to the cuspidal cross cap if and only if $\eta\lambda(p) \neq 0$, $\phi_f(0) = 0$ and $\phi'_f(0) \neq 0$.*

Here, $\eta\lambda : U \to \mathbb{R}$ means the directional derivative of λ by the vector field $\tilde{\eta}$, where $\tilde{\eta} \in \mathfrak{X}(U)$ is an extended vector field of η to U. Moreover, we have the following criterion for the cuspidal butterfly.

Theorem 8.2. *Let $f : U \to \mathbb{R}^3$ be a frontal and p a non-degenerate singular point of f and $\gamma : (\varepsilon, \varepsilon) \to U$, $\gamma(0) = p$ be a smooth parameterization of $S(f)$ near p. Then the map germ f at p is $\mathcal{A}$-equivalent to the cuspidal butterfly if and only if f to be a front near p and $\eta\lambda(p) = \eta\eta\lambda(p) = 0$ and $\eta\eta\eta\lambda(p) \neq 0$.*

A proof of this theorem is given in the appendix. Next we consider a degenerate singularity. Let p be a degenerate singularity of a front f. If $\mathrm{rank}(df)_p = 1$, then there exists a non-zero vector field η near p such that if $q \in S(f)$ then $\eta(q)$ generates the kernel of $df(q)$. A criterion for the degenerate singularity is given as follows.

Theorem 8.3. [*22*] *Let f be a front and p a degenerate singular point of f Then the following assertions hold.*

- *The map germ f at p is $\mathcal{A}$-equivalent to the cuspidal lips if and only if* $\mathrm{rank}(df)_p = 1$ *and* $\det \mathrm{Hess}\, \lambda(p) > 0$.
- *The map germ f at p is $\mathcal{A}$-equivalent to the cuspidal beaks if and only if* $\mathrm{rank}(df)_p = 1$, $\det \mathrm{Hess}\, \lambda(p) < 0$ *and $\eta\eta\lambda(p) \neq 0$.*

In order to study singularities of a front in pseudo-Riemannian space, we introduce the following notion.

Definition 8.4. ([13]) A lift $L_g : U \to T^*N$ of a C^∞-map $g : U \to N$ to be *admissible* if g never intersect to the zero-section and $g_*(T_pU) \subset \ker(L_g(p))$, where $\ker(L_g(p)) \subset T_{g(p)}N$ is the kernel of a linear map $L_g(p)$.

Using this notion, a criterion for the cuspidal cross cap is stated as follows.

Theorem 8.5. ([13, Theorem 1.4]) *Let $g : U \to N$ be a front and $L_g : U \to T^*N$ be an admissible lift of g. Let D be an arbitrary linear connection on N. Suppose that $\gamma(t)$ ($|t| < \varepsilon$) is a singular curve on U passing through a non-degenerate singular point $p = \gamma(0)$, and $\xi_g : (-\varepsilon, \varepsilon) \to TN$ is an arbitrarily fixed vector field along γ such that*
(1) $L(\xi_g)$ vanishes on U and
(2) ξ_g is transversal to $g_*(T_pU)$ at p.
We define a function $\psi_g(t)$ by

$$\psi_g(t) = L\Big(D^g_{\eta(t)}\xi_g\big(\gamma(t)\big)\Big), \tag{8.2}$$

where $\eta(t)$ is a null vector field on the singular curve parameterized by t. Then the germ g at p is $\mathcal{A}$-equivalent to the cuspidal cross cap if and only if $\psi_g(0) = 0$ and $\psi'_g(0) \neq 0$ hold, and $\eta(0)$ is transversal to $\gamma'(0)$.

8.2 Singularities of dual surfaces of $\boldsymbol{\ell}$

In this subsection, we apply the criteria in Subsection 8.1 for describing the conditions of singularities of dual surfaces of $\boldsymbol{\ell}$. We assume that $c_2 \equiv c_1 - c_4 \equiv 0$ in this section.

Theorem 8.6. *The singular set of $\boldsymbol{X}^h_\ell$ is $S(\boldsymbol{X}^h_\ell) = \{(s,t) \mid 2c_3(t) + 2sc_5(t) + s^2c_{36}(t) = 0\}$ and $\boldsymbol{X}^h_\ell$ is a frontal for any $p_0 = (s_0, t_0) \in S(\boldsymbol{X}^h_\ell)$. Then we have the following assertions:*

- *If $c_{36}(t_0) \neq 0$ holds, then $\boldsymbol{X}_\ell^h$ to be a front near p_0.*
- *$\boldsymbol{X}_\ell^h$ at p_0 is $\mathcal{A}$-equivalent to the cuspidal edge if and only if $c_{36} \neq 0$ and $\alpha_l^h := -2c_1(c_5 + sc_{36}) + 2c_3' + 2sc_5' + s^2 c_{36}' \neq 0$ hold at p_0.*
- *$\boldsymbol{X}_\ell^h$ at p_0 is $\mathcal{A}$-equivalent to the swallowtail if and only if $c_{36} \neq 0$, $c_5 + sc_{36} \neq 0$, $\alpha_l^h = 0$ and $c_1(\alpha_l^h)_s + (\alpha_l^h)' \neq 0$ hold at p_0.*
- *$\boldsymbol{X}_\ell^h$ at p_0 is $\mathcal{A}$-equivalent to the cuspidal butterfly if and only if $c_{36} \neq 0$, $c_5 + sc_{36} \neq 0$, $\alpha_l^h = 0$, $c_1(\alpha_l^h)_s + (\alpha_l^h)' \neq 0$ and $c_1^2(\alpha_l^h)_{ss} + 2c_1(\alpha_l^h)_s' + c_1'(\alpha_l^h)_s + (\alpha_l^h)'' = 0$ hold at p_0.*
- *$\boldsymbol{X}_\ell^h$ at p_0 is $\mathcal{A}$-equivalent to the cuspidal lips (resp. cuspidal beaks) if and only if $c_{36} \neq 0$, $c_5 + sc_{36} = 0$, $2c_3' + 2sc_5' + s^2 c_{36}' = 0$ and $\det H_\ell^h > 0$ (resp. $\det H_\ell^h < 0$ and $-2c_1(c_{36} + c_5' + sc_{36}') + (-2c_1(c_5 + sc_{36}) + c_3' + sc_5' + s^2 c_{36}')' \neq 0$) hold at p_0, where*

$$H_\ell^h = \begin{pmatrix} 2c_{36} & 2c_5' + 2sc_{36}' \\ 2c_5' + 2sc_{36}' & 2c_3'' + 2sc_5'' + s^2 c_{36}'' \end{pmatrix}.$$

- *$\boldsymbol{X}_\ell^h$ at p_0 is $\mathcal{A}$-equivalent to the cuspidal cross cap if and only if $c_{36} = 0$, $c_1 c_5 \neq 0$ and $c_{36}' \neq 0$ hold at p_0.*

Remark 8.7. Surfaces $\boldsymbol{X}_\ell^h$ satisfying $c_3 \equiv 0$ be a horo-flat horo-cyclic surfaces which is investigated in [22]. Substituting $c_3 \equiv 0$ in the formulae of Theorem 8.6, we have [22, Theorem 6.2].

Theorem 8.8. *The singular set of $\boldsymbol{X}_\ell^d$ is $S(\boldsymbol{X}_\ell^d) = \{(s,t) \mid -2c_6(t) - 2sc_5(t) + s^2 c_{36}(t) = 0\}$ and $\boldsymbol{X}_\ell^d$ is a frontal for any $(s_0, t_0) \in S(\boldsymbol{X}_\ell^d)$. Then we have the following assertions:*

- *If $c_{36}(t_0) \neq 0$ holds, then $\boldsymbol{X}_\ell^d$ to be a front near p_0.*
- *$\boldsymbol{X}_\ell^d$ at p_0 is $\mathcal{A}$-equivalent to the cuspidal edge if and only if $c_{36} \neq 0$ and $\alpha_l^d := 2c_1(c_5 - sc_{36}) + 2c_6' + 2sc_5' - s^2 c_{36}' \neq 0$ hold at p_0.*
- *$\boldsymbol{X}_\ell^d$ at p_0 is $\mathcal{A}$-equivalent to the swallowtail if and only if $c_{36} \neq 0$, $c_5 - sc_{36} \neq 0$, $\alpha_l^d = 0$ and $c_1(\alpha_l^d)_s + (\alpha_l^d)' \neq 0$ hold at p_0.*
- *$\boldsymbol{X}_\ell^d$ at p_0 is $\mathcal{A}$-equivalent to the cuspidal butterfly if and only if $c_{36} \neq 0$, $c_5 - sc_{36} \neq 0$, $\alpha_l^d = 0$, $c_1(\alpha_l^d)_s + (\alpha_l^d)' = 0$ and $c_1^2(\alpha_l^d)_{ss} + 2c_1(\alpha_l^d)_s' + c_1'(\alpha_l^d)_s + (\alpha_l^d)'' \neq 0$ hold at p_0.*
- *$\boldsymbol{X}_\ell^d$ at p_0 is $\mathcal{A}$-equivalent to the cuspidal lips (resp. cuspidal beaks) if and only if $c_{36} \neq 0$, $2c_6' + 2sc_5' - s^2 c_{36}' = 0$, $c_5 - sc_{36} = 0$ and $\det H_\ell^d > 0$ (resp. $\det H_\ell^d < 0$ and $c_1(-2c_1 c_{36} + 2c_5' - 2sc_{36}') + \big(2c_1(c_5 - sc_{36}) + 2c_6' + 2sc_5' - s^2 c_{36}'\big)' \neq 0$) hold at p_0, where*

$$H_\ell^d = \begin{pmatrix} -2c_{36} & 2c_5' - 2sc_{36}' \\ 2c_5' - 2sc_{36}' & 2c_6'' + 2sc_5'' - s^2 c_{36}'' \end{pmatrix}.$$

- *$\boldsymbol{X}_\ell^d$ at p_0 is $\mathcal{A}$-equivalent to the cuspidal cross cap if and only if $c_{36} = 0$, $c_1 c_5 \neq 0$ and $c_{36}' \neq 0$ hold at p_0.*

Theorem 8.9. *The singular set of $\boldsymbol{X}_\ell^\ell$ is $S(\boldsymbol{X}_\ell^\ell) = \{(s,t) \mid c_3(t)(s^2+1)+2sc_5(t)+c_6(t)(s^2-1)=0\}$ and $\boldsymbol{X}_\ell^\ell$ is a frontal for any $(s_0,t_0) \in S(\boldsymbol{X}_\ell^\ell)$. Then we have the following assertions:*

- *If $c_{36}(t_0) \neq 0$ holds, then $\boldsymbol{X}_\ell^\ell$ to be a front near p_0.*
- *$\boldsymbol{X}_\ell^\ell$ at p_0 is $\mathcal{A}$-equivalent to the cuspidal edge if and only if $c_{36} \neq 0$ and $\alpha_l^l := -2c_1(c_5 + sc_{36}) + c_3'(s^2+1) + 2sc_5' + c_6'(s^2-1) \neq 0$ hold at p_0.*
- *$\boldsymbol{X}_\ell^\ell$ at p_0 is $\mathcal{A}$-equivalent to the swallowtail if and only if $c_{36} \neq 0$, $c_5 + sc_{36} \neq 0$, $\alpha_l^\ell = 0$ and $-c_1(\alpha_l^\ell)_s(\alpha_l^\ell)' \neq 0$ hold at p_0.*
- *$\boldsymbol{X}_\ell^\ell$ at p_0 is $\mathcal{A}$-equivalent to the cuspidal butterfly if and only if $c_{36} \neq 0$, $c_5 + sc_{36} \neq 0$, $\alpha_\ell^\ell = 0$, $-c_1(\alpha_\ell^\ell)_s(\alpha_\ell^\ell)' = 0$ and $c_1^2(\alpha_\ell^\ell)_{ss} - 2c_1(\alpha_\ell^\ell)_s' - c_1'(\alpha_\ell^\ell)_s + (\alpha_\ell^\ell)'' \neq 0$ hold at p_0.*
- *$\boldsymbol{X}_\ell^\ell$ at p_0 is $\mathcal{A}$-equivalent to the cuspidal lips (resp. cuspidal beaks) if and only if $c_{36} \neq 0$, $c_3'(s^2+1) + 2sc_5' + c_6'(s^2-1) = 0$, $c_5 + sc_{36} = 0$ and $\det H_\ell^\ell > 0$ (resp. $\det H_\ell^\ell < 0$ and $-c_1\big(-2c_1c_{36} + 2(c_5 + sc_{36}')\big) + \big(-2c_1(c_5 + sc_{36}) + c_3'(s^2+1) + 2sc_5' + c_6'(s^2-1)\big)' \neq 0$) hold at p_0, where*

$$H_\ell^\ell = \begin{pmatrix} 2c_{36} & 2c_5' + 2sc_{36}' \\ 2c_5' + 2sc_{36}' & c_3''(s^2+1) + 2sc_5'' + c_6''(s^2-1) \end{pmatrix}.$$

- *$\boldsymbol{X}_\ell^\ell$ at p_0 is $\mathcal{A}$-equivalent to the cuspidal cross cap if and only if $c_{36} = 0$, $c_1c_5 \neq 0$ and $c_{36}' \neq 0$ hold at p_0.*

8.3 Singularities of dual surfaces of $\boldsymbol{a}_0$

In this subsection, we apply the criteria in Subsection 8.1 for describing the conditions of singularities of dual surfaces of $\boldsymbol{a}_0$. We assume that $c_2 \equiv c_3 \equiv 0$.

Theorem 8.10. *The singular set of $\boldsymbol{X}_h^d$ is $S(\boldsymbol{X}_h^d) = \{(s,t) \mid c_4(t)\cos s + c_5(t)\sin s = 0\}$ and $\boldsymbol{X}_h^d$ is a frontal for any $p_0 = (s_0,t_0)$. Then we have the following assertions:*

- *If $c_1 \neq 0$ holds, then $\boldsymbol{X}_h^d$ to be a front near p_0.*
- *$\boldsymbol{X}_h^d$ at p_0 is $\mathcal{A}$-equivalent to the cuspidal edge if and only if $c_1 \neq 0$ and $\alpha_h^d := -c_6(c_4\sin s - c_5\cos s) - (c_4'\cos s + c_5'\sin s) \neq 0$ holds at p_0.*
- *$\boldsymbol{X}_h^d$ at p_0 is $\mathcal{A}$-equivalent to the swallowtail if and only if $c_1 \neq 0$, $-c_4\sin s + c_5\cos s \neq 0$, $\alpha_h^d = 0$ and $-c_6(\alpha_h^d)_s + (\alpha_h^d)' \neq 0$.*
- *$\boldsymbol{X}_h^d$ at p_0 is $\mathcal{A}$-equivalent to the cuspidal butterfly if and only if $c_1 \neq 0$, $-c_4\sin s + c_5\cos s \neq 0$, $\alpha_h^d = 0$, $-c_6(\alpha_h^d)_s + (\alpha_h^d)' = 0$ and $c_6^2(\alpha_h^d)_{ss} - 2c_6(\alpha_h^d)_s' - c_6'(\alpha_h^d)_s + (\alpha_h^d)'' \neq 0$ holds at p_0.*
- *$\boldsymbol{X}_h^d$ at p_0 is $\mathcal{A}$-equivalent to the cuspidal lips (resp. cuspidal beaks) if and only if $c_1 \neq 0$, $c_4'\cos s + c_5'\sin s = 0$, $c_4\sin s - c_5\cos s = 0$, and $\det H_h^d > 0$ (resp. $\det H_h^d < 0$ and $-c_6(\alpha_h^d)_s + (\alpha_h^d)' \neq 0$) hold at p_0, where*

$$H_h^d = \begin{pmatrix} -c_4\cos s - c_5\sin s & c_4'\sin s - c_5'\cos s \\ c_4'\sin s - c_5'\cos s & c_4''\cos s + c_5''\sin s \end{pmatrix}.$$

- $\boldsymbol{X}_h^d$ *at* p_0 *is* $\mathcal{A}$*-equivalent to the cuspidal cross cap if and only if* $c_1 = 0$, $c_5c_6 \neq 0$ *and* $c_1' \neq 0$ *hold at* p_0.

Theorem 8.11. *The singular set of* $\boldsymbol{X}_h^\ell$ *is* $S(\boldsymbol{X}_h^\ell) = \{(s,t) \mid c_1(t) - c_4(t)\cos s - c_5(t)\sin s = 0\}$ *and* $\boldsymbol{X}_h^\ell$ *is a frontal for any* $p_0 = (s_0, t_0)$. *Then we have the following assertions:*

- *If* $c_1(t_0) \neq 0$ *holds, then* $\boldsymbol{X}_h^\ell$ *to be a front near* p_0.
- $\boldsymbol{X}_h^\ell$ *at* p_0 *is* $\mathcal{A}$*-equivalent to the cuspidal edge if and only if* $c_1 \neq 0$ *and* $\alpha_h^\ell := -c_6(c_4 \sin s - c_5 \cos s) + c_1' - c_4' \cos s - c_5' \sin s \neq 0$ *holds at* p_0.
- $\boldsymbol{X}_h^\ell$ *at* p_0 *is* $\mathcal{A}$*-equivalent to the swallowtail if and only if* $c_1 \neq 0$, $c_4 \sin s - c_5 \cos s \neq 0$, $\alpha_h^\ell = 0$ *and* $-c_6(\alpha_h^\ell)_s + (\alpha_h^\ell)' \neq 0$ *holds at* p_0.
- $\boldsymbol{X}_h^\ell$ *at* p_0 *is* $\mathcal{A}$*-equivalent to the cuspidal butterfly if and only if* $c_1 \neq 0$, $c_4 \sin s - c_5 \cos s \neq 0$, $\alpha_h^\ell = 0$ *and* $-c_6(\alpha_h^\ell)_s + (\alpha_h^\ell)' = 0$ *and* $c_6^2(\alpha_h^\ell)_{ss} - 2c_6(\alpha_h^\ell)_s' - c_6'(\alpha_h^\ell)_s + (\alpha_h^\ell)'' \neq 0$ *holds at* p_0.
- $\boldsymbol{X}_h^\ell$ *at* p_0 *is* $\mathcal{A}$*-equivalent to the cuspidal lips (resp. cuspidal beaks) if and only if* $c_1 \neq 0$, $c_4 \sin s - c_5 \cos s = 0$, $c_1' - c_4' \cos s - c_5' \sin s = 0$ *and* $\det H_h^\ell > 0$ *(resp.* $\det H_h^\ell$ *and* $-c_6(\alpha_h^\ell)_s + (\alpha_h^\ell)' \neq 0$*) holds at* p_0, *where*

$$H_h^\ell = \begin{pmatrix} c_4 \cos s + c_5 \sin s & c_4' \sin s - c_5' \cos s \\ c_4' \sin s - c_5' \cos s & c_1'' - c_4'' \cos s - c_5'' \sin s \end{pmatrix}.$$

- $\boldsymbol{X}_h^\ell$ *at* p_0 *is* $\mathcal{A}$*-equivalent to the cuspidal cross cap if and only if* $c_1 = 0$, $c_5c_6 \neq 0$ *and* $c_1' \neq 0$ *hold at* p_0.

Remark 8.12. Surfaces $\boldsymbol{X}_h^\ell$ satisfying $c_6 \equiv 0$ is called a *hyperbolic-flat tangent lightcone circular surface* which was investigated in [22]. Substituting $c_6 \equiv 0$ in the formulae of Theorem 8.11, we have [22, Theorem 8.2].

8.4 Singularities of dual surfaces of $\boldsymbol{a}_2$

In this subsection, we apply criteria in Subsection 8.1 for describing the conditions of singularities of dual surfaces of $\boldsymbol{a}_2$. In this section, we assume that $c_2 \equiv c_6 \equiv 0$.

Theorem 8.13. *The singular set of* $\boldsymbol{X}_d^h$ *is* $S(\boldsymbol{X}_d^h) = \{(s,t) \mid c_1(t)\cosh s - c_5(t)\sinh s = 0\}$ *and* $\boldsymbol{X}_d^h$ *is a frontal for any* $p_0 = (s_0, t_0)$. *Then we have the following assertions*:

- *If* $c_4 \neq 0$ *holds, then* $\boldsymbol{X}_d^h$ *is a front near* p_0.
- $\boldsymbol{X}_d^h$ *at* p_0 *is* $\mathcal{A}$*-equivalent to the cuspidal edge if and only if* $c_4 \neq 0$ *and* $\alpha_d^h := -c_3(c_1 \sinh s - c_5 \cosh s) + c_1' \cosh s - c_5' \sinh s \neq 0$ *hold at* p_0.
- $\boldsymbol{X}_d^h$ *at* p_0 *is* $\mathcal{A}$*-equivalent to the swallowtail if and only if* $c_4 \neq 0$, $c_1 \sinh s - c_5 \cosh s \neq 0$ *and* $\alpha_d^h = 0$ *and* $-c_3(\alpha_d^h)_s + (\alpha_d^h)' \neq 0$ *hold at* p_0.
- $\boldsymbol{X}_d^h$ *at* p_0 *is* $\mathcal{A}$*-equivalent to the cuspidal butterfly if and only if* $c_4 \neq 0$, $c_1 \sinh s - c_5 \cosh s \neq 0$ *and* $\alpha_d^h = -c_3(\alpha_d^h)_s + (\alpha_d^h)' = 0$, $-c_3(\alpha_d^h)_s + (\alpha_d^h)' = 0$ *and* $c_3^2(\alpha_d^h)_{ss} - 2c_3(\alpha_d^h)_s' - c_3'(\alpha_d^h)_s + (\alpha_d^h)'' \neq 0$ *hold at* p_0.

- $\boldsymbol{X}^h_d$ *at* p_0 *is* $\mathcal{A}$*-equivalent to the cuspidal lips (resp. cuspidal beaks) if and only if* $c_4 \neq 0$, $c_1 \sinh s - c_5 \cosh s = 0$, $c_1' \cosh s - c_5' \sinh s = 0$ $\det H^h_d > 0$ *(resp.* $\det H^h_d < 0$ *and* $-c_3(\alpha^h_d)_s + (\alpha^h_d)' \neq 0$*) hold at* p_0*, where*

$$H^h_d = \begin{pmatrix} c_1 \cosh s - c_5 \sinh s & c_1' \sinh s - c_5' \cosh s \\ c_1' \sinh s - c_5' \cosh s & c_1'' \cosh s - c_5'' \sinh s \end{pmatrix}.$$

- $\boldsymbol{X}^h_d$ *at* p_0 *is* $\mathcal{A}$*-equivalent to the cuspidal cross cap if and only if* $c_4 = 0$, $c_3 c_5 \neq 0$ *and* $c_4' \neq 0$ *hold at* p_0*.*

Theorem 8.14. *The singular set of* $\boldsymbol{X}^\ell_d$ *is* $S(\boldsymbol{X}^\ell_d) = \{(s,t) \mid -c_4(t) + c_1(t)\cosh s - c_5(t)\sinh s = 0\}$ *and* $\boldsymbol{X}^\ell_d$ *is a frontal for any* $p_0 = (s_0, t_0) \in S(\boldsymbol{X}^\ell_d)$*. Then we have following assertions*:

- *If* $c_4 \neq 0$ *holds, then* $\boldsymbol{X}^\ell_d$ *is a front near* p_0*.*
- $\boldsymbol{X}^\ell_d$ *at* p_0 *is* $\mathcal{A}$*-equivalent to the cuspidal edge if and only if* $c_4 \neq 0$ *and* $\alpha^\ell_d := -c_3(c_1 \sinh s - c_5 \cosh s) - c_4' + c_1' \cosh s - c_5' \sinh s \neq 0$ *hold at* p_0*.*
- $\boldsymbol{X}^\ell_d$ *at* p_0 *is* $\mathcal{A}$*-equivalent to the swallowtail if and only if* $c_4 \neq 0$, $c_1 \sinh s - c_5 \cosh s \neq 0$, $\alpha^\ell_d = 0$ *and* $-c_3(\alpha^\ell_d)_s + (\alpha^\ell_d)' \neq 0$ *hold at* p_0*.*
- $\boldsymbol{X}^\ell_d$ *at* p_0 *is* $\mathcal{A}$*-equivalent to the cuspidal butterfly if and only if* $c_4 \neq 0$, $c_1 \sinh s - c_5 \cosh s \neq 0$ $c_1 \sinh s - c_5 \cosh s \neq 0$, $\alpha^\ell_d = 0$ $-c_3(\alpha^\ell_d)_s + (\alpha^\ell_d)' = 0$ *and* $c_3^2(\alpha^\ell_d)_{ss} - 2c_3(\alpha^\ell_d)_s' - c_3'(\alpha^\ell_d)_s + (\alpha^\ell_d)'' \neq 0$ *holds at* p_0*.*
- $\boldsymbol{X}^\ell_d$ *at* p_0 *is* $\mathcal{A}$*-equivalent to the cuspidal lips (resp. cuspidal beaks)* $c_4 \neq 0$, $c_1 \sinh s - c_5 \cosh s = 0$, $-c_4' + c_1' \cosh s - c_5' \sinh s = 0$ *and* $\det H^\ell_d > 0$ *(resp.* $\det H^\ell_d < 0$ *and* $-c_3(\alpha^\ell_d)_s + (\alpha^\ell_d)' \neq 0$*) hold at* p_0*, where*

$$H^\ell_d = \begin{pmatrix} c_1 \cosh s - c_5 \sinh s & c_1' \sinh s - c_5' \cosh s \\ c_1' \sinh s - c_5' \cosh s & -c_4'' + c_1'' \cosh s - c_5'' \sinh s \end{pmatrix}.$$

- $\boldsymbol{X}^\ell_d$ *at* p_0 *is* $\mathcal{A}$*-equivalent to the cuspidal cross cap if and only if* $c_4 = 0$, $c_3 c_5 \neq 0$ *and* $c_4' \neq 0$ *hold at* p_0*.*

We now give proofs of these theorems.

Proof of Theorem 8.9. Since

$$\begin{aligned} \left(\boldsymbol{X}^\ell_\ell\right)_s &= 2s\boldsymbol{a}_0 + 2\boldsymbol{a}_1 + 2s\boldsymbol{a}_2 \\ \left(\boldsymbol{X}^\ell_\ell\right)' &= 2sc_1\boldsymbol{a}_0 + 2c_1\boldsymbol{a}_1 + 2c_1\boldsymbol{a}_2 + \big(c_3(s^2+1) + 2sc_5 + c_6(s^2-1)\big)\boldsymbol{a}_3, \end{aligned}$$

we have $S(\boldsymbol{X}^\ell_\ell) = \{(s,t) \mid c_3(t)(s^2+1) + 2sc_5(t) + c_6(t)(s^2-1) = 0\}$. Furthermore, an isotropic map $(\boldsymbol{X}^\ell_\ell, \boldsymbol{\ell}) : U \to \Delta_4$ is a Legendrian immersion if and only if $c_{36} \neq 0$ on $S(\boldsymbol{X}^\ell_\ell)$. In this case $\boldsymbol{X}^\ell_\ell$ is a front near p_0. Since $\boldsymbol{a}_0$ and $\boldsymbol{a}_2$ are linearly independent to TLC^*, we can choose the discriminant function λ as

$$\lambda = \det\left(\left(\boldsymbol{X}^\ell_\ell\right)_s, \left(\boldsymbol{X}^\ell_\ell\right)', \boldsymbol{a}_0, \boldsymbol{a}_2\right) = -2((s^2+1)c_3(t) + 2c_5(t) + 2sc_6(t)).$$

Since the kernel direction of $d\boldsymbol{X}_\ell^\ell$ on singular set is $\eta = -c_1\partial s + \partial t$ and we can take a transversal vector field ∂s, we have

$$\begin{array}{rcl}
\eta\lambda & = & -2c_1(c_5 + sc_{36}) + c_3'(s^2+1) + 2sc_5' + c_6'(s^2-1) \\
\eta\eta\lambda & = & -c_1\big(-2c_1c_{36} + 2(c_5 + sc_{36}')\big) \\
& & \qquad +\big(-2c_1(c_5 + sc_{36}) + c_3'(s^2+1) + 2sc_5' + c_6'(s^2-1)\big)' \\
\operatorname{Hess}\lambda & = & \begin{pmatrix} 2c_{36} & 2c_5' + 2sc_{36}' \\ 2c_5' + 2sc_{36}' & c_3''(s^2+1) + 2sc_5'' + c_6''(s^2-1) \end{pmatrix}
\end{array}$$

Hence we have all assertions of Theorem 8.9 except the case for the condition for the cuspidal cross cap. We give the proof of the condition for the cuspidal cross cap as follows: Let us define a lift $\omega : U \to T^*LC^*$ by

$$\omega_p(v) = \langle v, \boldsymbol{\ell}(p)\rangle\,,\ v \in T_{X_\ell^\ell(p)}LC^*,\ p \in U.$$

Then ω does not have intersection with the zero section. Since $(\pi\circ\omega)_*(z) = d\boldsymbol{X}_\ell^\ell(z)$ for any vector $z \in T_pU$, we have $\left\langle \boldsymbol{\ell}, d\boldsymbol{X}_\ell^\ell\right\rangle = \theta_4(\boldsymbol{a}_3, \boldsymbol{X}_\ell^\ell) = 0$. Thus we have $(\pi\circ\omega)_*(T_pU) \subset \ker\omega_p$. This means that ω is the admissible lift of $\boldsymbol{X}_\ell^\ell$. Under the assumption that $c_{36}(t_0) = 0$, $\lambda_s(s_0,t_0) \neq 0$ if and only if $c_5(t_0) \neq 0$. Then $S(\boldsymbol{X}_\ell^\ell)$ can be parameterized as $(s(t),t)$ for some function $s(t)$. Putting $\xi(t) = \boldsymbol{a}_3(t)$, then ξ is a non-zero vector field along $\boldsymbol{X}_\ell^\ell|_{S(X_\ell^\ell)}$. Since $\left\langle \xi, \boldsymbol{X}_\ell^\ell\right\rangle = 0$, vector field ξ satisfies the conditions of Theorem 8.5. Therefore the function $\psi_{X_\ell^\ell}(t)$ is equal to $\langle \eta\xi, \boldsymbol{\ell}\rangle(t) = -c_{36}(t)$. On the other hand, if $\lambda_s(s_0,t_0) = 0$ and $\lambda'(s_0,t_0) \neq 0$, then $(\boldsymbol{X}_\ell^\ell, (s_0,t_0))$ is not $\mathcal{A}$-equivalent to the cuspidal cross cap. This completes the proof of Theorem 8.9. $\square$

We can give the proofs of Theorems 8.6, 8.8, 8.10, 8.11, 8.13 and 8.14 by the same arguments as those of the above proof. We only state the fundamental data here, and omit the detailed proof. The discriminant function λ, null vector field η, the one-form ω and the vector field ξ for each dual surfaces are shown in the Table 1.

Surface	λ	η	ω	ξ
$\boldsymbol{X}_\ell^h$	$2c_3 + 2sc_5 + s^2c_{36}$	$(-c_1, 1)$	$\langle *, \boldsymbol{\ell}\rangle$	$\boldsymbol{a}_3$
$\boldsymbol{X}_\ell^d$	$-2c_6 - 2sc_5 + s^2c_{36}$	$(c_1, 1)$	$\langle *, \boldsymbol{\ell}\rangle$	$\boldsymbol{a}_3$
$\boldsymbol{X}_h^d$	$c_4\cos s + c_5\sin s$	$(-c_6, 1)$	$\langle *, \boldsymbol{a}_0\rangle$	$\boldsymbol{a}_1$
$\boldsymbol{X}_h^\ell$	$c_1 - c_4\cos s - c_5\sin s$	$(-c_6, 1)$	$\langle *, \boldsymbol{a}_0\rangle$	$\boldsymbol{a}_1$
$\boldsymbol{X}_d^h$	$c_1\cosh s - c_5\sinh s$	$(-c_3, 1)$	$\langle *, \boldsymbol{a}_2\rangle$	$\boldsymbol{a}_1$
$\boldsymbol{X}_d^\ell$	$-c_4 + c_1\cosh s - c_5\sinh s$	$(-c_3, 1)$	$\langle *, \boldsymbol{a}_2\rangle$	$\boldsymbol{a}_1$

Table 1: Fundamental data to recognize the conditions of singularities of dual surfaces

9 Dualities of singularities

Comparing Theorems 8.6, 8.8 and 8.9, when singular point is always $(0,t)$, with Theorems 8.10, 8.11, 8.13 and 8.14 we observe a certain duality between the swallowtail and the cuspidal cross cap. It can be summerized as follows.

Remark 9.1. The conditions that singular set is equal to the curve $(0,t)$ is

- $c_3 \equiv 0$ for $\boldsymbol{X}_\ell^h$,
- $c_6 \equiv 0$ for $\boldsymbol{X}_\ell^d$,
- $c_3 - c_6 \equiv 0$ for $\boldsymbol{X}_\ell^\ell$
- $c_4 \equiv 0$ for $\boldsymbol{X}_h^d$
- $c_1 - c_4 \equiv 0$ for $\boldsymbol{X}_h^\ell$,
- $c_1 \equiv 0$ for $\boldsymbol{X}_d^h$,
- $c_1 - c_4 \equiv 0$ for $\boldsymbol{X}_d^\ell$.

Moreover, if $c_2 \equiv c_3 \equiv c_1 - c_4 \equiv 0$, then $\boldsymbol{X}_\ell^h$ at $(0,t_0)$ is $\mathcal{A}$-equivalent to the swallowtail if and only if $c_6 = 0$ and $c_1 c_6' \neq 0$ at t_0. This condition is the same as the condition that $\boldsymbol{X}_h^\ell$ at $(0,t_0)$ is $\mathcal{A}$-equivalent to the cuspidal cross cap. Furthermore, $\boldsymbol{X}_\ell^h$ at $(0,t_0)$ is $\mathcal{A}$-equivalent to the cuspidal cross cap if and only if $c_1 = 0$ and $c_6 c_1' \neq 0$ at t_0. This condition is the same as the condition that $\boldsymbol{X}_h^\ell$ at $(0,t_0)$ is $\mathcal{A}$-equivalent to the swallowtail. Like as these arguments, we have the same type condition of singular points for dual surfaces when the singular set is equal to $(0,t)$.

We can summerize this situation on the Table 2. In the table, S means the singular set.

	duality	$S = \{(0,t)\}$	cuspidal edge	swallowtail	cuspidal cross cap
$\boldsymbol{X}_\ell^h$	$c_2 \equiv 0$ $c_1 - c_4 \equiv 0$	$c_3 \equiv 0$	$c_6 \neq 0$, $c_1 c_5 \neq 0$	$c_6 c_5 \neq 0$, $c_1 = 0,\ c_1' \neq 0$,	$c_1 c_5 \neq 0$, $c_6 = 0,\ c_6' \neq 0$,
$\boldsymbol{X}_\ell^d$	$c_2 \equiv 0$ $c_1 - c_4 \equiv 0$	$c_6 \equiv 0$	$c_3 \neq 0$, $c_1 c_5 \neq 0$	$c_3 c_5 \neq 0$, $c_1 = 0,\ c_1' \neq 0$	$c_1 c_5 \neq 0$, $c_3 = 0,\ c_3' \neq 0$,
$\boldsymbol{X}_\ell^\ell$	$c_2 \equiv 0$ $c_1 - c_4 \equiv 0$	$c_3 - c_6 \equiv 0$	$c_{36} \neq 0$, $c_1 c_5 \neq 0$	$c_{36} c_5 \neq 0$, $c_1 = 0,\ c_1' \neq 0$	$c_1 c_5 \neq 0$, $c_{36} = 0,\ c_{36}' \neq 0$,
$\boldsymbol{X}_h^d$	$c_2 \equiv 0$ $c_3 \equiv 0$	$c_4 \equiv 0$	$c_1 \neq 0$ $c_5 c_6 \neq 0$	$c_1 c_5 \neq 0$ $c_6 = 0,\ c_6' \neq 0$	$c_5 c_6 \neq 0$ $c_1 = 0,\ c_1' \neq 0$
$\boldsymbol{X}_h^\ell$	$c_2 \equiv 0$ $c_3 \equiv 0$	$c_1 - c_4 \equiv 0$	$c_1 \neq 0$ $c_5 c_6 \neq 0$	$c_1 c_5 \neq 0$ $c_6 = 0,\ c_6' \neq 0$	$c_5 c_6 \neq 0$ $c_1 = 0,\ c_1' \neq 0$
$\boldsymbol{X}_d^h$	$c_2 \equiv 0$ $c_6 \equiv 0$	$c_1 \equiv 0$	$c_4 \neq 0$ $c_3 c_5 \neq 0$	$c_4 c_5 \neq 0$ $c_3 = 0,\ c_3' \neq 0$	$c_3 c_5 \neq 0$ $c_4 = 0,\ c_4' \neq 0$
$\boldsymbol{X}_d^\ell$	$c_2 \equiv 0$ $c_6 \equiv 0$	$c_1 - c_4 \equiv 0$	$c_4 \neq 0$ $c_3 c_5 \neq 0$	$c_4 c_5 \neq 0$ $c_3 = 0,\ c_3' \neq 0$	$c_3 c_5 \neq 0$ $c_4 = 0,\ c_4' \neq 0$

Table 2: Dualities of condition for singularity.

We can observe there are some dual relations of conditions for singularities of the swallowtail and the cuspidal cross cap on each dual points of surfaces. Furthermore, the condition of holding the duality and that the singular set is $\{(0,t)\}$ are the same between $\boldsymbol{X}^h_\ell$ and $\boldsymbol{X}^\ell_h$ (resp. between $\boldsymbol{X}^d_\ell$ and $\boldsymbol{X}^\ell_d$):

$$\begin{array}{ccc} \boldsymbol{X}^h_\ell & \xrightarrow{\text{taking singular value}} & \boldsymbol{a}_0 \\ {\scriptstyle \Delta_2\text{-dual}}\uparrow & & \downarrow{\scriptstyle \Delta_2\text{-dual}} \\ \boldsymbol{\ell} & \xleftarrow{\text{taking singular value}} & \boldsymbol{X}^\ell_h \end{array} \quad \text{and} \quad \begin{array}{ccc} \boldsymbol{X}^d_\ell & \xrightarrow{\text{taking singular value}} & \boldsymbol{a}_2 \\ {\scriptstyle \Delta_3\text{-dual}}\uparrow & & \downarrow{\scriptstyle \Delta_3\text{-dual}} \\ \boldsymbol{\ell} & \xleftarrow{\text{taking singular value}} & \boldsymbol{X}^\ell_d. \end{array}$$

Like as the remark, a duality between the swallowtail and the cuspidal cross cap have been pointed out in many researches, for example, [33, 13, 22]. In this section, we give an interpretation for this duality. Firstly, we prove the following lemma.

Lemma 9.2. *Let M_i $(i = 1, 2)$ be three dimensional manifolds and $\Delta \subset M_1 \times M_2$ a five dimensional submanifold with the contact structure. Assume that the canonical projection $\pi_1 : \Delta \to M_1$ is a Legendre fibrations. If an isotropic map $L_1 = (f_1, \nu_1)$ and a frontal $f_2 : U \to M_2$ satisfies that p is a non-degenerate singular point of both f_i $(i = 1, 2)$ and ν_1 degenerates a curve such that $\nu_1 = f_2 \circ \sigma$, where σ is a submersion $U \to S(f)$. If the null direction of f_1 does not parallel to the kernel of σ, then the following two conditions are equivalent.*

- *L_1 is a Legendrian immersion.*
- *The null direction of f_2 at p is transversal to $S(f_1)$.*

Proof. Since p is a non-degenerate singular point, L_1 is a Legendrian immersion if and only if the directional derivative $\eta_1\nu_1$ does not vanish. this is equivalent to the condition that $df_2(\eta_1)(\sigma)$ does not vanish. This is equivalent to that the tangential direction of $S(f_1)$ does not parallel to η_2. This is equivalent to the condition that η_2 is transversal to $S(f_1)$. This completes the proof. $\square$

Theorem 9.3. *Let p be a non-degenerate singular point of a frontal f. Then we have the following criteria of singularities by using the function ψ_f defined in* (8.2)*.*

1. *If $\psi_f(p) \neq 0$, then f at p is $\mathcal{A}$-equivalent to the cuspidal edge.*
2. *Assume that f is a front. If $\psi_f(p) = 0$ and $(d/dt)\psi_f(p) \neq 0$, then f at p is $\mathcal{A}$-equivalent to the swallowtail.*
3. *Assume that the null direction at p is transversal to $S(f)$. If $\psi_f(p) = 0$ and $(d/dt)\psi_f(p) \neq 0$, then f at p is $\mathcal{A}$-equivalent to the cuspidal cross cap.*

Proof. Since the conditions are independent of the choice of coordinates, we take the coordinate system (u, v) satisfying $S(f) = \{v = 0\}$. Under this conditions, ψ_f is proportional to ϕ_f, where ϕ_f is defined in (8.1). Firstly, we prove (1). The condition $\phi_f \neq 0$ implies that f_u and $\eta\nu$ are linearly independent. Since ν points the kernel direction of df, this implies that f to be a front. Moreover, we have $f_u \neq \mathbf{0}$, this implies that η does not tangent to $S(f)$. By Theorem 8.1, we have (1).

Next, we assume that f to be a front and $\phi_f = 0$ at p. Then this condition implies $f_u(p) = \mathbf{0}$, namely, η tangents $S(f)$ at p. Thus we can take a function $\beta(u)$ such that

$\eta(u) = \partial/\partial u + \beta(u)\partial/\partial v$, $\beta(0) = 0$. By Theorem 8.1, f at p is $\mathcal{A}$-equivalent to the swallowtail if and only if $\beta'(0) \neq 0$. On the other hand, $(d/dt)\phi_f(p) \neq 0$ implies that $\det(f_{uu}, \nu, \nu_u)(p) \neq 0$. Since f_v, ν and ν_u are linear independent at p, this is equivalent to $\langle f_{uu}, f_v\rangle(p) \neq 0$. Since η is the null vector field on the u-axis, $f_u + \beta(u) + f_v = \mathbf{0}$ holds on the u-axis. Thus $\langle f_{uu}, f_v\rangle(p) \neq 0$ implies $\beta'(0) \neq 0$. This completes the proof. The assertion (3) directly holds from Theorem 8.1. $\square$

We can give the alternative proof of Theorem 8.11 in the special case of $c_1 - c_4 \equiv 0$ and $c_5 \neq 0$.

Proof of Theorem 8.11. If $c_2 \equiv c_3 \equiv c_1 - c_4 \equiv 0$, and $c_5(t_0) \neq 0$, then by Theorem 8.6 $\boldsymbol{X}_h^\ell$ at $(t_0, 0)$ is $\mathcal{A}$-equivalent to the swallowtail if and only if $c_6 \neq 0$, $c_1 = 0$ and $c_1' \neq 0$ at t_0. Furthermore, $\boldsymbol{X}_h^\ell$ at $(t_0, 0)$ is $\mathcal{A}$-equivalent to the cuspidal cross cap if and only if $c_1 \neq 0$, $c_6 = 0$ and $c_6' \neq 0$ at t_0.

Under the assumptions $c_2 \equiv c_3 \equiv c_1 - c_4 \equiv 0$ and $c_5(t_0) \neq 0$, it holds that $S(\boldsymbol{X}_\ell^h) = S(\boldsymbol{X}_h^\ell) = \{(t, 0)\}$ near $(t_0, 0)$. Hence we can apply Lemma 9.2 and Theorem 9.3. This means that the conditions for the swallowtail and the cuspidal cross cap of the dual surface are obtained by only interchanging the conditions for the cuspidal cross cap and the swallowtail of the original surface. Thus we have that $\boldsymbol{X}_h^\ell$ at $(t_0, 0)$ is $\mathcal{A}$-equivalent to the cuspidal cross cap if and only if $c_6 \neq 0$, $c_1 = 0$ and $c_1' \neq 0$ at t_0. Furthermore, $\boldsymbol{X}_h^\ell$ at $(t_0, 0)$ is $\mathcal{A}$-equivalent to the swallowtail if and only if $c_1 \neq 0$, $c_6 = 0$ and $c_6' \neq 0$ at t_0. This is the same as Theorem 8.11 under the assumption $c_1 - c_4 \equiv 0$. $\square$

A A criterion for the cuspidal butterfly

In this section, we give a proof of Theorem 8.2. The main tool for the proof is the notion of generating families. Let $G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, \mathbf{0})$ be a function germ which we call an *unfolding of* $g(q) = G(q, \mathbf{0})$. We say that G is a *Morse family of hypersurfaces* if the mapping

$$\Delta^* G = \left(G, \frac{\partial G}{\partial q_1}, \dots, \frac{\partial G}{\partial q_k}\right) : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R} \times \mathbb{R}^k, \mathbf{0})$$

is non-singular, where $(q, x) = (q_1, \dots, q_k, x_1, \dots, x_n) \in (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$. In this case we have a smooth $(n-1)$-dimensional submanifold

$$\Sigma_*(G) = \left\{ (q, x) \in (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \;\middle|\; G(q, x) = \frac{\partial G}{\partial q_1}(q, x) = \cdots = \frac{\partial G}{\partial q_k}(q, x) = 0 \right\}$$

and the map germ $\Phi_G : (\Sigma_*(G), \mathbf{0}) \longrightarrow PT^*\mathbb{R}^n$ defined by

$$\Phi_G(q, x) = \left(x, \left[\frac{\partial G}{\partial x_1}(q, x) : \cdots : \frac{\partial G}{\partial x_n}(q, x)\right]\right)$$

is a Legendrian immersion germ. The fundamental result of Arnol'd-Zakalyukin [2, 34] assets that all Legendrian submanifold germs in $PT^*\mathbb{R}^n$ are constructed by the above method. We call G a *generating family* of $\Phi_G(\Sigma_*(G))$. Therefore the wave front of $\Phi_G(\Sigma_*(G))$ is

$$W(\Phi_G) = \left\{ x \in \mathbb{R}^n \;\middle|\; \exists q \in \mathbb{R}^k \text{ such that } G(q, x) = \frac{\partial G}{\partial q_1}(q, x) = \cdots = \frac{\partial G}{\partial q_k}(q, x) = 0 \right\}.$$

We also write $\mathcal{D}_G = W(\Phi_G)$ and call it the *discriminant set* of G.

We now introduce an equivalence relation among Legendrian submanifold germs. Let $i : (L, p) \subset (PT^*\mathbb{R}^n, p)$ and $i' : (L', p') \subset (PT^*\mathbb{R}^n, p')$ be Legendrian submanifold germs. Then we say that i and i' are *Legendrian equivalent* if there exists a contact diffeomorphism germ $H : (PT^*\mathbb{R}^n, p) \longrightarrow (PT^*\mathbb{R}^n, p')$ such that H preserves fibers of π and that $H(L) = L'$.

Since the Legendrian lift $i : (L, p) \subset (PT^*\mathbb{R}^n, p)$ is uniquely determined on the regular part of the wave front $W(i)$, we have the following simple but significant property of Legendrian immersion germs[35]:

Proposition A.1. *Let $i : (L, p) \subset (PT^*\mathbb{R}^n, p)$ and $i' : (L', p') \subset (PT^*\mathbb{R}^n, p')$ be Legendrian immersion germs such that the representative of both the regular sets of the projections $\pi \circ i$ and $\pi \circ i'$ are dense. Then i and i' are Legendrian equivalent if and only if wave front sets $W(i)$ and $W(i')$ are diffeomorphic as set germs.*

The assumption in the above proposition is a generic condition for i and i'.

We can interpret the Legendrian equivalence by using the notion of generating families. We denote $\mathcal{E}_n$ the local ring of function germs $(\mathbb{R}^n, \mathbf{0}) \longrightarrow \mathbb{R}$ with the unique maximal ideal $\mathfrak{M}_n = \{h \in \mathcal{E}_n \mid h(\mathbf{0}) = 0\ \}$. Let $G_1, G_2 : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ be function germs. We say that G_1 and G_2 are *P-$\mathcal{K}$-equivalent* if there exists a diffeomorphism germ $\Psi : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$ of the form $\Psi(q, x) = (\psi_1(q, x), \psi_2(x))$ for $(q, x) \in (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$ such that $\Psi^*(\langle G_1 \rangle_{\mathcal{E}_{k+n}}) = \langle G_2 \rangle_{\mathcal{E}_{k+n}}$. Here $\Psi^* : \mathcal{E}_{k+n} \longrightarrow \mathcal{E}_{k+n}$ is the pull back $\mathbb{R}$-algebra isomorphism defined by $\Psi^*(h) = h \circ \Psi$.

Let $\bar{G} : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, \mathbf{0})$ be a function germ. We say that $\bar{G}$ is a *$\mathcal{K}$-versal unfolding of* $g = \bar{G}|\mathbb{R}^k \times \{\mathbf{0}\}$ if for any unfolding $G : (\mathbb{R}^k \times \mathbb{R}^m, \mathbf{0}) \longrightarrow (\mathbb{R}, \mathbf{0})$ of g (i.e., $G(q, \mathbf{0}) = g(q)$), there exists a map germ $\phi : (\mathbb{R}^m, \mathbf{0}) \longrightarrow (\mathbb{R}^n, \mathbf{0})$ such that $\phi^*\bar{G}$ and G are P-$\mathcal{K}$-equivalent, where $\phi^*\bar{G}(q, u) = \bar{G}(q, \phi(u))$. For an unfolding $G(t, x)$ of a function $g(t)$ of one-variable, we have the following useful criterion on the $\mathcal{K}$-versal unfoldings in (cf., [4], 6.10): We say that g has an *A_r-singularity* at t_0 if $g^{(p)}(t_0) = 0$ for all $1 \leq p \leq r$, and $g^{(r+1)}(t_0) \neq 0$. We have the following lemma

Lemma A.2. *Let G be an unfolding of g and $g(t)$ has an A_r-singularity ($r \geq 1$) at t_0. We denote the $(r-1)$-jet of the partial derivative $\partial G/\partial x_i$ at t_0 by*

$$j^{(r-1)}\left(\frac{\partial G}{\partial x_i}(t, x_0)\right)(t_0) = \sum_{j=0}^{r-1} \alpha_{ji}(t - t_0)^j$$

for $i = 1, \ldots, n$. Then G is a $\mathcal{K}$-versal unfolding if and only if the $r \times n$ matrix of coefficients (α_{ji}) has rank r ($r \leq n$).

It follows from the above lemma that the function germ defined by

$$t^{r+1} + x_1 t^{r-1} + x_2 t^{r-2} + \cdots + x_{r-1} t + x_r$$

is a $\mathcal{K}$-versal unfolding of $g(t) = t^{r+1}$. One of the main results in the theory of Legendrian singularities is the following theorem:

Theorem A.3. *Let $G_1, G_2 : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ be Morse families of hypersurfaces. Then Φ_{G_1} and Φ_{G_2} are Legendrian equivalent if and only if G_1 and G_2 are P-$\mathcal{K}$-equivalent.*

As a corollary of Proposition A.1 and Theorem A.3, we have the following proposition.

Proposition A.4. *Let $G_1, G_2 : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ be Morse families of hypersurfaces. Suppose that both the regular sets of the representative of projections $\pi \circ \Phi_{G_1}, \pi \circ \Phi_{G_2}$ are dense. Then $(W(\Phi_{G_1}), 0)$ and $(W(\Phi_{G_2}), 0)$ are diffeomorphic as set germs if and only if G_1 and G_2 are P-$\mathcal{K}$-equivalent.*

The following Lemma roles the key of the proof for the criteria. Two function germs $g_i : (\mathbb{R}, \mathbf{0}) \to (\mathbb{R}, 0)$ $(i = 1, 2)$ are $\mathcal{R}$-equivalent if there exists a diffeomorphism germ $\alpha : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ such that $\alpha \circ g_1 = g_2$ holds.

Lemma A.5. *Let $g : ((\mathbb{R}; t), \mathbf{0}) \to (\mathbb{R}, 0)$ be a function germ such that $\mathcal{R}$-equivalent to t^5. If an unfolding $G : ((\mathbb{R} \times \mathbb{R}^3; t, x, y, z), \mathbf{0}) \to (\mathbb{R}, 0)$ of g is a Morse family and a function $\bar{G}(t, x, y, z, w) = G(t, x, y, z) + wt^3$ is a $\mathcal{K}$-versal unfolding of g, then $G(t, x, y, z)$ is P-$\mathcal{K}$-equivalent to $t^5 + xt^2 + yt + z$.*

Proof. Since the condition does not depend on the parameter transformation of t, we can assume that $g(t) = t^5$. Moreover, since the map $t^5 + wt^3 + xt^2 + yt + z$ is the versal unfolding of t^5, there is a diffeomorphism $(\phi_1, \phi_2, \phi_3, \phi_4) : \mathbb{R}^4 \to \mathbb{R}^4$ such that G is P-$\mathcal{K}$-equivalent to

$$t^5 + \phi_1(x, y, z, w)t^3 + \phi_2(x, y, z, w)t^2 + \phi_3(x, y, z, w)t + \phi_4(x, y, z, w).$$

Since $\overline{G} = G + wt^3$ is a $\mathcal{K}$-versal unfolding, and the condition of lemma only depend on the P-$\mathcal{K}$-equivalent class, we can rechoose (x, y, z) such that $\overline{G}$ is P-$\mathcal{K}$-equivalent to

$$t^5 + \phi_1(x, y, z, w)t^3 + xt^2 + yt + z.$$

Furthermore, since $\overline{G}$ is a versal unfolding and $\partial\phi_1/\partial w(\mathbf{0}) = 0$, we rechoose w such that $\overline{G}$ is P-$\mathcal{K}$-equivalent to

$$t^5 + (w - h(x, y, z))t^3 + xt^2 + yt + z$$

for some function h. Summerizing up these argument, we can assume that $\overline{G}$ is

$$\overline{G}_h(t, x, y, z) := t^5 + (w - h(x, y, z))t^3 + xt^2 + yt + z.$$

We have the following Zakalyukin's lemma

Lemma A.6. [35, Theorem 1.4] *Let $\mathcal{V} : (\mathbb{R} \times \mathbb{R}^4, \mathbf{0}) \to (\mathbb{R}, 0)$ be a $\mathcal{K}$-versal unfolding of the form*

$$\mathcal{V}(t, x, y, z, w) = t^5 + wt^3 + xt^2 + yt + z \tag{A.1}$$

and $\sigma : (\mathbb{R}^4, \mathbf{0}) \to (\mathbb{R}, 0)$ be a function germ with (x, y, z, w)-variables such that $\partial\sigma/\partial w(\mathbf{0}) \neq 0$. Then there exists a diffeomorphism germ $\Theta : (\mathbb{R}^4, \mathbf{0}) \to (\mathbb{R}^4, \mathbf{0})$ such that

$$\Theta(D_{\mathcal{V}}) = D_{\mathcal{V}} \text{ and } \sigma \circ \Theta(x, y, z, w) = w.$$

Let us continue to prove of Lemma A.5. We apply Lemma A.6 to $\mathcal{V}$ of (A.1) and $w - h(x, y, z)$. Then there exists a diffeomorphism germ Θ such that

$$\Theta(D_{\mathcal{V}}) = D_{\mathcal{V}} \text{ and } (w - h(x, y, z)) \circ \Theta(x, y, z, w) = w.$$

We define a diffeomorphism germ

$$\Psi(x, y, z, w) = (x, y, z, w - h(x, y, z)),$$

then it holds that $\Psi^*\mathcal{V} = \overline{G}_h$. Define a new diffeomorphism germ $\tilde{\Theta}$ by $\tilde{\Theta} = \Psi \circ \Theta$ then we have

$$\tilde{\Theta}(D_{\mathcal{V}}) = \Psi \circ \Theta(D_{\mathcal{V}}) = \Psi(D_{\mathcal{V}}) = D_{\overline{G}_h}.$$

Hence $D_{\mathcal{V}}$ and $D_{\overline{G}}$ are diffeomorphic. On the other hand, let us define $\pi : (\mathbb{R}^4, \mathbf{0}) \to (\mathbb{R}, 0)$ by $\pi(x, y, z, w) = w$. Since

$$\pi \circ \tilde{\Theta}(x, y, z, w) = \pi \circ \Psi \circ \Theta = (w - h(x, y, z)) \circ \Theta = w,$$

we have $\pi \circ \tilde{\Theta} = \pi$. Since the set of regular points of $D_{\mathcal{V}}$ is dense, by the Zakalyukin theorem ([35], see also [28, Appendix]), there exist a diffeomorphism germ $\Xi : \mathbb{R} \times \mathbb{R}^4 \to \mathbb{R} \times \mathbb{R}^4$ of the form

$$\begin{aligned}&\Xi(t, x, y, z, w)\\ &= \big(\xi(t, x, y, z, w), \zeta_1(x, y, z, w), \zeta_2(x, y, z, w), \zeta_3(x, y, z, w), \zeta_4(w)\big)\end{aligned}$$

such that $\Xi^*(\langle \mathcal{V} \rangle_{\mathcal{E}_{1+4}}) = \langle \overline{G}_h \rangle_{\mathcal{E}_{1+4}}$.

If we restrict the above map to $w = 0$, we complete the proof Lemma A.2. □

Using these results, we give the criterion of the A_4-singularity of wave fronts.

Let $f : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ be a front and ν be the normal vector field of f. Let $\mathbf{0}$ be a non-degenerate singular point of f. Needless to say, the conditions of Theorem 8.2 do not depend on the choice of coordinates and choice of ν. One can prove the following lemma.

Lemma A.7. *One can choose the coordinate systems (u, v) of $(\mathbb{R}^2, \mathbf{0})$ and (X_1, X_2, Z) of $(\mathbb{R}^3, \mathbf{0})$ satisfying*

- $\eta \equiv \partial v$.
- $f(u, v) = (f_1(u, v), f_2(u, v), u)$ *and* $(f_1)_u(\mathbf{0}) = (f_2)_u(\mathbf{0}) = 0$.
- $\nu(\mathbf{0}) = (1, 0, 0)$.

Under this coordinate system, we prove that if $f : \mathbb{R}^2 \to \mathbb{R}^3$ satisfies $\eta\lambda = \eta\eta\lambda = 0$ and $\eta\eta\eta\lambda \neq 0$ at $\mathbf{0}$ then f at $\mathbf{0}$ is $\mathcal{A}$-equivalent to the cuspidal butterfly.

Proof of Theorem 8.2. Let us fix a small number u and consider a family of plane curves $\Gamma^u(v) = \Gamma(u, v) = (f_1(u, v), f_2(u, v), u)$ in the plane $\Pi_u = \{(X_1, X_2, Z) | Z = u\}$ and show that these are fronts near $\mathbf{0}$. Denote $\nu = (\nu_1, \nu_2, \nu_3)$ and put

$$[N^u(v)] = [(\nu_1(u, v), \nu_2(u, v), 0)].$$

Then $[N^u(v)]$ is well-defined near $\mathbf{0}$. We put

$$\gamma(u, v) = (f_1(u, v), f_2(u, v)) \text{ and } n(u, v) = (\nu_1(u, v), \nu_2(u, v)).$$

Then, since $\langle \gamma'(u, v), n(u, v) \rangle \equiv 0$, $(\gamma, [n])$ is an isotropic map for all u, where $'$ denotes $\partial/\partial v$ and $\langle \cdot, \cdot \rangle$ is the canonical inner product of $\mathbb{R}^3$. Since $\nu_3'(\mathbf{0})$, we have $n'(\mathbf{0}) \neq \mathbf{0}$. This implies that for each u near 0, $(\gamma, [n])$ is a Legendrian immersion germ.

We define two functions $\Psi : \mathbb{R} \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ and $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ as follows:

$$\Psi(v, X_1, X_2, Z) = n_1(Z, v)(X_1 - f_1(Z, v)) + n_2(Z, v)(X_2 - f_2(Z, v)), \quad \psi(v) = \Psi(v, 0, 0, 0).$$

Then we have $D_\Psi = f(U)$. Hence by Lemma A.6 and the arguments in the above, it is sufficient to prove that ψ has an A_4-singularity and Ψ satisfies the conditions of Lemma A.5. In the following context, we put $Z = u$.

Lemma A.8. *It holds that* $f'(\mathbf{0}) = f'(\mathbf{0}) = f'''(\mathbf{0}) = \mathbf{0}$, $f''''(\mathbf{0}) \neq \mathbf{0}$ *and* $(\overline{f})'(\mathbf{0}) = (\overline{f})''(\mathbf{0}) = (\overline{f})'''(\mathbf{0}) = \mathbf{0}$, $(\overline{f})''''(\mathbf{0}) \neq \mathbf{0}$.

Proof. Since ∂_v is the null vector field, so that we have $f'(\mathbf{0}) = \mathbf{0}$ and $S(f) = \{f_v = \mathbf{0}\}$. By $\eta\lambda = 0$, since $(\partial_v =)\eta_0 \in T_0S(f)$, it holds that $f''(\mathbf{0}) = \mathbf{0}$. Furthermore, by $\lambda''(\mathbf{0}) = 0$ and $f'(\mathbf{0}) = f''(\mathbf{0}) = \mathbf{0}$, we have $\lambda''(\mathbf{0}) = \det(f_u, f''', \nu)(\mathbf{0})$. Hence it holds that $f'''(\mathbf{0}) \in \operatorname{span}\{f_u(\mathbf{0}), \nu(\mathbf{0})\}$.

On the other hand, we have $\langle f_u, f''' \rangle (\mathbf{0}) = \langle (0,0,1), (*,*,0) \rangle = 0$. Differentiating $\langle \nu, f' \rangle = 0$, we have $\langle \nu, f'' \rangle = \langle \nu, f' \rangle' - \langle \nu', f' \rangle$ and $\langle \nu, f''' \rangle = \langle \nu, f'' \rangle' - \langle \nu', f'' \rangle$. Hence $\langle \nu, f'' \rangle \equiv 0$ holds on $S(f)$. Since $\eta_0 \in T_0S(f)$, it holds that $\langle \nu, f'' \rangle' (\mathbf{0}) = 0$ and $\langle \nu, f''' \rangle (\mathbf{0}) = 0$. Thus we have $f'''(\mathbf{0}) = 0$.

Since $\lambda'''(\mathbf{0}) \neq 0$ and $f'(\mathbf{0}) = f''(\mathbf{0}) = f'''(\mathbf{0}) = \mathbf{0}$, it holds that $0 \neq \lambda'''(\mathbf{0}) = \det(f_u, f'''', \nu)(\mathbf{0})$. In particular, $f''''(\mathbf{0}) \neq \mathbf{0}$ holds. □

To prove Theorem 8.2, firstly we show that ψ has the A_4-singularity at $\mathbf{0}$. Differentiating $\langle (\overline{f})', n \rangle \equiv 0$ and by Lemma A.8, we have $\langle (\overline{f})'''', n \rangle (\mathbf{0}) = 0$ and $4 \langle (\overline{f})'''', n' \rangle (\mathbf{0}) + \langle (\overline{f})''''', n \rangle (\mathbf{0}) = 0$.

By these formulae and Lemma A.8, we have $\psi'(0) = \psi''(0) = \psi'''(0) = 0$, $\psi''''(0) = -\langle n, (\overline{f})'''' \rangle (\mathbf{0}) = 0$ and $\psi'''''(0) = -\langle n', (\overline{f})'''' \rangle (\mathbf{0})$.

On the other hand, since n, n' is linearly independent at $\mathbf{0}$ and $\langle n, (\overline{f})'''' \rangle (\mathbf{0}) = 0$, we have

$$\langle n', (\overline{f})'''' \rangle (\mathbf{0}) \neq 0 \Longleftrightarrow (\overline{f})''''(\mathbf{0}) \neq \mathbf{0} \Longleftrightarrow f''''(\mathbf{0}) \neq \mathbf{0}.$$

Hence ψ has the A_4 singularity at $\mathbf{0}$.

Next, we show that (Ψ, Ψ', Ψ'') is non-singular. If this is satisfied, Ψ satisfies the condition of Lemma A.5 namely, Ψ is a Morse family and $\Psi(v, X_1, X_2, u) + wv^3$ is a $\mathcal{K}$-versal unfolding of ψ. Remark that the discriminant set of an unfolding $t^5 + xt^2 + yt + z$ of a function t^5 is diffeomorphic to the image of the canonical cuspidal butterfly $(u, v) \mapsto (u, 5v^4 + 2uv, 4v^5 + uv^2 - u^2)$ at $\mathbf{0}$ as set germs. Therefore by Proposition A.4 and Lemma A.5, we can show that f at $\mathbf{0}$ is $\mathcal{A}$-equivalent to the cuspidal butterfly.

Since $\Psi_X(\mathbf{0}) = -n_1(\mathbf{0}), \Psi_Y(\mathbf{0}) = -n_2(\mathbf{0})$ and $\Psi_u = \sum_{i=1,2} \langle (n_i)_u, X_i - f_i \rangle - \langle n_i, (f_i)_u \rangle$, it holds that $\Psi_u(\mathbf{0}) = 0$. By a direct calculation, we have $\Psi'_X(\mathbf{0}) = -n'_1(\mathbf{0}), \Psi'_Y(\mathbf{0}) = -n'_2(\mathbf{0})$ and $\Psi'_u = \sum_{i=1,2} \langle (n_i)'_u, X_i - f_i \rangle - \langle n_i, f' \rangle - \langle n'_i, (f_i)_u \rangle - \langle n_i, (f_i)'_u \rangle$. Since $\langle n_i, (f_i)'_u \rangle = \langle n, (\overline{f})' \rangle_u - \langle n_u, (\overline{f})' \rangle = 0$ holds at $\mathbf{0}$, we have $\Psi'_u(\mathbf{0}) = 0$.

Thus it is sufficient to prove that the matrix

$$\begin{pmatrix} \partial(\Psi,\ \Psi',\ \Psi'')/\partial t \\ \partial(\Psi,\ \Psi',\ \Psi'')/\partial X_1 \\ \partial(\Psi,\ \Psi',\ \Psi'')/\partial X_2 \\ \partial(\Psi,\ \Psi',\ \Psi'')/\partial u \end{pmatrix} (\mathbf{0}) = \begin{pmatrix} 0 & 0 & 0 \\ n_1 & n'_1 & * \\ n_2 & n'_2 & * \\ 0 & 0 & \Psi''_u \end{pmatrix} (\mathbf{0})$$

is of full rank. Hence we show that $\Psi''_u(\mathbf{0}) \neq 0$.

Differentiating $\left\langle n, (\overline{f})' \right\rangle \equiv 0$ by u and v, and by Lemma A.8, we have

$$\left\langle n, (\overline{f})''_u \right\rangle (\mathbf{0}) = \left\langle n', (\overline{f})'_u \right\rangle (\mathbf{0}). \tag{A.2}$$

Differentiating Ψ by u and v two times, and by Lemma A.8 and (A.2), we have

$$\Psi''_u(\mathbf{0}) = -\left\langle n', (\overline{f})'_u \right\rangle (\mathbf{0}).$$

On the other hand, since $\left\langle n, (\overline{f})'_u \right\rangle (\mathbf{0}) = 0$,

$$\left\langle n', (\overline{f})'_u \right\rangle (\mathbf{0}) \neq 0 \Longleftrightarrow (\overline{f})'_u(\mathbf{0}) \neq 0 \Longleftrightarrow f'_u(\mathbf{0}) \neq 0.$$

holds. By $\lambda_u(\mathbf{0}) \neq 0$ and $f_v(\mathbf{0}) = \mathbf{0}$, we have

$$0 \neq \lambda_u(\mathbf{0}) = \det(f_u, f'_u, \nu)(\mathbf{0}).$$

In particular, $f'_u(\mathbf{0}) \neq \mathbf{0}$ holds. This implies the desired result.

The converse pert of the theorem is obvious since the conditions and assertions of Theorem 8.2 are independent of the choice of coordinates and the choice of ν, and the canonical A_4 singularity satisfies the condition of theorem. $\square$

Remark that since $\mathbf{0}$ is a non-degenerate singular point, we have the parameterization $\gamma(t)$ of $S(f)$. Take the null vector field on γ as $\eta(t)$. Define a function of t by

$$\mu(t) = \det(\gamma'(t),\ \eta(t)).$$

One can easily show that $\mu(0) = \mu'(0) = 0$ and $\mu''(0) \neq 0$ and $\eta\lambda(\mathbf{0}) = \eta\eta\lambda(\mathbf{0}) = 0$ and $\eta\eta\eta\lambda(\mathbf{0}) \neq 0$ are equivalent, as a corollary, the following assertion holds.

Corollary A.9. *A front germ f at $\mathbf{0}$ is $\mathcal{A}$-equivalent to the A_4-singularity if and only if $\mathbf{0}$ is a non-degenerate singular point and $\mu(0) = \mu'(0) = 0$ but $\mu''(0) \neq 0$ holds.*

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Journal *of* Singularities
Volume 2 (2010), 128-142

Proc. of Singularities in Aarhus
received 1 March 2010
DOI: 10.5427/jsing.2010.2h

A Short Note on Hauser's Kangaroo Phenomena and Weak Maximal Contact in Higher Dimensions

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Abstract

Currently there are several approaches to resolution of singularities in positive characteristic all of which have hit some obstruction. One natural idea is to try to construct new meaningful examples at this point to gain a wider range of experience. To produce such examples we mimic the characteristic zero approach and focus on cases where it fails. In particular, this short note deals with an example-driven study of failure of maximal contact and the search for an appropriate replacement.

1 Introduction

Hypersurfaces of maximal contact are one of the key concepts in Hironaka's inductive proof of desingularization in characteristic zero, but unfortunately they need not even exist locally in positive characteristic as e.g. Narasimhan's example [13] shows. In [5] and [9] Hauser replaces hypersurfaces of maximal contact by the characteristic-free notion of hypersurfaces of weak maximal contact, i.e. hypersurfaces which maximize the order of the subsequent coefficient ideal, but which do not necessarily contain the equiconstant points after all sequences of blowing ups in permissible centers. In the corresponding approach ([8], [10]) to resolution of surface singularities in positive characteristic, this modification of the concept of maximal contact turns out to be sufficient to enter into an approach in the flavour of Hironaka's original induction on the dimension of the ambient space. To obtain desingularisation of surfaces along those lines, this is, of course, not the only change to the characteristic zero arguments; important further modifications to certain components of the desingularisation invariant are required. Considering higher dimensions, however, the first step toward a construction of a desingularization similar to the characteristic zero approach or even toward new meaningful examples illustrating the obstructions against it again needs to be a reconsideration of the right generalization of maximal contact.

For readers convenience, we briefly recall some key concepts in section 2. Here one focus will be on the question of recognition of a potential kangaroo. In section 3, we start by considering an example where the original definition of weak maximal contact does not suffice for the description of a kangaroo phenomenon and then suggest a slightly modified version which is suitable for any dimension and not just surfaces. Using this new notion of a flag of weak maximal contact, section 4 is then devoted to examples of the different roles which the hypersurfaces originating from the flag can play in the course of a sequence of permissible blowing ups.

The author would like to thank Herwig Hauser, Vincent Cossart, Dominique Wagner and Santiago Encinas for fruitful discussions which originally started her interest in desingularisation in positive characteristic. The author is also indebted to the referee of the article whose comments were a great help in revising a previous version of this article. All examples appearing in this article originated from structured experiments using the computer algebra system SINGULAR.

2 Basic facts and definitions

A section of just a few pages is obviously not sufficient to even give a brief overview of the tools and general philosophy of algorithmic desingularization, let alone all the delicacies of the case of positive characteristic. On the other hand, more than just 5 pages would be by far too long compared to the following two sections. Hence we do not attempt this here, but only very briefly sketch the idea of the characteristic zero resolution process to give a context, subsequently recalling the notions of hypersurfaces of weak maximal contact and of kangaroo points in positive characteristic. For additional background information on the characteristic zero case, we would like to point to more thorough discussions in section 4.2 of [6] from the practical point of view and in [5] embedded in a detailed treatement of the resolution process. For a detailed introduction to characteristic p phenomena and kangaroo points see [8].

2.1 The philosophy of the characteristic zero approach

In Hironaka's original work [11] and in all algorithmic approaches based on it, e.g. [3],[1],[5], the general approach is that of a finite sequence of blow-ups at appropriate non-singular centers. The very heart of these proofs is the choice of center which is controlled by a tuple of invariants assigned to each point; it is of a structure similar[1] to the following one

$$(ord, n; ord, n; \dots)$$

with lexicographic comparison, the upcoming center being the set of maximal value of the invariant. Here *ord* stands for an order of an appropriate (auxiliary) ideal (see below), n for a counting of certain exceptional divisors. At each ';' a new auxiliary ideal of smaller ambient dimension, a coefficient ideal, is created by means of a hypersurface of maximal contact.

To fix notation, let W be a smooth equidimensional scheme over an algebraically closed field K of characteristic zero and $X \subset W$ a subscheme thereof. We now immediately focus on one affine chart U with coordinate ring R and denote the maximal ideal at $x \in U$ by $\mathfrak{m}_x$. The order of the ideal $I_X = \langle g_1, \dots, g_r \rangle \subset R$ at a point $x \in U$ is defined as

$$ord_x(I) := \max\{m \in \mathbb{N} | \ I \subset \mathfrak{m}_x^m\}.$$

In characteristic zero, the order of the non-monomial part of an ideal can never increase under blow-ups which makes it a good ingredient for the controlling invariant of the resolution

[1] In the case of Bierstone and Milman, the very first entry is a finer invariant, the Hilbert-Samuel function.

process whose decrease marks the improvement of the singularities.

For the descent in ambient dimension, hypersurfaces of maximal contact are required; these locally contain all points of maximal order, satisfy certain normal crossing conditions and continue to contain all points at which the maximal order did not yet drop after any permissible sequence of blow-ups. In characteristic zero, they always exist locally and can be computed in a rather straight-forward way. The construction of the coefficient ideal for I at X w.r.t. a hypersurface of maximal contact $Z = V(z)$ is then performed in the following way:

$$Coeff_Z(I) = \sum_{k=0}^{ord_x(I)-1} I_k^{\frac{k!}{k-i}}$$

where I_k is the ideal generated by all polynomials which appear as coefficients of z^k in some element of I. Given this notion of coefficient ideal, it is possible to rephrase the condition on a hypersurface of maximal contact from 'containing all points of maximal order' to 'maximizing the order of the non-monomial part of the arising coefficient ideal under all choices of hypersurfaces'.

2.2 Weak maximal contact and kangaroos

In positive characteristic, there are well known examples of failure of maximal contact in the sense that eventually the equiconstant points will leave the strict transform of any chosen smooth hypersurface (see [13]). Using the characteristic free formulation of the first condition for maximal contact, i.e. that it should maximize the order of the non-monomial part of the subsequent coefficient ideal, and dropping the condition that this should hold after any permissible sequence of blow-ups, we obtain Hauser's definition of weak maximal contact. In this way, Hauser and Wagner [10] then allow passage to a new hypersurface of weak maximal contact, if the previously chosen one happens to fail to have the maximizing property at some moment in the resolution process.

Additionally there are examples (see [12]) in which the order of the non-monomial part of the first coefficient ideal can increase under a sequence of blow-ups in positive characteristic. In [8] Hauser shows that these two phenomena are closely related in the sense that both arise in the same rather rare settings and gives an explicit criterion for the possibility of such a phenomenon, which he calls a kangaroo point focusing on the point where this occurs. In this article, we often choose to refer to this as a kangaroo phenomenon, emphasizing the fact that not the point itself is in the center of interest, but the deviation from the characteristic zero case. Using the same notation for W, X etc. as in the previous section, we now recall Hauser's definition:

Definition 1 ([8]) *Let $\pi : W' \longrightarrow W$ be a blow-up at a permissible center Z, and $x \in Z$ a point of maximal order c for I_X. Denoting the weak transform of X under π by X', let $x' \in X' \cap \pi^{-1}(x)$ be a point at which $ord_{x'}(I_{X'}) = c$. Then x' is called a kangaroo point, if the order of the non-monomial part of the coefficient ideal of I_X at x w.r.t. a hypersurface of weak maximal contact is less than the order of the non-monomial part of the coefficient ideal of $I_{X'}$ w.r.t. a (possibly newly chosen) hypersurface of weak maximal contact.*

Definition 2 *Generalizing Hauser's notion of a kangaroo point, we shall call a blowing up, at which such an increase in order occurs for one of the coefficient ideals at some level in the descent of ambient dimension, a kangaroo phenomenon.*

Remark 3 ([8]) *A kangaroo point can only occur, if the following conditions are satisfied:*

(a) the order c of the ideal I_X at x does not exceed the order of $I_{X'}$ at x' and is divisible by the charateristic of the ground field.

(b) The order of the non-monomial part of the coefficient ideal is a multiple of c. [2]

(c) The exceptional multiplicities of the coefficient ideal need to satisfy a certain numerical inequality (whose specification would need to much room here).

This remark does not yield a sufficient criterion of detection of kangaroos. However, if a kangaroo phenomenon occurs, then its effect is an increase of order of the non-monomial part of the coefficient ideal by means of leaving at least two exceptional divisors at the same time and a suitable change of hypersurface of weak maximal contact (see examples in sections 3 and 4 for details).

Combining the above observations of Hauser with well-known observations by Hironaka and Giraud, condition (a) can be made a bit more precise. To this end, we need to recall another singularity invariant, the ridge (french: la faîte). Following the exposition of [14], let us consider the tangent cone $C_{X,x}$ of I_X at x and the largest subgroup scheme $A_{X,x}$ of the tangent space $T_{W,x}$ satisfying the conditions that it is homogeneous and leaves the tangent cone stable w.r.t. the translation action. $A_{X,x}$ is called the ridge of the tangent cone of I_X at x.

It is a well-known, important fact that the ridge can be generated by additive polynomials, i.e. by polynomials of the form

$$\sum_{i=1}^{n} a_i x_i^{p^e}$$

where p is the characteristic of the underlying field. In characteristic zero the ridge is always generated by polynomials of degree one; in positive characteristic the occurrence of a ridge not generated by polynomials of degree one marks a point for which the reasoning of characteristic zero might break down. Following the exposition of [2] the ridge can also be phrased as the smallest set of additive polynomials $\{p_1, \ldots, p_r\}$ generating the smallest algebra $k[p_1, \ldots, p_r]$ such that

$$I_X = (I_X \cap k[p_1, \ldots, p_r])k[\underline{x}].$$

Combining this with Hauser's condition (a), we obtain a refined version for hypersurfaces, which, of course, still requires $ord_{x'}(I_{X'}) = c = ord_x(I_X)$ and, additionally, that the ridge must at least have one generator in higher degree, i.e. in some degree p^e. This sharpens the

[2] For kangaroo phenomena, this condition should analogously read 'one of the coefficient ideals occurring in the descent of ambient dimension'.

condition of divisibility of the order by a p-th power to the fact that some variable actually only occurs as p-th powers in the tangent cone and is implicitly already present in [8]. According to Hauser's condition (b), the degree of the non-monomial part of the first coefficient ideal is required to be a multiple of the degree c. In contrast to condition (a), this can not be made more precise by simply adding the condition that the ridge of the non-monomial part of this coefficient ideal is not generated in degree 1, because higher order generators of the coefficient ideal might introduce lower degree polynomials into the ridge which allow dropping of certain contributions arising from the lowest order generators of the ideal. To illustrate the role of the ridge, we give 3 examples:

Example 1 Over a field K of characteristic 3, consider an affine chart $U = \mathbb{A}^4_K$ (with variables named x, y, z, w) which already results from a sequence of 2 blow-ups and contains exceptional divisors $E_1 = V(w)$ and $E_2 = V(z)$, born from the first and second blow-up respectively. (These two blow-ups are indeed necessary for the possibility of an occurrence of a kangaroo point after the subsequent blowing up, according to Hauser's technical condition (c) which was not formulated explicitly in the previously stated remark.)
Locally at the coordinate origin of this chart, consider the three subvarieties of $\mathbb{A}^4_K$ defined by the following ideals:

- $I_{X_1} = \langle x^3 + z^{14}w^{10}(z^6 - w^6)\rangle$
  This is the strict transform[3] of $\langle x^3 + z^{13} - zw^{18}\rangle$ under the two blow-ups. The ridge of I_{X_1} can obviously be described by $\{x^3\}$, the non-monomial part of its first coefficient ideal is
  $$\langle z^{12} + z^6w^6 + w^{12}\rangle,$$
  with ridge $\{z^3, w^3\}$.
  After blowing up again at the origin, we obtain (in the $E_3 = V(w)$-chart) the strict transform
  $$I_{X_1'} = \langle x^3 + z^{14}w^{27}(z^6 - 1)\rangle$$
  which after a coordinate change $z_{new} = z - 1$ and a passage to a new hypersurface of weak maximal contact $V(x + z_{new}^2 w^9) = V(x_{new})$ reads as $I_{transf.} = \langle x_{new}^3 + z_{new}^6 w^{27}(-z_{new} + h.o.t.)\rangle$. Since z_{new} does not correspond to an exceptional divisor, this has a non-monomial part of the first coefficient ideal of the form
  $$\langle z_{new}^{14} + h.o.t.\rangle.$$
  This ideal is of order 14 as compared to the corresponding order 12 before the last blowing up which clearly indicates the occurrence of a kangaroo point.

- $I_{X_2} = \langle x^2y + z^{14}w^{10}(z^6 - w^6)\rangle$
  This is the strict transform of $\langle x^2y + z^{13} - zw^{18}\rangle$ under the two blow-ups.[4] The ridge of

[3] Actually this is the weak transform of I_{X_1} which in the principal ideal case happens to coincide with the strict transform.

[4] Here we are actually already deviating a bit from Hauser's original definition, because we consider an initial part involving 2 variables and then descend in ambient dimension in one step of 2 to $V(x, y)$ seen as a hypersurface in $V(x)$ which is in turn a hypersurface in $\mathbb{A}^4$. This is possible by collecting all coefficients of monomials of the form $x^a y^b$ with $a + b = k$ into the ideal I_k; for more details on this see e.g. [7], where this has been used in a very explicit way.

I_{X_2} is obviously $\{x,y\}$, the non-monomial part of its coefficient ideal w.r.t. the descent in ambient dimension to $V(x,y)$ is

$$\langle z^{12} + z^6w^6 + w^{12}\rangle$$

as before with ridge $\{z^3, w^3\}$.
After blowing up again at the origin, we obtain (in the $E_3 = V(w)$-chart) the strict transform

$$I_{X_2'} = \langle x^2y + z^{14}w^{27}(z^6 - 1)\rangle$$

for which even a coordinate change $z_{new} = z - 1$ cannot lead to a kangaroo point, because no suitable passage to new hypersurfaces of weak maximal contact killing the term $z_{new}^6w^{27}$ is available. This could already be expected at the beginning due to the fact that the ridge of I_{X_2} is generated in degree 1.

The third example is of a different flavor and only serves to illustrate, how higher order generators of the ideal might influence the ridge in a way which is not desirable for the consideration of coefficient ideals:

- $I_{X_3} = \langle x^3 + z^{14}w^{10}(z^6 - w^6), z^{30}w^{17}(y^{19} + y^5z^7w^3)\rangle$
This is the weak transform of $\langle x^3 + z^{13} - zw^{18}, y^5z^{18} + y^{19}w\rangle$ under the two blow-ups. The ridge of I_{X_3} is obviously $\{x^3, y, z, w\}$, whereas only the hypersurface $V(x)$ can be chosen as hypersurface of weak maximal contact. The non-monomial part of the first coefficient ideal is

$$\begin{gathered}\langle z^{12} + z^6w^6 + w^{12}, (z^6 - w^6)(z^{16}w^7(y^{19} + y^5z^7w^3)),\\ z^{32}w^{14}(y^{38} - y^{24}z^7w^3 + y^{10}z^{14}w^6)\rangle.\end{gathered}$$

The ridge can be computed to be $\{y, z, w\}$, e.g. by the algorithm of [2].
After blowing up again at the origin, we obtain (in the $E_3 = V(w)$-chart) the weak transform

$$I_{X_3'} = \langle x^3 + z^{14}w^{27}(z^6 - 1), \dots\rangle$$

which after a coordinate change $z_{new} = z - 1$ and a passage to a new hypersurface of weak maximal contact $V(x + z_{new}^2w^9) = V(x_{new})$ has the same first generator of order 14 as in example 1, the second generator does not have effect on the order of the non-monomial part of the first coefficient ideal as can be checked by explicit computation. Comparing this to the first example, we see that the higher order generator, which does not actually influence the order of the non-monomial part of the coefficient ideal, masked the situation in the computation of the ridge.

From these three examples, we see the usefulness of the ridge for anticipating kangaroo points in the case of hypersurfaces, whereas in the case of ideals this may be hidden by contributions of higher order generators. However, if we only consider the ridge of the ideal which is generated precisely by the lowest-order generators of the original ideal (instead of the ridge of the whole ideal), then there is hope to use this new ridge for ideals and maybe even to slightly sharpen item (b) in Hauser's condition for kangaroo points.

Remark 4 *These considerations already suggest a strategy for finding interesting examples by constructing hypersurfaces for which the ridge is not generated in degree 1 and, additionally, at least once during the iterated descents in ambient dimension the ridge of the ideal generated by the lowest order generators (denoted from now on as n-ridge for short) of the non-monomial part of the respective coefficient ideal is also not generated in degree one. In the experiments, which lead to the examples of the subsequent sections, an additional heuristic in the choice of hypersurfaces of weak maximal contact was used: When given the choice between different hypersurfaces, more precisely between linearly independent initial parts of possible hypersurfaces, we try to minimize the degree of the generator of the ridge/n-ridge corresponding to the chosen hypersurface. The reasoning behind this heuristic is to force the unpleasant, but interesting behaviour into the lowest possible ambient dimension and hence keep a clearer view of the occurring phenomena.*

Remark 5 *Similar examples to those of the subsequent sections can easily be constructed in any positive characteristic. For section 3 this is straight forward, for section 4 it is best achieved by starting in the middle, i.e. precisely where the first kangaroo has just occurred and construct from there by blowing down and blowing up.*

3 In higher dimension not all hypersurfaces of weak maximal contact are suitable

The following example shows that the property of maximizing the order of the non-monomial part of the upcoming coefficient ideal is not sufficient to properly cover all kangaroo phenomena in higher dimensions. It is stated in characteristic 2 to allow considerations in rather low degrees, but similar examples can be constructed for any positive characteristic.

Example 2 We consider a sequence of three blow ups of the hypersurface $V(x^2 + w^3 + y^{25} + yz^{16}) \subset \mathbb{A}^4_K$, $char(K) = 2$, $K = \overline{K}$. At each step the respective maximal orders, chosen hypersurfaces of weak maximal contact and coefficient ideals are specified. In the presence of exceptional divisors, we make use of Bodnar's trick [4], which allows skipping the intersection with exceptional divisors in intermediate levels of the descent in ambient dimension, if we have normal crossing between the upcoming hypersurface of weak maximal contact and the exceptional divisors.

To keep the whole rather lengthy sequence of blowing ups more readable, we only give rather scarce comments. A more commented version of a single blowing up step was already stated at the end of the previous section.

original hypersurface:
$I = \langle f \rangle = \langle x^2 + w^3 + y^{25} + yz^{16} \rangle$

- in ambient space $\mathbb{A}^4_K$
  $I = \langle x^2 + w^3 + y^{25} + yz^{16} \rangle$
  The maximal order 2 is attained at $V(x, y, z, w)$.
  The ridge of this ideal corresponds to $\{x^2\}$.
  As hypersurface of weak maximal contact we may use $H_1 = V(x) \subset \mathbb{A}^4_K$.

- in ambient space H_1
  $I_{H_1} = \langle w^3 + y^{25} + yz^{16} \rangle$.
  The maximal order of 3 is then again attained at the origin of H_1.
  The n-ridge (in the short-hand notation introduced in section 2) is $\{w\}$
  As hypersurface of weak maximal contact we now use
  $H_2 = V(x, w) \subset H_1 \subset \mathbb{A}^4_K$.

- in ambient space H_2
  $I_{H_2} = \langle y^{50} + y^2 z^{32} \rangle = \langle (y^{25} + yz^{16})^2 \rangle$.
  The maximal order of 34 is again attained at the origin of H_2 and the n-ridge is $\{y^2, z^{32}\}$.

- The only possible choice of center is $V(x, y, z, w)$.

As a sideremark to the coefficient ideal in ambient space H_2: Here it becomes evident that there are 2 mechanisms which can cause the n-ridge to have generators in higher degree: on one hand, it may be an honest generator in higher degree, on the other hand, it might have arisen from taking powers of contributing ideals I_k when forming the coefficient ideal (see section 2). However, taking powers can not accidentally cause the degree of a generator of the ridge to drop. Hence the degree of the generators of the ridge can still be used as a rather weak indicator for the possibility of new phenomena in characteristic p. Moreover, a higher degree generator of the n-ridge arising from mechanism 2 is only likely to occur, if the contributing ideals I_k are principal, because otherwise mixed products of generators would exist in the set of generators of the power of I_k.

after first blowing up, chart $E_1 = V(y)$:
$I_{strict} = \langle x^2 + yw^3 + y^{23} + y^{15} z^{16} \rangle$

- in ambient space $\mathbb{A}^4_K$
  $I_{strict} = \langle x^2 + y(w^3 + y^{22} + y^{14} z^{16}) \rangle$
  The maximal order is again 2, attained at the origin and the ridge is again $\{x^2\}$. We can keep the strict transform of H_1 as our hypersurface of weak maximal contact. (As $\{E_1, H_{1strict}\}$ has normal crossings, we may use Bodnàr's trick [4] and hand the exceptional divisor down to the lower dimension instead of intersecting with it at this point.)

- in ambient space $H_{1strict}$
  The non-monomial part of the coefficient ideal[5] is $\langle w^3 + y^{22} + y^{14} z^{16} \rangle$.
  The maximal order of 3 is again attained at the origin and the n-ridge is $\{w\}$ as before.
  We may also use the strict transform of H_2 again for the descent in ambient dimension.
  (Here we have normal crossing of $\{E_1, H_{1strict}, H_{2strict}\}$ and can again use Bodnàr's trick.)

- in ambient space $H_{2strict}$
  non-monomial part of coefficient ideal: $\langle y^{16} + z^{32} \rangle = \langle (y^8 + z^{16})^2 \rangle$
  maximal order 16 attained at the origin
  n-ridge: $\{y^{16}\}$

[5]As taking the coefficient ideal and subsequently calculating the controlled transform under the blowing up on one hand and calculating the weak transform of the ideal followed by computing the new coefficient ideal on the other hand are known (e.g. [5]) to lead to the same ideal, we won't go into details on this point.

- It is easy to check that here again the choice of center has to be the origin.

after second blowing up, chart $E_2 = V(z)$:
$I_{strict} = \langle x^2 + yz^2w^3 + y^{23}z^{21} + y^{15}z^{29}\rangle$

- in ambient space $\mathbb{A}^4_K$
  $I_{strict} = \langle x^2 + yz^2(w^3 + y^{22}z^{19} + y^{14}z^{27})\rangle$
  maximal order: 2 at $V(x, zw, yz)$
  ridge: $\{x^2\}$
  hypersurface of weak maximal contact: strict transform of H_1
  ($\{E_{1strict}, E_2, H_{1strict}\}$ n.cr.)

- in ambient space $H_{1strict}$
  non-monomial part of coefficient ideal: $\langle w^3 + y^{14}z^{19}(y^8 + z^{16})\rangle$
  maximal order: 3 at $V(w, yz)$
  n-ridge: $\{w\}$
  hypersurface of weak maximal contact: strict transform of H_2
  ($\{E_{1strict}, E_2, H_{2strict}\}$ n.cr.)

- in ambient space $H_{2strict}$
  non-monomial part of coefficient ideal: $\langle y^{16} + z^{16}\rangle = \langle (y^8 + z^8)^2\rangle$
  maximal order: 16 at $V(y + z)$
  n-ridge: $\{y^{16} + z^{16}\}$

- center needs to be $V(x, y, z, w)$ as the locus of maximal order after the second descent in ambient dimension is not normal crossing with the exceptional divisors

after third blowing up, chart $E_3 = V(z)$:
$I_{strict} = \langle x^2 + yz^4w^3 + y^{23}z^{42} + y^{15}z^{42}\rangle$

- in ambient space $\mathbb{A}^4_K$
  $I_{strict} = \langle x^2 + yz^4(w^3 + y^{22}z^{38} + y^{14}z^{38})\rangle$
  maximal order: 2 at $V(x, zw)$
  ridge: $\{x^2\}$
  hypersurface of weak maximal contact: strict transform of H_1
  (E_1 does not meet this chart, $\{E_{2strict}, E_3, H_{1strict}\}$ n.cr.)

- in ambient space $H_{1strict}$
  non-monomial part of coefficient ideal: $\langle w^3 + y^{12}z^{42}(y^8 + 1)\rangle$
  maximal order: 3 at $V(w, yz(y + 1))$ n-ridge: $\{w\}$
  hypersurface of weak maximal contact: strict transform of H_2
  ($\{E_{2strict}, E_3, H_{2strict}\}$ n.cr.)

- in ambient space $H_{2strict}$
  non-monomial part of coefficient ideal: $\langle y^{16} + 1\rangle = \langle (y^8 + 1)^2\rangle$
  maximal order: 16 at $V(y + 1)$

Changing the hypersurface for the first descent in ambient dimension from $H_{1strict}$ to $V(x + (y+1)^4z^{21})$, however, we may increase the order of the coefficient ideal in ambient dimension 2. For simplicity of notation, we first make a coordinate change which translates the point of maximal order to the coordinate origin:

- in ambient space $\mathbb{A}^4_K$
  $\langle x^2 + z^4(w^3 + y_{new}w^3 + y_{new}^8 z^{38} + y_{new}^9 z^{38} + h.o.t.)\rangle$
  maximal order 2 at $V(x, zw)$
  ridge: $\{x^2\}$
  new hypersurface of weak maximal contact: $H_1' = V(x + y^4 z^{21})$
  (E_1 does not meet this chart, $\{E_{2strict}, E_3, H_1'\}$ n.cr.)

- in ambient space H_1'
  non-monomial part of coefficient ideal: $\langle w^3 + y_{new}^9 z^{38} + h.o.t.\rangle$
  maximal order 3 at $V(w, y_{new}z)$
  n-ridge: $\{w\}$
  hypersurface of weak maximal contact: $H_2' = V(w)$
  ($\{E_{2strict}, E_3, H_2'\}$ n.cr.)

- in ambient space H_2'
  non-monomial part of the coefficient ideal: $\langle y_{new}^{18} + h.o.t.\rangle$
  maximal order: 18 **exceeds previous order** 16
  kangaroo phenomenon

Here the new phenomenon is that the change of the hypersurface of weak maximal contact was not forced by the first coefficient ideal, but by one of the later ones which would not be covered by the standard definition of weak maximal contact.

In the light of the previous example, we suggest a slightly modified version of weak maximal contact:

Definition 6 *Consider a given point x of a scheme X (possibly in the presence of an exceptional divisor E) and pass to an affine chart U containing this point. We call a flag*

$$\mathcal{H} = H_1 \supset H_2 \supset \cdots \supset H_s$$

admissible at x, if the following properties hold:

(a) H_1 is a smooth hypersurface in the ambient space U. H_{i+1} is a smooth hypersurface in H_i.

(b) H_i is a hypersurface of weak maximal contact for the coefficient ideal obtained by descent of the ambient space through $H_1, \ldots, H_{i-1}$.

(c) $x \in H_s$.

$\mathcal{H}$ is called a flag of weak maximal contact for I_X at x if it maximizes the resolution invariant lexicographically among all choices of admissible flags at x.

This definition obviously behaves well under passage to a coefficient ideal w.r.t. H_1 by omitting the first entry H_1 from $\mathcal{H}$ to obtain the new flag $\mathcal{H}_{H_1}$. This is again a flag of maximal contact, since conditions (a)-(c) and maximality follow trivially from the respective conditions on $\mathcal{H}$. Hence considering a flag of weak maximal contact instead of a hypersurface of weak maximal contact does not change any of the key properties, but allows more flexibility for dealing with lower level kangaroos.

4 Two different kinds of double kangaroos

It is a well known fact that the situation in positive characteristic can only differ from the one in characteristic zero in rather special situations. Hauser studied such phenomena in great detail in [8] by considering precisely the two levels involved in a kangaroo point. For surfaces, he and Wagner extended these considerations to a general treatment of the purely inseparable case in [10]. The situation in higher dimension differs from this easiest case in the sense that there might be more than just two levels at which the ridge is not generated in degree 1 at some time during the process of blowing ups. The following two examples illustrate three different roles of the different levels of the flag of weak maximal contact in such a setting.

Definition 7 *Let $\mathcal{H}$ be a flag of weak maximal contact for an ideal $I_X \subset W$ at the point x which we assume for simplicity to be the origin of our coordinate chart. We denote the i-th coefficient ideal, which arises when descending to H_i, by $J_i \subset \mathcal{O}_{H_i}$. If the ideal generated by the lowest order generators of J_{i-1} is not a principal ideal, H_i is called*

- **neutral**, *if the degree 1 part of the generator of the principal ideal $I_{H_i} \subset \mathcal{O}_{H_{i-1},0}$ is in the $\mathbb{C}$-span of the degree 1 elements of the ridge/n-ridge of J_{i-1}.*
- **active**, *if it is the H_i of lowest index i which is not neutral.*
- **dormant**, *if it is neither active nor neutral.*

If, on the other hand, the ideal generated by the lowest order generators of J_{i-1} is principal, it is of the form $g^{\frac{b!}{b-k}}$ for some $k < b$ and we change the notions of neutral, active and dormant by replacing the ridge/n-ridge of J_{i-1} by the one of $\langle g \rangle$.

Remark 8

1. *According to Hauser's description of the process leading to kangaroo points, at least one active H_i and one dormant H_j are necessary to produce a kangaroo phenomenon.*
2. *If the ideal generated by the lowest order generators of J_{i-1} is not principal, there is at least one ideal among the contributing I_k, of which the ideal generated by its lowest order generators is itself not principal, e.g. generated by f_1 and f_2. Hence taking the $\frac{b!}{b-k}$-th power of of this I_k upon forming the coefficient ideal, we obtain all mixed products of the form $f_1^a f_2^b$, $a + b = \frac{b!}{b-k}$. This implies that higher degree generators of the n-ridge can only occur if they would also occur for $\langle f_1, f_2 \rangle$.*
   If on the other hand, the ideal generated by the lowest order generators of J_{i-1} is principal, the generator is of the form $g^{\frac{b!}{b-k}}$ for some k and hence masks the true situation of the (n-)ridge of g. This is the reason for the special treatment of this case in the above definition.

Both of the following examples were constructed in a straight forward way, combining two occurrences of kangaroos at two different levels. Similar examples can be constructed in any positive characteristic and for any ambient dimension exceeding 4. However, these examples involve several blow-ups between the first and the second occurrence, basically making a fresh start after the first. Here no effort is made to reduce this number of blow-ups, since the context of this article is the study of the roles of the hypersurfaces of weak maximal contact.

To keep these rather lengthy examples more readable, we only state the blow-ups, the weak transform at each step and the flag of weak maximal contact, whenever the latter changes, but omit all data which is related to coefficient ideals, since these can easily be computed for these examples.

Example 3 In this example, a hypersurface in $\mathbb{A}^5_K$, $char K = 3$, we shall see 2 occurrences of kangaroo points on two different levels of coefficient ideals. For both occurrences, the active hypersurface of weak maximal contact is the first one in the flag. Note that the two blowing ups with chart $E = V(y)$ after the first kangaroo are only used for setting up the degrees for the following kangaroo.[6]

- before 1st blowing up
  $I = \langle w^3 + y^6z^3v^2 + x^9y^8 + x^{18}y^2 + x^{18}v^2 \rangle$
  Flag:
  $V(w)$ active, $V(w,v)$ neutral, $V(w,v,z)$ dormant, $V(w,v,z,y)$ neutral
- after blowing up at the origin, chart $E_{new} = V(x)$
  $I = \langle w^3 + x^8(y^6z^3v^2 + x^6y^8 + x^9y^2 + x^9v^2) \rangle$
- after blowing up at the origin, chart $E_{new} = V(y)$
  $I = \langle w^3 + x^8y^{16}(z^3v^2 + x^6(x^3 + y^3) + x^9v^2) \rangle$
  Flag: $V(w)$ active, $V(w,v)$ neutral, $V(w,v,z)$ dormant, $V(w,v,z,y)$ dormant
- after blowing up at the origin, chart $E_{new} = V(x)$
  $I = \langle w^3 + x^{26}y^{16}(z^3v^2 + x^4 + x^4y^3 + x^6v^2) \rangle$
  coordinate change: $y_{new} = y_{old} + 1$, $w_{new} = w_{old} + x^{10}y$
  $I = \langle w^3 + x^{26}((y-1)^{16}(z^3v^2 + x^6v^2) - x^4y^4 + h.o.t.) \rangle$
  Flag in new coordinates:
  $V(w)$ active, $V(w,v)$ neutral, $V(w,v,z)$ dormant, $V(w,v,z,y)$ neutral
  Kangaroo at 3rd coefficient ideal
- after blowing up at the origin, chart $E_{new} = V(x)$
  $I = \langle w^3 + x^{28}((xy-1)^{16}(z^3v^2 + x^3v^2) - x^3y^4 + h.o.t.) \rangle$
- after blowing up at the origin, chart $E_{new} = V(v)$
  $I = \langle w^3 + x^{28}v^{30}(z^3 + x^3 - x^4yv^2 + h.o.t.) \rangle$
- after blowing up at the origin, chart $E_{new} = V(y)$
  $I = \langle w^3 + x^{28}y^{58}v^{30}(x^3 + z^3 - x^4y^4v^2 + h.o.t.) \rangle$
- after blowing up at the origin, chart $E_{new} = V(y)$
  $I = \langle w^3 + x^{28}y^{116}v^{30}(x^3 + z^3 - x^4y^7v^2 + h.o.t.) \rangle$

[6] Whenever we write 'h.o.t.' we want to indicate that there are further terms of higher degree, which are irrelevant for the further considerations. In this case only the first non-relevant term is stated, even if this does not happen to be the term originating from the previous first non-relevant term

- after blowing up at the origin, chart $E_{new} = V(z)$
  $I = \langle w^3 + x^{28}y^{116}z^{174}v^{30}(1 + x^3 - xy^7z^{10}v^2 + h.o.t.)\rangle$
  coord. change: $x_{new} = x_{old} + 1$, $y_{new} = y_{old} + 1$, $w_{new} = w_{old} + z^{58}v^{10}x^3$
  $I = \langle w^3 + z^{174}v^{30}(x^4 + x^3y + h.o.t)\rangle$
  Flag: $V(w)$, $V(w, x)$, ...
  Kangaroo at 1st coefficient ideal

Example 4 In this example, again in the same affine space as before, we shall see 2 occurrences of kangaroo points on two different levels of coefficient ideals. For the first occurrence, a dormant hypersurface of weak maximal contact acts as the active one, for the second it is the top-level active hypersurface of weak maximal contact. This example again basically consists of two regular kangaroo phenomena in a row, occurring on two different levels, but in a different flavor than example 4.

- before first blowing up
  $I = \langle w^3 + xy^9z^9v + x^7y^{20}v + x^{34}y^2v + x^{46}v\rangle$
  Flag:
  $V(w)$ active, $V(w, v)$ neutral, $V(w, v, z)$ dormant, $V(w, v, y, z)$ neutral

- after blowing up at the origin, chart $E_{new} = V(x)$
  $I = \langle w^3 + x^{17}(y^9z^9v + x^8y^{20}v + x^{17}y^2v + x^{27}v)\rangle$

- after blowing up at the origin, chart $E_{new} = V(y)$
  $I = \langle w^3 + x^{17}y^{33}(z^9v + x^8y^{10}v + x^{17}yv + x^{27}y^9v)\rangle$
  Flag: $V(w)$ active, $V(w, v)$ neutral, $V(w, v, z)$ dormant, $V(w, v, y, z)$ dormant

- after blowing up at the origin, chart $E_{new} = V(x)$
  $I = \langle w^3 + x^{57}y^{33}(z^9v + x^9y^{10}v + x^9yv + x^{27}y^9v)\rangle$
  coord. change: $y_{new} = y_{old} + 1$, $z_{new} = z_{old} + xy$
  $I = \langle w^3 + x^{57}(y-1)^{33}(z^9v + x^9y^{10}v - x^{27}v + x^{27}y^9v)\rangle$
  Flag in new coordinates:
  $V(w)$ active, $V(w, v)$ neutral, $V(w, v, z)$ dormant, $V(w, v, y, z)$ neutral
  Kangaroo at 3rd coefficient ideal

- after blowing up at the origin, chart $E_{new} = V(x)$
  $I = \langle w^3 + x^{64}(xy-1)^{33}(z^9v + x^{10}y^{10}v - x^{18}v + x^{27}y^9v)\rangle$

- after blowing up at the origin, chart $E_{new} = V(x)$
  $I = \langle w^3 + x^{71}(x^2y-1)^{33}(z^9v + x^{11}y^{10}v - x^9v + x^{27}y^9v)\rangle$

- after blowing up at the origin, chart $E_{new} = V(v)$
  $I = \langle w^3 + x^{71}(x^2yv^3-1)^{33}v^{78}(z^9 - x^9 + h.o.t)\rangle$
  Flag: $V(w)$ active, $V(w, z)$ dormant, $V(w, x, z)$ dormant, ...

- after blowing up at the origin, chart $E_{new} = V(y)$
  $I = \langle w^3 + x^{71}y^{155}v^{78}(x^2y^6v^3-1)^{33}(z^9 - x^9 + h.o.t)\rangle$

- after blowing up at the origin, chart $E_{new} = V(y)$
  $I = \langle w^3 + x^{71}y^{310}v^{78}(x^2y^6v^3 - 1)^{33}(z^9 - x^9 + h.o.t)\rangle$

- after blowing up at the origin, chart $E_{new} = V(z)$
  $I = \langle w^3 + x^{71}y^{310}v^{78}z^{465}(x^2y^6v^3z^{16} - 1)^{33}(1 - x^9 + h.o.t.)\rangle$
  coord. change: $x_{new} = x_{old} - 1$, $y_{new} = y_{old} + 1$, $w_{new} = w_{old} - v^{26}z^{155}x^3$
  Kangaroo at 1st coefficient ideal

In both examples the relevant order of the first respectively second coefficient ideal dropped significantly after the first kangaroo phenomenon, but before the occurrence of the kangaroo on this level. The examples have been constructed to illustrate roles of hypersurfaces of maximal contact in multiple kangaroos and not to specifically illustrate the increase in order. Nevertheless the observed behaviour raises several questions, which seem to be natural starting points for further experiments in the search for new meaningful examples:

- Is it possible to find an occurrence of two kangaroo phenomena whose 'distance' is less than 3 blow ups?
- Is it possible to find an occurrence of two kangaroo phenomena for which the drop of order between the first and the second kangaroo does not outweigh the increase of order?
- If one of the previous question has an affirmative answer, what is the smallest dimension in which this occurs?

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Journal *of* Singularities
Volume 2 (2010), 143-159

Proc. of Singularities in Aarhus
received 18 December 2009
DOI: 10.5427/jsing.2010.2i

WHITNEY STRATIFIED MAPPING CYLINDERS

CLAUDIO MUROLO

To Andrew du Plessis for his 60th birthday.

Abstract. In this paper we investigate (b)-regularity for stratified mapping cylinders $C_{\mathcal{W}'}(\mathcal{W})$ of a stratified submersion $f : \mathcal{W} \to \mathcal{W}'$ between two Whitney stratifications. We show how Goresky's condition (D) for f is sufficient to obtain (b)-regularity of $C_{\mathcal{W}'}(\mathcal{W})$.

Revisiting some ideas of Goresky we give different proofs, a finer analysis and new equivalent properties.

1. Introduction.

Let $\mathcal{X} = (A, \Sigma)$ be a stratified set of support A and stratification Σ (see §2 for the definition) contained in a Euclidean space $\mathbb{R}^N$. A substratified object of $\mathcal{X}$ is a stratified space $\mathcal{W} = (W, \Sigma_{\mathcal{W}})$, where W is a subset of A, such that each stratum in $\Sigma_{\mathcal{W}}$ is contained in a single stratum of $\mathcal{X}$. In this paper we study the (b)-regularity of the stratified mapping cylinder $M(f_{\mathcal{W}})$ of a stratified surjective submersion $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ when $\mathcal{W}$ and $\mathcal{W}'$ are (b)-regular.

Since $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ is surjective, $M(f_{\mathcal{W}})$ will be a cone that we will denote by $C_{\mathcal{W}'}(\mathcal{W})$.

Our motivation comes from the works of Goresky **[6, 7]** which followed his thesis **[5]**.

In 1976 and 1978 Goresky **[5, 6]** proved an important triangulation theorem for Thom-Mather abstract stratified sets $\mathcal{X}$. The proof was obtained by a double induction on $k \leq \dim \mathcal{X}$, first by triangulating, for each k-stratum X of $\mathcal{X}$, a boundary k-manifold $X_d^o \subseteq X$, and then using a stratified mapping cylinder $C_{\mathcal{W}'}(\mathcal{W})$ to glue a triangulation of X_d^o with a triangulation of a submanifold of the singular part $\partial X = \overline{X} - X = \sqcup_{X'<X} X'$ of $\overline{X}$. This method allowed one to extend the triangulation to the part $X - X_d^o$ of X near the singularity ∂X of $\overline{X}$.

Such mapping cylinders produce cellular (but not necessarily triangulated) stratified sets.

In this context to know how to obtain Whitney (i.e. (b)-) regularity of such mapping cylinders would be very useful in order to obtain a proof of the following:

Conjecture 1. 1. *Every compact Whitney stratified space $\mathcal{X}$ admits a Whitney cellularisation.*

This would be also a first important step of a possible proof of the celebrated Thom conjecture:

Conjecture 1. 2. *Every compact Whitney stratified space $\mathcal{X}$ admits a Whitney triangulation.*

Let us recall that in 2005 M. Shiota proved that semi-algebraic sets admit a Whitney triangulation **[16]** and more recently M. Czapla announced a new proof of this result **[2]** as a corollary of a more general triangulation theorem for definable sets. On the other hand, our motivation being the applications to Goresky's geometric homology theory, we are interested in the stronger Conjectures 1.1 and 1.2 for stratifications having C^1-strata.

In 1981 Goresky defines for a Whitney stratification $\mathcal{X}$, two geometric homology and cohomology theories $WH_k(\mathcal{X})$ and $WH^k(\mathcal{X})$ whose cycles and cocycles are substratified Whitney objects of $\mathcal{X}$ and proves the following representation theorems (**[7]**, Theorems 3.4. and 4.7) :

Key words and phrases. Stratified sets and maps, Whitney Condtions (a) and (b), regular cellularisations.

Theorem 1. 1. *If $\mathcal{X} = (M, \{M\})$ is the trivial stratification of a compact C^1-manifold M, the homology representation map $R_k : WH_k(\mathcal{X}) \to H_k(M)$ is a bijection.*

Theorem 1. 2. *If $\mathcal{X} = (A, \Sigma)$ is a compact Whitney stratification, the cohomology representation map $R^k : WH^k(\mathcal{X}) \to H^k(A)$ is a bijection.*

Here the Goresky maps R_k and R^k are the analogues of the Thom-Steenrod representation maps between the differential bordism of a space and its singular homology.

In 1994 such theories were improved by the author of this paper by introducing a sum operation in $WH_k(\mathcal{X})$ and $WH^k(\mathcal{X})$, geometrically meaning transverse union of stratified cycles [**12, 13**], with which the bijections R_k and R^k become group isomorphisms.

The possibility of constructing Whitney cellularisations of Whitney cycles and cocycles using mapping cylinders ([**7**], Appendices 1,2,3) was the main tool of Goresky to obtain two such important representation theorems.

We underline here that in the homology case the main result $R_k : WH_k(\mathcal{X}) \to H_k(M)$ was established *only* when $\mathcal{X} = (M, \{M\})$ is a trivial stratification of a compact manifold M and that the complete homology statement for $\mathcal{X}$ an arbitrary compact (b)-regular stratification remains a famous problem of Goresky, still unsolved ([**7**] p.178) :

Conjecture 1. 3. *If $\mathcal{X} = (A, \Sigma)$ is a compact <u>Whitney stratification</u>, the homology representation map $R_k : WH_k(\mathcal{X}) \to H_k(A)$ is a bijection.*

Hovewer, the proof of Conjecture 1.3 would follow as a corollary if one proves Conjecture 1.1.

In conclusion Whitney regularity of the mapping cylinders of stratified submersions would play an extremely important role in answering affirmatively the Conjectures 1.1, 1.2 and 1.3.

The content of the paper is the following.

In §2 we review the most important classes of regular stratifications concerned by our analysis: the abstract stratified sets of Thom-Mather [**17, 8, 9**], and the Whitney (b)-regular stratifications [**19**], and we briefly recall the relation between them.

Then we recall the definition of condition (D), introduced by Goresky in his thesis [**5, 6**] for stratified submersions $f_{|\mathcal{W}} : \mathcal{W} \subseteq M \to \mathcal{W}' \subseteq M$, as a technical tool to obtain (b)-regularity of stratified mapping cylinders, and recall the results of Goresky of 1976-81 [**5, 7**] about it.

In §3 we study relations between condition (D) and stratified mapping cylinders.

The section is an exploration of some ideas of Goresky [**5, 7**] of which we give a finer analysis, different proofs, and some new equivalent properties.

For $\mathcal{X} = (A, \Sigma)$ a Whitney stratification, we consider the important case in which the stratified submersion $f_{|\mathcal{W}} : \mathcal{W} \subseteq M \to \mathcal{W}' \subseteq M$ is the restriction of a projection $\pi_X : T_X \to X$ on a stratum X of an system of control data $\mathcal{F} = \{(\pi_X, \rho_X) : T_X \to X \times \mathbb{R}\}_{X \in \Sigma}$ of $\mathcal{X}$ [**8, 9**].

The stratified mapping cylinder of $\pi_{X\,|\mathcal{W}}$ has then as embedded model the cone $C_{\mathcal{W}'}(\mathcal{W})$ equipped with its natural stratification $\bigsqcup_{S \subseteq \mathcal{W}\ ,\ S' = \pi_X(S)} \left[S \sqcup C^o_{S'}(S) \sqcup S' \right]$ (Proposition 3.4).

First, in Proposition 3.5 we explain what incidence relations in $C_{\mathcal{W}'}(\mathcal{W})$ are always (b)-regular, then using a convenient horizontal distribution $\{\mathcal{D}(y)\}_y$ in Theorem 3.3 and in Corollary 3.1.3) we prove that, if $\pi_{X\,|\mathcal{W}} : \mathcal{W} \to \pi_X(\mathcal{W})$ satisfies Condition (D), all remaining incidence relations $R' < C^o_{S'}(S)$ (with $R < S$ in $\mathcal{W}$) are (a)-regular, and thanks to this in Proposition 3.6 and Theorem 3.4 we prove that the naturally stratified cone $C_{\mathcal{W}'}(\mathcal{W})$ is a Whitney (b)-regular stratification.

In Corollary 3.2 we conclude that if $\mathcal{W}$ is a Whitney cellularisation of a compact subset $W \subseteq S_X(1) \subseteq T_X(1)$ such that $\pi_{\mathcal{W}}$ is cellular then $C_{\mathcal{W}'}(\mathcal{W})$ is a Whitney cellularisation too.

2. Stratified Spaces and Maps and Condition (D).

We recall that a *stratification* of a topological space A is a locally finite partition Σ of A into C^1 connected manifolds (called the *strata* of Σ) satisfying the *frontier condition* : if X and Y are disjoint strata such that X intersects the closure of Y, then X is contained in the closure of Y. We write then $X < Y$ and $\partial Y = \sqcup_{X<Y} X$ so that $\overline{Y} = Y \sqcup \big(\sqcup_{X<Y} X\big) = Y \sqcup \partial Y$ and $\partial Y = \overline{Y} - Y$ ($\sqcup$ = disjoint union).

The pair $\mathcal{X} = (A, \Sigma)$ is called a *stratified space* with *support* A and *stratification* Σ.

The *k-skeleton* of $\mathcal{X}$ is the stratified space $\mathcal{X}_k = (A_k, \Sigma_{|A_k})$ of support $A_k = \sqcup_{\dim X \leq k} X$.

A *stratified map* $f : \mathcal{X} \to \mathcal{X}'$ between stratified spaces $\mathcal{X} = (A, \Sigma)$ and $\mathcal{X}' = (B, \Sigma')$ is a continuous map $f : A \to B$ which sends each stratum X of $\mathcal{X}$ into a unique stratum X' of $\mathcal{X}'$, such that the restriction $f_X : X \to X'$ is C^1.

A *stratified submersion* is a stratified map f such that each $f_X : X \to X'$ is a C^1-submersion.

2.1. Regular Stratified Spaces and Maps.

Extra conditions may be imposed on the stratification Σ, such as to be an *abstract stratified set* in the sense of Thom-Mather **[17, 8, 9]** or, when A is a subset of a C^1 manifold, to satisfy conditions (a) or (b) of Whitney **[19]**, or (c) of K. Bekka **[1]** or, when A is a subset of a C^2 manifold, to satisfy conditions (w) of Kuo-Verdier **[20]**, or (L) of Mostowski **[15]**.

In this paper we will consider essentially Whitney (i.e. (b)-regular) stratifications :

Definition 2. 1. Let Σ be a stratification of a subset $A \subseteq \mathbb{R}^N$, $X < Y$ strata of Σ and $x \in$.

One says that $X < Y$ *is* (b)*-regular* (or that it satisfies *Condition* (b) *of Whitney*) at x if for every pair of sequences $\{y_i\}_i \subseteq Y$ and $\{x_i\}_i \subseteq X$ such that $\lim_i y_i = x \in X$ and $\lim_i x_i = x$ and moreover $\lim_i T_{y_i} Y = \tau$ and $\lim_i [y_i - x_i] = L$ in the appropriate Grassmann manifolds (here $[v]$ denotes the vector space spanned by v) then $L \subseteq \tau$.

The pair $X < Y$ *is called* (b)*-regular* if it is (b)-regular at every $x \in X$.

Σ is called a (b)*-regular* (or a *Whitney)* stratification if all $X < Y$ in Σ are (b)-regular.

For a C^1-retraction $\pi : U \to X$ defined on a neighbourhood U of x, one says that $X < Y$ is (b^π)*-regular at* x (or that it satisfies *Condition* (b^π) *at* x) if $L = \lim_i [y_i - \pi(y_i)]$ implies $L \subseteq \tau$.

One says that $X < Y$ is (a)*-regular at* x (or that it satisfies *Condition* (a) *at* x) if $T_x X \subseteq \tau$.

We recall that $X < Y$ is (b)-regular (at x) if and only if it is (a)- and (b^π)-regular (at x) for some C^1-retraction $\pi : U_x \to X$ defined in a neighbourhood U of x **[18]**.

Most important properties of Whitney stratifications follow because they are in particular abstract stratified sets **[8, 9]**. It is then helpful to recall the definition below.

Definition 2. 2. (Thom-Mather 1970) Let $\mathcal{X} = (A, \Sigma)$ be a stratified space.

A family $\mathcal{F} = \{(\pi_X, \rho_X) : T_X \to X \times [0, \infty[)\}_{X \in \Sigma}$ is called *a system of control data (SCD)* of $\mathcal{X}$ if for each stratum $X \in \Sigma$ we have that:

1) T_X is a neighbourhood of X in A (called a *tubular neighbourhood of* X);

2) $\pi_X : T_X \to X$ is a continuous retraction of T_X onto X (called *projection on* X);

3) $\rho_X : T_X \to [0, \infty[$ is a continuous function : $X = \rho_X^{-1}(0)$ (called *distance function from* X)

and, furthermore, for every pair of adjacent strata $X < Y$, by considering the restriction maps $\pi_{XY} = \pi_{X|T_{XY}}$ and $\rho_{XY} = \rho_{X|T_{XY}}$, on the subset $T_{XY} = T_X \cap Y$, we have that :

5) the map $(\pi_{XY}, \rho_{XY}) : T_{XY} \to X \times]0, \infty[$ is a C^1-submersion (it follows in particular that :

$\dim X < \dim Y$);

6) for every stratum Z of $\mathcal{X}$ such that $Z > Y > X$ and for every $z \in T_{YZ} \cap T_{XZ}$ the following *control conditions* are satisfied :

i) $\pi_{XY}\pi_{YZ}(z) = \pi_{XZ}(z)$ (called the π-*control condition*)

ii) $\rho_{XY}\pi_{YZ}(z) = \rho_{XZ}(z)$ (called the ρ-*control condition*).

In what follows we will pose $T_X(\epsilon) = \rho_X^{-1}([0,\epsilon[)\,, \forall\, \epsilon \geq 0$, and without loss of generality will assume $T_X = T_X(1)$ [**8, 9**].

The pair $(\mathcal{X}, \mathcal{F})$ is called an *abstract stratified set* if A is Hausdorff, locally compact and admits a countable basis for its topology.

Since one usually works with a unique SCD $\mathcal{F}$ of $\mathcal{X}$, in what follows we will omit $\mathcal{F}$.

If $\mathcal{X}$ is an abstract stratified set, then A is metrizable and the tubular neighbourhoods $\{T_X\}_{X\in\Sigma}$ may (and will always) be chosen such that: "$T_{XY} \neq \emptyset \Leftrightarrow X \leq Y$" and "$T_X \cap T_Y \neq \emptyset \Leftrightarrow X \leq Y$ or $X \geq Y$" (where both implications $\Leftarrow$ automatically hold for each $\{T_X\}_X$) as in [**8, 9**], pp. 41-46.

The notion of system of control data of $\mathcal{X}$, introduced by Mather, is very important because it allows one to obtain good extensions of (stratified) vector fields [**8, 9**] which are the fundamental tool in showing that a stratified (controlled) submersion $f : \mathcal{X} \to M$ into a manifold, satisfies Thom's First Isotopy Theorem : the stratified version to Ehresmann's fibration theorem [**17, 8, 9, 3**]. Moreover by applying it to the projections $\pi_X : T_X \to X$ it follows in particular that $\mathcal{X}$ has a *locally trivial structure* and so also a locally trivial topologically conical structure.

Since Whitney (b)-regular) stratification are abstract stratified sets [**8, 9**], they are locally trivial.

2.2. **Condition** (D) **and Goresky's results.** The following definition was introduced by Goresky first in [**5**] (1976) and [**7**] (1981).

Definition 2. 3. Let $f : M \to M'$ be a C^1 map between C^1-manifolds and $\mathcal{W} \subseteq M$ and $\mathcal{W}' \subseteq M'$ Whitney stratifications such that the restriction $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ is a surjective stratified submersion (so f takes each stratum Y of $\mathcal{W}$ to only one stratum $Y' = f(Y)$ of $\mathcal{W}' = f(\mathcal{W})$).

One says that $f : M \to M'$ *satisfies condition* (D) *with respect to* $\mathcal{W}$ *and* $\mathcal{W}'$ and we will say for short that the restriction $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ *satisfies the condition* (D) if the following holds :

for every pair of adjacent strata $X < Y$ of $\mathcal{W}$ and every point $x \in X$ and every sequence $\{y_i\}_i \subseteq Y$ such that $\lim_i y_i = x \in X$ and moreover $\lim_i T_{y_i}Y = \tau$ and $\lim_i T_{f(y_i)}Y' = \tau'$ in the appropriate Grassmann manifolds then $f_{*x}(\tau) \supseteq \tau'$.

Later on we will also consider given, with the obvious restricted meaning of the definition 2.3, what one intends by : *"$f : M \to M'$ satisfies condition (D) with respect to $X < Y$"* and *"$f : M \to M'$ satisfies condition (D) with respect to $X < Y$ at $x \in X$"* (*"at $x \in X < Y$"*).

In the whole of the paper we will denote $Y' = f(Y)$ and $y' = f(y)\,,\ \forall\, y \in Y$.

Two simple examples of f satisfying and not-satisfying the condition (D) are the following.

Example 2. 1. Let M be the horizontal plane $M = \{z = 1\} \subseteq \mathbb{R}^3$, $M' = L(0,1,0) = y$-axis $\subseteq \mathbb{R}^3$ and $f : M \to M'$ the standard projection $f(x, y, z) = y$.

Let $\mathcal{W}$ be the stratified space of support the half parabola $W = \{y = x^2, x \geq 0\} \cap M$ in M and stratification $\Sigma_{\mathcal{W}} = \{R, S\}$ where $R = \{(0,0,1)\}$ and $S = W \cap \{x > 0\}$. Then $R < S$.

Let $\mathcal{W}$ be the strat ed space of support the half y-axis, $W = M \quad y \quad 0$ in M and strati cation $\mathcal{W} = R \ S$ where $R = (0\ 0\ 0)$ and $S = M \quad y > 0$. Then $R < S$.

Then for every sequence $s_n \ n \quad S$ such that $\lim_n s_n = (0\ 0\ 1) \quad R$ one has : $= \lim_n T_{s_n} S = x$-axis $\quad ker\, f$ and $= \lim_n T_{s_n} S = y$-axis. Thus $f\ (\ )$.

Hence $f_{\mathcal{W}} : \mathcal{W} \quad \mathcal{W}$ does not satis es the condition (D) at $(0\ 0\ 1) \quad R < S$. $\square$

Example 2. 2. Let consider the same strati ed spaces of the example 2.1 but using now $W = \ y = tan(x) \ x \quad 0 \quad M$ the half graph of the tangent map in M.

Then for every sequence $s_n \ n \quad S$ such that $\lim_n s_n = (0\ 0\ 1) \quad R$ one has : $= \lim_n T_{s_n} S = L(1\ 1\ 0) \quad \ker f$ and $= \lim_n T_{s_n} S =$ the y-axis line. Thus $f\ (\ )$

Hence $f_{\mathcal{W}} : \mathcal{W} \quad \mathcal{W}$ satis es the condition (D) at $(0\ 0\ 1) \quad R < S$. $\square$

Below Figure 1a represents the case of Example 2.1 while Figure 1b the case of Example 2.2

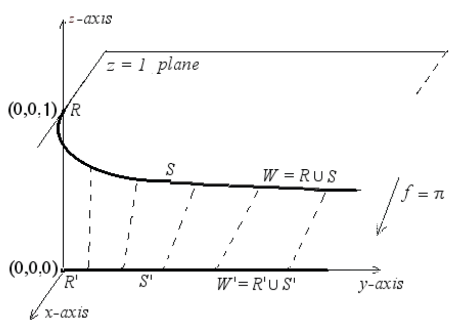

Figure 1a of Example 2.1

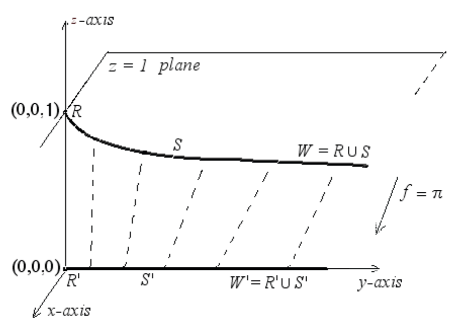

Figure 1b of Example 2.2

An important case in which condition (D) is satis ed is given by the following ([**5**] 3.7.4):

Example 2. 3. *Let $h : \mathbb{R}^N \quad \mathbb{R}^l \quad 0^k$ be a surjective submersion and $\mathcal{H} \quad \mathbb{R}^N$ and $\mathcal{H} \quad \mathbb{R}^l \quad 0^k$ linear cellular complexes such that the restriction $h_{\mathcal{H}} : \mathcal{H} \quad \mathcal{H} = f(\mathcal{H})$ is a cellular map.*

Then $h_{\mathcal{H}} : \mathcal{H} \quad \mathcal{H}$ satis es the condition (D).

Proof. Obviously, $\mathcal{H}$ and $\mathcal{H}$ are Whitney strati cations whose strata are their linear cells.

Let $R < S$ be cells of $\mathcal{H}$, $s_i \ i \quad S$ a sequence such that $\lim_i s_i = r \quad R \quad \overline{S}$, and let us denote $R = h(R)$, $S = h(S)$ and $s_i = h(s_i)$ and $r = h(r)$.

Since S and S are linear cells, then $T_{s_i} S$ and $T_{s_i} S$ are always the same two vector subspaces independently of $i \quad \mathbb{N}$: namely $[S] \quad \mathbb{R}^N$ and $[S\ ] \quad \mathbb{R}^l \quad 0^k$.

So $\lim_i T_{s_i} S = [S]$ and $\lim_i T_{s_i} S = [S\ ]$.

Similarly since $h : \overline{S} \quad \overline{S}$ is a cellular map, it is the restriction of a linear a ne map and then $h \ {}_{s_i} : T_{s_i} S \quad T_{s_i} S$ is independently of $i \quad \mathbb{N}$ always the same linear surjective map $H : [S] \quad [S\ ]$.

Thus

$$h \ {}_r(\lim_i T_{s_i} S) = h \ {}_r([S]) = H([S]) = [S\ ] = \lim_i T_{s_i} S = \lim_i h \ {}_{s_i}([s]) \qquad \square$$

Example 2. 4. *Let $f : M \quad M$ be a surjective C^1-submersion and h and h two C^1 cellularisations of two subsets $\mathcal{K} \quad M$ and $\mathcal{K} \quad M$ making the following diagram*

$$\begin{array}{ccccc} \mathcal{H} & \overset{h}{} & \mathcal{K} & & M \\ & & & & \\ g \sim & & & & \sim f \\ & & & & \\ \mathcal{H} & \overset{h}{} & \mathcal{K} & & M \end{array}$$

commutative where $g : \mathcal{H} \quad \mathcal{H}$ is a cellular map of cellular complexes.

Then $f_{\mathcal{K}} : \mathcal{K} \quad \mathcal{K}$ satis es the condition (D).

Proof. Since h is a C^1 cellularisation of $\mathcal{K}$, then by de nition **[6]**, p in a simplex $<$ of $\mathcal{H}$, the map h admits a C^1 extension h, a di eomorphism on a neighbourhood U_p of p in the a ne plane spanned by the linear cell .

Similarly, h being a C^1 cellularisation of $\mathcal{K}$ it admits a C^1 extension h, a di eomorphism on a neighbourhood U_p of $p = g(p)$ in the a ne plane spanned by the linear cell $= g(\ )$.

Therefore, $q = h(p) \quad \mathcal{K}$ (h a bijection), with the two isomorphisms $(h_{\ p})^{\ 1}$ and $h_{\ p}$ one has:

$$f_{\ q} = h_{\ p} \quad g_{\ p} \quad (h_{\ p})^{\ 1}$$

Finally, since by Example 2.3 g satis es Condition (D) at $p \qquad <$, then f satis es Condition (D) at $q = f(p) \quad f(\ ) < f(\ )$. □

The main reason for which Goresky introduced Condition (D) is that it provided the (b)-regularity for the natural strati cations on the mapping cylinder of a strati ed submersion.

Proposition 2. 1. *Let $\ : E \quad M$ be a C^1 riemannian vector bundle and $M = S_M$ the τ-sphere bundle of E. If $\mathcal{W} \quad M$, $\mathcal{W} = (\mathcal{W}) \quad M$ are two Whitney strati cations such that $_{\mathcal{W}} : \mathcal{W} \quad \mathcal{W}$ is a strati ed submersion which satis es condition (D), then the closed strati ed mapping cylinder*

$$\overline{C_M(\mathcal{W})} = \underset{Y \quad \mathcal{W}}{} (C_M(Y) \geq \ _M(Y)) \leq \ _M(Y) \leq Y$$

is a Whitney (i.e. (b)-regular) strati ed space.

Proof. **[7]** Appendix A.1. Lemma (i). □

Our work in 3 will be essentially to give a new proof, together with a ner analysis, of the following important statement which is the key property in proving the Proposition below :

Proposition 2. 2. *Every Whitney strati cation $\mathcal{W}$ with conical singularities and conical control data admits a Whitney cellularisation.*

Proof. **[7]** Appendix A.2. Proposition. □

Propositions 2.1 and 2.2 are the main properties which allowed Goresky to prove Proposition below and, thanks to this, his two homology representation theorems, Theorem 1.1 and Theorem 1.2, recalled in the introduction.

Proposition 2. 3. *Every Whitney strati cation $\mathcal{W}$ in a manifold M is cobordant in M to one $\mathcal{W}$ having conical singularities and control data, and which is hence (b)-regular.*

Proof. [**7**] Appendix A.3. Proposition. □

We end this section recalling that a detailed account of condition (D) including new analytic su cient conditions in terms of limits of a new distance function between tangent spaces is given in [**14**].

3. Condition (D) and stratified mapping cylinders.

Let $\mathcal{X} = (A \quad)$ be a Whitney strati ed space with strati cation and support A closed in $\mathbb{R}^N$.

In this section we consider the important case in which $f_{\mathcal{W}} : \mathcal{W} \quad M \quad \mathcal{W} \quad M$ is obtained as the restriction of a projection $_X : T_X \quad X$ on a stratum X of an SCD $\mathcal{F} = (\ _X \ \sigma_X) : T_X \quad X \quad \mathbb{R} \ _X$ of $\mathcal{X}$.

For our analysis it will be convenient to add to the strati cation $\mathcal{X}$ all strata of $\mathbb{R}^N \geq A$.

The connected components of $\mathbb{R}^N \geq A$ being N-manifolds this will again give a Whitney strati cation, namely again $\mathcal{X}$ of $A \supseteq (\mathbb{R}^N \geq A) = \mathbb{R}^N$ and then we will not lose generality.

It is well known that each neighbourhood T_X of an SCD of $\mathcal{X}$ can be obtained as a tubular neighbourhood of X in $\mathbb{R}^N$ and $_X : T_X \quad X$ as a C^1 map [**8**].

On the other hand T_X remains equipped with the induced Whitney strati cation by its intersections with all strata $Y > X$ of $\mathcal{X}$; that is : $T_X = \leq_Y \ _X T_{XY}$ (as usual $T_{XY} = T_X \quad Y$).

Similarly the τ-sphere bundle $S_X = \sigma_X{}^1(\tau)$ of T_X, remains equipped with a natural induced Whitney strati cation $S_X = \leq_{Y>X} S_{XY}$ where $S_{XY} = S_X \quad Y$.

Let consider then for $f : M \quad M$ the restriction map $f = \ _{X\, S_X} : S_X \quad X$ between the C^1-manifolds $M = S_X$ and $M = X$ which is a C^1-submersion [**8**].

We will consider for $\mathcal{W}$ a Whitney strati cation of a compact set $W \quad S_X$ *stratifying* $_X$ as de ned below.

De nition 3. 4. Let $\mathcal{W} = (W \quad)$ be a Whitney strati cation of a compact set $W \quad S_X$.

We will say that $\mathcal{W}$ *strati es* $_X$ if the image $W = \ _X(W)$ has a natural Whitney strati cation $\mathcal{W} \ = \leq_S S$ (where $S = \ _X(S)$, and S ranges over all strata of $\mathcal{W}$) which makes $_{X\,\mathcal{W}} : \mathcal{W} \quad \mathcal{W}$ a strati ed surjective submersion (denoted $_\mathcal{W}$).

We will investigate the condition (D) for the restriction $f_\mathcal{W} = \ _\mathcal{W} : \mathcal{W} \quad S_X \quad \mathcal{W} \quad X$.

A very important example occurs when $\mathcal{W}$ is a Whitney triangulation of $S_X \geq \supseteq_{X\ <X} T_X$ for which the restriction $_X : S_X \geq \supseteq_{X\ <X} T_X \quad X \geq \supseteq_{X\ <X} T_X$ is a PL map [**5**] : this case will be treated in Corollary 3.2.

Let $l = \dim X$. The analysis of condition (D) at a point $x \quad R$ for every stratum R of $\mathcal{W}$ is local and invariant by C^1-di eomorphisms, hence starting from now we will suppose [**18**] that $\tau = 1$, $X = \mathbb{R}^l \quad 0^k$ $(l + k = N)$ and $_X = \ $, $\sigma_X = \sigma$ are the standard data :

$$\sigma(z) = \ z_{l+1}^2 + \quad + z_N^2 \ ^{\frac{1}{2}} \qquad (z) = (z_1 \qquad z_l \ 0^k) \quad \text{where} \quad z = (z_1 \qquad z_N) \quad \mathbb{R}^N$$

Thus $S_X = S_X^1 = \ z \quad \mathbb{R}^N \quad z_{l+1}^2 + \quad + z_N^2 = 1 \ = \mathbb{R}^l \quad S^{k\ 1}$ and the C^1-submersion $f = \ _{X\, S_X}$ is the canonical projection : $\mathbb{R}^l \quad S^{k\ 1} \quad \mathbb{R}^l \quad 0^k$ (also denoted $_X$).

In particular $\mathcal{W}$ will be a Whitney strati cation $\quad S_X^1 = \mathbb{R}^l \quad S^{k\ 1}$ stratifying $_X$.

With these hypotheses the closed cone with straight lines in $\mathbb{R}^N$:

$$C_\mathcal{W} \ (\mathcal{W}) = \ tp + (1 \geq t) \ (p) \quad p \quad \mathcal{W} \quad t \quad [0 \ 1]$$

with its natural strati cation, gives a di erential model of the strati ed mapping cylinder of the strati ed submersion $_{\mathcal{W}} : \mathcal{W} \quad \mathcal{W}$ as follows.

For every subset $H \quad S^1_X$, written $H = \ _X(H)$ let us denote by :

$$C_H\ (H) = \ tp + (1 \geq t)\ \ (p)\ \ p \quad H \quad t \quad [0\ \ 1]$$

$$C^o_H\ (H) = \ tp + (1 \geq t)\ \ (p)\ \ p \quad H \quad t \quad]0\ \ 1[$$

respectively the closed and the open cone of H induced by .

The natural strati cation of $C_{\mathcal{W}}\ (\mathcal{W})$ is then given by :

$$C_{\mathcal{W}}\ (\mathcal{W}) = \underset{S \ \ \mathcal{W}}{\quad} S \leq C^o_S\ (S) \leq S$$

Proposition below says that $C_{\mathcal{W}}\ (\mathcal{W})$ can be strati ed as the strati ed image of an appropriate globally C^1 strati ed map F which makes it into a di erential model of the strati ed mapping cylinder $M(\ _{\mathcal{W}}) = (\mathcal{W} \quad [0\ \ 1] \leq \mathcal{W}\ ) \quad (z\ \ 0) \subseteq \ (z)$.

Proposition 3. 4. *Let F be the map*

$$F\ : S^1_X \quad [0\ \ 1] \quad C_X(S^1_X) \qquad F(z\ \ t) = tz + (1 \geq t)z \qquad z = \ _X(z)$$

1) F is a homotopy satisfying $F_0(z) = 1_{S^1_X}(z)$ and $F_1(z) = \ _{X\ S^1_X}(z)$ whose restriction o $F(S^1_X \quad 0\ ) = X$, that is $F\ : S^1_X \ \ ]0\ \ 1] \quad C_X(S^1_X) \geq X = C^o_X(S^1_X) \leq S^1_X$, is a C^1-isotopy.

2) $C_{\mathcal{W}}\ (\mathcal{W}) = F(\mathcal{W} \quad [0\ \ 1])$.

Proof. Immediate. □

Looking at the regularity of the incidence relations in $C_{\mathcal{W}}\ (\mathcal{W})$ we have :

Proposition 3. 5. *Let $\mathcal{W}$ be a Whitney strati cation in $S^1_X = \mathbb{R}^l \quad S^{k\ \ 1}$ which strati es the canonical projection $_X : S^1_X \quad X = \mathbb{R}^l \quad 0^k$ and let $\mathcal{W} = \ _X(\mathcal{W})$.*

For every pair of strata $R < S$ of $\mathcal{W}$, by denoting $S = \ _X(S)$, $R = \ _X(R)$, the cone

$$C_{R\quad S}\ (R \supseteq S) = \ R \leq C^o_R\ (R) \leq R \quad \leq \quad S \leq C^o_S\ (S) \leq S$$

satis es (b)-regularity for all incidence relations $<$ below :

$$\begin{array}{ccccc} R & < & S & \mathcal{W} & S^1_X \\ & \nwarrow & & & \\ C^o_R\ (R) & < & C^o_S\ (S) & C_{\mathcal{W}}\ (\mathcal{W}) & \mathbb{R}^N \\ R & < & S & \mathcal{W} & X \end{array}$$

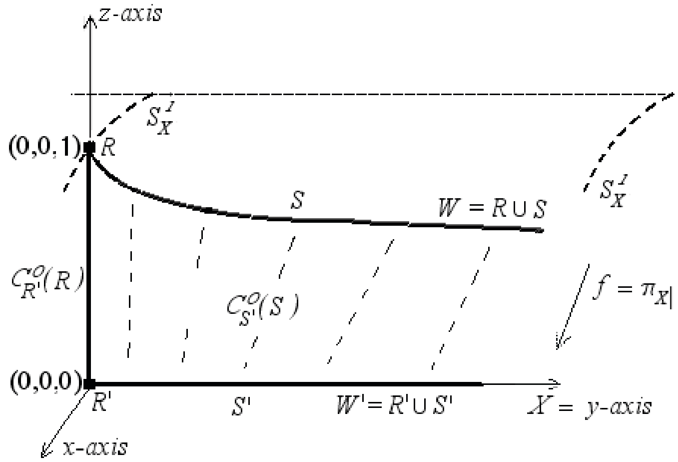

Figure 2

Proof. Since $\mathcal{W}$ and $\mathcal{W}$ are Whitney (b)-regular strati cations the pair of strata $R < S$ in $\mathcal{W}$ and $R < S$ in $\mathcal{W}$ are trivially (b)-regular.

Since the proofs of (b)-regularity for the pairs $R < C^o_R(R)$ and $S < C^o_S(S)$ are obviously the same and this also holds for the pairs $R < C^o_R(R)$ and $S < C^o_S(S)$ it will be su cient to prove the (b)-regularity of the following adjacent pairs of strata :

$$\begin{array}{ccc} R & & S \\ & \nwarrow & \\ C^o_R(R) & < & C^o_S(S) \\ & & \\ & & S \end{array}$$

The restriction of the C^1-homotopy F to $S^1_X \quad]0\ 1]$ (namely again F) :

$$F \ : \ S^1_X \quad]0\ 1] \qquad C_X(S^1_X) \geq X \qquad F(z\ t) = \ (z) + t(z \geq \ (z))$$

is a C^1 di eomorphism of manifolds with boundary such that :

$$C^o_S(S) = F(S \quad]0\ 1[) \qquad S = F(S \qquad 1\ ) \quad \text{and} \quad C^o_R(R) = F(R \quad]0\ 1[)$$

Hence the (b)-regularity of

$$R < C^o_S(S) \qquad S < C^o_S(S) \qquad \text{and} \qquad C^o_R(R) < C^o_S(S)$$

follows via F respectively by the (b)-regularity in $\mathbb{R}^N$ of

$$R < S \quad]0\ 1[\qquad S < S \quad]0\ 1[\qquad \text{and} \qquad R \quad]0\ 1[< S \quad]0\ 1[$$

Then, it only remains to prove that $S < C^o_S(S)$ is (b)-regular.

It is well known that (b)-regularity is satis ed for a pair of strata $S < Y$ if and only if (a)-regularity and $(b^{\ S\ Y})$-regularity are satis ed for the restriction $\ _{S\ Y} : T_S \quad Y \quad S$ of a C^1-retraction $\ _S : T_S \quad S$ de ned on a neighbourhood T_S of S **[18]**.

We will show then that $S < Y = C^o_S(S)$ is (a)- and $(b^{\ S\ Y})$-regular.

<u>*(a)-regularity.*</u> For every point $z \quad S^1_X$, by denoting $z = (x\ x\ )$ with $x \quad \mathbb{R}^l$ and $x \quad \mathbb{R}^k$ then $(z) = (x\ 0^k)$ and $z \geq \ (z) = (0^l\ x\ )$ and $F(x\ x\ \ t) = (x\ tx\ )$. Similarly for every $v \quad \mathbb{R}^{l+k}$,

$v = (u, u')$, and at every point $(z,t) = (x, x', t) \in S^1_X \times]0,1[$ the image of the differential map F

$$F_{*(z,t)} \quad : \quad T_{(z,t)}(S^1_X \times [0,1]) \quad \to \quad T_{F(z,t)} C_X(S^1_X)$$

is given by :

$$F_{*(z,t)}(v,\lambda) = \begin{pmatrix} 1_{\mathbb{R}^l} & 0 & 0 \\ & & \\ 0 & t \cdot 1_{\mathbb{R}^k} & x' \end{pmatrix} \cdot \begin{pmatrix} u \\ u' \\ \lambda \end{pmatrix} = (u, tu') + \lambda(0, x') =$$

$$= \pi(v) + t(v - \pi(v)) + \lambda(z - \pi(z)) \,.$$

By considering the submanifold $Y_t = F(S \times \{t\})$ of $Y = C^o_{S'}(S) = F(S \times]0,1[)$ and a point $y = F(s,t) \in Y_t \subseteq Y$ one finds :

$$T_{F(s,t)} Y_t \;\; = \;\; F_{*(s,t)}\big(T_{(s,t)}(S \times \{t\})\big) \;\; = \;\; F_{*(s,t)}(T_s S \times \{0\}) \;\; = \;\; \big\{ F_{*(s,t)}(v,0) \mid v \in T_s S \big\}$$

with

$$F_{*(s,t)}(v,0) = (tu, u') = \pi(v) + t(v - \pi(v))$$

and so for every $s_0 \in S$, if $s'_0 = \pi(s_0)$, F being a C^1 map at $(s_0, 0)$ one has :

$$\lim_{(s,t)\to(s_0,0)} T_{F(s,t)} Y_t = \lim_{(s,t)\to(s_0,0)} F_{*(s,t)}(T_s S \times \{0\}) = F_{*(s_0,0)}(T_s S \times \{0\}) = \pi_{*s_0}(T_{s_0} S) = T_{s'_0} S' .$$

Consequently, for each point $s_0 \in S$:

$$\lim_{(s,t)\to(s_0,0)} T_{(s,t)}\, C^o_{S'}(S) \quad \supseteq \quad \lim_{(s,t)\to(s_0,0)} T_{F(s,t)} Y_t \quad = \quad T_{s'_0} S'$$

which proves the (a)-regularity $S' < C^o_{S'}(S)$.

<u>$(b^{\pi_{S'Y}})$*-regularity.*</u> To prove that $S' < C^o_{S'}(S)$ is $(b^{\pi_{S'Y}})$-regular, it is natural to take for $\pi_{S'}$ the restriction of the canonical projection $\pi : \mathbb{R}^N \to \mathbb{R}^l \times 0^k$, and denote it again by π.

Let us consider a sequence $\{F(s_n,t_n)\}_n \subseteq C^o_{S'}(S)$ such that $\lim_n F(s_n,t_n) = s'_0 \in S'$ and there exist both limits of lines and tangent spaces :

$$L = \lim_n \overline{F(s_n,t_n)\, \pi(F(s_n,t_n))} \in G^1_n \quad \text{and} \qquad \tau = \lim_n T_{F(s_n,t_n)}\, C^o_{S'}(S) \in \mathbb{G}^h_n \,, \quad (h = \dim S + 1) \,.$$

Then $\{s_n\} \subseteq S$ is a convergent sequence, $\lim_n s_n = s_0 \in S$, such that if $s'_n = \pi(s_n)$ then $\lim_n s'_n = s'_0 = \pi(s_0)$ and $\lim_n t_n = 0$.

Since $C^o_{S'}(S) = F(S \times]0,1[) = C_{S'}(S) - S \cup S'$, with $S' = \pi(S)$ and $\pi(F(s_n,t_n)) = \pi(s_n) = s'_n$, then for every line $L_n = \overline{F(s_n,t_n)\, \pi(F(s_n,t_n))}$ we have :

$$L_n \;\; = \;\; \overline{F(s_n,t_n)\, \pi(F(s_n,t_n))} \;\; = \;\; \overline{s_n\, s'_n} \;\; = \;\; [s_n - s'_n] \,,$$

where $[v]$ denotes the vector subspace spanned by $v \in \mathbb{R}^N$, so that

$$L \;\; = \;\; \lim_n L_n \;\; = \;\; \lim_n\, [s_n - s'_n] \;\; = \;\; [s_0 - s'_0] \,.$$

On the other hand, for every $n \in \mathbb{N}$, by decomposing in a direct sum

$$T_{(s_n,t_n)} \;\; S \times]0,1[\;\; = \;\; T_{s_n} S \;\times\; \mathbb{R} \;\; = \;\; T_{s_n} S \times \{0\} \quad + \quad \{0^h\} \times \mathbb{R}$$

one also has :

$$F_{(s_n\ t_n)}\ T_{(s_n\ t_n)}S\ \ ]0\ 1[\quad =\quad F_{(s_n\ t_n)}\ T_{s_n}S\quad 0\ )\quad +\quad F_{(s_n\ t_n)}\ \ 0^h\quad \mathbb{R}\ =$$

$$(v) + t_n(v \geq\ (v))\quad v\quad T_{s_n}S\quad +\quad \rho(s_n \geq s_n)\ \ \rho\quad \mathbb{R}\quad =$$

as in the previous proof of (a)-regularity :

$$=\ T_{F(s_n\ t_n)}Y_{t_n}\ +\ [s_n \geq s_n]$$

Finally, since

$$\lim_n (T_{F(s_n\ t_n)}Y_{t_n}\ +\ [s_n \geq s_n])\qquad \lim_n T_{F(s_n\ t_n)}Y_{t_n}\ +\ \lim_n\ [s_n \geq s_n]$$

one nds :

$$=\quad \lim_n\ T_{F(s_n\ t_n)}\,C^o_S\,(S)\quad =\quad \lim_n\ F_{(s_n\ t_n)}\ T_{(s_n\ t_n))}S\ \ ]0\ 1)[\quad =$$

$$=\quad \lim_n\quad T_{F(s_n\ t_n)}Y_{t_n}\ +\ [s_n \geq s_n]\qquad T_{s_0}S\quad +\quad [s_0 \geq s_0]$$

This proves L and concludes the proof of $(b\ )$-regularity of $S\ < C^o_S\,(S)$. ☐

If we consider as in Proposition 3.5 for $_{S\ \ Y} : Y \supseteq S\qquad S$ the restriction of $: \mathbb{R}^N$ $\mathbb{R}^l\quad 0$ and similarly for the distance function to S the restriction of the standard distance $\sigma(z_1\qquad z_N) = z^2_{l+1} + \qquad + z^2_N\ ^{\frac{1}{2}}$, then the strati cation of only two strata $S\ < C^o_S\,(S) = Y$ remains equipped with an SCD $(\ _S\ \ \sigma_S\ )$. With such an SCD one can consider the canonical distribution $\mathcal{D}_{S\ Y} : S \supseteq Y\qquad \mathbb{G}^{\dim S}_N$ relative to the (a)-regular pair of strata $S\ < Y = C^o_S\,(S) =$ $F(S\ \ ]0\ 1[)$ as de ned in **[10, 11]**, by the subspace of T_yY closest to $T_s\,S$:

$$\mathcal{D}_{S\ Y}(y) =\quad (\ker(\ _{S\ Y}\ \ \sigma_{S\ Y})\ _y\ ;\ \ker \sigma_{S\ Y\ \ y})$$

where the notation $(U\ V)$ means the orthogonal complement of a vector subspace V in a vector space U and $V\quad U\quad \mathbb{R}^N$ are considered with the standard Euclidian scalar product.

Remark 3. 1. *By Proposition 3.5, $S\ < C^o_S\,(S)$ is (a)-regular, hence the canonical distribution $\mathcal{D}_{S\ Y}(y)$ relative to $S\ < C^o_S\,(S) = Y$ satis es :* $\lim_{y\quad s\quad S} \mathcal{D}_{S\ Y}(y)\quad T_s\,S$ **[10, 11]**. ☐

Now, for every $t\quad]0\ 1]$, the di eomorphism

$$F_t : S = Y_1\qquad Y_t = F(S\qquad t\ )\qquad y = F_t(s) = F(s\ \ t) =\ (s) + t(s \geq\ (s))$$

induces (as in the proof of 3.3) an isomorphism between the tangent spaces and their subspaces

$$F_{t\ \ s} : T_sS\qquad T_yY_t\qquad F_{t\ \ s}(v) =\ (v) + t(v \geq\ (v))$$

By considering for the Whitney strati cation $\mathcal{W}\quad S^1_X = \mathbb{R}^l\quad S^{k\ \ 1}$ stratifying the canonical projection $_X\ :\ S^1_X\ =\ \mathbb{R}^l\quad S^{k\ \ 1}\qquad X\ =\ \mathbb{R}^l\quad 0^k$ (i.e. such that the map $_{\mathcal{W}}\ :\ \mathcal{W}$ $\mathcal{W}\ =\ _X(\mathcal{W})$ is a strati ed surjective submersion) and for each stratum S of $\mathcal{W}$ the canonical distribution $\mathcal{D}(s)\ _s$ of $_{\mathcal{W}\ S}$ (see also **[14]** 3) de ned by, $\mathcal{D}(s) =\quad (\ker\ \ _{X\ S\ \ s}\ \ T_sS)$ we have :

Lemma 3. 1. *The strati cation* $S\ < Y = C_S\ (S)$*, with the SCD* $(\quad_{S\ Y}\ \sigma_{S\ Y})\ $*, satis es:*

1) Each hypersurface $Y_t = F_t(S)$ *of* Y*, coincides with the hypersurface* $\sigma_{S\ Y}^{\ 1}(t)$ *:* $Y_t = \sigma_{S\ Y}^{\ 1}(t)$*.*

2) If $y = F(s\ \ 1)$*, so that* $y = s \quad Y_1 = S$ *the distributions* $\mathcal{D}(s) = \mathcal{D}_{S\ Y}(y)$ *coincide.*

3) $F_t : S \qquad Y_t$*, carries the distribution* $\mathcal{D}(s)$ *into* $\mathcal{D}_{S\ Y}(y)$ *:*

$$F_{t\ \ s}(\mathcal{D}(s)) = \mathcal{D}_{S\ Y}(y)$$

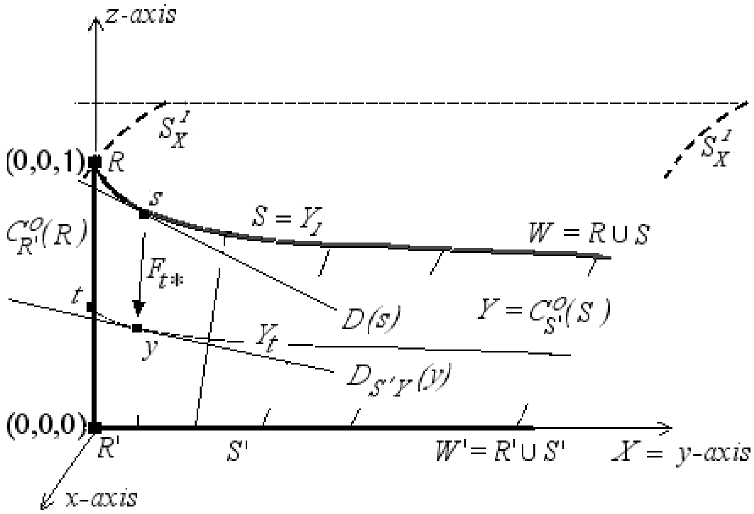

Figure 3

Proof 1). If $y = F(s \quad) \quad Y$, being $\ y \geq \ \ (y) = \ \ (s \geq \ \ (s))$ and $\ s \geq \ \ (s) \ \ = 1$ one has :

$$\sigma_{S\ Y}(y) = \ \ y \geq \ \ (y) \ \ = \ \ \ (s \geq \ \ (s)) \ \ = \ \ \ \ s \geq \ \ (s) \ \ = \qquad\qquad \text{and so :}$$

$$y \quad Y_t \quad \circ \qquad = t \quad \circ \quad \sigma_{S\ Y}(y) = t \quad \circ \quad y \quad \sigma_{S\ Y}^{\ 1}(t)$$

Proof 2). If $y = F(s\ \ 1)$, so $s = y$ and $S = Y_1 = \sigma_{S\ Y}^{\ 1}(1) \quad Y$ (by i)) one has :

$$T_sS = T_yY_1 = T_y\sigma_{S\ Y}^{\ 1}(1) = \ker \sigma_{S\ Y\ \ y} \quad T_yY$$

and since $\quad_{X\ S} = \quad_{S\ Y\ Y_1}$ we also have

$$\ker \quad_{X\ S\ \ s} = \ker \quad_{S\ Y\ Y_1\ \ y} = \ker \quad_{S\ Y\ \ y} \quad T_yY_1 = \ker \quad_{S\ Y\ \ y} \quad \ker \sigma_{S\ Y\ \ y}$$

so that, using again $T_sS = \ker \sigma_{S\ Y\ \ y}$, one concludes :

$$\mathcal{D}(s) = \quad (\ker \quad_{X\ S\ \ s}\ T_sS) = \quad (\ker \quad_{S\ Y\ \ y} \quad \ker \sigma_{S\ Y\ \ y}\ ;\ \ker \sigma_{S\ Y\ \ y}) = \mathcal{D}_{S\ Y}(y)$$

Proof 3). First remark that, for every point $y = F(t\ \ s)$ and vector $v \quad \mathcal{D}(s)$, one has :

$$F_{t\ s}(v) \quad F_{t\ s}(T_sS) = T_{F(t\ s)}F_t(S) = T_yY_t = T_y\sigma_{S\ Y}^{\ 1}(t) = \ker \sigma_{S\ Y\ y}$$

By $\ker \ _s \quad \ker \ _{S\ Y\ y} \quad \ker \ _{S\ Y\ y} \quad \ker \sigma_{S\ Y\ y}$ it follows :

$$(v) \quad \mathbb{R}^l \quad 0 = (\ker \ _s) \qquad (\ker \ _{S\ Y\ y})$$

and since $v \geq \ (v) \quad \ker \ _{S\ Y\ y} = (\ker \sigma_{S\ Y\ y})$ we nd :

$$F_{t\ s}(v) = \ (v) \geq t(v \geq \ (v)) \quad (\ker \ _{S\ Y\ y}) \ + \ (\ker \sigma_{S\ Y\ y}) \ = (\ker \ _{X\ y} \quad \ker \sigma_{S\ Y\ y})$$

and nally thanks to $F_{t\ s}(v) \quad \ker \sigma_{S\ Y\ y}$ we deduce that $F_{t\ s}(v)$ also lies in :

$$[\ker \ _{X\ y} \quad \ker \sigma_{S\ Y\ y}] \qquad \ker \sigma_{S\ Y\ y} \qquad (\ker \ _{S\ Y\ y} \quad \ker \sigma_{S\ Y\ y} \ \ker \sigma_{S\ Y\ y}) = \mathcal{D}_{S\ Y}(y)$$

In conclusion $F_{t\ s}(\mathcal{D}(s)) \quad \mathcal{D}_{S\ Y}(y)$ and having the same dimension (by 2)) they coincide. □

Proposition 3.5 proves the (b)-regularity of each pair of adjacent strata of the cone $C_{R\ \ S}$ $(R \supseteq S)$ except for $R \ < C_S^o\,(S)$.

Therefore, to have nally the global (b)-regularity of a cone $C_{\mathcal{W}}\ (\mathcal{W}\,)$ one needs to obtain the (b)-regularity of the pair $R \ < C_S^o\,(S)$ for each stratum $R = \ _X(R)$ and $R < S$.

This property will be described in terms of condition (D) in Theorem below.

Theorem 3. 3. *Let* $\mathcal{W}$ *be a Whitney strati cation in* $S_X^1 = \mathbb{R}^l \quad S^{k\ \ 1}$ *stratifying the canonical projection* $\ _X : S_X^1 = \mathbb{R}^l \quad S^{k\ \ 1} \qquad X = \mathbb{R}^l \quad 0^k$ *and let* $\mathcal{W} \ = \ _X(\mathcal{W}\,)$.

Let $R < S$ *be two strata of* $\mathcal{W}$ *and* $r \quad R$, $S = \ _X(S)$, $R = \ _X(R)$ *and* $s = \ _X(s) \quad s \quad S$.

The following conditions are equivalent :

1) $\ _{\mathcal{W}} : \mathcal{W} \qquad \mathcal{W}$ *satis es the condition* (D) *at* $r \quad R < S$ *;*

2) $\ _{X\ r}(\lim_i \mathcal{D}(s_i)) \quad \lim_i \ _{X\ s_i}(\mathcal{D}(s_i))$ *for every sequence* $\ s_i\ _i \quad S : \lim_i s_i = r \quad R < S$.

3) The cone $C_{R\ \ S}$ $(R \supseteq S)$ *has the strata* $S \ < Y = C_S^o\,(S)$ *such that the canonical distribution* $\mathcal{D}_{S\ Y}(y)$ *satis es : for every sequence* $\ y_i = F(s_i\ t_i)\ _i \quad Y$ *such that* $\lim_i y_i = r \quad R$ $\lim_i \mathcal{D}_{S\ Y}(y_i) \quad \lim_i \ _{S\ Y\ y_i}(\mathcal{D}_{S\ Y}(y_i))$.

Proof. Let $\ s_i \quad S$ be a sequence such that $\lim_i s_i = r \quad R$ and both limits $\lim_i T_{s_i}S =$ and $\lim_i \ _{s_i}(T_{s_i}S) =$ exist in the appropriate Grassmann manifolds.

Since $\mathcal{W}$ strati es $\ _X : \mathcal{W} \qquad \mathcal{W}$ then the restriction $\ _S : S \qquad _X(S) = S$ is a C^1 submersion and in particular $T_{s_i}S = \ _{s_i}(T_{s_i}S)$.

$(1 \circ \ 2)$. It is $(1 \circ \ 4)$ of Theorem 4.1 **[14]** for the strati ed submersion $\ _{\mathcal{W}} : \mathcal{W} \qquad \mathcal{W}$

$(2 \circ \ 3)$. Statement 2) above is obviously intended for every sequence $\ s_i \qquad S$ such that both limits $\lim_i \mathcal{D}(s_i) = \mathcal{D}$ and $\lim_i \ _{s_i}(\mathcal{D}(s_i)) = \mathcal{D}$ exist in the appropriate Grassmann manifold and similarly for the limits in the statement 3).

By Lemma 3.1 $\mathcal{D}_{S\ Y}(y_i) = F_{t_i\ s_i}(\mathcal{D}(s_i))$ and because the homotopy $F : id \subseteq$ is a C^1 map such that $F_0 = \ _X$, if $(r\ 0) = \lim_i(s_i\ t_i)$ we have :

$$\lim_i \mathcal{D}_{S\ Y}(y_i) = \lim_i F_{t_i\ s_i}(\mathcal{D}(s_i)) = F_{0\ r}(\lim_i \mathcal{D}(s_i)) = \ _{X\ r}(\lim_i \mathcal{D}(s_i))$$

By the submersivity of $\ _{X\ S} : S \qquad S$ and of $\ _{S\ Y} : Y \qquad S$ (**[11]**), for every i we have both: $\ _{X\ s_i}(\mathcal{D}(s_i)) = T_{s_i}S = \ _{S\ Y\ y_i}(\mathcal{D}_{S\ Y}(y_i))$ and in conclusion :

$$_{X\ r}(\lim_i \mathcal{D}(s_i)) \quad \lim_i \ _{X\ s_i}(\mathcal{D}(s_i)) \qquad \lim_i \mathcal{D}_{S\ Y}(y_i) \quad \lim_i \ _{S\ Y\ y_i}(\mathcal{D}_{S\ Y}(y_i)) \quad \square$$

Corollary 3. 1. *Let $\mathcal{W}$ be a Whitney strati cation in $S^1_X = \mathbb{R}^l \quad S^{k\ \ 1}$ stratifying the canonical projection $_X : S^1_X = \mathbb{R}^l \quad S^{k\ \ 1} \quad X = \mathbb{R}^l \quad 0^k$ and let $\mathcal{W} \ = \ _X(\mathcal{W})$.*

Let $R < S$ be two strata of $\mathcal{W}$ and $r \quad R$, $S = \ _X(S)$, $R = \ _X(R)$ and $s = \ _X(s) \quad s \quad S$.

If the strati ed submersion $_{\mathcal{W}} : \mathcal{W} \quad \mathcal{W}$ satis es condition (D) at $r \quad R < S$ then :

1) The cone $C_{R\ \ S}\ (R \supseteq S)$ has strata $Y = C^o_S\ (S) > S$ such that for every sequence of points $y_i = F(s_i\ \ t_i) \quad Y$ such that $\lim_i y_i = r \quad R$ one has $\lim_i \mathcal{D}_{S\ Y}(y_i) \quad T_r\ R$.

2) The cone $C_{R\ \ S}\ (R \supseteq S)$ has the strata $Y = C^o_S\ (S) > S$ such that for every sequence of points $y_i = F(s_i\ \ t_i) \quad Y$ such that $\lim_i y_i = r \quad R$ one has $\lim_i T_{y_i} Y \quad \lim_i T_{s_i} S$.

3) The cone $C_{R\ \ S}\ (R \supseteq S)$ has the pair of strata $Y = C^o_S\ (S) > R$ which is (a)-regular.

Proof 1). By hypothesis the strati ed submersion $_{\mathcal{W}} : \mathcal{W} \quad \mathcal{W}$ satis es the condition (D) at $r \quad R < S$ so by Theorem 3.3 :

$$\lim_i \mathcal{D}_{S\ Y}(y_i) \qquad \lim_i \ _{S\ Y\ y_i}\mathcal{D}_{S\ Y}(y_i) \quad = \quad \lim_i T_{s_i} S$$

and moreover $R < S$ being (a)-regular by hypothesis on $\mathcal{W}$ one also has

$$\lim_i T_{s_i} S \qquad T_r\ R \qquad \text{and so} \qquad \lim_i \mathcal{D}_{S\ Y}(y_i) \qquad T_r\ R$$

Proof 2). From the proof of 1) one has : $\lim_i T_{y_i} Y \quad \lim_i \mathcal{D}_{S\ Y}(y_i) \quad \lim_i T_{s_i} S$.

Proof 3). Thanks to 2) and 1), $\quad y_i = F(s_i\ \ t_i) \quad Y$ such that $\lim_i y_i = r \quad R$ one has :

$$\lim_{y_i\ \ r} T_{y_i} Y \qquad \lim_i \mathcal{D}_{S\ Y}(y_i) \qquad \lim_i T_{s_i} S \qquad T_r\ R \qquad \square$$

Proposition 3. 6. *Let $\mathcal{W}$ be a Whitney strati cation in $S^1_X = \mathbb{R}^l \quad S^{k\ \ 1}$ stratifying the canonical projection $_X : S^1_X = \mathbb{R}^l \quad S^{k\ \ 1} \quad X = \mathbb{R}^l \quad 0^k$ and let $\mathcal{W} \ = \ _X(\mathcal{W})$.*

Let $R < S$ be two strata of $\mathcal{W}$ and $r \quad R$, $S = \ _X(S)$, $R = \ _X(R)$ and $s = \ _X(s) \quad s \quad S$.

If the strati ed submersion $_{\mathcal{W}} : \mathcal{W} \quad \mathcal{W}$ satis es the condition (D) at $r \quad R < S$ then the following conditions are equivalent :

1) The cone $C_{R\ \ S}\ (R \supseteq S)$ is (a)-regular at $r \quad R < C^o_S\ (S)$.

2) The cone $C_{R\ \ S}\ (R \supseteq S)$ is (b)-regular at $r \quad R < C^o_S\ (S)$.

Proof. 1) $\quad$ 2). As in Proposition 3.5 we use that condition (b) holds if and only if the conditions (a) and $(b\ ^R)$ hold for some C^1-retraction $_R$ de ned on an open neighbourhood of R .

The proof reduces then to proving that $(b\ ^R)$ holds with respect to the pair $R < Y = C^o_S\ (S)$.

As in Proposition 3.5 if $y = ts + (1 \geq t)s \quad Y$, $_S\ (y) = s$ since $C^o_S\ (S)$ is a cone, then :

$$\overline{y\ \ _S\ (y)} = [s \geq s\ ]$$

Let us x a sequence $y_i = t_i s_i + (1 \geq t) s_{i\ \ i} \quad Y$ converging to a point $r \quad R < S$ such that both limits exist in the appropriate Grassmann manifolds :

$$= \lim_i T_{y_i} C^o_S\ (S) \quad \text{and} \quad L = \lim_i \overline{y_i\ \ _R\ (y_i)} = \lim_i\ [y_i \geq \ _R\ (y_i)]$$

Splitting every vector $y_i \geq \ _R(y_i)$ in the following orthogonal sum :

$$y_i \geq \ _R(y_i) \ = \ (y_i \geq s_i) \ + \ (s_i \geq \ _R(y_i))$$

every 1-dimensional vector space $\overline{y_i \ \ _R(y_i)} = [y_i \geq \ _R(y_i)]$ is contained in the 2-dimensional vector space spanned by the two orthogonal 1-dimensional vector space as follows :

$$\overline{y_i \ \ _R(y_i)} \ = \ [y_i \geq \ _R(y_i)] \qquad [y_i \geq s_i] \ + \ [s_i \geq \ _R(y_i)]$$

Obviously $\lim_i y_i = r$ if and only if $\lim_i t_i = 0$, $\lim_i s_i = r$ and so $\lim_i s_i = r$. Hence :

$$\lim_i [y_i \geq s_i] \ = \ [r \geq r\ ]$$

By hypothesis, $R\ < S$ being (b)-regular the condition $(b\ ^R\ )$ holds with respect to $R\ < S$, up to taking a subsequence if necessary, such that $\lim_i [s_i \geq \ _R(y_i)]$ exists in $\mathbb{G}^1_N$, we have :

$$\lim_i [s_i \geq \ _R(y_i)] \qquad \lim_i T_{s_i} S$$

Every $[y_i \geq s_i]$ $\quad [s_i \geq \ _R(y_i)]$ being orthogonal, then

$$\lim_i \quad [y_i \geq s_i] + [s_i \geq \ _R(y_i)] \ \ = \lim_i [y_i \geq s_i] \ + \ \lim_i \ \ [s_i \geq \ _R(y_i)]$$

and by Theorem 3.3, since the strati ed submersion $\ _{\mathcal{W}} : \mathcal{W} \qquad \mathcal{W}$ satis es condition (D) at $r \quad R < S$ then $\lim_i T_{s_i} S \quad \lim_i \mathcal{D}_{S\ Y}(y_i)$. Therefore one nds :

$$\lim_i \overline{y_i \ \ _R(y_i)} \qquad \lim_i \quad [y_i \geq s_i] \ + \ [s_i \geq \ _R(y_i)] \ \ =$$

$$= \ \lim_i [y_i{\geq}s_i] \ + \ \lim_i [s_i{\geq} \ _R(y_i)] \qquad \lim_i \ \ [y_i{\geq}s_i] + \lim_i \ \mathcal{D}_{S\ Y}(y_i) =$$

and nally, again since $[y_{ii} \geq s_i]$ $\quad \mathcal{D}_{S\ Y}(y_i)$ are orthogonal for every i one concludes :

$$= \ \lim_i \quad [y_i \geq s_i] \ + \ \ \mathcal{D}_{S\ Y}(y_i) \qquad \lim_i T_{y_i} Y$$

That is $R\ < Y = C^o_S\ (S)$ satis es the condition $(b\ ^R\ )$ at $r \quad R$.

Proof. 2) $\quad$ 1). The (b)-regularity always implies the (a)-regularity **[19, 3]**. ☐

We nd then the following equivalent version of Goresky s result Proposition 2.1 :

Theorem 3. 4. *Let $\mathcal{W}$ be a Whitney strati cation in $S^1_X = \mathbb{R}^l \quad S^{k\ \ 1}$ which strati es the canonical projection $\ _X : S^1_X = \mathbb{R}^l \quad S^{k\ \ 1} \qquad X = \mathbb{R}^l \quad 0^k$ and let $\mathcal{W} \ = \ \ _X(\mathcal{W}\,)$.*

If $\ _{\mathcal{W}} : \mathcal{W} \quad S^1_X \quad \mathcal{W} \ = \ \ _X(\mathcal{W}\,) \quad X$ satis es the condition (D), then :

1) *The closed cone $C_{\mathcal{W}}\ (\mathcal{W}\,) = \ \ tp + (1 \geq t) \ \ (p) \quad p \quad \mathcal{W} \quad t \quad [0\ \ 1]$ is (a)-regular.*

2) *The closed cone $C_{\mathcal{W}}\ (\mathcal{W}\,) = \ \ tp + (1 \geq t) \ \ (p) \quad p \quad \mathcal{W} \quad t \quad [0\ \ 1]$ is (b)-regular.*

Proof. Every incidence relation in $C_{\mathcal{W}}\ (\mathcal{W}\,)$ comes from some strata $R < S$ of $\mathcal{W}$ in a cone $C_{R\ \ S}\,(R \supseteq S) \quad C_{\mathcal{W}}\ (\mathcal{W}\,)$ as treated in Proposition 3.5, Corollary 3.1 and Proposition 3.6.

By Proposition 3.5, all incidence relations on $C_{R\ \ S}\,(R \supseteq S)$ are (a)- and (b)-regular except possibly for the pairs $R\ < C^o_S\ (S)$.

Since by hypothesis $_{\mathcal{W}} : \mathcal{W} \quad S^1_X \quad \mathcal{W} \;=\; _X(\mathcal{W}) \quad X$ satis es the condition (D), every pair $R \; < C^o_S \; (S)$ is (a)-regular by Corollary 3.1 and so also (b)-regular by Propostion 3.6. $\square$

We also nd, when $\mathcal{W}$ and $\mathcal{W}$ are Whitney triangulations (or cellularisations), the following important corollary which is helpful as an approach to Conjectures 1.1 and 1.2. :

Corollary 3. 2. *If $\mathcal{W}$ and $\mathcal{W}$ are Whitney triangulations (resp. cellularisations) of compact sets $W \quad S_X(1)$ and $W \quad X$ such that $_{X\,\mathcal{W}} : \mathcal{W} \quad \mathcal{W}$ is a simplicial (resp. cellular) map, then the strati ed closed cone $C_{\mathcal{W}} \; (\mathcal{W}\,)$ is a Whitney cellularisation of $C_W \; (W\,)$.*

Proof. Since $_{X\,\mathcal{W}} : \mathcal{W} \quad \mathcal{W}$ is a simplicial (resp. cellular) map, thanks to Example 2.4 it satis es Condition (D) and so the closed cone $C_{\mathcal{W}} \; (\mathcal{W}\,)$ is (b)-regular thanks to Theorem 3.4. $\square$

Condition (D) for $_{\mathcal{W}} : \mathcal{W} \quad \mathcal{W}$ is however su cient for (b) regularity but not necessary :

Example 3. 5. Let us consider a quarter of the Whitney umbrella :

$$C_{\mathcal{W}} \; (\mathcal{W}\,) = \quad (x \;\; y \;\; z) \quad \mathbb{R}^3 \quad yz^2 = x^2 \quad x \quad 0 \quad z \quad 0$$

where $\mathcal{W} \; = R \leq S$ and $\mathcal{W} \; = R \; \leq S$ are strati ed by :

$R = \;\; (0 \;\; 0 \;\; 1) \;\; < S =$ half parabola $\quad S_X(1)$;
$R \; = \;\; (0 \;\; 0 \;\; 0) \;\; < S \; = \;\; 0 \quad\; [0 + \quad [\quad 0 \quad\; X = \;\; 0 \quad \mathbb{R} \quad 0$.

Then as in Example 2.1, $_{\mathcal{W}} : \mathcal{W} \quad \mathcal{W}$ does not satisfy condition (D), but $R \; = \;\; 0$ is a point, so $R \; < Y = C^o_S \; (S)$ is automatically (a) regular and easily also (b)-regular. $\square$

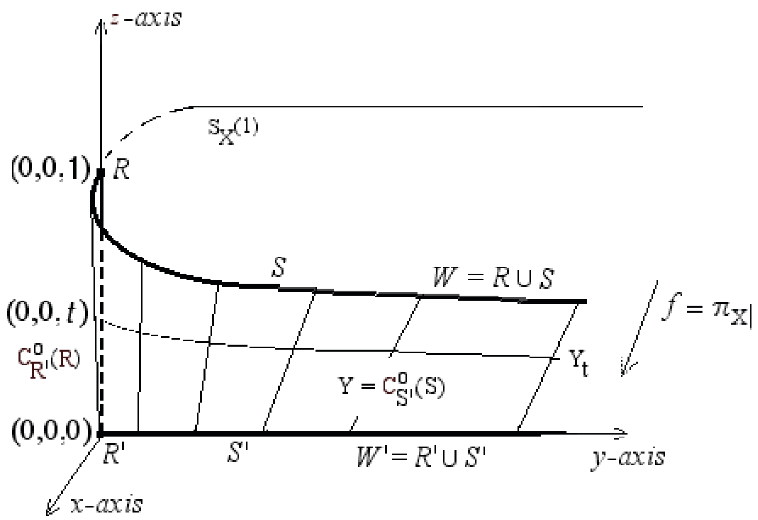

Figure 4

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Journal *of* Singularities
Volume 2 (2010), 160-169

Proc. of Singularities in Aarhus
received 1 December 2009
DOI: 10.5427/jsing.2010.2j

SINGULARITIES OF ONE-PARAMETER PEDAL UNFOLDINGS OF SPHERICAL PEDAL CURVES

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Abstract. In this paper, we present the concept of one-parameter pedal unfoldings of a pedal curve in the unit sphere S^2, and we classify their generic singularities with respect to $\mathcal{A}$-equivalence.

1. Introduction

Let I be an open interval containing zero, and let S^2 be the unit sphere in Euclidean space $\mathbb{R}^3$. A C^∞ map $\mathbf{r} : I \to S^2$ is called a *spherical unit speed curve* if $\left\|\frac{d\mathbf{r}}{ds}(s)\right\|$ is 1 for any $s \in I$. For a given spherical unit speed curve $\mathbf{r} : I \to S^2$, we put

$$\mathbf{t}(s) = \frac{d\mathbf{r}}{ds}(s),\ \mathbf{n}(s) = \mathbf{r}(s) \times \mathbf{t}(s),$$

where $\mathbf{r}(s) \times \mathbf{t}(s)$ denotes the vector product of $\mathbf{r}(s)$ and $\mathbf{t}(s)$. The construction clearly shows that the vector $\mathbf{t}(s)$ is perpendicular to the vector $\mathbf{r}(s)$ and that the vector $\mathbf{n}(s)$ is perpendicular to both $\mathbf{r}(s)$ and $\mathbf{t}(s)$. The map $\mathbf{n} : I \to S^2$ is called the *spherical dual* of $\mathbf{r}$; the singularities of spherical dual curves are Legendrian singularities that are relatively well investigated [1, 2, 3, 4, 5, 21].

For a point $P \in S^2$, let E_P denote the set $\{X \in S^2 \mid P \cdot X = 0\}$, where $P \cdot X$ denotes the scalar product of P and X. For a given spherical unit speed curve $\mathbf{r} : I \to S^2$, consider a point P of $S^2 - \{\pm\mathbf{n}(s) \mid s \in I\}$, where $\mathbf{n}$ is the spherical dual of $\mathbf{r}$. The *spherical pedal curve relative to the point* P for a given spherical unit speed curve $\mathbf{r} : I \to S^2$ is a curve obtained by mapping $s \in I$ to the nearest point in $E_{\mathbf{n}(s)}$ from P. The pedal curve relative to P for $\mathbf{r}$ is denoted by $ped_{\mathbf{r},P}$, and the point P is called the *pedal point of* the pedal curve $ped_{\mathbf{r},P}$. Note that all points in $E_{\mathbf{n}(s)}$ are equidistant from $\pm\mathbf{n}(s)$; hence, the point P must lie outside $\{\pm\mathbf{n}(s) \mid s \in I\}$ to satisfy the definition of $ped_{\mathbf{r},P}$. The classification of singularities of spherical pedal curves can be found in literature [17, 18, 19].

Suppose that the location of the pedal point P moves smoothly, depending on one-parameter $\lambda \in J$, where J is an open interval containing zero in $\mathbb{R}$. In other words, suppose that there exist an open interval J containing zero and a C^∞ immersion $P : J \to S^2$. Then, the *pedal unfolding of the pedal curve* $ped_{\mathbf{r},P(0)}$ can be defined as the map $Un\text{-}ped_{\mathbf{r},P} : I \times J \to S^2 \times J$, given by

$$Un\text{-}ped_{\mathbf{r},P}(s,\lambda) = (ped_{\mathbf{r},P(\lambda)}(s), \lambda).$$

Two C^∞ map-germs $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ are said to be $\mathcal{A}$-*equivalent* if there exist germs of C^∞-diffeomorphisms $h_1 : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ and $h_2 : (\mathbb{R}^p, 0) \to (\mathbb{R}^p, 0)$ such that $f \circ h_1 = h_2 \circ g$. For a spherical unit speed curve germ $\mathbf{r} : (I, 0) \to S^2$, we put $\kappa(s) = \mathbf{n}(s) \cdot \mathbf{t}'(s)$, where $\mathbf{t}'$ denotes

2010 *Mathematics Subject Classification.* 58K40, 58K50.
Key words and phrases. spherical pedal curve, pedal unfolding, cross-cap, $S_k^\pm$ singularity, Chen-Matumoto-Mond singularity, cuspidal edge, cuspidal cross-cap, cuspidal $S_k^\pm$ singularity, cuspidal Chen-Matumoto-Mond singularity.

TABLE 1. Normal forms of $\mathcal{A}$-simple monogerms $(\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ ([15])

Germ	*Name*
$f(s, \lambda) = (s, s^2, \lambda)$	*Immersion*
$f(s, \lambda) = (s^3 + \lambda s, s^2, \lambda)$	*Cross-cap* (S_0)
$f(s, \lambda) = (s^3 \pm \lambda^{k+1} s, s^2, \lambda)$, $(k \geq 1)$	$S_k^{\pm}$
$f(s, \lambda) = (\lambda^2 s \pm s^{2k+1}, s^2, \lambda)$, $(k \geq 2)$	$B_k^{\pm}$
$f(s, \lambda) = (\lambda s^3 \pm \lambda^k s, s^2, \lambda)$, $(k \geq 3)$	$C_k^{\pm}$
$f(s, \lambda) = (\lambda^3 s + s^5, s^2, \lambda)$	F_4
$f(s, \lambda) = (\lambda s + s^{3k-1}, s^3, \lambda)$, $(k \geq 2)$	H_k

the derivative of $\mathbf{t}$. Then, the point $\mathbf{r}(0)$ is called the *inflection point* (resp., *ordinary inflection point*) if $\kappa(0) = 0$ holds (resp., $\kappa(0) = 0$ and $\kappa'(0) \neq 0$ hold). For any $k \geq 0$, a C^∞ immersed curve germ $P : (J, 0) \to S^2$ is said to *have* $(k+1)$*-point contact with* $\mathbf{r} : (I, 0) \to S^2$ at $P(0) = \mathbf{r}(0)$ if $P(0) = \mathbf{r}(0)$, $F \circ P(0) = (F \circ P)'(0) = \cdots = (F \circ P)^{(k)}(0) = 0$, and $(F \circ P)^{(k+1)}(0) \neq 0$ hold for any neighbourhood U of $\mathbf{r}(0)$ and any non-singular C^∞ function $F : U \to \mathbb{R}$ such that $F \circ \mathbf{r}(s) = 0$ (for details on $(k+1)$-point contact, see [5]). It can be clearly seen that a C^∞ immersed curve germ $P : (J, 0) \to S^2$ has 1-point contact with $\mathbf{r} : (I, 0) \to S^2$ at $P(0) = \mathbf{r}(0)$ if and only if P and $\mathbf{r}$ are transverse at $P(0) = \mathbf{r}(0)$.

Theorem 1. *Let I, J be open intervals containing $0 \in \mathbb{R}$, and let $\mathbf{r} : I \to S^2$ be a spherical unit speed curve such that $\mathbf{r}(0)$ is not an inflection point. Furthermore, let $P : J \to S^2$ be a C^∞ immersion. Then, the following hold:*

(1) *The germ of pedal unfolding $Un\text{-}ped_{\mathbf{r},P} : (I \times J, (0, 0)) \to S^2 \times J$ is immersive if and only if $P(0) \neq \mathbf{r}(0)$.*
(2) *The germ of pedal unfolding $Un\text{-}ped_{\mathbf{r},P} : (I \times J, (0, 0)) \to S^2 \times J$ is $\mathcal{A}$-equivalent to the cross-cap in Table 1 if und only if $P(0) = \mathbf{r}(0)$ and $P, \mathbf{r}$ are transverse at $P(0) = \mathbf{r}(0)$.*
(3) *The germ of pedal unfolding $Un\text{-}ped_{\mathbf{r},P} : (I \times J, (0, 0)) \to S^2 \times J$ is $\mathcal{A}$-equivalent to $S_k^{\pm}$ in Table 1 if and only if $P(0) = \mathbf{r}(0)$ and P has $(k+1)$-point contact with $\mathbf{r}$ at $0 \in J$ $(k \geq 1)$.*
(4) *The $\mathcal{A}$-equivalence classes of map-germs $B_k^{\pm}, C_k^{\pm}, F_4$, and H_k in Table 1 can never be realized as singularities of the pedal unfolding $Un\text{-}ped_{\mathbf{r},P}$.*
(5) *The germ of pedal unfolding $Un\text{-}ped_{\mathbf{r},P} : (I \times J, (0, 0)) \to S^2 \times J$ is $\mathcal{A}$-equivalent to the cuspidal edge in Table 2 if and only if $P(0) = \mathbf{r}(0)$ and $(P(J), P(0))$ coincides with $(\mathbf{r}(I), \mathbf{r}(0))$ as set-germs.*

If k is even, then it can be clearly seen that S_k^+ is $\mathcal{A}$-equivalent to S_k^- [15]. On the other hand, S_k^+ is not $\mathcal{A}$-equivalent to S_k^- if k is odd. Figure 2 shows that the curvature of $\mathbf{r}$ at zero is greater than the curvature of P at zero if and only if the pedal unfolding $Un\text{-}ped_{\mathbf{r},P}$ is $\mathcal{A}$-equivalent to S_k^-. Since $S_1^{\pm}$ has been investigated independently in [6], it is reasonable to classify the $\mathcal{A}$-equivalence class of $S_1^{\pm}$ as *Chen-Matumoto-Mond singularity.*

Theorem 2. *Let I, J be open intervals containing $0 \in \mathbb{R}$, and let $\mathbf{r} : I \to S^2$ be a spherical unit speed curve such that $\mathbf{r}(0)$ is an ordinary inflection point. Furthermore, let $P : J \to S^2$ be a C^∞ immersion. Then, the following hold:*

(1) *The germ of pedal unfolding $Un\text{-}ped_{\mathbf{r},P} : (I \times J, (0, 0)) \to S^2 \times J$ is $\mathcal{A}$-equivalent to the cuspidal edge in Table 2 if and only if $P(0) \notin E_{\mathbf{n}(0)}$.*

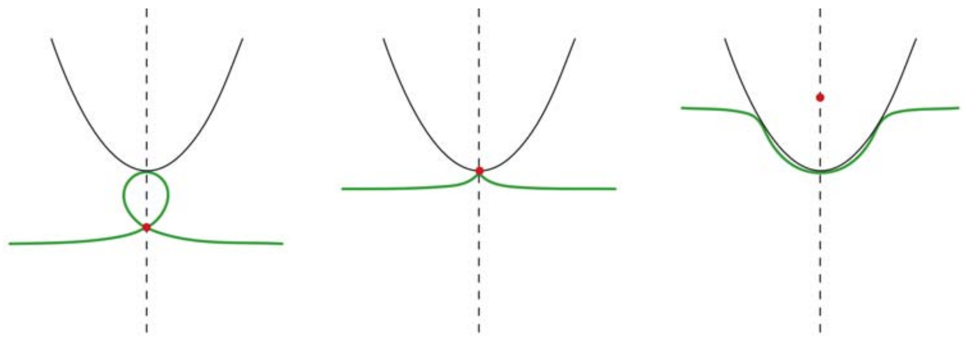

FIGURE 1. Cross-cap. Left: $\lambda = -\varepsilon$, Center: $\lambda = 0$, Right: $\lambda = \varepsilon$.

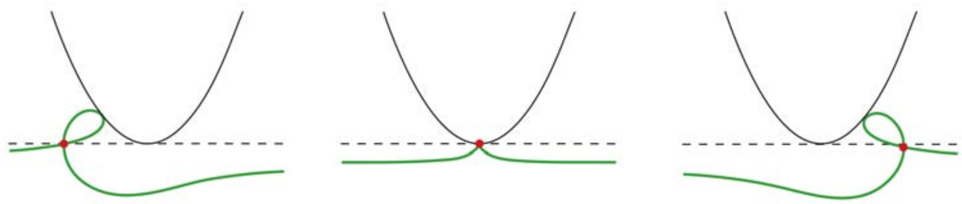

FIGURE 2. S_1^-. Left: $\lambda = -\varepsilon$, Center: $\lambda = 0$, Right: $\lambda = \varepsilon$.

TABLE 2.

Germ	*Name*
$g(s,\lambda) = (s^3, s^2, \lambda)$	*Cuspidal edge*
$g_0^+(s,\lambda) = (s^5 + \lambda s^3, s^2, \lambda)$	*Cuspidal cross-cap (Cuspidal S_0)*
$g_k^\pm(s,\lambda) = (s^5 \pm \lambda^{k+1} s^3, s^2, \lambda),\ (k \geq 1)$	*Cuspidal $S_k^\pm$*

(2) *The germ of pedal unfolding $Un\text{-}ped_{\mathbf{r},P} : (I \times J, (0,0)) \to S^2 \times J$ is $\mathcal{A}$-equivalent to the cuspidal cross-cap in Table 2 if und only if $P(0) \in E_{\mathbf{n}(0)} - \{\mathbf{r}(0)\}$ and P is transverse to $E_{\mathbf{n}(0)}$ at $P(0)$.*

(3) *The germ of pedal unfolding $Un\text{-}ped_{\mathbf{r},P} : (I \times J, (0,0)) \to S^2 \times J$ is $\mathcal{A}$-equivalent to cuspidal $S_k^\pm$ $(k \geq 1)$ in Table 2 if and only if $P(0) \in E_{\mathbf{n}(0)} - \{\mathbf{r}(0)\}$ and P has $(k+1)$-point contact with $E_{\mathbf{n}(0)}$ $(k \geq 1)$.*

As in the case of $S_k^\pm$ singularities, it can be clearly seen that cuspidal S_k^+ singularity is $\mathcal{A}$-equivalent to cuspidal S_k^- singularity if k is even. On the other hand, cuspidal S_k^+ singularity is not $\mathcal{A}$-equivalent to cuspidal S_k^- singularity if k is odd. Figure 4 shows that for a sufficiently small positive real number ε, there exists a positive real number δ such that the union of tangent lines $\cup_{s\in(-\varepsilon,\varepsilon)} E_{\mathbf{n}(s)}$ contains the images $P((-\delta,\delta))$ if and only if the map-germ $Un\text{-}ped_{\mathbf{r},P}$: $(I \times J, (0,0)) \to S^2 \times J$ is $\mathcal{A}$-equivalent to cuspidal S_k^- singularity. Since map-germ g_0^+ singularity

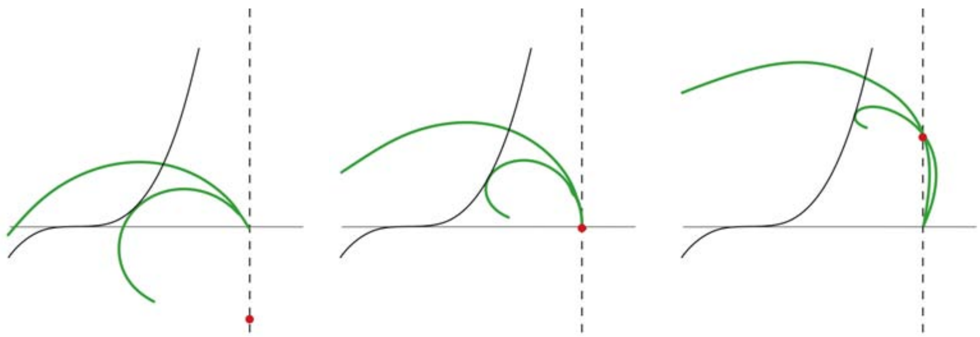

FIGURE 3. Cuspidal cross-cap. Left: $\lambda = -\varepsilon$, Center: $\lambda = 0$, Right: $\lambda = \varepsilon$.

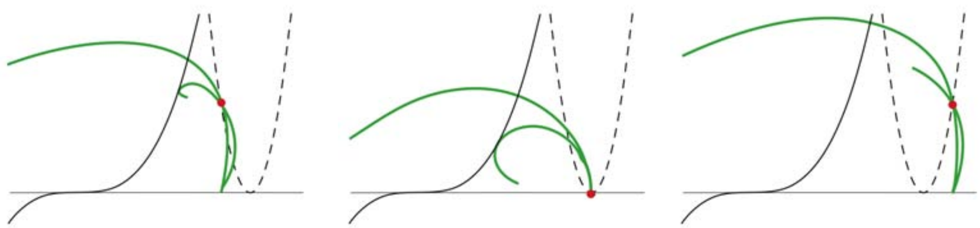

FIGURE 4. Cuspidal S_1^-. Left: $\lambda = -\varepsilon$, Center: $\lambda = 0$, Right: $\lambda = \varepsilon$.

is known as the normal form of the *cuspidal cross-cap* (see [11]), it is reasonable to classify the $\mathcal{A}$-equivalence class of the map-germ $g_{k,\pm}$ (resp., $g_{1,\pm}$) as *cuspidal $S_k^\pm$ singularity* (resp., *cuspidal Chen-Matumoto-Mond singularity*).

It can be clearly seen that the cuspidal edge, cuspidal cross-cap, and cuspidal $S_k^\pm$ are not finitely $\mathcal{A}$-determined (but finitely $\mathcal{K}$-determined) by the Mather-Gaffney geometric characterization of finite determinacy, even though $S_k^\pm$ singularity is $(k+2)$-$\mathcal{A}$-determined [15] (for the definition of finite determinacy and Mather-Gaffney geometric characterization, see [23]). Thus, in order to prove Theorems 1 and 2 in a unified manner, it is difficult to directly use the standard techniques of the finite determinacy theory developed in [8, 9, 10, 13, 14, 15, 20, 23].

On the other hand, Saji succeeded in obtaining simple criteria for Chen-Matumoto-Mond singularity and cuspidal $S_k^\pm$-singularities [22]. Although Saji's criteria are useful, the criteria for $S_k^\pm$ singularities ($k \geq 2$) have not been provided by him; therefore, Saji's criteria are not suited to our purpose. In this study, we plan to develop a unified method for proving Theorems 1 and 2; hence, we adopt a recognition criterion for map-germs that appear as singularities of pedal unfoldings. It is important to note that this criterion has already been presented in a suitable form in [15].

The preliminary work required to prove Theorems 1 and 2 is presented in Section 2. Theorems 1 and 2 are proved in Sections 3 and 4, respectively.

The author would like to extend his gratitude to Y. Sakemi for providing useful graphics.

2. Preliminaries

2.1. **Spherical pedal curves.** Let I, S^2, and $\mathbf{r} : I \to S^2$ be an interval containing zero, the unit sphere in $\mathbb{R}^3$, and a spherical unit speed curve respectively. Furthermore, let $\mathbf{t} : I \to S^2$, $\mathbf{n} : I \to S^2$ be map-germs, as described in Section 1. Then, we have the following Serret-Frenet type formula.

Lemma 2.1 ([17])**.**

$$\begin{pmatrix} \mathbf{r}'(s) \\ \mathbf{t}'(s) \\ \mathbf{n}'(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & \kappa(s) \\ 0 & -\kappa(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{r}(s) \\ \mathbf{t}(s) \\ \mathbf{n}(s) \end{pmatrix}.$$

By Lemma 2.1, the dual curve germ $\mathbf{n} : (I, 0) \to S^2$ is non-singular at 0 if and only if $\kappa(0) \neq 0$. By using Lemma 2.1 recursively, we obtain the following:

Lemma 2.2. (1) *Suppose that $\kappa(0) \neq 0$. Then, the properties $\mathbf{r}(0) \cdot \mathbf{n}'(0) = 0$, $\mathbf{r}(0) \cdot \mathbf{n}''(0) \neq 0$, and $\mathbf{t}(0) \cdot \mathbf{n}'(0) \neq 0$ hold.*
(2) *Suppose that $\kappa(0) = 0$ and $\kappa'(0) \neq 0$. Then, the properties $\mathbf{r}(0) \cdot \mathbf{n}'(0) = \mathbf{r}(0) \cdot \mathbf{n}''(0) = 0$, $\mathbf{r}(0) \cdot \mathbf{n}^{(3)}(0) \neq 0$, $\mathbf{t}(0) \cdot \mathbf{n}'(0) = 0$, and $\mathbf{t}(0) \cdot \mathbf{n}''(0) \neq 0$ hold.*

Let P be a point of $S^2 - \{\pm \mathbf{n}(s) \mid s \in I\}$.

Lemma 2.3 ([17])**.** *The pedal curve of $\mathbf{r}$ relative to the pedal point P is given by the following expression:*

$$ped_{\mathbf{r},P}(s) = \frac{1}{\sqrt{1 - (P \cdot \mathbf{n}(s))^2}}(P - (P \cdot \mathbf{n}(s))\mathbf{n}(s)).$$

Let Ψ_P be the C^∞ map from $S^2 - \{\pm P\}$ to S^2, given by

$$\Psi_P(X) = \frac{1}{\sqrt{1 - (P \cdot X)^2}}(P - (P \cdot X)X).$$

The map Ψ_P, which has been introduced and used in [17, 18, 19] (the hyperbolic version of Ψ_P has been introduced and investigated independently in [12]), has the following distinctive properties :

(1) $X \cdot \Psi_P(X) = 0$ for any $X \in S^2 - \{\pm P\}$.
(2) $\Psi_P(X) \in \mathbb{R}P + \mathbb{R}X$ for any $X \in S^2 - \{\pm P\}$.
(3) $P \cdot \Psi_P(X) > 0$ for any $X \in S^2 - \{\pm P\}$.

By property 3, $\Psi_P(S^2 - \{\pm P\})$ lies inside the open hemisphere centered at P. By properties 1 and 2, $\Psi_P(E_P) = P$. Let the open hemisphere centered at P be denoted by H_P, and put $B_P = \pi(S^2 - \{\pm P\})$, where $\pi : S^2 \to P^2(\mathbb{R})$ is the canonical projection. Since $\Psi_P(X) = \Psi_P(-X)$, the map Ψ_P canonically induces the map $\widetilde{\Psi}_P : B_P \to H_P$. Then, by Lemma 2.3, $ped_{\mathbf{r},P}$ is factored into three maps as follows:

$$ped_{\mathbf{r},P}(s) = \widetilde{\Psi}_P \circ \pi \circ \mathbf{n}(s).$$

Let $p : B \to \mathbb{R}^2$ be the blow up centered at the origin in $\mathbb{R}^2$.

Lemma 2.4 ([17])**.** *Let P be a point of S^2. Then, there exist C^∞ diffeomorphisms $h_1 : B_P \to B$ and $h_2 : H_P \to \mathbb{R}^2$ such that the equality $h_2 \circ \widetilde{\Psi}_P = p \circ h_1$ holds, and the set $\pi(E_P)$ is mapped to the exceptional set of p by h_1.*

2.2. **Criterion for recognition problem due to Mond.** Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation of the form $T(s, \lambda) = (-s, \lambda)$. Two C^∞ function germs $p_1, p_2 : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ are said to be $\mathcal{K}^T$-*equivalent* if there exist a germ of C^∞ diffeomorphism $h : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ of the form $h \circ T = T \circ h$ and a C^∞ function-germ $M : (\mathbb{R}^2, (0, 0)) \to \mathbb{R}$ of the form $M \circ T = M$, $M(0, 0) \neq 0$ such that $p_1 \circ h(s, \lambda) = M(s, \lambda)p_2(s, \lambda)$ ([15]).

Theorem 3 ([15]). *Two C^∞ map-germs of the following form*

$$f_i(s, \lambda) = (sp_i(s^2, \lambda), s^2, \lambda) \quad \textit{where } p_i(s^2, \lambda) \notin m_2^\infty, \quad (i = 1,\ 2)$$

are $\mathcal{A}$-equivalent if and only if the function-germs $p_i(s^2, \lambda)$ are $\mathcal{K}^T$-equivalent.

Note that Theorem 3 provides a criterion for the $\mathcal{A}$-equivalence of C^∞ map-germs of the forms $(s, \lambda) \mapsto (\varphi(s, \lambda), s^2, \lambda)$ ($\varphi : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ is a C^∞ function-germ) on the basis of the Malgrange preparation theorem (for the Malgrange preparation theorem, see [4, 23]).

3. Proof of Theorem 1

Since $\mathbf{r}(0)$ is not an inflection point, the dual germ $\mathbf{n} : (I, 0) \to S^2$ is a C^∞ immersive germ.

Proof of assertion 1 of Theorem 1.

Suppose that $P(0)$ does not belong to $E_{\mathbf{n}(0)}$. Then, by Lemma 2.4, the restriction $\Psi_{P(0)}|_{S^2-\{\pm P(0)\}-E_{P(0)}}$ is C^∞ immersive. Thus, by Lemma 2.3, the map-germ $ped_{\mathbf{r},P(0)} : (I, 0) \to S^2$ is also C^∞ immersive. Therefore, the map-germ $Un\text{-}ped_{\mathbf{r},P} : (I \times J, (0, 0)) \to S^2 \times J$ is also C^∞ immersive. Next, suppose that $P(0) \in E_{\mathbf{n}(0)} - \mathbf{r}(0)$. Then, the image of the dual $\mathbf{n}$ and $E_{P(0)}$ intersect transeversely at $\mathbf{n}(0)$. Thus, by Lemmata 2.3 and 2.4, the map-germ $ped_{\mathbf{r},P(0)} : (I, 0) \to S^2$ is C^∞ immersive. Therefore, the map-germ $Un\text{-}ped_{\mathbf{r},P} : (I \times J, (0, 0)) \to S^2 \times J$ is also C^∞ immersive.

Conversely, suppose that the map-germ $Un\text{-}ped_{\mathbf{r},P} : (I \times J, (0, 0)) \to S^2 \times J$ is C^∞ immersive. Then, in particular, the map-germ $ped_{\mathbf{r},P(0)} : (I, 0) \to S^2$ is also C^∞ immersive. In order to conclude the proof of assertion 1 of Theorem 1, it is sufficient to show that the assumption $P(0) = \mathbf{r}(0)$ implies a contradiction. The assumption $P(0) = \mathbf{r}(0)$ implies that the image of $\mathbf{n}$ is tangent to $E_{P(0)}$ at $\mathbf{n}(0)$. By Lemma 2.4, the map-germ $ped_{\mathbf{r},P(0)} : (I, 0) \to S^2$ must be singular; this is a contradiction. □

Proof of assertion 5 of Theorem 1.

Suppose that both $P(0) = \mathbf{r}(0)$ and $(P(J), P(0)) = (\mathbf{r}(I), \mathbf{r}(0))$ as set-germs hold. Then, for any $\lambda \in J$, $ped_{\mathbf{r},P(\lambda)} : (I, 0) \to S^2$ is $\mathcal{A}$-equivalent to the ordinary cusp $s \mapsto (s^3, s^2)$ by [17] (also, see [19]). Thus, by using the Malgrange preparation theorem and Theorem 3, the map-germ $Un\text{-}ped_{\mathbf{r},P} : (I \times J, (0, 0)) \to S^2 \times J$ is $\mathcal{A}$-equivalent to the cuspidal edge $(s, \lambda) \mapsto (s^3, s^2, \lambda)$.

Conversely, suppose that the map-germ $Un\text{-}ped_{\mathbf{r},P} : (I \times J, (0, 0)) \to S^2 \times J$ is $\mathcal{A}$-equivalent to the cuspidal edge. Then, in particular, for any sufficiently small $\lambda_0 \in J$, there exists a sufficiently small $s_0 \in I$ such that the map-germ $ped_{\mathbf{r},P(\lambda_0)} : (I, s_0) \to S^2$ is singular. Since $\mathbf{r}(0)$ is not an inflection point, by Lemma 2.4, $E_{P(\lambda_0)} = S^2 \cap (\mathbb{R}\mathbf{t}(s_0) + \mathbb{R}\mathbf{n}(s_0))$. Therefore, $P(\lambda_0) = \mathbf{r}(s_0)$. □

Proof of assertions 2 and 3 of Theorem 1.

By composing an appropriate rotation without the loss of generality, it can be assumed that $\mathbf{r}(0) = (0, 1, 0)$, $\mathbf{t}(0) = (0, 0, 1)$, $\mathbf{n}(0) = (-1, 0, 0)$. For a point Q of S^2, put $H(Q) = \{X \in S^2 \mid Q \cdot X \geq 0\}$, and let $\alpha_{\mathbf{n}(0)} : H(\mathbf{n}(0)) - E_{\mathbf{n}(0)} \to \{-1\} \times \mathbb{R}^2$ be the central projection relative to $\mathbf{n}(0)$. Then, by Lemma 2.2, the germ of composition $\alpha_{\mathbf{n}(0)} \circ \mathbf{n}$ is of the form

$$\alpha_{\mathbf{n}(0)} \circ \mathbf{n}(s) = \big(s + \varphi_1(s), s^2 + \varphi_2(s)\big) \quad (\varphi_j(s) = o(s^j)).$$

Since $\varphi_2(s) = o(s^2)$, the map-germ given by $h\left(s\sqrt{1+\frac{\varphi_2(s)}{s^2}}\right) = s$ is a well-defined germ of local C^∞ diffeomorphism. Thus, there exists a C^∞ map-germ $\widetilde{\varphi}_1 : (I, 0) \to \mathbb{R}$ such that

$$\alpha_{\mathbf{n}(0)} \circ \mathbf{n} \circ h(s) = \left(s + \widetilde{\varphi}_1(s), s^2\right) \quad (\widetilde{\varphi}_1(s) = o(s)).$$

Let $\alpha_{P(0)} : H(P(0)) - E_{P(0)} \to \mathbb{R} \times \{1\} \times \mathbb{R}$ be the central projection relative to $P(0)$. By the form mentioned above and Lemma 2.4, the germ of composition $\alpha_{P(0)} \circ ped_{\mathbf{r},P(0)}$ is $\mathcal{A}$-equivalent to a map-germ of the following form:

$$s \mapsto \left((s + \widetilde{\varphi}_1(s))s^2, s^2\right).$$

Next, we investigate the influence of moving the pedal points $P(\lambda)$. Suppose that $P(0) = \mathbf{r}(0)$ and P has $(k+1)$-point contact with $\mathbf{r}$ at $0 \in J$ $(k \geq 0)$. In other words, suppose that there exist a sufficiently small neighborhood U of $\mathbf{r}(0)$ in S^2 and a C^∞ function $F : U \to \mathbb{R}$ such that $F \circ \mathbf{r}(s) \equiv 0$ $(\forall s \in I \cap \mathbf{r}^{-1}(U))$, $F \circ P(0) = (F \circ P)'(0) = \cdots = (F \circ P)^{(k)}(0) = 0$, and $(F \circ P)^{(k+1)}(0) \neq 0$. Since $\mathbf{r} : I \to S^2$ is a unit speed curve, it can be assumed that F is non-singular provided that I (resp., U) is a sufficiently small neighborhood of 0 (resp., $\mathbf{r}(0)$). Then, there exists a sufficiently small neighborhood $\widetilde{U} \subset U$ of $\mathbf{r}(0)$ such that for any $X \in \widetilde{U}$, the integral curve of $-\mathrm{grad}(F)$ starting from X lies within $\widetilde{U}$ until it reaches the image of the unit speed curve $\mathbf{r}(I)$. Let this reaching point be denoted by $\gamma(X)$ and define the map $\Gamma : \widetilde{U} \to I$ as $\Gamma(X) = \mathbf{r}^{-1} \circ \gamma(X)$. Then, $(\widetilde{U}, (\Gamma, F))$ can be used as a chart at $\mathbf{r}(0)$ since the map $(\Gamma, F) : \widetilde{U} \to I \times \mathbb{R}$ is non-singular. By using the chart $(\widetilde{U}, (\Gamma, F))$ and by the proof of assertion 5 of Theorem 1, the germ of composition

$$(s, \lambda) \mapsto \left(\alpha_{P(0)} \circ ped_{\mathbf{r},P(\lambda)} \circ h(s), \lambda\right)$$

is $\mathcal{A}$-equivalent to a map-germ of the following form:

$$(a) \qquad (s, \lambda) \mapsto \left((s + \widetilde{\varphi}_1(s))\left(s^2 \pm F \circ P(\lambda)\right), s^2 \pm F \circ P(\lambda), \lambda\right).$$

Furthermore, by the Malgrange preparation theorem and Theorem 3, a map-germ of the form (a) must be $\mathcal{A}$-equivalent to the map-germ $f_k^\pm(s, \lambda) = \left(s\left(s^2 \pm \lambda^{k+1}\right), s^2, \lambda\right)$.

Conversely, we suppose that the germ of pedal unfolding $Un\text{-}ped_{\mathbf{r},P} : (I \times J, (0,0)) \to S^2 \times J$ is $\mathcal{A}$-equivalent to $S_k^\pm$ $(k \geq 0)$, $P(0) = \mathbf{r}(0)$ and that P does not have $(k+1)$-point contact with $\mathbf{r}$ at $0 \in J$. Then, by the proof presented above, for any positive integer ℓ, P does not have $(\ell+1)$-point contact with $\mathbf{r}$ at $0 \in J$. In particular, there exists a C^∞ immersion $\widetilde{P} : J \to S^2$ such that $\widetilde{P}$ is sufficiently near P under the Whitney C^∞ topology, and $\widetilde{P}$ has $(k+2)$-point contact with $\mathbf{r}$ at $0 \in J$. By the proof of the implication described above, it can be concluded that $S_k^\pm$ singularity is adjacent to $S_{k+1}^\pm$ singularity; however, this contradicts the adjacency diagram given in [15]. □

Proof of assertion 4 of Theorem 1.

Suppose that the map-germ $Un\text{-}ped_{\mathbf{r},P} : (I \times J, (0,0)) \to S^2 \times J$ is $\mathcal{A}$-equivalent to one of $B_k^\pm, C_k^\pm, F_4$, and H_k. Then, by assertions 1, 2, and 3 of Theorem 1, the given immersion $P : J \to S^2$ must satisfy not only $P(0) = \mathbf{r}(0)$ but also the condition that for any positive integer ℓ, P does not have $(\ell+1)$-point contact with $\mathbf{r}$ at $0 \in J$. Thus, for any positive integer ℓ, there exists a C^∞ immersion $\widetilde{P} : J \to S^2$ such that $\widetilde{P}$ is sufficiently near P under the Whitney C^∞ topology, and $\widetilde{P}$ has the $(\ell+1)$-contact with $\mathbf{r}$ at $0 \in J$. Hence, it can be concluded that one of $B_k^\pm, C_k^\pm, F_4$, and H_k singularity is adjacent to $S_\ell^\pm$ singularity for any positive integer ℓ; however, this contradicts the adjacency diagram given in [15]. □

4. Proof of Theorem 2

Since $\mathbf{r}(0)$ is an ordinary inflection point, by Lemma 2.2 and the Malgrange preparation theorem, the dual germ $\mathbf{n} : (I, 0) \to S^2$ is $\mathcal{A}$-equivalent to the ordinary cusp $s \mapsto (s^3, s^2)$.

Proof of assertion 1 of Theorem 2.

Suppose that $P(0)$ does not belong to $E_{\mathbf{n}(0)}$. Then, for any sufficiently small $\lambda_0 \in J$, $P(\lambda_0)$ lies outside $E_{\mathbf{n}(0)}$. This implies that by Lemma 2.4, the map-germ $\Psi_{P(\lambda_0)}$ at $\mathbf{n}(0)$ is non-singular. Thus, by Lemma 2.3, the map-germ $ped_{\mathbf{r},P(\lambda_0)} : (I, 0) \to S^2$ is also $\mathcal{A}$-equivalent to the ordinary cusp. Therefore, by Theorem 3, the map-germ $Un\text{-}ped_{\mathbf{r},P} : (I \times J, (0, 0)) \to S^2 \times J$ is $\mathcal{A}$-equivalent to the cuspidal edge.

Conversely, suppose that the map-germ $Un\text{-}ped_{\mathbf{r},P} : (I \times J, (0, 0)) \to S^2 \times J$ is $\mathcal{A}$-equivalent to the map-germ $g(s, \lambda) = (s^3, s^2, \lambda)$; we show that $P(0) \in E_{\mathbf{n}(0)}$ implies a contradiction under this assumption. The property $P(0) \in E_{\mathbf{n}(0)}$ implies that $\mathbf{n}(0) \in E_{P(0)}$. Since the dual germ $\mathbf{n} : (I, 0) \to S^2$ is $\mathcal{A}$-equivalent to the ordinary cusp $s \mapsto (s^3, s^2)$, by Lemma 2.4, $\mathbf{n}(0) \in E_{P(0)}$ implies that $j^3(Un\text{-}ped_{\mathbf{r},P})(0)$ is not $\mathcal{A}^3$-equivalent to $j^3g(0)$. This contradicts the assumption that $Un\text{-}ped_{\mathbf{r},P} : (I \times J, (0, 0)) \to S^2 \times J$ is $\mathcal{A}$-equivalent to the map-germ $g(s, \lambda) = (s^3, s^2, \lambda)$. □

Proof of "if" parts of assertions 2 and 3 of Theorem 1.

Since $P(0)$ belongs to $E_{\mathbf{n}(0)} - \{\mathbf{r}(0)\}$, by composing an appropriate rotation without the loss of generality, it can be assumed that $\mathbf{n}(0) = (-1, 0, 0)$ and $P(0) = (0, 0, 1)$. Let $\alpha_{\mathbf{n}(0)} : H(\mathbf{n}(0)) - E_{\mathbf{n}(0)} \to \{-1\} \times \mathbb{R}^2$ be the central projection relative to $\mathbf{n}(0)$. Then, by Lemma 2.2, the germ of composition $\alpha_{\mathbf{n}(0)} \circ \mathbf{n}$ is of the form

$$\alpha_{\mathbf{n}(0)} \circ \mathbf{n}(s) = (as^2 + bs^3 + \varphi_1(s), cs^2 + ds^3 + \varphi_2(s)),$$

where $bc \neq 0$ and $\varphi_j(s) = o(s^3)$. Since $c \neq 0$, there exists a germ of C^∞ diffeomorphism $h : (I, 0) \to (I, 0)$ such that

$$\alpha_{\mathbf{n}(0)} \circ \mathbf{n} \circ h(s) = (\widetilde{a}s^2 + \widetilde{b}s^3 + \widetilde{\varphi}_1(s), s^2),$$

where $\widetilde{b} \neq 0$ and $\widetilde{\varphi}_1(s) = o(s^3)$. Let $\alpha_{P(0)} : H(P(0)) - E_{P(0)} \to \mathbb{R}^2 \times \{1\}$ be the central projection relative to $P(0)$. By the form mentioned above and Lemma 2.4, the germ of composition $\alpha_{P(0)} \circ ped_{\mathbf{r},P(0)}$ is $\mathcal{A}$-equivalent to a map-germ of the following form:

$$s \mapsto \left((\widetilde{a}s^2 + \widetilde{b}s^3 + \widetilde{\varphi}_1(s))s^2, s^2\right).$$

Next, we investigate the influence of moving the pedal points $P(\lambda)$. Suppose that $P(0) = \mathbf{r}(0)$ and P has $(k+1)$-point contact with $E_{\mathbf{n}(0)}$ at $0 \in J$ $(k \geq 0)$. Since $E_{\mathbf{n}(0)}$ is defined by the equation $\mathbf{n}(0) \cdot X = 0$, the assumption of $(k + 1)$-point contact implies that $\mathbf{n}(0) \cdot P(0) = \mathbf{n}(0) \cdot P'(0) = \cdots = \mathbf{n}(0) \cdot P^{(k)}(0) = 0$ and $\mathbf{n}(0) \cdot P^{(k+1)}(0) \neq 0$. Then, as in the proof of assertions 2 and 3 of Theorem 1, the germ of composition

$$(s, \lambda) \mapsto (\alpha_{P(0)} \circ ped_{\mathbf{r},P(\lambda)}(s), \lambda)$$

is $\mathcal{A}$-equivalent to the germ of the following form:

$$(b) \qquad (s, \lambda) \mapsto \left(\left(\widetilde{a}s^2 + \widetilde{b}s^3 + \varphi_1(s)\right)\left(s^2 \pm \mathbf{n}(0) \cdot P(\lambda)\right), s^2 \pm \mathbf{n}(0) \cdot P(\lambda), \lambda\right).$$

Furthermore, by the Malgrange preparation theorem and Theorem 3, a map-germ of the form mentioned in (b) must be $\mathcal{A}$-equivalent to the map-germ $g_k^{\pm}(s, \lambda) = \left(s^3(s^2 \pm \lambda^{k+1}), s^2, \lambda\right)$. □

The "only if" parts of assertions 2 and 3 of Theorem 2 can be proved as follows. Put $\widetilde{g}(s,\lambda) = s^2$, $\widetilde{g}_0(s,\lambda) = s^4 + \lambda s^2$, $\widetilde{g}^{\pm}_{2i+1}(s,\lambda) = s^4 \pm \lambda^{2i+2}s^2$, and $\widetilde{g}^{+}_{2i}(s,\lambda) = s^4 + \lambda^{2i+1}s^2$. Then, it can be clearly seen that any two distinct elements of the following set are not $\mathcal{K}^T$-equivalent.

$$\left\{\widetilde{g}, \widetilde{g}_0, \widetilde{g}^{+}_1, \widetilde{g}^{-}_1, \widetilde{g}^{+}_2, \widetilde{g}^{+}_3, \widetilde{g}^{-}_3, \cdots\right\}.$$

Hence, by Theorem 3, any two distinct elements of the set of the cuspidal edge, cuspidal cross-cap, cuspidal S_1^+, cuspidal S_1^-, cuspidal S_2^+, cuspidal S_3^+, cuspidal S_3^- $\cdots$ are not $\mathcal{A}$-equivalent. Furthermore, by Theorem 3 and the form of $g_0, g_1^{\pm}, g_2^{\pm}, \cdots$ in Table 2, the following adjacency diagram is obtained.

$$(c) \qquad \cdots \longrightarrow \text{cuspidal } S_k \longrightarrow \cdots \longrightarrow \text{cuspidal } S_1 \longrightarrow \text{cuspidal } S_0.$$

Proof of "only if" parts of the assertions 2, 3 of Theorem 2.

As in the proof of the "only if" parts of assertions 2 and 3 of Theorem 1, we suppose that $Un\text{-}ped_{\mathbf{r},P} : (I \times J, (0,0)) \to S^2 \times J$ is $\mathcal{A}$-equivalent to cuspidal $S_k^{\pm}$ $(k \geq 0)$, $P(0) \in E_{\mathbf{n}(0)} - \{\mathbf{r}(0)\}$, and P does not have $(k+1)$-point contact with $E_{\mathbf{n}(0)}$ at $0 \in J$. Then, by the "if" parts of assertions 2, 3 of Theorem 2, for any non-negative integer ℓ, P does not have $(\ell+1)$-point contact with $E_{\mathbf{n}(0)}$ at $0 \in J$. In particular, for any non-negative integer ℓ, there exists a C^∞ immersion $\widetilde{P} : J \to S^2$ such that $\widetilde{P}$ is sufficiently near P under the Whitney C^∞ topology, and $\widetilde{P}$ has $(\ell+1)$-point contact with $E_{\mathbf{n}(0)}$ at $0 \in J$. Hence, it can be concluded that cuspidal $S_k^{\pm}$ singularity is adjacent to cuspidal $S_\ell^{\pm}$ singularity for any positive integer ℓ; however, this contradicts diagram (c).

Next, suppose that $Un\text{-}ped_{\mathbf{r},P} : (I \times J, (0,0)) \to S^2 \times J$ is $\mathcal{A}$-equivalent to cuspidal $S_k^{\pm}$ $(k \geq 0)$ and $P(0) = \{\mathbf{r}(0)\}$. In this case, the tangent cone of $\mathbf{n}(I)$ at $\mathbf{n}(0)$ coincides with E_P. Thus, by Lemma 2.4, $j^2(Un\text{-}ped_{\mathbf{r},P})(0)$ is not $\mathcal{A}^2$-equivalent to $j^2 g_k^{\pm}(0)$; this contradicts the assumption that $Un\text{-}ped_{\mathbf{r},P} : (I \times J, (0,0)) \to S^2 \times J$ is $\mathcal{A}$-equivalent to the map-germ $g_k^{\pm}(s,\lambda) = (s^5 \pm \lambda^{k+1}s^3, s^2, \lambda)$. $\square$

Remarks

It is possible to adopt the criteria given in [16] or an argument similar to that given in [7] to prove Theorems 1 and 2. However, the criteria in [16] are too general to be directly applied to our study, and the argument in [7] seems to be somewhat ad hoc. Thus, in order to apply them to our study, considerable preliminary work is required, the proofs of which are time-consuming and complicated. On the other hand, Theorem 3 is the most suitable criterion for our study. Moreover, the calculations with respect to $\mathcal{K}^T$-equivalence are relatively straightforward; hence, by using Theorem 3, we can prove both Theorem 1 and Theorem 2 in a coherent and unified manner.

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Journal *of* Singularities
Volume 2 (2010), 170-190

Proc. of Singularities in Aarhus
received 20 November 2009
DOI: 10.5427/jsing.2010.2k

GEOMETRY OF IRREDUCIBLE PLANE QUARTICS AND THEIR QUADRATIC RESIDUE CONICS

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Dedicated to Professor Du Plessis on his sixtieth birthday.

Abstract. Let D be an irreducible plane curve in $\mathbb{P}^2$. In this article, we first introduce a notion of a quadratic residue curve mod$D$, and study quadratic residue conics C mod an irreducible quartic curve Q. As an application, we study a dihedral cover of $\mathbb{P}^2$ with branch locus $C+Q$ and give two examples of Zariski pairs as by-products.

Introduction

In this article, we study the geometry of irreducible plane quartic Q and a smooth conic C which is tangent to Q with even order at each point in $C \cap Q$. The geometry of a smooth plane quartic and its bitangent lines is a classical object and well studied by many mathematicians from various points of view. We hope that this article adds another interesting topic to geometry of plane quartics. All varieties throughout this paper are defined over the field of complex numbers, $\mathbb{C}$. In order to explain our motivation and results on the above subject, let us start with introducing some notions and definitions.

Let Σ be a smooth projective surface. Let $f' : Z' \to \Sigma$ be a double cover of Σ, i.e., Z' is a normal surface and f' is a finite surjective morphism of degree 2. We denote its canonical resolution by $\mu : Z \to Z'$ (see [7] for the canonical resolution). Note that μ is the identity if Z' is smooth. We put $f := f' \circ \mu$. We denote the involution on Z induced by the covering transformation of f' by σ_f. The branch locus $\Delta_{f'}$ of f' is the subset of Σ consisting of points x such that f' is not locally isomorphic over x. Similarly we define the branch locus Δ_f of f. Note that $\Delta_{f'} = \Delta_f$.

Definition 0.1. Let D be an irreducible curve on Σ. We call D a splitting curve with respect to f if f^*D is of the form

$$f^*D = D^+ + D^- + E,$$

where $D^+ \neq D^-$, $\sigma_f^* D^+ = D^-$, $f(D^+) = f(D^-) = D$ and $\mathrm{Supp}(E)$ is contained in the exceptional set of μ. If the double cover $f : Z \to \Sigma$ is determined by its branch locus Δ_f, i.e., any double cover with branch locus Δ_f is isomorphic to Z' over Σ, and D is a splitting curve with respect to f, we say that "Δ_f *is a quadratic residue curve mod* D".

Remark 0.1.

- Note that if Σ is simply connected, then any double cover of Σ is determined by its branch locus.
- In our previous results on dihedral covers and their application to the study of the topology of the complements of plane curves, we see that splitting curves play important roles and that it is indispensable to know their properties of them. (see [2], [17], [18], for example). This is our first motivation to study splitting curves.
- Our terminology comes from elementary number theory. Let m be a square free positive integer, let p be an odd prime with $p \not| m$ and let $\mathcal{O}_{\mathbb{Q}(\sqrt{m})}$ be the integer ring of $\mathbb{Q}(\sqrt{m})$. It

is well known that the ideal (p) generated by p in $\mathcal{O}_{\mathbb{Q}(\sqrt{m})}$ satisfies the following properties (See [8, Proposition 13.1.3], p.190, for example):

- If m is a quadratic residue mod p, then $(p) = \mathfrak{p}_1\mathfrak{p}_2$, where $\mathfrak{p}_i$ $(i = 1, 2)$ are distinct prime ideals.
- If m is not a quadratic residue mod p, then (p) is a prime ideal.

Suppose that $f : Z \to \Sigma$ is uniquely determined by Δ_f. Likewise the Legendre symbol in elementary number theory, we here introduce a notation to describe if Δ_f is a quadratic residue mod D or not. For an irreducible curve D on Σ, we put

$$(\Delta_f/D) = \begin{cases} 1 & \text{if } \Delta_f \text{ is a quadratic residue curve mod } D \\ -1 & \text{if } \Delta_f \text{ is not a quadratic residue curve mod } D \end{cases}$$

As $\mathbb{P}^2$ is simply connected, any double cover of $\mathbb{P}^2$ is just determined by its branch locus. On the other hand, any reduced plane curve B of even degree can be the branch locus of a double cover. Hence for any irreducible plane curve D, one can consider (B/D).

In this article, we consider the case when *any point $x \in B \cap D$ is a smooth point of both B and D*. For such a case, if the intersection multiplicity at some point in $B \cap D$ is odd, then we infer that $(B/D) = -1$. This leads us to introduce a notion of *even tangential curve*.

Definition 0.2. Let D_1 and D_2 are reduced divisors on a smooth projective surface without any common irreducible component. We say that D_1 and D_2 are even tangential or D_1 (resp. D_2) is even tangential to D_2 (resp. D_1) if

(i) For $\forall P \in D_1 \cap D_2$, $P \notin \mathrm{Sing}(D_1) \cup \mathrm{Sing}(D_2)$, and
(ii) the intersection multiplicity of D_1 and D_2 at P, $I_P(D_1, D_2)$, is even for $\forall P \in D_1 \cap D_2$.

Note that we do not pay attention to $\sharp(D_1 \cap D_2)$ to define even tangential curves.

Now our basic problem can be formulated as follows:

Problem 0.1. Let B be a reduced plane curve of even degree.

(i) Find an even tangential curve D to B and determine the value of (B/D).
(ii) What can we say about the topology of $\mathbb{P}^2 \setminus (B + D)$ from the value of (B/D)?

As a first step, we consider the case when B is a smooth conic C. Suppose that D is an irreducible plane curve which is even tangential to C. We easily see the following:

- If $\deg D = 1, 2$, we have $(C/D) = 1$.
- If $\deg D = 3$, we have
  (i) $(C/D) = -1$ if D is smooth, and
  (ii) $(C/D) = 1$ if D is a nodal cubic.
  Note that there is no even tangential cuspidal cubic to C.

Hence the case of $\deg D = 4$ seems to be the first interesting case. Now let us restate our exact problems which we consider in this article:

Problem 0.2. Fix an irreducible quartic Q.

(i) Find even tangential conics C to Q and determine the value (C/Q).
(ii) Does the value (C/Q) affect the topology of $\mathbb{P}^2 \setminus (C + Q)$?

In this article, we first consider Problem 0.2 (i) and give a formula to determine (C/Q) (see Theorem 2.1). We next count the number of even tangential conics passing through a smooth point x on Q. Now our result is as follows:

Theorem 0.1. *Choose a smooth point x of Q and let l_x be the tangent line to Q at x. There exist finitely many (possibly no) even tangential conics C to Q through x and we have the following table:*

- Ξ_Q: *the set of types of singularities of* Q. *Note that* Q *has at worst simple singularities and we use the notation in* [3] *in order to describe the type of a singularity.*
- $l_x \cap Q$: *This shows how* l_x *meets* Q. *We use the following notation to describe it.*
  - *s:* $I_x(l_x, Q) = 2$ *or* 3, *and* l_x *meets* Q *transversely at other point(s).*
  - *b:* l_x *is either bitangent line through* x *or* $I_x(l_x, Q) = 4$.
  - *sb:* $I_x(l_x, Q) = 2$ *and* l_x *passes through a double point of* Q.
- ETC*: the set of even tangential conics passing through* x *and* $\sharp$ETC *denotes its cardinality.*
- QRETC*: the set of even tangential conics passing through* x *with* $(C/Q) = 1$ *and* $\sharp$QRETC *denotes its cardinality.*
- *We omit cases of* $(\Xi_Q,\, l_x \cap Q)$ *which do not occur. For example, the case of* $(\Xi_Q, l_x \cap Q) = (A_6, b)$ *is omitted, as such a case does not occur.*

No.	Ξ_Q	$l_x \cap Q$	$\sharp$ETC	$\sharp$QRETC
1	A_6	s	0	0
2	A_6	sb	0	0
3	E_6	s	0	0
4	E_6	b	0	0
5	A_5	s	1	1
6	A_5	b	1	1
7	A_5	sb	0	0
8	D_5	s	1	1
9	D_5	b	0	0
10	D_4	s	3	3
11	D_4	b	0	0
12	$A_4 + A_2$	s	0	0
13	$A_4 + A_2$	sb	0	0
14	$A_4 + A_1$	s	0	0
15	$A_4 + A_1$	b	0	0
16	$A_4 + A_1$	sb	0	0
17	$A_4 + A_1$	sb	0	0
18	$A_3 + A_2$	s	1	1
19	$A_3 + A_2$	sb	0	0
20	$A_3 + A_2$	sb	1	1
21	$A_3 + A_1$	s	2	2
22	$A_3 + A_1$	b	1	1
23	$A_3 + A_1$	sb	1	1
24	$A_3 + A_1$	sb	0	0
25	$3A_2$	s	0	0
26	$3A_2$	b	0	0
27	$2A_2 + A_1$	s	0	0
28	$2A_2 + A_1$	b	0	0
29	$2A_2 + A_1$	sb	0	0
30	$A_2 + 2A_1$	s	1	1
31	$A_2 + 2A_1$	b	0	0
32	$A_2 + 2A_1$	sb	0	0
33	$A_2 + 2A_1$	sb	1	1

No.	Ξ_Q	$l_x \cap Q$	$\sharp$ETC	$\sharp$QRETC
34	$3A_1$	s	4	4
35	$3A_1$	b	1	1
36	$3A_1$	sb	2	2
37	A_4	s	3	0
38	A_4	b	1	0
39	A_4	sb	1	0
40	A_3	s	7	1
41	A_3	b	2	0
42	A_3	sb	4	1
43	$2A_2$	s	3	0
44	$2A_2$	b	3	0
45	$2A_2$	sb	1	0
46	$A_2 + A_1$	s	6	0
47	$A_2 + A_1$	b	3	0
48	$A_2 + A_1$	sb	3	0
49	$A_2 + A_1$	sb	2	0
50	$2A_1$	s	13	1
51	$2A_1$	b	6	0
52	$2A_1$	sb	7	1
53	A_2	s	15	0
54	A_2	b	6	0
55	A_2	sb	10	0
56	A_1	s	30	0
57	A_1	b	15	0
58	A_1	sb	20	0
59	$\emptyset$	s	63	0
60	$\emptyset$	b	36	0

Note that there exist both quadratic and non-quadratic residue even tangential conics to Q for the cases 40, 42, 50 and 52. These cases are interesting when we consider Problem 0.2 (ii). In fact, we study dihedral covers of $\mathbb{P}^2$ whose branch loci are $C + Q$, and have the following result (see §3 for the notations on dihedral covers):

Theorem 0.2. *Let Q be an irreducible quartic, let C be an even tangential conic to Q and let $f_C : Z_C \to \mathbb{P}^2$ be a double cover with $\Delta_{f_C} = C$. If there exists a $\mathcal{D}_{2p}$-cover $\pi : S \to \mathbb{P}^2$ with $\Delta_\pi = C + Q$ for an odd prime $p \geq 5$, then we have the following:*

(i) $D(X/\mathbb{P}^2) = Z_C \cong \mathbb{P}^1 \times \mathbb{P}^1$, i.e., π is branched at $2C + pQ$.

(ii) $(C/Q) = 1$. Moreover, if we put $f_C^*Q = Q^+ + Q^-$, then $Q^+ \sim Q^- \sim (2, 2)$.

Conversely, if the second condition holds, then there exist $\mathcal{D}_{2n}$-covers $\pi_n : S_n \to \mathbb{P}^2$ branched at $2C + nQ$ for any $n \geq 3$.

Since both of $\deg C$ and $\deg Q$ are even, we infer that there exists a $(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$-cover of $\mathbb{P}^2$ with branch locus $C + Q$. Hence, from Theorem 0.2, we have the following corollaries:

Corollary 0.1. *If there exists a $\mathcal{D}_{2p}$-cover of $\mathbb{P}^2$ with $\Delta_\pi = C + Q$ for some odd prime $p \geq 5$, then there exists a $\mathcal{D}_{2n}$-cover $\mathbb{P}^2$ with $\Delta_\pi = C + Q$ for any $n \geq 2$.*

Corollary 0.2. (*i*) *Let p be an odd prime ≥ 5. If there exists an epimorphism from the fundamental group $\pi_1(\mathbb{P}^2 \setminus (C+Q), *)$ to $\mathcal{D}_{2p}$, then $(C/Q) = 1$ and $Q^+ \sim Q^-$.*

(*ii*) *If there exists an epimorphism $\pi_1(\mathbb{P}^2 \setminus (C+Q), *)$ to $\mathcal{D}_{2p}$, then there exists an epimorphism $\pi_1(\mathbb{P}^2 \setminus (C+Q), *)$ to $\mathcal{D}_{2n}$ for any $n \geq 2$.*

This paper consists of 5 sections. In §1, we start with preliminaries on theory of elliptic surface. We prove Theroem 0.1 in §2. In §3, we give a summary on branched Galois covers, mainly dihedral covers. We prove Theorem 0.2 in §4. In §5, we consider an application of Theorem 0.2 and give two examples of Zariski pairs.

Acknowledgement. Most of this article was written during the author's visit to Ruhr Universität Bochum under the support of SFB/TR 12. The author thanks Professor A. Huckleberry for his arrangement and hospitality. The author also thanks the organizers of the symposium "Singularities in Aarhus" for giving the author an opportunity to give a talk on the subject in this article. Finally he thanks the referee for valuable comments on the first version of this article.

1. Preliminaries on elliptic surfaces

1.1. **Elliptic surfaces.** We review some general facts on elliptic surfaces. For details, we refer to [9], [10] and [14]. Let $\varphi : \mathcal{E} \to C$ be an elliptic surface over a smooth projective curve C with a section O. Throughout this article, we always assume that

(i) φ is relatively minimal and
(ii) there exists at least one singular fiber.

Let $\mathrm{NS}(\mathcal{E})$ be the Néron-Severi group of $\mathcal{E}$ and let T_φ be the subgroup of $\mathrm{NS}(\mathcal{E})$ generated by O and all the irreducible components of fibers of φ. T_φ has a canonical basis as follows:

O, a general fiber $\mathfrak{f}$, and $\{\Theta_{v,1}, \ldots, \Theta_{v,m_v-1}\}_{v\in R_\varphi}$, where

- $R_\varphi := \{v \in C \mid \varphi^{-1}(v) \text{ is reducible.}\}$, and
- we label the irreducible components of $\varphi^{-1}(v)$ as follows: $\Theta_{v,0}, \Theta_{v,1}, \ldots, \Theta_{v,m_v-1}$, $\Theta_{v,0}O = 1$.

Let $\mathrm{MW}(\mathcal{E})$ be the Mordell-Weil group, the group of sections, of $\mathcal{E}$, O being the zero sections. Under these circumstances, we have

Theorem 1.1. [14, Theorem 1.3] *There is a natural isomorphism*

$$\mathrm{MW}(\mathcal{E}) \cong \mathrm{NS}(\mathcal{E})/T_\varphi.$$

Also in [14], a symmetric bilinear form $\langle\, , \rangle$, called the height pairing, on $\mathrm{MW}(\mathcal{E})$ is defined by using the intersection pairing as follows:

For any $s \in \mathrm{MW}(\mathcal{E})$, $\langle s, s\rangle \geq 0$ and $= 0$ if and only if s is a torsion. More explicitly, for $s_1, s_2 \in \mathrm{MW}(\mathcal{E})$, $\langle s_1, s_2\rangle$ is given by

$$\langle s_1, s_2\rangle = \chi(\mathcal{O}_\mathcal{E}) + s_1O + s_2O - s_1s_2 - \sum_{v\in R_\varphi} \mathrm{Corr}_v(s_1, s_2),$$

where $\mathrm{Corr}_v(s_1, s_2)$ is given by

$$\mathrm{Corr}_v(s_1, s_2) = (s_2\Theta_{v,1}, \ldots, s_2\Theta_{v,m_{v-1}})(-A_v^{-1}) \begin{pmatrix} s_1\Theta_{v,1} \\ \cdot \\ s_1\Theta_{v,m_v-1} \end{pmatrix},$$

and A_v is the intersection matrix $(\Theta_{v,i}\Theta_{v,j})$ $(1 \leq i, j \leq m_v - 1)$. As for explicit values of $\mathrm{Corr}_v(s_1, s_2)$, see Table 8.16 in [14].

1.2. **A "reciprocity" between sections and trisections on rational ruled surfaces.** Let Σ_d be the Hirzebruch surface of degree d (d: even positive integer). We denote its section with self-intersection number $-d$ and its fiber of the ruling by $\Delta_{0,d}$ and F_d, respectively. Let Γ_d be an irreducible curve on Σ_d such that

(1) $\Gamma_d \sim 3(\Delta_{0,d} + dF_d)$ and
(2) Γ_d has at worst simple singularities.

Let Δ be a section on Σ_d such that (i) $\Delta \sim \Delta_{0,d} + dF_d$ and (ii) Δ and Γ_d are even tangential.

Let $p'_d : S'_d \to \Sigma_d$ be the double cover with branch locus $\Delta_{0,d} + \Gamma_d$ and $\mu_d : S_d \to S'_d$ be the canonical resolution and put $p_d := p'_d \circ \mu_d$. Since $\Delta_{0,d} + \Gamma_d$ meets a general fiber $F_d \cong \mathbb{P}^1$ in 4 distinct points, one can easily see that S_d has an elliptic fibration $\varphi_d : S_d \to \mathbb{P}^1$ over $\mathbb{P}^1$. Moreover, by its construction, we infer that

(a) φ_d is relatively minimal,
(b) the preimage of $\Delta_{0,d}$ gives a section which we denote by O, and
(c) Δ gives rise to two sections s_Δ^+ and s_Δ^- of φ_d.

Let $\mathrm{MW}(S_d)$ be the group of sections , the Mordell-Weil group, of φ_d, where O is the zero element. Let $q_d : W_d \to \Sigma_d$ be a double cover with branch locus $\Delta_{0,d} + \Delta$. Note that q_d is uniquely determined by $\Delta_{0,d} + \Delta$ as Σ_d is simply connected and that $W_d \cong \Sigma_{d/2}$. Then we have

Theorem 1.2.
$$((\Delta_{0,d} + \Delta)/\Gamma_d)) = (-1)^{\varepsilon(s_\Delta^+)}$$
where, for a section $s \in \mathrm{MW}(S_d)$, $\varepsilon(s)$ is defined as follows:
$$\varepsilon(s) = \begin{cases} 0 & \exists s_o \in \mathrm{MW}(S_d) \text{ such that } s = 2s_o \\ 1 & \nexists s_o \in \mathrm{MW}(S_d) \text{ such that } s = 2s_o \end{cases}$$
Note that $\varepsilon(s_\Delta^+) = \varepsilon(s_\Delta^-)$ as $s_\Delta^+ = -s_\Delta^-$ on $\mathrm{MW}(S_d)$.

Proof. It is enough to show
$$((\Delta_{0,d} + \Delta)/\Gamma_d)) = 1 \Leftrightarrow s_\Delta^\pm \in 2\,\mathrm{MW}(S_d).$$
($\Rightarrow$) As we have seen, $W_d \cong \Sigma_{d/2}$. We choose affine open subsets $V \subset W_d (\cong \Sigma_{d/2})$, and $U \subset \Sigma_d$ as follows:

(i) Both U and V are $\mathbb{C}^2$.
(ii) We choose affine coordinates (t, u) and $(\tilde{t}, \zeta)$ of U and V, respectively, in such a way that q_d is given by
$$q_d : (\tilde{t}, \zeta) \mapsto (t, u) = (\tilde{t}, \zeta^2 + f(t)),$$
where $f(t)$ is a polynomial of degree $\leq d$. Note that with respect to these coordinates (t, u) and $(\tilde{t}, \zeta)$, $\Delta \cap U : u - f(t) = 0$, $\Delta_{0,d}$ corresponds to the section given by $u = \infty$ and the involution σ_{q_d} is given by $(\tilde{t}, \zeta) \mapsto (\tilde{t}, -\zeta)$.

Since $((\Delta_{0,d} + \Delta)/\Gamma_d) = 1$, $q_d^*\Gamma_d$ is of the form $\Gamma^+ + \Gamma^-$. Since $\sigma_{q_d}^*\Gamma^+ = \Gamma^-, \sigma_{q_d}^*\Delta_{0,d/2} = \Delta_{0,d/2}$ and $\sigma_{q_d}^* F_{d/2} = F_{d/2}$, $\Gamma^+ \sim \Gamma^- \sim 3(\Delta_{0,d/2} + d/2F_{d/2})$. Hence we may assume
$$\begin{aligned} \Gamma^+ &: \quad F(\tilde{t}, \zeta) = \zeta^3 + a_1(\tilde{t})\zeta^2 + a_2(\tilde{t})\zeta + a_3(\tilde{t}) = 0 \\ \Gamma^- &: \quad -F(\tilde{t}, -\zeta) = \zeta^3 - a_1(\tilde{t})\zeta^2 + a_2(\tilde{t})\zeta - a_3(\tilde{t}) = 0, \end{aligned}$$
where $\deg a_k(\tilde{t}) \leq kd/2$. Since $\zeta^2 = u - f(t), t = \tilde{t}$, we have
$$F(\tilde{t}, \zeta) = (a_1(t)u - a_1(t)f(t) + a_3(t)) + (u - f(t) + a_2(t))\zeta.$$

As $q_d^*\Gamma = \Gamma^+ + \Gamma^-$, we may assume that Γ_d is given by

$$-F(\tilde{t},\zeta)F(\tilde{t},-\zeta) = (a_1(t)u - a_1(t)f + a_3(t))^2 - (u - f(t) + a_2(t))^2(u - f(t)) = 0.$$

On the other hand, over U is S'_d is given by

$$S'_d|_{{p'_d}^{-1}} : y^2 = (a_1(t)u - a_1(t)f + a_3(t))^2 - (u - f(t) + a_2(t))^2(u - f(t)),$$

and the above equation considered as a Weierstrass equation of the generic fiber, $S_{d,\eta}$, of φ_d. By our construction, $s^{\pm}_{\Delta}$ is given by

$$s^{\pm}_{\Delta} : (f(t), \pm a_3(t)).$$

Put

$$s^{\pm}_o : (\mp(f(t) - a_2(t)), \pm(a_1(t)a_2(t) - a_3(t)).$$

Then $s^{\pm}_o \in \mathrm{MW}(S_d)$ and by the definition of the group law, we have

$$2s^{\pm}_o = s^{\pm}_{\Delta}.$$

($\Leftarrow$) We use the affine open subsets of Σ_d and W_d as before. Suppose that Γ_d is given by

$$\Gamma_d : F_{\Gamma_d}(t,u) = u^3 + c_1(t)u^2 + c_2(t)u + c_3(t) = 0$$

where $c_k(t)(i = 1,2,3)$ are polynomials of degrees $\le kd$. Then S'_d over U is given by $y^2 = F_{\Gamma_d}(t,u)$ and, as we have seen, this equation can be regarded as a Weierstrass equation of the generic fiber $S_{d,\eta}$. Since $s^+_{\Delta}O = 0$ and $p_d(s^+_{\Delta}) = \Delta$, $s^+_{\Delta} \in \mathrm{MW}(S_d)$ is given by

$$s^+_{\Delta} : (u,y) = (f(t), g(t)),$$

where $g(t)$ is a polynomial of degree $\le 3d/2$. Let $s_o \in \mathrm{MW}(S_d)$ such that $2s_o = s^+_{\Delta}$. Since s_o is a $\mathbb{C}(\mathbb{P}^1)(= \mathbb{C}(t))$-rational point of $S_{d,\eta}$, there exist $f_o(t), g_o(t) \in \mathbb{C}(t)$ such that

$$s_o : (u,y) = (f_o(t), g_o(t)).$$

Since $s^+_{\Delta}O = 0$, by [9, Theorem 9.1], we infer that $s_oO = 0$. Therefore $f_o(t), g_o(t) \in \mathbb{C}[t]$ and $\deg f_o \le d, \deg g_o \le 3d/2$. Now let

$$y = \alpha(t)u + \beta(t), \alpha(t), \beta(t) \in \mathbb{C}(t)$$

be the tangent line of the elliptic curve $S_{d,\eta}$ over $\mathbb{C}(t)$ at s_o. By the definition of the group law on $S_{d,\eta}$, we have

$$F(t,u) = (\alpha(t)u + \beta(t))^2 + (u - f_o(t))^2(u - f(t)).$$

As $F(t,u), f, f_o \in \mathbb{C}[t,u]$, we infer that $\alpha(t), \beta(t) \in \mathbb{C}[t]$. Thus we may assume that $\Gamma_d \cap U$ is given by

$$(\alpha(t)u + \beta(t))^2 + (u - f_o(t))^2(u - f(t)) = 0.$$

As $q_d^*\Gamma_d$ on V is given by

$$\begin{aligned} & (\alpha(t)u + \beta(t))^2 + (u - f_o(t))^2\zeta^2 \\ = \quad & \{(\alpha(t)u + \beta(t)) + \sqrt{-1}(u - f_o(t))\zeta\} \times \{(\alpha(t)u + \beta(t)) - \sqrt{-1}(u - f_o(t))\zeta\}, \end{aligned}$$

Γ_d is splitting with respect to q_d, i.e., $((\Delta_{0,d} + \Delta)/\Gamma_d) = 1$. □

Remark 1.1. Theorem 1.2 can be generalized to the case when S_d has a hyperelliptic fibration under some restriction. See [19].

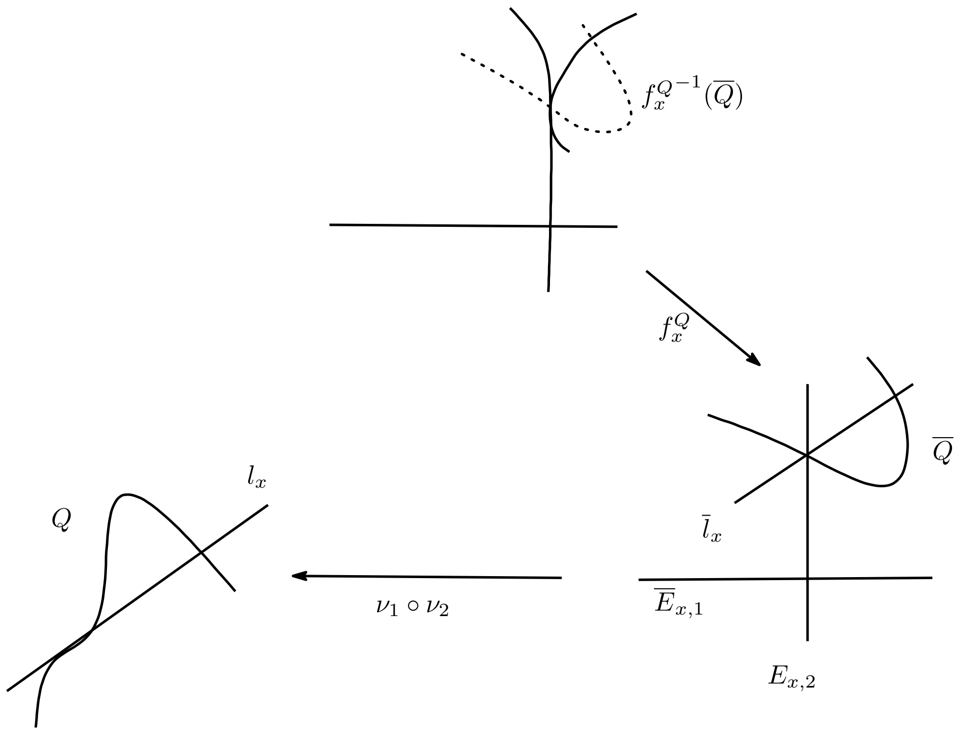
1.3. **Double covers of $\mathbb{P}^2$ branched along quartics and rational elliptic surfaces.** An elliptic surface $\mathcal{E}$ is said to be rational, if $\mathcal{E}$ is a rational surface. Hence it is an elliptic surface over $\mathbb{P}^1$. Analogously to [17], we associate a rational elliptic surface $\mathcal{E}_x^Q$ to a reduced quartic Q in $\mathbb{P}^2$ with a distinguished smooth point $x \in Q$ as follows:

Let $\nu_1 : \mathbb{P}_x^2 \to \mathbb{P}^2$ be a blowing-up at x. We denote the proper transform of the tangent line l_x at x by $\bar{l}_{x,1}$, and the exceptional curve of ν_1 by $E_{x,1}$. We next consider another blowing up $\nu_2 : \widehat{\mathbb{P}}^2 \to \mathbb{P}_x^2$ at $\bar{l}_{x,1} \cap E_{x,1}$, and denote the proper transforms of $\bar{l}_{x,1}, E_{x,1}$ and the exceptional curve of ν_2 by $\bar{l}_x$, $\overline{E}_{x,1}$, and $E_{x,2}$, respectively. Let $f' : \mathcal{E}' \to \widehat{\mathbb{P}}^2$ be a double cover with branch locus $\overline{E}_{x,1}$ and $\overline{Q}$, where $\overline{Q}$ is the proper transform of Q with respect to $\nu_2 \circ \nu_1$. Let $\mu_x^Q : \mathcal{E}_x^Q \to \mathcal{E}'$ be the canonical resolution of $\mathcal{E}'$ and put $f_x^Q := f' \circ \mu_x^Q$. Then we see that $\mathcal{E}_x^Q$ satisfies the following properties:

(i) The pencil Λ_x of lines through x induces a relatively minimal elliptic fibration $\varphi_x^Q : \mathcal{E}_x^Q \to \mathbb{P}^1$.
(ii) The preimage of $\overline{E}_{x,1}$ gives rise to a section O of φ_x^Q, and the generic fiber has a group structure, O being the zero element. Moreover the covering transformation of $\mathcal{E}_x^Q$ coincides with the involution induced by the inversion of the group law.
(iii) The preimages of $E_{x,2}$ and $\bar{l}_x$ in $\mathcal{E}_x^Q$ are irreducible components of singular fibers. The types of the singular fiber cointainig the preimages of $E_{x,2}$ and $\bar{l}_x$ are as follows:

I_2	l_x meets Q at x and at another two distinct points.
III	l_x is a 3-fold tangent point.
I_3	l_x is a bitangent line.
IV	l_x is a 4-fold tangent point.
I_n $(n \geq 4)$	l_x passes through a singular point of type $A_n (n \geq 1)$.

We use here Kodaira's notation ([9]) in order to describe the types of singular fibers. The following picture describes the case that l_x is a 3-fold tangent line at x.



(iv) Other singular fibers of $\mathcal{E}_x^Q$ correspond to lines in Λ_x not meeting Q at 4 distinct points. We refer to [11, Table 6.2] for details.

Remark 1.2. Note that any rational elliptic surface $\mathcal{E}$ with at least one reducible singular fiber is obtained above. Namely $\mathcal{E} = \mathcal{E}_x^Q$ for some Q and a smooth point x on Q.

1.4. **The Mordell-Weil lattices of $\mathcal{E}_x^Q$.** In this subsection, we give a table of types of singularities of Q, the relative position of l_x and Q, and the Mordell-Weil lattices of $\mathcal{E}_x^Q$. We first note that $\mathrm{MW}(\mathcal{E}_x^Q)$ has no 2-torsion, since we assume that Q is irreducible. Also we omit cases which never occur. As for the structure of the Mordell-Weil lattices for rational elliptic surfaces, we refer to [12] and to [15] for the correction of the misprints in [12]. Let us explain notations used in the table.

- Ξ_Q and $l_x \cap Q$ are the same as those in the table Theorem 0.1
- $R_{Q,x}$: the subgroup of $\mathrm{NS}(\mathcal{E}_x^Q)$ generated by $\{\Theta_{v,1}, \ldots, \Theta_{v,m_v-1}\}_{v \in R_{\varphi_x^Q}}$. Note that $R_{Q,x}$ is isomorphic to a direct sum of root lattices of A-D-E type, and we describe $R_{Q,x}$ as a direct sum of them.
- $\mathrm{MW}(\mathcal{E}_x^Q)$: the lattice structure of $\mathrm{MW}(\mathcal{E}_x^Q)$. To describe them, we use the notation in [12]. Namely $\bullet^*$ means the dual lattice of the lattice $\bullet$ and $\langle m \rangle$ denotes a lattice of rank 1, $\mathbb{Z}x$ with $\langle x, x \rangle = m$. Also a matrix means the intersection matrix with respect to a certain basis. Note that the lattice structure is determined by $R_{Q,x}$ as $\mathrm{MW}(\mathcal{E}_x^Q)$ has no 2-torsion.
- $\mathrm{MW}^0(\mathcal{E}_x^Q)$: the narrow part of $\mathrm{MW}(\mathcal{E}_x^Q)$, i.e., the subgroup of $\mathrm{MW}(\mathcal{E}_x^Q)$ consisting of sections s with $s\Theta_{v,0} = 1$.

No.	Ξ_Q	$l_x \cap Q$	$R_{Q,x}$	$\mathrm{MW}(\mathcal{E}_x^Q)$	$\mathrm{MW}^0(\mathcal{E}_x^Q)$
1	A_6	s	$A_6 \oplus A_1$	$\langle 1/14 \rangle$	$\langle 14 \rangle$
2	A_6	sb	A_8	$\mathbb{Z}/3\mathbb{Z}$	$\{0\}$
3	E_6	s	$E_6 \oplus A_1$	$\langle 1/6 \rangle$	$\langle 6 \rangle$
4	E_6	b	$E_6 \oplus A_2$	$\mathbb{Z}/3\mathbb{Z}$	$\{0\}$
5	A_5	s	$A_5 \oplus A_1$	$A_1^* \oplus \langle 1/6 \rangle$	$A_1 \oplus \langle 6 \rangle$
6	A_5	b	$A_5 \oplus A_2$	$A_1^* \oplus \mathbb{Z}/3\mathbb{Z}$	A_1
7	A_5	sb	A_7	$\langle 1/8 \rangle$	$\langle 8 \rangle$
8	D_5	s	$D_5 \oplus A_1$	$A_1^* \oplus \langle 1/4 \rangle$	$A_1 \oplus \langle 4 \rangle$
9	D_5	b	$D_5 \oplus A_2$	$\langle 1/12 \rangle$	$\langle 12 \rangle$
10	D_4	s	$D_4 \oplus A_1$	$(A_1^*)^{\oplus 3}$	$A_1^{\oplus 3}$
11	D_4	b	$D_4 \oplus A_2$	$\frac{1}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$	$\begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix}$
12	$A_4 + A_2$	s	$A_4 \oplus A_2 \oplus A_1$	$\langle 1/30 \rangle$	$\langle 30 \rangle$
13	$A_4 + A_2$	sb	$A_4 \oplus A_4$	$\mathbb{Z}/5\mathbb{Z}$	$\{0\}$
14	$A_4 + A_1$	s	$A_4 \oplus A_1^{\oplus 2}$	$\frac{1}{10}\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$	$\begin{pmatrix} 6 & -2 \\ -2 & 4 \end{pmatrix}$
15	$A_4 + A_1$	b	$A_4 \oplus A_2 \oplus A_1$	$\langle 1/30 \rangle$	$\langle 30 \rangle$
16	$A_4 + A_1$	sb	$A_4 \oplus A_3$	$\langle 1/20 \rangle$	$\langle 20 \rangle$
17	$A_4 + A_1$	sb	$A_6 \oplus A_1$	$\langle 1/14 \rangle$	$\langle 14 \rangle$
18	$A_3 + A_2$	s	$A_3 \oplus A_2 \oplus A_1$	$A_1^* \oplus \langle 1/12 \rangle$	$A_1 \oplus \langle 12 \rangle$
19	$A_3 + A_2$	sb	$A_4 \oplus A_3$	$\langle 1/20 \rangle$	$\langle 20 \rangle$
20	$A_3 + A_2$	sb	$A_5 \oplus A_2$	$A_1^* \oplus \mathbb{Z}/3\mathbb{Z}$	A_1
21	$A_3 + A_1$	s	$A_3 \oplus A_1^{\oplus 2}$	$(A_1^*)^{\oplus 2} \oplus \langle 1/4 \rangle$	$A_1^{\oplus 2} \oplus \langle 4 \rangle$
22	$A_3 + A_1$	b	$A_3 \oplus A_2 \oplus A_1$	$A_1^* \oplus \langle 1/12 \rangle$	$A_1 \oplus \langle 12 \rangle$
23	$A_3 + A_1$	sb	$A_5 \oplus A_1$	$A_1^* \oplus \langle 1/6 \rangle$	$A_1 \oplus \langle 6 \rangle$
24	$A_3 + A_1$	sb	$A_3 \oplus A_3$	$\langle 1/4 \rangle^{\oplus 2}$	$\langle 4 \rangle^{\oplus 2}$
25	$3A_2$	s	$A_2^{\oplus 3} \oplus A_1$	$\langle 1/6 \rangle \oplus \mathbb{Z}/3\mathbb{Z}$	$\langle 6 \rangle$
26	$3A_2$	b	$A_2^{\oplus 4}$	$(\mathbb{Z}/3\mathbb{Z})^{\oplus 2}$	$\{0\}$
27	$2A_2 + A_1$	s	$A_2^{\oplus 2} \oplus A_1^{\oplus 2}$	$\langle 1/6 \rangle^{\oplus 2}$	$\langle 6 \rangle^{\oplus 2}$
28	$2A_2 + A_1$	b	$A_2^{\oplus 3} \oplus A_1$	$\langle 1/6 \rangle \oplus \mathbb{Z}/3\mathbb{Z}$	$\langle 6 \rangle$
29	$2A_2 + A_1$	sb	$A_4 \oplus A_2 \oplus A_1$	$\langle 1/30 \rangle$	$\langle 30 \rangle$

No.	Ξ_Q	$l_x \cap Q$	$R_{Q,x}$	$\mathrm{MW}(\mathcal{E}_x^Q)$	$\mathrm{MW}^0(\mathcal{E}_x^Q)$
30	A_2+2A_1	s	$A_2 \oplus A_1^{\oplus 3}$	$A_1^* \oplus \frac{1}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$	$A_1 \oplus \begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix}$
31	A_2+2A_1	b	$A_2^{\oplus 2} \oplus A_1^{\oplus 2}$	$\langle 1/6 \rangle^{\oplus 2}$	$\langle 6 \rangle^{\oplus 2}$
32	A_2+2A_1	sb	$A_4 \oplus A_1^{\oplus 2}$	$\frac{1}{10}\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$	$\begin{pmatrix} 6 & -2 \\ -2 & 4 \end{pmatrix}$
33	A_2+2A_1	sb	$A_3 \oplus A_2 \oplus A_1$	$A_1^* \oplus \langle 1/12 \rangle$	$A_1 \oplus \langle 12 \rangle$
34	$3A_1$	s	$A_1^{\oplus 4}$	$(A_1^*)^{\oplus 4}$	$A_1^{\oplus 4}$
35	$3A_1$	s	$A_2 \oplus A_1^{\oplus 3}$	$A_1^* \oplus \frac{1}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$	$A_1 \oplus \begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix}$
36	$3A_1$	sb	$A_3 \oplus A_1^{\oplus 2}$	$(A_1^*)^{\oplus 2} \oplus \langle 1/4 \rangle$	$A_1^{\oplus 2} \oplus \langle 4 \rangle$
37	A_4	s	$A_4 \oplus A_1$	$\frac{1}{10}\begin{pmatrix} 3 & 1 & -1 \\ 1 & 7 & 3 \\ -1 & 3 & 7 \end{pmatrix}$	$\begin{pmatrix} 4 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$
38	A_4	b	$A_4 \oplus A_2$	$\frac{1}{15}\begin{pmatrix} 2 & 1 \\ 1 & 8 \end{pmatrix}$	$\begin{pmatrix} 8 & -1 \\ -1 & 2 \end{pmatrix}$
39	A_4	sb	A_6	$\frac{1}{7}\begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$	$\begin{pmatrix} 4 & -1 \\ -1 & 2 \end{pmatrix}$
40	A_3	s	$A_3 \oplus A_1$	$A_3^* \oplus A_1^*$	$A_3 \oplus A_1$
41	A_3	b	$A_3 \oplus A_2$	$\frac{1}{12}\begin{pmatrix} 7 & 1 & 2 \\ 1 & 7 & 2 \\ 2 & 2 & 4 \end{pmatrix}$	$\begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 4 \end{pmatrix}$
42	A_3	sb	A_5	$A_2^* \oplus A_1^*$	$A_2 \oplus A_1$
43	$2A_2$	s	$A_2^{\oplus 2} \oplus A_1$	$A_2^* \oplus \langle 1/6 \rangle$	$A_2 \oplus \langle 6 \rangle$
44	$2A_2$	b	$A_2^{\oplus 3}$	$A_2^* \oplus \mathbb{Z}/3\mathbb{Z}$	A_2
45	$2A_2$	sb	$A_4 \oplus A_2$	$\frac{1}{15}\begin{pmatrix} 2 & 1 \\ 1 & 8 \end{pmatrix}$	$\begin{pmatrix} 8 & -1 \\ -1 & 2 \end{pmatrix}$
46	A_2+A_1	s	$A_2 \oplus A_1^{\oplus 2}$	$\frac{1}{6}\begin{pmatrix} 2 & 1 & 0 & -1 \\ 1 & 5 & 3 & 1 \\ 0 & 3 & 6 & 3 \\ -1 & 1 & 3 & 5 \end{pmatrix}$	$\begin{pmatrix} 4 & -1 & 0 & 1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 1 & 0 & -1 & 2 \end{pmatrix}$
47	A_2+A_1	b	$A_2^{\oplus 2} \oplus A_1$	$A_2^* \oplus \langle 1/6 \rangle$	$A_2 \oplus \langle 6 \rangle$
48	A_2+A_1	sb	$A_4 \oplus A_1$	$\frac{1}{10}\begin{pmatrix} 3 & 1 & -1 \\ 1 & 7 & 3 \\ -1 & 3 & 7 \end{pmatrix}$	$\begin{pmatrix} 4 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$
49	A_2+A_1	sb	$A_4 \oplus A_1$	$\frac{1}{12}\begin{pmatrix} 7 & 1 & 2 \\ 1 & 7 & 2 \\ 2 & 2 & 4 \end{pmatrix}$	$\begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 4 \end{pmatrix}$
50	$2A_1$	s	$A_1^{\oplus 3}$	$D_4^* \oplus A_1^*$	$D_4 \oplus A_1$
51	$2A_1$	b	$A_2 \oplus A_1^{\oplus 2}$	$\frac{1}{6}\begin{pmatrix} 2 & 1 & 0 & -1 \\ 1 & 5 & 3 & 1 \\ 0 & 3 & 6 & 3 \\ -1 & 1 & 3 & 5 \end{pmatrix}$	$\begin{pmatrix} 4 & -1 & 0 & 1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 1 & 0 & -1 & 2 \end{pmatrix}$
52	$2A_1$	sb	$A_3 \oplus A_1$	$A_3^* \oplus A_1^*$	$A_3 \oplus A_1$

No.	Ξ_Q	$l_x \cap Q$	$R_{Q,x}$	$\mathrm{MW}(\mathcal{E}_x^Q)$	$\mathrm{MW}^0(\mathcal{E}_x^Q)$
53	A_2	s	$A_2 \oplus A_1$	A_5^*	A_5
54	A_2	b	$A_2^{\oplus 2}$	$(A_2^*)^{\oplus 2}$	$A_2^{\oplus 2}$
55	A_2	sb	A_4	A_4^*	A_4
56	A_1	s	$A_1^{\oplus 2}$	D_6^*	D_6
57	A_1	b	$A_2 \oplus A_1$	A_5^*	A_5
58	A_1	sb	A_3	D_5^*	D_5
59	$\emptyset$	s	A_1	E_7^*	E_7
60	$\emptyset$	b	A_2	E_6^*	E_6

2. Proof of Theorem 0.1

We keep the same notations as before. Our result will be proved case-by-case. Let us start with the following lemma.

Lemma 2.1. *Let C be an even tangential conic to Q through x. The preimage of C in $\mathcal{E}_x^Q$ consists of two sections s_C^+ and s_C^- such that*

(i) $\langle s_C^+, s_C^+\rangle = \langle s_C^-, s_C^-\rangle = 2$
(ii) $s_C^+O = s_C^-O = 0$
(iii) $s_C^+\Theta_{v,0} = s_C^-\Theta_{v,0} = 1$ *for all* $v \in R_{\varphi_x^Q}$*, i.e,* $s_C^\pm \in \mathrm{MW}^0(\mathcal{E}_x^Q)$.

Coversely, for any section s in $\mathrm{MW}(\mathcal{E}_x^Q)$ satisfying two of the above three properties, the image of s in $\mathbb{P}^2$ is an even tangential conic to Q.

Proof. We first note that two of the properties (i), (ii) and (iii) imply the remaining. This follows from the formula

$$\langle s, s\rangle = 2 + 2sO - \sum_{v\in R_\varphi} \mathrm{Corr}_v(s,s)$$

for the rational elliptic surface $\mathcal{E}_x^Q$ and $s \in \mathrm{MW}(\mathcal{E}_x^Q)$.

Let $\overline{C}$ be the proper transform of C in $\widehat{\mathbb{P}}^2$. Since $\overline{C}$ is tangent to $\overline{Q}$ at each intersection point and $\overline{C}\cap\overline{E}_{x,1} = \emptyset$, the preimage of $\overline{C}$ in $\mathcal{E}_x^Q$ consists of 2 irreducible components s_C^+ and s_C^- so that $s_C^\pm O = 0$. Since $\overline{C}$ meets the proper transform of a general member in Λ_x at one point, both s_C^+ and s_C^- are sections of $\varphi_x^Q : \mathcal{E}_x^Q \to \mathbb{P}^1$. The property (iii) follows from the fact that $\overline{C}$ meets $E_{x,2}$ and $\overline{C}$ does not pass through singularities of $\overline{Q}$. Now the property (i) is straightforward from the explicit formula for $\langle\, ,\, \rangle$.

Conversely, suppose that we have a section s satisfying two of the properties (i), (ii) and (iii). Let C_s be the image of s in $\mathbb{P}^2$. By our construction of $\mathcal{E}_x^Q$, we infer that C_s is a conic tangent to Q at x. Since C_s is also the image of $\sigma^*_{f_x^Q}s$, we infer that C_s is an even tangent conic to Q. □

Theorem 2.1. *Let C be an even tangential conic to Q and let s_C^+ be the section as above.*

$$(C/Q) = (-1)^{\varepsilon(s_C^+)},$$

where the symbol $\varepsilon(s_C^+)$ is the same as that defined in Theorem 1.2.

Proof. Let $\widehat{\mathbb{P}}^2$ as before. Since $\bar{l}_x$ is a (-1) curve, by blowing down $\bar{l}_x$, we obtain Σ_2 with the following properties:

(i) The image of $\overline{Q}$ is a trisection $\Gamma_Q \sim 3(\Delta_{0,d} + 2F)$.
(ii) Singularities of Γ_Q are the same as those of Q except the A_1 singularity caused by blowing down $\bar{l}_x$.
(iii) The image of $\overline{E}_{x,1} = \Delta_{0,d}$.

(iv) The image of $\overline{C}$ is a section Δ_C such that $\Delta_C \sim (\Delta_{0,d} + 2F)$ and Δ_C is even tangent to Γ_Q.

Let $f_o : Z_o \to \Sigma_2$ be the induced double cover by $f_C : Z_C \to \mathbb{P}^2$, i.e., the $\mathbb{C}(Z_C)$-normalization of Σ_2. One easily see that $\Delta_{f_o} = \Delta + \Delta_C$.

$$\begin{tikzcd} Z_C \arrow[d] & \mathcal{E}_x^Q \arrow[d] & Z_o \arrow[d] \\ \mathbb{P}^2 & \widehat{\mathbb{P}}^2 \arrow[l] \arrow[r] & \Sigma_2 \end{tikzcd}$$

Since Δ_C is the image of $\overline{C}$, it is also the image of $s_C^{\pm}$. Hence we infer that

$$(C/Q) = 1 \Leftrightarrow (\Delta_{0,d} + \Delta_C/\Gamma_Q) = 1.$$

Hence by Theorem 1.2, we infer that $(C/Q) = 1$ if and only if $s_C^+ \in 2\,\mathrm{MW}(\mathcal{E}_x^Q)$. □

Remark 2.1. Suppose that $s_C^+ \in 2\,\mathrm{MW}(\mathcal{E}_x^Q)$. Let s_o be an element in $\mathrm{MW}(\mathcal{E}_x^Q)$ such that $2s_o = s_C^+$. By Lemma 2.1 (i), we have $\langle s_o, s_o\rangle = 1/2$. Hence if $\mathrm{MW}(\mathcal{E}_x^Q)$ has no section s with $\langle s, s\rangle = 1/2$, there is no quadratic residue even tangential conic to Q through x.

Lemma 2.2. *Let $\widetilde{Q}$ be the normalization of Q and we denote the genus of $\widetilde{Q}$ by $g(\widetilde{Q})$.*

(i) *No even tangential conic to Q is quadratic residue* $\mathrm{mod}Q$ *if $g(\widetilde{Q}) \geq 2$.*

(ii) *All even tangential conic to Q are quadratic residue* $\mathrm{mod}Q$ *if $g(\widetilde{Q}) = 0$.*

Proof. (i) Let C be an even tangential conic to Q and suppose that $(C/Q) = 1$. Let $f_C : Z_C \to \mathbb{P}^2$ be a double cover with $\Delta_{f_C} = C$. Then f_C^*Q is of the form $Q^+ + Q^-$. Since $Z_C = \mathbb{P}^1 \times \mathbb{P}^1, \mathrm{Pic}(Z_C) \cong \mathbb{Z} \oplus \mathbb{Z}$ and the covering transformation induces an involution $(a, b) \mapsto (b, a)$ on $\mathrm{Pic}(Z_C)$, we infer that $Q^+ \sim Q^- \sim (2, 2)$. Since Q^+, Q^- and Q are birationally equivalent, we have $g(\widetilde{Q}) \leq 1$ and the result follows.

(ii) Since the induced double cover on $\widetilde{Q}$ is unramified, $(C/Q) = 1$. □

Now we easily have the following theorem:

Theorem 2.2. *Let Q be an irreducible quartic. Choose a smooth point $x \in Q$ and let $\mathcal{E}_x^Q$ be the rational elliptic surface as in §1. Then we have the following:*

(i) *Let* ETC *be the set of conics passing through x. Then*

$$\begin{aligned} \sharp\mathrm{ETC} &= \sharp\{s \in \mathrm{MW}(\mathcal{E}_x^Q) \mid \langle s, s\rangle = 2, sO = 0\}/2 \\ &= \sharp\{s \in \mathrm{MW}^0(\mathcal{E}_x^Q) \mid \langle s, s\rangle = 2\}/2 \end{aligned}$$

(ii) *Let* QRETC *be the set of even tangential conics passing through x with $(C/Q) = 1$. Then*

$$\begin{aligned} \sharp\mathrm{QRETC} &= \sharp\{s \in 2\,\mathrm{MW}(\mathcal{E}_x^Q) \mid \langle s, s\rangle = 2, sO = 0\}/2 \\ &= \sharp\{s \in 2\,\mathrm{MW}(\mathcal{E}_x^Q) \cap \mathrm{MW}^0(\mathcal{E}_x^Q) \mid \langle s, s\rangle = 2\}/2 \end{aligned}$$

Proof. Our statements (i) and (ii) are immediate from Lemma 2.1 and Theorem 2.1. □

We now prove Theorem 0.1 case-by-case. We first compute $\sharp$ETC. By Lemma 2.1, it is enough to see the number of sections s in the narrow part $\mathrm{MW}^0(\mathcal{E}_x^Q)$ of $\mathrm{MW}(\mathcal{E}_x^Q)$ with $\langle s, s\rangle = 2$.

For the lattices of A-D-E types, it is nothing but the number of roots, and the following table is well known (see [6])

A_n	D_n $(n \geq 4)$	E_6	E_7
$n(n+1)$	$2n(n-1)$	72	126

From the above table and that in §2, our statement on $\sharp$ETC is straightforward except for the cases $11, 14, 30, 32, 35, 37, 38, 39, 41, 45, 46, 48, 49, 51$. For the rank 2 cases among the exceptional cases, our statement follows easily by direct computation. For the cases of rank > 2, we make use of [12, Lemma 3.8], which is as follows:

$$\begin{pmatrix} 4 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} \cong A_1^{\perp} \text{ in } A_4, \quad \begin{pmatrix} 4 & -1 & 0 & 1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 1 & 0 & -1 & 2 \end{pmatrix} \cong A_1^{\perp} \text{ in } A_5$$

$$\begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \cong A_2^{\perp} \text{ in } D_5$$

where the terminology $\bullet^{\perp}$ in $\blacksquare$ means that we embed a lattice $\bullet$ into $\blacksquare$ and $\bullet^{\perp}$ is the orthogonal complement of $\bullet$ in $\blacksquare$. Also, by [12, Lemma 3.8], the embedding is determined up to isomorphism. Hence we just count the number of roots which are orthogonal to the embedded lattices. To be more precise, we explain the case $A_1^{\perp}$ in A_5. We first consider the realization of A_5 as follows:

$$A_5 = \{(x_1, \dots, x_6) \mid \sum_i x_i = 0, x_i \in \mathbb{Z}\} \subset \mathbb{R}^6$$

and the pairing is induced from the Euclidean metric $\sum_i x_i^2$ in $\mathbb{R}^6$. Under these circumstances, the roots are given by a vector $(1, -1, 0, 0, 0, 0)$ and those obtained by permutations of the coordinates. We fix an embedding of A_1 given by $\mathbb{Z}(1, -1, 0, 0, 0, 0) \subset A_5$. Then roots in $A_1^{\perp}$ are

$$\begin{array}{lll} (0,0,\pm1,\mp1,0,0) & (0,0,\pm1,0,\mp1,0) & (0,0,\pm1,0,0,\mp1) \\ (0,0,0,\pm1,\mp1,0) & (0,0,0,\pm1,0,\mp1) & (0,0,0,0,\pm1,\pm1). \end{array}$$

Since the remaining cases are similar, we omit them. Thus we have a list for $\sharp$ETC.

We now go on to compute $\sharp$QRETC. We first note that $\sharp\mathrm{QRETC} = 0$ if $\sharp\mathrm{ETC} = 0$. In the following, we only cosider the case of $\sharp\mathrm{ETC} \neq 0$.

Since Q is irreducible, $\mathrm{MW}(\mathcal{E}_x^Q)$ has no 2-torsion. Hence for each $s \in 2\,\mathrm{MW}(\mathcal{E}_x^Q)$, there exists a unique $s_o \in \mathrm{MW}(\mathcal{E}_x^Q)$ such that $2s_o = s$. For distinct $C_1, C_2 \in$ QRETC, $s_{C_1}^+$ and $s_{C_2}^+$ are distinct in $\mathrm{MW}^0(\mathcal{E}_x^Q)$. Hence it is enough to compute

$$\sharp\{s_o \in \mathrm{MW}(\mathcal{E}_x^Q) \mid \langle s_o, s_o\rangle = 1/2, 2s_o \in \mathrm{MW}^0(\mathcal{E}_x^Q)\}$$

Now Theorem 0.1 follows from the following claim:

Claim. *Suppose that $\sharp\mathrm{ETC} \neq 0$. If $\mathrm{MW}(\mathcal{E}_x^Q)$ has an A_1^* as a direct summand, then two generators $\pm\tilde{s}$ of A_1^* are sections such that $\langle \tilde{s}, \tilde{s}\rangle = 1/2, 2\tilde{s} \in \mathrm{MW}^0(\mathcal{E}_x^Q)$. Conversely if there exists $s_o \in \mathrm{MW}(\mathcal{E}_x^Q)$ such that $\langle s_o, s_o\rangle = 1/2, 2s_o \in \mathrm{MW}^0(\mathcal{E}_x^Q)$, then $\mathbb{Z}s_o(\cong A_1^*)$ is a direct summand of $\mathrm{MW}(\mathcal{E}_x^Q)$.*

Proof of Claim. Suppose that A_1^* is a direct summand of $\mathrm{MW}(\mathcal{E}_x^Q)$ and let $\tilde{s}$ be a section such that $\mathbb{Z}\tilde{s} = A_1^*$. Then $\langle \tilde{s}, \tilde{s}\rangle = 1/2$ and $2\tilde{s} \in \mathrm{MW}^0(\mathcal{E}_x^Q)$ by [14, Theorem 9.1].

We now go on to show the converse. Let s_o be a section with $\langle s_o, s_o\rangle = 1/2, 2s_o \in \mathrm{MW}^0(\mathcal{E}_x^Q)$. As for the dual lattices of A-D-E type, we have the following table:

Type	A_n^*	D_n^* $(n \geq 4)$	E_6^*	E_7^*
Minimum norm	$\frac{n}{(n+1)}$	1	$\frac{4}{3}$	$\frac{3}{2}$

Hence we easily see that $\mathrm{MW}(\mathcal{E}_x^Q)$ has an A_1^* direct summand except for the cases 37, 38, 39, 41, 45, 46, 48, 49 and 51. We see that there is no section s with $\langle s, s\rangle = 1/2$ for these exceptional cases.

Cases 38, 39 and 45. In these cases, the paring $\langle\, ,\, \rangle$ takes its value in $1/15\mathbb{Z}$ (Cases 38 and 45), and $1/7\mathbb{Z}$ (Case 39), where $1/m\mathbb{Z} = \{a/m \mid a \in \mathbb{Z}\}$. Hence there is no section s with $\langle s, s\rangle = 1/2$.

Cases 37 and 48. Let s be any element of $\mathrm{MW}(\mathcal{E}_x^Q)$. In these cases,

$$\langle s, s\rangle = 2(1 + sO) - \frac{k_1(5 - k_1)}{5} - \frac{1}{2}k_2,$$

where $k_1 \in \{0, 1, 2, 3, 4\}, k_2 \in \{0, 1\}$. Hence we infer that there is no s with $\langle s, s\rangle = 1/2$.

Cases 41 and 49. Let s be any element of $\mathrm{MW}(\mathcal{E}_x^Q)$. In these cases,

$$\langle s, s\rangle = 2(1 + sO) - \frac{k_1(4 - k_1)}{4} - \frac{2}{3}k_2,$$

where $k_1 \in \{0, 1, 2, 3\}, k_2 \in \{0, 1\}$. Hence we infer that there is no s with $\langle s, s\rangle = 1/2$.

Cases 46 and 51. Let s be any element of $\mathrm{MW}(\mathcal{E}_x^Q)$. In these cases,

$$\langle s, s\rangle = 2(1 + sO) - \frac{2}{3}k_1 - \frac{1}{2}k_2 - \frac{1}{2}k_3,$$

where $k_1, k_2, k_3 \in \{0, 1\}$. Hence we infer that there is no s with $\langle s, s\rangle = 1/2$.

After checking each case we see that s_o generates an A_1^* direct summand.

3. Preliminaries from theory of Galois covers

3.1. **Galois covers.** In this subsection, we summarize some facts and terminologies on Galois covers. For details, see [1, §3]. Let X and Y be normal projective varieties. We call X a cover if there exists a finite surjective morphism $\pi : X \to Y$. Let $\mathbb{C}(X)$ and $\mathbb{C}(Y)$ be rational function fields of X and Y, respectively. If X is a cover of Y, then $\mathbb{C}(X)$ is an algebraic extension of $\mathbb{C}(Y)$ with $\deg \pi = [\mathbb{C}(X) : \mathbb{C}(Y)]$. Let G be a finite group. A G-cover is a cover $\pi : X \to Y$ such that $\mathbb{C}(X)/\mathbb{C}(Y)$ is a Galois extension with $\mathrm{Gal}(\mathbb{C}(X)/\mathbb{C}(Y)) \cong G$. For a cover $\pi : X \to Y$, the branch locus Δ_π of π is a subset of Y as follows:

$$\Delta_\pi = \{y \in Y \mid \pi \text{ is not locally isomorphic over } y\}.$$

If Y is smooth, Δ_π is an algebraic subset of pure codimention 1 ([21]). Let $\pi : X \to Y$ be a G-cover of a smooth projective variety Y. Let $\Delta_\pi = \Delta_{\pi,1} + \ldots + \Delta_{\pi,r}$ denote the irreducible decomposition of Δ_π. We say that $\pi : X \to Y$ is branched at $e_1\Delta_{\pi,1} + \ldots + e_r\Delta_{\pi,r} (e_i \geq 2, i = 1, \ldots, r)$ if the ramification index along $\Delta_{\pi,i}$ is e_i for each i.

Let B be a reduced divisor on a smooth projective variety Y and $B = B_1 + \ldots + B_r$ denote its irreducible decomposition. It is known that the existence of a G-cover $\pi : X \to Y$ at $\sum_i e_iB_i$ can be characterized as follows:

Theorem 3.1. *There exists a G-cover of Y branched at $\sum_i e_iB_i$ if and only if there exists an epimorphism $\phi : \pi_1(Y \setminus B, *) \to G$ such that for each meridian γ_i of B_i, the image of its class $[\gamma_i]$, $\phi([\gamma_i])$, has order e_i.*

3.2. **Dihedral covers.** Let $\mathcal{D}_{2n}$ be the dihedral group of order $2n$ ($n \geq 3$) given by $\langle \sigma, \tau \mid \sigma^2 = \tau^n = (\sigma\tau)^2 = 1\rangle$. In [17], we developed a method to deal with $\mathcal{D}_{2n}$-covers, and some variants of the results in [17] have been studied since then. We summarize here some results which we need later. Let us start with introducing some notation in order to explain them.

Let $\pi : X \to Y$ be a $\mathcal{D}_{2n}$-cover. By its definition, $\mathbb{C}(X)$ is a D_{2n}-extension of $\mathbb{C}(Y)$. Let $\mathbb{C}(X)^\tau$ be the fixed field by τ. We denote the $\mathbb{C}(X)^\tau$- normalization by $D(X/Y)$. We denote the induced morphisms by $\beta_1(\pi) : D(X/Y) \to Y$ and $\beta_2(\pi) : X \to D(X/Y)$. Note that X is a $\mathbb{Z}/n\mathbb{Z}$-cover of $D(X/Y)$ and $D(X/Y)$ is a double cover of Y such that $\pi = \beta_1(\pi) \circ \beta_2(\pi)$:

$$\begin{tikzcd} X \arrow[dd, "\pi"'] \arrow[dr, "\beta_2(\pi)"] & \\ & D(X/Y) \arrow[dl, "\beta_1(\pi)"] \\ Y & \end{tikzcd}$$

Generic $\mathcal{D}_{2n}$-covers. A $\mathcal{D}_{2n}$-covers $\pi : S \to \Sigma$ is said to be *generic* if $\Delta(\pi) = \Delta(\beta_1(\pi))$. As for conditions for the existence of generic $\mathcal{D}_{2n}$-covers with prescribed branch loci, we have the following:

Let B be a reduced divisor on Σ with at worst simple singularities. Suppose that there exists a double cover $f'_B : Z'_B \to \Sigma$ with branch locus B and let $\mu_B : Z_B \to Z'_B$ be the canonical resolution. We define the subgroup R_B of $\mathrm{NS}(Z_B)$ as follows:

$$R_B := \oplus_{b \in \mathrm{Sing}(B)} R_b,$$

where R_b is the subgroup in $\mathrm{NS}(Z_B)$ generated by the exceptional divisor of the singularity ${f'}_B^{-1}(x)$. Then we have the following result:

Theorem 3.2. [1, Theorem 3.27] *Let p be an odd prime and suppose that Z_B is simply connected. There exists a generic $\mathcal{D}_{2p}$-cover $\pi : S \to \Sigma$ with branch locus B if and only if $\mathrm{NS}(Z_B)/R_B$ has p-torsion.*

Let $R_b^\vee = \mathrm{Hom}_{\mathbb{Z}}(R_b, \mathbb{Z})$. R_b can be regarded as a subgroup of $R_b^\vee$ by using the intersection pairing. Since the torsion subgroup of $\mathrm{NS}(Z_B)/R_B$ can be considered as a subgroup of $\oplus_{b \in \mathrm{Sing}(B)} R_b^\vee / R_b$, we have the following corollary:

Corollary 3.1. *If there exists no b such that $p | \sharp(R_b^\vee / R_b)$, then there exists no generic $\mathcal{D}_{2p}$-cover with branch locus B.*

Non-generic $\mathcal{D}_{2n}$-covers. A $\mathcal{D}_{2n}$-cover is said to be *non-generic* if $\Delta(\beta_1(\pi))$ is a proper subset of $\Delta(\pi)$. We consider a non-generic $\mathcal{D}_{2n}$-cover of Σ under the following setting:

Let $B = B_1 + B_2$ be a reduced divisor on Σ such that:

(i) there exists a double cover $f'_{B_1} : Z'_{B_1} \to \Sigma$ with $\Delta_{f'_{B_1}} = B_1$, and
(ii) B_2 is irreducible.

Let $f_{B_1} : Z_{B_1} \to \Sigma$ be the canonical resolution of Z'_{B_1}.

Proposition 3.1. [1, Proposition 3.31] *Suppose that Σ is simply connected and the preimage of the strict transform of B_2 consists of two distinct irreducible components B_2^+ and B_2^-. If there exist an effective divisor D and a line bundle $\mathcal{L}$ on Z_{B_1} satisfying conditions*

(i) $D = B_2^+ + D'$; D' and $\sigma_{f_{B_1}}^* D'$ have no common components,
(ii) $\mathrm{Supp}(D' + \sigma_{f_{B_1}}^* D')$ is contained in the exceptional set of $\mu_{f'_{B_1}}$ and
(iii) $D - \sigma_{f_{B_1}}^* D \sim n\mathcal{L}$ $(n \geq 3)$, where $\sim$ denotes linear equivalence,

then there exists a $\mathcal{D}_{2n}$-cover $\pi : S \to \Sigma$ branched at $2B_1 + nB_2$ such that $\Delta_{\beta_1(\pi)} = B_1$.

Corollary 3.2. *If $\sigma_{f_{B_1}}^* B_2^+ \sim B_2^-$ and there exists a $\mathcal{D}_{2n}$-cover of Σ branched at $2B_1 + nB_2$ for any $n \geq 3$.*

Proposition 3.2. [1, Proposition 3.32] *Under the notation above, if a $\mathcal{D}_{2n}$-cover $\pi : S \to \Sigma$ branched at $2B_1 + nB_2$ exists, then the following holds:*

(i) $D(S/\Sigma) = Z'_{B_1}$. *The preimage of the porper transform of* B_2 *in* Z_{B_1} *consists of two irreducible components,* $B_2^{\pm}$.

(ii) There exist effective divisors D_1 *and* D_2, *and a line bundle* $\mathcal{L}$ *on* Z_{B_1} *such that*

- $\mathrm{Supp}(D_1 + \sigma^*_{f_{B_1}} D_1 + D_2)$ *is contained in the exceptional set of* μ,
- D_1 *and* $\sigma^*_{f_{B_1}} D_1$ *have no common components,*
- *if* $D_2 \neq \emptyset$, *then* n *is even,* D_2 *is reduced, and* $D' = \sigma^*_{f_{B_1}} D'$ *for each irreducible component* D' *of* D_2, *and*
- $(B_2^+ + D_1 + \frac{n}{2} D_2) - (B_2^- + \sigma^*_{f_{B_1}} D_1) \sim n\mathcal{L}$.

Corollary 3.3. *If a* $\mathcal{D}_{2n}$*-cover* $\pi : S \to \Sigma$ *branched at* $2B_1 + nB_2$ *exists, then* B_2 *is a splitting curve with respect to* f_{B_1}.

4. Proof of Theorem 0.2

We first note that there are 3 possibilities for $\beta_1(\pi) : D(S/\mathbb{P}^2) \to \mathbb{P}^2$:

Case 1. $D(S/\mathbb{P}^2) = Z_C, \beta_1(\pi) = f_C$.

Case 2. $D(S/\mathbb{P}^2) = Z'_Q, \beta_1(\pi) = f'_Q$.

Case 3. $D(S/\mathbb{P}^2) = Z'_{C+Q}, \beta_1(\pi) = f'_{C+Q}$.

Note that $f'_\bullet : Z_\bullet \to \mathbb{P}^2$ denotes a double cover with branch locus $\bullet$. We show that our statements (i) and (ii) hold for Case 1 and neither Cases 2 nor 3 occur.

Case 1. In this case, π is branched at $2C + pQ$. Hence, by Corollary 3.3, we infer that $(C/Q) = 1$. Put $f_C^* Q = Q^+ + Q^-$. By Proposition 3.2, $Q^+ - Q^-$ is p-divisible in $\mathrm{Pic}(Z_C)$. Since $Q^+ + Q^- \sim (4,4)$, Q^+ is linearly equivalent to either $(3,1), (1,3)$ or $(2,2)$. Hence, $Q^+ \sim Q^- \sim (2,2)$ if $p \geq 3$.

Case 2. Let Σ_2, Δ_C and Γ_Q be the Hirzebruch surface of degree 2 and the divisors obtained as in §2. By considering the $\mathbb{C}(S)$-normalization of Σ_2, we have a D_{2p}-cover branched at $2(\Delta_{0,d} + \Gamma_Q) + p\Delta_C$. As in [18], we reduce our problem on the existence of $\mathcal{D}_{2p}$-covers to that on a linear equation on $\mathrm{MW}(\mathcal{E}_x^Q)$. By [18, Proposition 4.1], the following proposition is straightforward:

Proposition 4.1. *If there exists a* $\mathcal{D}_{2p}$*-cover of* $\mathbb{P}^2$ *branched at* $pC + 2Q$, *then* $s_C^+ \in p\,\mathrm{MW}(\mathcal{E}_x^Q)$.

Let s_o be an element in $\mathrm{MW}(\mathcal{E}_x^Q)$ such that $ps_o = s_C^+$. Then we have $\langle s_o, s_o \rangle = 2/p^2$. On the other hand, by the table in §1, the value of $\langle s_o, s_o \rangle \in 1/(2^3 \cdot 3 \cdot 5 \cdot 7)\mathbb{Z}$. Therefore Case 2 does not occur.

Case 3. Our statement may follow from the results in [13]. However, we prove our statement without using the fact that Z_B is a $K3$ surface. Put $B = C + D$. In this case, the canonical resolution of $D(S/\mathbb{P}^2)$ is Z_B. Hence by Theorem 3.2, $\mathrm{NS}(Z_B)/R_B$ has p-torsion. By Corollary 3.1 and Theorem 0.1, it is enough to show that there exists no $\mathcal{D}_{10}$-cover in the case when Q has one A_4 singularity and C is an even bitangential conic to Q. Let D be an element of $\mathrm{NS}(Z_B)$ such that D gives rise to 5-torsion in $\mathrm{NS}(Z_B)/R_B$. By using the intersection pairing, D can be regarded as an element of $R_B^\vee = \oplus_{b \in \mathrm{Sing}(B)} R_b^\vee$. Since $R_b^\vee$ can be embedded into $R_b \otimes \mathbb{Q}$ canonically, D can be expressed as an element in $\oplus_{b \in \mathrm{Sing}(B)} R_b \otimes \mathbb{Q}$. Let b_o be the unique A_4 singularity, and put

$$D \approx_{\mathbb{Q}} \sum_{b \in \mathrm{Sing}(Q)} D_b, \quad D_b \in R_b \otimes \mathbb{Q}$$

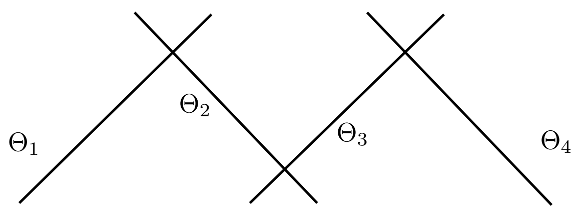
and let $\gamma(D_b)$ be the class of D_b in $R_b^\vee/R_b$. Since the type of singularity of B other than b_o is either A_3, A_7, A_{11} or A_{15}, $\gamma(D_b) = 0$ if $b \neq b_o$. As $R_{b_o}^\vee/R_{b_o}$ is generated by

$$\frac{1}{5}(4\Theta_1 + 3\Theta_2 + 2\Theta_3 + \Theta_1),$$

we have

$$D - \sum_{b\in \mathrm{Sing}(B)\setminus\{b_o\}} D_b \approx_{\mathbb{Q}} \frac{k}{5}(4\Theta_1 + 3\Theta_2 + 2\Theta_3 + \Theta_1) \bmod R_B,$$

for some $k \in \{\pm 1, \pm 2\}$. Here we label the irreducible components as follows:



By modifying D with an element in R_B suitably, we may assume $D \approx_{\mathbb{Q}} k/5(4\Theta_1+3\Theta_2+2\Theta_3+\Theta_1)$. This shows that

$$D^2 = -\frac{4k^2}{5}.$$

This leads us to a contradiction, as $D^2 \in \mathbb{Z}$. Therefore Case 3 does not occur.

The remaining part of Theorem 0.2 is immediate from Corollary 3.2. □

Remark 4.1.

(1) $(C/Q) = 1$ is not enough for the existence of D_{2n}-covers. In fact, for Q with $3A_1$ singularities, there exists an even tangential conic C such that $(C/Q) = 1$ but $Q^+ \not\sim Q^-$ (see [2]).

(2) By [13], there exists an irreducible quartic Q with one A_5 singularity and an even tangential conic C to Q such that
- $C \cap Q = \{x_1, x_2\}, I_{x_1}(C,Q) = 2, I_{x_2}(C,Q) = 6$, and
- $\mathrm{NS}(Z_B)/R_B$ has 3-torsion.

By Theorem 3.2, there exists a $\mathcal{D}_6$-cover branched at $2(C+Q)$. In this case, $(C/Q) = 1$, but $Q^+ \not\sim Q^-$. In fact, if $Q^+ \sim Q^-$, then Q^+ is a rational curve with one singularity whose type is either A_1 or A_2. This singularity must give rise to another singularity of Q, which is impossible.

5. Application to the study of Zariski pairs

Let (B_1, B_2) be a pair of reduced plane curves. We call (B_1, B_2) a Zariski pair if

(1) both of B_1 and B_2 have the same combinatorial type (see [1] for the precise definition of combinatorial type), and
(2) there exists no homeomorphism $h : \mathbb{P}^2 \to \mathbb{P}^2$ such that $h(B_1) = B_2$.

In the case of an irreducible quartic Q and its even tangential conic, the combinatorial type of $C + Q$ is determined by Ξ_Q, $\sharp C \cap Q$ and $I_P(C,Q)$ for each $P \in C \cap Q$.

As an application of the previous sections, we have

Proposition 5.1. *Let Q_1 and Q_2 be irreducible quartics and let C_1 and C_2 be their even tangential conics, respectively. Suppose that $C_i + Q_i$ $(i = 1, 2)$ have the same combinatorial type.*

(i) If $(C_1/Q_1) = 1$ and $(C_2/Q_2) = -1$, then $(C_1 + Q_1, C_2 + Q_2)$ is a Zariski pair.

(ii) If $(C_i/Q_i) = 1$ $(i = 1, 2)$, $Q_1^+ \sim Q_1^-$ and $Q_2^+ \not\sim Q_2^-$, then $(C_1 + Q_1, C_2 + Q_2)$ is a Zariski pair.

Proof. (i) As $C_1 + Q_1$ and $C_2 + Q_2$ have the same combinatorial type, $\Xi_{Q_1} = \Xi_{Q_2}$. Since $(C_1/Q_1) = 1$ and $(C_2/Q_2) = -1$, by Theorem 0.1, we see that $\Xi_{Q_1} = \Xi_{Q_2} = 2A_1$ or A_3. Therefore $Q_1^+ \sim Q_1^- \sim (2, 2)$. Hence by Corollary 0.2, we infer that $\pi_1(\mathbb{P}^2 \setminus (C_1 + Q_1), *) \not\cong \pi_1(\mathbb{P}^2 \setminus (C_2 + Q_2), *)$, i.e., $(C_1 + Q_1, C_2 + Q_2)$ is a Zariski pair.

(ii) Our statement is immediate from [2, Proposition 2].

□

An example for Proposition 5.1 (ii) can be found in [2]. We end this section by giving examples for Proposition 5.1 (i). Let $\mathcal{E}_x^Q$ be the rational elliptic surface corresponding to either No. 40 or No.50 in Theorem 0.1. Choose sections s_1 and s_2 in $\mathrm{MW}(\mathcal{E}_x^Q)$ in such a way that

- $\langle s_i, s_i \rangle = 2, s_i O = 0$ $(i = 1, 2)$ and
- $s_1 \in 2\,\mathrm{MW}(\mathcal{E}_x^Q)$, while $s_2 \notin 2\,\mathrm{MW}(\mathcal{E}_x^Q)$.

By Lemma 2.1, there exist even tangential conics C_{s_1} and C_{s_2} arising from s_1 and s_2, respectively. By Theorem 2.1, we have $(C_{s_1}/Q) = 1$ and $(C_{s_2}/Q) = -1$. Hence if C_{s_1} and C_{s_2} intersects Q in the same manner, we have an example for Proposition 5.1 (i). Now we go on to give explicit examples.

Example 5.1. (*cf.* [16, Example, p.198]) Let Q be an irreducible quartic given by the affine equation

$$f(t, u) = u^3 + (271350 - 98t)u^2 + t(t - 5825)(t - 2025)u + 36t^2(t - 2025)^2 = 0.$$

By taking homogeneous coordinates, $[U, T, V]$, of $\mathbb{P}^2$ in such a way that $u = U/V, t = T/V$, we easily see that $[1, 0, 0]$ is a smooth point of Q. Choose $[1, 0, 0]$ as the distinguished point x. We easily see that the tangent line l_x is given by $V = 0$, and $I_x(l_x, Q) = 3$. The elliptic surface $\varphi_x^Q : \mathcal{E}_x^Q \to \mathbb{P}^1$ corresponding to Q and x is given by a Weierstrass equation

$$y^2 = f(t, u).$$

By [16, Example, p.198], $\mathcal{E}_x^Q$ satisfies the following properties:

(i) φ_x^Q has 3 reducible singular fibers over $t = 0, 2025, \infty$, whose types are: I_2 over $t = 0, 2025$ and III over $t = \infty$. This implies Q has $2A_1$ as its singularities.
(ii) $\mathrm{MW}(\mathcal{E}_x^Q) \cong D_4^* \oplus A_1^*$.

Choose three sections of $\mathcal{E}_x^Q$ given by [16] as follows:

$$s_o : (0, 6t^2 - 12150t),\ \tilde{s}_1 : (-32t, 2t^2 - 6930t),\ \tilde{s}_2 : (-20t, 4t^2 - 4500t).$$

For these sections, $s_o \in A_1^*$ and $\tilde{s}_i \in D_4^*$ $(i = 1, 2)$ and we have

$$\langle s_o, s_o \rangle = \frac{1}{2},\ \langle \tilde{s}_i, \tilde{s}_i \rangle = 1\,(i = 1, 2),\ \langle \tilde{s}_1, \tilde{s}_2 \rangle = 0,$$

and there is no other section s with $\langle s, s \rangle = 1/2$ other than $\pm s_o$.

The sections given by $s_1 := 2s_o$ and $s_2 := \tilde{s}_1 + \tilde{s}_2$ are

$$s_1 = \left(\frac{1}{144}t^2 + \frac{1231}{72}t - \frac{5143775}{144}, -\frac{1}{1728}t^3 - \frac{2335}{576}t^2 + \frac{13493375}{576}t - \frac{29962489375}{1728}\right)$$

$$s_2 = \left(\frac{1}{36}t^2 + \frac{435}{2}t - \frac{921375}{4}, -\frac{1}{216}t^3 - \frac{1181}{24}t^2 - \frac{41625}{8}t + \frac{373156875}{8}\right).$$

Since $s_2 \in D_4^*$, we infer that s_1 is 2-divisible, while s_2 is not 2-divisible. Also, both s_1 and s_2 do not meet the zero section O and $\langle s_1, s_1\rangle = \langle s_2, s_2\rangle = 2$. Let C_1 and C_2 be conics given by

$$\begin{aligned} C_1 : u &= \frac{1}{144}t^2 + \frac{1231}{72}t - \frac{5143775}{144} \\ C_2 : u &= \frac{1}{36}t^2 + \frac{435}{2}t - \frac{921375}{4}. \end{aligned}$$

We infer that C_1 and C_2 are the even tangent conics corresponding to s_1 and s_2, respectively. It is a straightforward computation that, for each i, C_i is tangent to Q at four distinct points. Hence $(C_1 + Q, C_2 + Q)$ is an example for Proposition 5.1 (i).

Example 5.2. (*cf.* [16, Example, p. 210]) Let Q be an irreducible quartic given by the affine equation

$$f(t, u) = u^3 + (25t + 9)u^2 + (144t^2 + t^3)u + 16t^4 = 0.$$

We take a homogeneous coordinate $[U, T, V]$ as in the previous example. With this coordinate $[1, 0, 0]$ is a smooth point and choose $[1, 0, 0]$ as the distinguished point x. The tangent line l_x is again given by $V = 0$ and $I_x(l_x, Q) = 3$. The elliptic surface $\varphi_x^Q : \mathcal{E}_x^Q \to \mathbb{P}^1$ corresponding to Q and x is given by a Weierstrass equation

$$y^2 = f(t, u).$$

Note that we change the equation slightly. The original Weierstrass equation in [16] is $y^2 - 6uy = u^3 + 25tu^2 + (144t^2 + t^3)u + 16t^4$. By [16, Example, p. 210], $\mathcal{E}_x^Q$ satisfies the following properties:

(i) φ_x^Q has 2 reducible singular fibers over $t = 0, \infty$, whose types are: I_4 over $t = 0$ and III over $t = \infty$. This implies Q has A_3 as its singularity.

(ii) $\mathrm{MW}(\mathcal{E}_x^Q) \cong A_3^* \oplus A_1*$.

By modifying the sections given [16] slightly, take three sections of $\mathcal{E}_x^Q$ as follows:

$$s_o : (0, 4t^2),\ \tilde{s}_1 : (-16t, -48t),\ \tilde{s}_2 : (-15t, t^2 + 45t).$$

For these sections, $s_o \in A_1^*$ and $\tilde{s}_i \in A_3^*$ $(i = 1, 2)$ and we have

$$\langle s_o, s_o\rangle = \frac{1}{2},\ \langle \tilde{s}_i, \tilde{s}_i\rangle = \frac{3}{4}\,(i = 1, 2),\ \langle \tilde{s}_1, \tilde{s}_2\rangle = \frac{1}{4},$$

and there is no other section s with $\langle s, s\rangle = 1/2$ other than $\pm s_o$. The sections given by $s_1 := 2s_0$ and $s_2 := \tilde{s}_1 + \tilde{s}_2$ are

$$\begin{aligned} s_1 &= \left(\frac{1}{64}t^2 - \frac{41}{2}t + 315, -\frac{1}{512}t^3 - \frac{55}{32}t^2 + \frac{2637}{8}t - 5670\right) \\ s_2 &= \left(t^2 + 192t + 8640, -t^3 - 301t^2 - 27936t - 803520\right). \end{aligned}$$

Since $s_2 \in A_3^*$, we infer that s_1 is 2-divisible, while s_2 is not 2-divisible. Also, both $2s_o$ and $s_1 + s_2$ do not meet the zero section O and $\langle s_1, s_1\rangle = \langle s_2, s_2\rangle = 2$. Let C_1 and C_2 be conics given by

$$\begin{aligned} C_1 : u &= \frac{1}{64}t^2 - \frac{41}{2}t + 315 \\ C_2 : u &= t^2 + 192t + 8640. \end{aligned}$$

We infer that C_1 and C_2 are even tangential conics to Q corresponding to s_1 and s_2, respectivly. A straightforward computation shows that, for each i, C_i is tangent to Q at four distinct points. Hence $(C_1 + Q, C_2 + Q)$ is an example for Proposition 5.1 (i).

Remark 5.1.

(1) Zariski pairs in Examples 5.1 and 5.2 can be found in [13]. Hence our examples are not new. Our justification lies in a new point of view: quadratic residue curves.

(2) For Zariski pairs in Examples 5.1 and 5.2, there exists a Z-spitting conic for $C_1 + Q_1$, while there exists no such conic for $C_2 + Q_2$ (see [13] for the definition of Z-splitting conics). Moreover precisely, for an irreducible quartic Q with $\Xi_Q = 2A_1$ or A_3 and its even tangential conic C, one can show $(C/Q) = 1$ if and only if there exists a Z-splitting conic for $C + Q$ whose class order is 4 ([20]).

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Journal *of* Singularities
Volume 2 (2010), 191-209

Proc. of Singularities in Aarhus
received 3 September 2009
DOI: 10.5427/jsing.2010.2l

GENERIC SPACE CURVES, GEOMETRY AND NUMEROLOGY

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Abstract. A projective curve $\Gamma \in P^3(C)$ defines a stratification of P^3 according to the types of the singularities of the projection of Γ from the variable point. In this paper we calculate the degrees of these strata, assuming that Γ is projection-generic in the sense of [8].

We use geometrical properties of the stratifications of P^3 and of the blow-up B_Γ of P^3 along Γ (with exceptional set denoted E_Γ) introduced in [9] to introduce several auxiliary curves: more precisely, there are three 2-dimensional strata: the surface of tangents to Γ, the surface of T-secants (i.e. lines joining two points with coplanar tangents), and the surface of 3-secants. We obtain three plane curves by intersecting these with a generic plane. Three curves in E_Γ were introduced in [9]. We also have three curves in $\Gamma \times \Gamma$ closely related to them. Our numerical results are obtained by applying the genus and related formulae to these curves.

Introduction

A projective curve $\Gamma \subset P^3(\mathbb{C})$ defines a stratification of P^3 according to the types of the singularities of the projection of Γ from the variable point. The object of this paper is to calculate the degrees of (the closures of) these strata, and the number of special points of each type on Γ, in terms of the degree d and genus g of Γ. We will assume throughout that Γ is projection-generic in the sense of [8]. This has two advantages: in [9] we obtained local normal forms for this stratification; and we will see that we obtain precise answers, without needing to interpret our numbers as being counted with multiplicities.

Our techniques are essentially classical. We will use geometrical properties of the stratifications of P^3 and of the blow-up B_Γ of P^3 along Γ (with exceptional set denoted E_Γ) introduced in [9] to introduce several auxiliary curves: the study of these and of their interrelations is itself of some interest. More precisely, there are three 2-dimensional strata: the surface of tangents to Γ, the surface of T-secants (i.e. lines joining two points with coplanar tangents), and the surface of 3-secants. We obtain 3 plane curves Π_* by intersecting these with a generic plane. Three curves E_* in E_Γ were introduced in [9]. We also have three curves T_* in $\Gamma \times \Gamma$ closely related to them. Our numerical results are obtained by applying the genus and related formulae to these curves.

We introduce our notation in §1, and describe the singularities of the auxiliary curves in §2 (in the case of the T_* the proofs are deferred to §7). In §3 we begin calculations, and in Lemma 3.2 express many of our degrees in terms of parameters k_i. In §4 we analyse the correspondences T_*, give the degrees of the 2-dimensional strata in Proposition 5.1, and calculate the k_i in Proposition 4.2. In §5 we analyse the E_* and in Proposition 5.2 complete the count of points of special types on Γ. In §6 we analyse the Π_*, and determine the degrees of the remaining curve strata, which we list in Theorem 6.1. After some experiment, I have settled on collecting all these formulae by powers of g, since it turns out that in all but two cases, the term not involving g factors over $\mathbb{Q}[d]$ into linear factors.

2000 *Mathematics Subject Classification.* 14H50, 14N10.

1. Recall of results and notations of [8]

If Γ is a smooth space curve, the projections Γ_P of Γ from a variable point P of space give a 3-parameter family of maps to the plane. In [8], we analysed this situation in the real C^∞ case, and gave explicit genericity conditions, (PG1)-(PG6) below, defining a set of space curves which is open and dense in the family of all such curves. These conditions also make sense in the complex case, and (as in [9]) we will assume them throughout the paper. Among projective algebraic curves, it is not clear that those satisfying the conditions form a dense set, though this should not be hard to establish at least for rational curves of high enough degree. In practice, this fails for degree 1 or 2, and seems to hold for degrees ≥ 3.

The projection along a line L through P has a singular point if L meets Γ in more than 1 point, or if L is tangent to Γ. Call a line meeting Γ in r points an $r-$secant; it is a $T-r-$secant if the tangents at 2 of the points are coplanar (we omit r if $r = 2$). Write $T_Q\Gamma$ for the tangent at a point $Q \in \Gamma$, and $O_Q\Gamma$ for the osculating plane at Q.

Hypothesis (PG1) is that the family of projections of Γ from points not on Γ is generic in the sense that it versally unfolds the singularities of any curve of the set. More precisely, the induced map is transverse to each stratum; thus the unfolding is versal in each case except that of the X_9 stratum (quadruple points), where we only have topological versality.

It follows that the curvature of Γ is non-zero, but the torsion may vanish: if it vanishes at Q, we call Q a stall. Equivalently, here the local intersection number of Γ with $O_Q\Gamma$ exceeds 3; the hypothesis implies that it is at most 4.

It also follows that for $P \in P^3 \setminus \Gamma$, the types of singularities of the projection have codimension ≤ 3, and the points P such that the sum of the codimensions of singularities of Γ_P is c form smooth $(3 - c)$-dimensional manifolds which regularly stratify $P^3 \setminus \Gamma$; normal forms are given by model versal unfoldings of the singularities that occur (except for the X_9 stratum; a precise normal form for this case was given in [9, Lemma 7.2]).

We partition $P^3 \setminus \Gamma$ as follows. If Σ denotes a list of singularities, $S^o(\Sigma)$ consists of points P such that Σ is the list of singularities of Γ_P. Define also

$S(\Sigma)$ is the closure of $S^o(\Sigma)$ in P^3, and

$n(\Sigma)$ is the degree of $S(\Sigma)$.

We will calculate these degrees in all cases where $S(\Sigma)$ has dimension 1 or 2.

For a codimension 0 set of projections we have only normal crossing (A_1) singularities. Apart from these, in codimension 1, Γ_P can have

a cusp (A_2) if, for some $Q \in \Gamma$, $P \in T_Q\Gamma$;

a tacnode (A_3) if P lies on a T-secant QR of Γ; or

a triple point (D_4) if P lies on a 3-secant QRS of Γ.

For a codimension 2 set of points P, Γ_P can have two codimension 1 singularities, or one of A_4, A_5, D_5, D_6 or X_9 (in Arnold's notation, but here we have maps $\mathbb{C} \to \mathbb{C}^2$, not functions on $\mathbb{C}^2$). In codimension 3 we have A_6, A_7, D_8 and combinations of singularities of lower codimension.

In particular, any T-secant contains a unique point, its T-centre, projection of Γ from which gives an A_5, rather than an A_3 singularity; for a T-3-secant a D_8 rather than a D_6.

We also contemplate projections of Γ from points of itself. Since Γ has nowhere zero curvature, each projection $\Gamma_P := \pi_P(\Gamma)$ with $P \in \Gamma$ is well defined and is again given by a smooth map. We can, at least locally, regard $\{\Gamma_P\}$ as a 1-parameter family of parametrised plane curves. Hypothesis (PG2) is that the family of projections Γ_P of Γ from points $P \in \Gamma$ has generic singularities.

To fit these into the family of projections from points $P \notin \Gamma$, write $\pi_\Gamma : B_\Gamma \to P^3$ for the blow-up along Γ and E_Γ for the exceptional set; thus a point of E_Γ is a pair (P, Π) with $P \in \Gamma$

and Π a plane through $T_P\Gamma$. There is a natural projection $\pi_E : E_\Gamma \to \Gamma$, which is a fibre bundle with fibre a projective line. Now define a family of curves $\{\Phi_z : z \in B_\Gamma\}$ by:

if $z \notin E_\Gamma$, so $z \in B_\Gamma \setminus \Gamma$, set $\Phi_z := \Gamma_z$,

if $z = (P, \Pi) \in E_\Gamma$, set $\Phi_z := \Gamma_P \cup L$, where $L := \pi_P(\Pi)$.

Thus the line L goes through the point $Y_P := \pi_P(T_P\Gamma)$. This is a flat family: near any point there is a smooth function whose zero set meets the fibre over z in Φ_z.

If Φ, Φ' are plane curves and $P \in \Phi \cap \Phi'$, define $\kappa_P(\Phi, \Phi')$ to be the local intersection number at P minus 1; write $\kappa(\Phi, \Phi')$ for the sum over all $P \in \Phi \cap \Phi'$. We will call $P \in \Gamma$ a *special point* if either Γ_P fails to have normal crossings or, for some line L through Y_P, $\kappa(L, \Gamma_P) \geq 2$: condition (PG4) is that we always have $\kappa(L, \Gamma_P) \leq 2$.

We next list the types of special point on Γ and our notation for them (we follow [9] rather than [8]). If Γ_P itself fails to have normal crossings, it has a singular point Z_P. There are three cases, according to the type of the singularity.

α: type A_2: $P \in T_Q\Gamma$ for some $Q \in \Gamma$,

β: type A_3: we have a T-3-secant $P(QR)$,

γ: type D_4: we have a 4-secant $PQRS$.

Hypothesis (PG3) is that for cases α, β and γ, Y_PZ_P is transverse to Γ_P at all points.

We say P has type δ if Y_P is a double point on Γ_P, i.e. if $Q \in T_P\Gamma$ for some $Q \neq P$.

If Γ_P has normal crossings and $\Gamma_P \cup L$ does not, then $\kappa(L, \Gamma_P) > 0$. Excluding cases α, β, γ, δ, we have $\kappa(L, \Gamma_P) = 1$ if either

a: L touches Γ_P at Y_P ($\Pi = O_P\Gamma$),

b: L touches Γ_P elsewhere (Π contains a tangent line $T_Q\Gamma$), or

c: L passes through a node of Γ_P (Π contains a trisecant PQR).

These cases occur when the point (P, Π) lies on certain curves in E_Γ. We denote these curves by E_a, E_b, E_c respectively.

In all cases, $\kappa(L, \Gamma_P) \leq 2$. The cases $\kappa(L, \Gamma_P) = 2$ are enumerated as follows, where $q, r, \ldots$ denote the images of $Q, R, \ldots$ under projection from P.

ab: L touches Γ_P at p and q: $T_Q\Gamma \subset O_P\Gamma = \Pi$.

ac: L touches Γ_P at p and goes through a double point $r = s$: $PRS \subset O_P\Gamma = \Pi$.

bb: L touches Γ_P at q and r: $T_P\Gamma$, $T_Q\Gamma$ and $T_R\Gamma$ lie in Π.

bc: L touches Γ_P at q and passes through a double point $r = s$: Π contains $T_P\Gamma$, $T_Q\Gamma$ and the trisecant PRS.

cc: L_P passes through double points $q = r$ and $s = t$: Π contains $T_P\Gamma$ and the trisecants PQR and PST.

a_2: L_P is an inflexional tangent at p: P is a stall on Γ, $\Pi = O_P\Gamma$.

b_2: L_P is an inflexional tangent at q: $T_P\Gamma \subset O_Q\Gamma = \Pi$.

c_2: L is tangent at q to a double point $q = r$ of Γ_P: we have a T-trisecant PQR with $T_P\Gamma$, $T_Q\Gamma$ both in Π.

Hypothesis (PG5) states that the curve E_c has transverse intersections with E_a, E_b and E_c at $S(ac)$, $S(bc)$ and $S(cc)$ respectively.

Finally, hypothesis (PG6) is that for no $P \in \Gamma$ can we have more than one of the cases α, β, γ, δ, ab, ac, bb, bc, cc, a_2, b_2, c_2.

If $P \in \Gamma$ is not of type α, β, γ or δ, there is at most one line L through Y_P with $\kappa(L, \Gamma_P) \geq 2$. Thus if there is a special point of E_Γ in $\pi_E^{-1}(P)$, it is unique and we denote its type by the same symbol as above.

If Γ_P has a singular point Z_P, and (P, Π) such that $L = Y_PZ_P$, we also denote the type of (P, Π) by the same letter as for P. For P of type δ we distinguish (P, Π) of type δ_1, with

$\Pi = O_P\Gamma$ and type δ_2, when Π passes through $T_Q\Gamma$: in each of these, L touches Γ_P at Y_P.

If X is any of α, β, γ, δ, ab, ac, bb, bc, cc, a_2, b_2, c_2 we write $S(X)$ for the set of points of Γ of type X, and $\#(X)$ for its cardinality. We calculate all the numbers $\#(X)$ below. We extend the stratification of $P^3 \setminus \Gamma$ to P^3 by declaring the strata in Γ to be the $S(X)$ just defined, and the rest of Γ to be a single stratum. We proved in [9] that this stratification is regular, and obtained normal forms at all points of Γ. We will make crucial use of these in this paper.

We now define our auxiliary curves. First, we have the three curves E_a, E_b and E_c in E_Γ defined above.

Next we have three curves in $\Gamma \times \Gamma$:

$(P, Q) \in T_a$: if $P \in O_Q\Gamma$,

$(P, Q) \in T_b$: if PQ is a T-secant,

$(P, Q) \in T_c$: if PQ is a trisecant.

For $Y \subset \Gamma \times \Gamma$, denote by Y^t the image of Y under interchange of factors.

There are just three 2-dimensional strata in P^3, which we will denote by $A := S(A_2)$, the surface of tangents to Γ; by $B := S(A_3)$, the surface of T-secants; and by $C := S(D_4)$ the surface of trisecants. Each of these is, by definition, a ruled surface. Choose a plane Π_0 transverse to all strata of the stratification of P^3, and define three curves in Π_0 by

$\Pi_a := \Pi_0 \cap A$, $\Pi_b := \Pi_0 \cap B$, $\Pi_c := \Pi_0 \cap C$.

2. Singularities of the auxiliary curves

We first consider the curves in E_Γ, which were analysed in our previous work. We are interested in singularities of the curves in relation to the projection π_E, which is a submersion on Γ. The result is as follows.

Lemma 2.1. [9, Theorem 1.2, Addendum 4.3, Lemma 4.4, Addendum 5.2] *The curves E_a, E_b and E_c in E_Γ are smooth, disjoint and submerse on Γ except as below.*

At a point of type ab, ac, bb, bc or cc, the two curves meet transversely.

At $S(a_2)$, E_a and E_b touch (simply).

At $S(b_2)$, E_b has a cusp with non-vertical tangent.

At $S(c_2)$, E_b and E_c touch.

At $S(\alpha)$, E_b and E_c meet transversely.

At $S(\beta)$, E_c touches the fibre.

At $S(\gamma)$, E_c has 3 transverse branches.

At $S(\delta_1)$, E_a and E_c meet transversely.

At $S(\delta_2)$, E_b touches the fibre.

Next we consider the plane sections Π_a, Π_b, Π_c of A, B and C. These have the same degrees as the corresponding surfaces, and have singularities only where the plane meets a singular set of the surface. If Γ has degree d, Π meets Γ in d points, and for any curve stratum $S(\Sigma)$ in $P^3 \setminus \Gamma$, Π meets $S(\Sigma)$ transversely in $n(\Sigma)$ points, and the local picture of strata at each is given by that in a versal unfolding of Σ. Denote by m_a, m_b and m_c the respective multiplicities of A, B and C along Γ.

We recall that B is the surface of tangents to $S(A_5)$, so has a cuspidal edge along $S(A_5)$. We do not need to list intersection points of these plane sections, since we obtain the same information from the mutual intersections of the surfaces A, B and C.

Lemma 2.2. *The curve Π_a has d A_2 singularities and $n(2A_2)$ A_1 singularities.*

The curve Π_b has $n(2A_3)$ A_1 singularities, $n(A_5)$ A_2 singularities, and d ordinary singular points of multiplicity m_b.

The curve Π_c has $n(2D_4)$ A_1 singularities, $n(X_9)$ X_9 singularities, and d ordinary singular points of multiplicity m_c.

To describe the singularities of the curves in $\Gamma \times \Gamma$ we need further notations. First we define notation for the points of $\Gamma \times \Gamma$ which play a rôle: these are related to special lines $PQ \in P^3$.

A special line of type A_4 is tangent to Γ at a stall P: a point of $S(a_2)$. Define $(P, P) \in W_1$, and W_1' to consist of points (Q, P) with $Q \neq P$, $Q \in O_P\Gamma$.

A special line of type D_5 is a secant PQ with $Q \in T_P\Gamma$: P has type δ, Q has type α. Define $(P, Q) \in W_2$, $(P, P) \in W_2'$.

A special line of type D_6 is a T-trisecant PQR with $T_Q\Gamma$, $T_R\Gamma$ coplanar: P has type β, Q, R have type c_2. Define $(P, Q) \in W_3$, $(Q, R) \in W_3'$.

A special line of type X_9 is a 4-secant $PQRS$: P, Q, R, S have type γ. Define $(P, Q) \in W_4$.

We also have special lines PQ where $T_P\Gamma \subset O_Q\Gamma$: P has type b_2, Q has type ab. Define $(P, Q) \in W_5$.

Finally, we have special lines which are trisecants $PQR \subset O_P\Gamma$: P has type ac. Here Q, R are not special in the sense of [8], but the projections Γ_Q, Γ_R each have a flecnode. Define $(P, Q) \in W_6$.

For $T \subset \Gamma \times \Gamma$, write $I_1(T)$ for the set of singular points of projection on the first factor (classically known as coincidence points), and $I_2(T)$ for singular points of the second projection, so that $I_1(T^t) = (I_2(T))^t$. We call a point $(P, Q) \in I_1(T)$ simple if T is smooth at (P, Q) and the local intersection number of T with $\{P\} \times \Gamma$ is 2; similarly for $I_2(T)$. We now describe these points, and also the intersections of the T_* with the diagonal $\Delta \subset \Gamma \times \Gamma$ (classically known as united points) and with each other. The proofs will require detailed calculations, which we defer to §7.

Theorem 2.3. *(i) We have $T_a \cap \Delta = T_b \cap \Delta = W_1$, $T_c \cap \Delta = W_2'$. At W_1, the tangent to T_a is $3t_p + t_q = 0$, and to T_b is $t_p + t_q = 0$.*

(ii) We have $I_1(T_a) = W_5^t$, $I_2(T_a) = W_1'^t \cup W_2$, $I_1(T_b) = W_2 \cup W_5$, $I_1(T_c) = W_2^t \cup W_3 \cup W_4$, $I_2(T_c) = W_2 \cup W_3^t \cup W_4$. All coincidence points except W_4 are simple.

(iii) The curves T_a and T_b are smooth; the singularities of T_c are simple nodes at points of type W_4, with 2 transverse branches, each tangent to neither fibre.

(iv) We have $T_a \cap T_b = W_1 \cup W_2^t \cup W_5$, $T_a \cap T_c = W_2^t \cup W_6$, and $T_b \cap T_c = W_2 \cup W_2^t \cup W_3'$.

(v) The intersection number at each of these common points is $+1$, except that at W_2^t the intersection number of T_a^t and T_c is 2.

3. Preliminaries

We denote the degrees of the 2-dimensional strata by $d_a = n(A_2)$, $d_b = n(A_3)$ and $d_c = n(D_4)$. Denote also by m_a, m_b and m_c the respective multiplicities of A, B and C along Γ.

Lemma 3.1. *The multiplicities along Γ in P^3 are $m_a = 2$, $m_b = 2(d - 3 + g)$ and $m_c = \frac{1}{2}(d - 2)(d - 3) - g$.*

Proof. Since, as is well known, the tangent surface A has a cuspidal edge along Γ, $m_a = 2$.

The projection of Γ from a general point P of itself is a plane curve Γ_P with degree $d-1$ and genus g, whose only singularities are simple nodes (type A_1). It follows from Plücker's formulae that such a curve has class $2(d - 2 + g)$ and that the number of nodes is $\frac{1}{2}(d - 2)(d - 3) - g$.

Now T-secants of Γ through P project to tangents from Y_P to Γ_P, hence there are just $2(d - 3 + g)$ of them; and trisecants through P project to lines joining Y_P to nodes of Γ_P. The result follows. ☐

The strata $S(A_4)$, $S(D_5)$, $S(D_6)$, $S(X_9)$ are unions of straight lines: write k_1, k_2, k_3, k_4 for the numbers of these lines, k_5 for the number of tangents to Γ which lie in an osculating plane at a different point, and k_6 for the number of trisecants PQR lying in the osculating plane $O_P\Gamma$. Several of our degrees can easily be expressed in terms of the k_i.

Lemma 3.2. *We have*

(i) $\#(a_2) = n(A_4) = n(A_6) = \#(W_1) = k_1$, *and* $\#(W_1') = (d-4)k_1$.
(ii) $\#(\alpha) = \#(\delta) = n(D_5) = \#(W_2) = \#(W_2') = k_2$.
(iii) $\#(\beta) = n(D_6) = n(D_8) = k_3$, $\#(c_2) = \#(W_3) = \#(W_3') = 2k_3$.
(iv) $\#(\gamma) = 4n(X_9) = 4k_4$, $\#(W_4) = 12k_4$.
(v) $\#(ab) = \#(b_2) = \#(W_5) = k_5$.
(vi) $\#(ac) = k_6$ *and* $\#(W_6) = 2k_6$.

Proof. (i) holds since $S(A_4)$ consists of k_1 lines, each of which is a tangent at a stall P, and contains a unique point of $S(A_6)$, and there are $\#(a_2)$ stalls. Moreover, each contributes one point $(P,P) \in W_1$, and $O_P\Gamma$ has intersection number 4 with Γ at P, hence there are $d-4$ further intersections Q with $(Q,P) \in W_1'$ (note that $O_P\Gamma$ cannot be tangent at a further point).

(ii) holds since $S(D_5)$ consists of k_2 lines, each of which is a tangent at a point $P \in S(\delta)$, meeting Γ again at a point $Q \in S(\alpha)$; it contributes one point (P,Q) to W_2 and one point (Q,Q) to W_2'.

(iii) holds since $S(D_6)$ consists of k_3 lines, each of which is a T-trisecant, meeting Γ in two points P, Q with coplanar tangents, each in $S(c_2)$, and one other point $R \in S(\beta)$, and contains a unique point of $S(D_8)$. It also contributes 2 points (R,P), (R,Q) to W_3 and 2 points (P,Q), (Q,P) to W_3'.

(iv) holds since $S(X_9)$ consists of k_4 4-secants $PQRS$, each of which meets Γ in 4 points of $S(\gamma)$. Any ordered pair from $PQRS$ gives a point of W_4.

(v) holds since k_5 counts the secants PQ with $T_P\Gamma \subset O_Q\Gamma$, and the point $P \in S(b_2)$, $Q \in S(ab)$, and $(P,Q) \in W_5$.

(vi) holds since there are k_6 trisecants $PQR \subset O_P\Gamma$, and $P \in S(ac)$, (P,Q) and (P,R) belong to W_6. $\square$

We will make frequent use of the genus formula for a curve on an algebraic surface. The following version is the most convenient for us, since it does not assume the curve irreducible:

If M is a reduced curve on a smooth surface S with canonical class K_S, we have

$$[M].([M] + K_S) = \mu(M) - \chi(M).$$

Here $\mu(M)$ denotes the total Milnor number and $\chi(M)$ the (topological) Euler characteristic of the Riemann surface M. This formula is easily deduced from the traditional version (see e.g. [1, 1.15]). It also follows from a routine topological argument (see e.g. [7, Theorem 6.4.1]) that the numbers $\mu(M_t) - \chi(M_t)$ are constant in a family of curves M_t.

In particular, if M is a plane curve of degree d, we obtain the Plücker relation $\mu(M) - \chi(M) = d(d-3)$.

Next we need the Plücker relations for space curves which are given in [3, p.270]. Let Δ be a reduced and non-planar space curve (it need not be projection-generic). We have invariants:

$\chi(\Delta)$, the Euler characteristic of Δ,
$r_0(\Delta)$, the degree, the number of points in which Δ meets a general plane,
$r_1(\Delta)$, the rank, the number of tangent lines to Δ meeting a general line,
$r_2(\Delta)$, the class, the number of osculating planes of Δ containing a general point.

At a point P where, in some local co-ordinates, we have local parametrisations

$$x_1 = a_1 t^{b_1} + \dots, \quad x_2 = a_2 t^{b_2} + \dots, \quad x_3 = a_3 t^{b_3} + \dots,$$

with the $a_i \neq 0$ and $0 < b_1 < b_2 < b_3$, we set $b_0 := 0$ and define

$$s_i(P) := b_{i+1} - b_i - 1 \quad (i = 0, 1, 2).$$

At all but finitely many $P \in \Delta$, $s_0 = s_1 = s_2 = 0$, so we can define

$$s_i(\Delta) = \sum_{P \in \Delta} s_i(P).$$

In fact, in the cases arising below, we do not encounter points P with $\sum_i s_i(P) > 1$. We call s_0 the number of cusps, s_1 the number of flexes, and s_2 the number of stalls.

The set of osculating planes to Δ is a curve $\Delta^\vee$ in the dual projective space $P^\vee$. We call $\Delta^\vee$ the *dual* curve of Δ. The elementary projective characters of $\Delta^\vee$ are $r_i^\vee = r_{2-i}$, $s_i^\vee = s_{2-i}$ $(i = 0, 1, 2)$.

The following are partial analogues for space curves of the Plücker formulae.

Lemma 3.3. [5, Cor.5.3, p.491] [3, p.270] *For Δ a reduced and non-planar space curve, we have $-\chi(\Delta) - s_0 = -2r_0 + r_1$, $-\chi(\Delta) - s_1 = r_0 - 2r_1 + r_2$, $-\chi(\Delta) - s_2 = r_1 - 2r_2$.*

Proof. We claim that projecting Δ from a general point P gives a plane curve with Euler characteristic $-\chi(\Delta)$, degree r_0, class r_1, with s_0 cusps and $s_1 + r_2$ flexes. Applying the Plücker formulas to this projection gives the first two relations; the same argument for the dual curve yields the third.

To justify the claim, note that if $P \notin \Delta$, projection does not change the degree. If P lies on no tangent, projection introduces no new cusp. A general P lies on the osculating planes at just r_2 distinct ordinary points; thus projecting from P adds r_2 to the number of flexes. A general line through a general point P is a general line; through it pass r_1 tangent planes to C, so its projected image lies on r_1 tangents to C_P. □

Apply Lemma 3.3 to the curve Γ. Here $s_0 = s_1 = 0$ since Γ is smoothly embedded and the curvature does not vanish, and $r_0 = d$. Since Γ is smooth and connected of genus g, $\chi(\Gamma) = 2-2g$. It thus follows from the lemma that $r_1 = 2d + 2g - 2$, $r_2 = 3d + 3g - 6$ and $s_2 = 4d + 12g - 12$. Now it follows from the definitions that $d_a = r_1$ and $k_1 = s_2$. Hence we have

$$d_a = 2d - 2 + 2g, \quad k_1 = 4d - 12 + 12g. \tag{1}$$

4. Correspondences in $\Gamma \times \Gamma$

In this section we evaluate the constants k_i by studying the correspondences T_*.

In general, a curve $T \subset \Gamma \times \Gamma$ is called a *correspondence* on Γ. We denote the degrees of the projections on the factors by $d_1(T)$, $d_2(T)$, thus $d_1(T^t) = d_2(T)$. We need the notion of *valence*: see e.g. [3, pp 284] for further details.

For $P \in \Gamma$, write $T(P) = \{Q \mid (P, Q) \in T\}$: we can consider this as a divisor on Γ if we count multiplicities appropriately. Then T has valence k if the linear equivalence class of $T(P) + kP$ is independent of P. We will denote the valence of T by $v(T)$; we have $v(T^t) = v(T)$. For the above cases, we have

Lemma 4.1. *(compare* [3, pp 291-5]*)*
T_a has $d_1(T_a) = d - 3$, $d_2(T_a) = 3d + 6g - 9$ and $v(T_a) = 3$.
T_b has $d_1(T_b) = m_b = 2d + 2g - 6$ and $v(T_b) = 4$.
T_c has $d_1(T_c) = 2m_c = (d - 2)(d - 3) - 2g$ and $v(T_c) = d - 4$.

Proof. Consider the projection π_P of Γ from P with image Γ_P; recall that Y_P denotes the image of the tangent at P. As in the proof of Lemma 3.1, Γ_P has degree $d-1$ and genus g, and hence by Lemma 3.3 has class $2(d-2+g)$, $m_c = \frac{1}{2}(d-2)(d-3)-g$ nodes, and $3(d+2g-3)$ flexes. Now

$(P,Q) \in T_a$ if $\pi_P(Q)$ is a flex of Γ_P,
$(P,Q) \in T_a^t$ if $\pi_P(Q)$ lies on the tangent at Y_P,
$(P,Q) \in T_b$ if Y_P lies on the tangent to Γ_P at $\pi_P(Q)$,
$(P,Q) \in T_c$ if $\pi_P(Q)$ is a node of Γ_P.

Hence $d_1(T_a)$ is the number of further intersections with Γ_P of the tangent at Y_P, so is 2 less than the degree; $d_2(T_a)$ is equal to the number of flexes of Γ_P; $d_1(T_b)$ is the number of tangents from Y_P, which is equal to the class, diminished by 2 to allow for the tangent at Y_P itself; and $d_1(T_c)$ is equal to double the number of nodes of Γ_P, since each contributes two points Q.

For the valences we argue following [3, p 295].

For T_a, consider the projection $\pi_P : \Gamma \to P^2$ from P. The canonical class $K_\Gamma = \pi_P^*(-3H_{P^2}) + T_a(P)$, and $\pi_P^* H_{P^2} = H_{P^3} - P$, so $T_a(P) + 3P = K_\Gamma + 3H_{P^3}$.

For T_b, consider the projection $\pi_L : \Gamma \to P^1$ from $T_P\Gamma$. Then the canonical class $K_\Gamma = \pi_L^*(-2H_{P^1}) + T_b(P)$, and $\pi_L^* H_{P^1} = H_{P^3} - 2P$, so $T_b(P) + 4P = K_\Gamma + 2H_{P^3}$.

For T_c, as on [3, p 291] we have $K_\Gamma = \pi_P^*((d-4)H_{P^2}) - D$, where D is the preimage of the double points of Γ_P, and hence is $T_c(P)$. Again using $\pi_P^* H_{P^2} = H_{P^3} - P$, we find $T_c(P) + (d-4)P = (d-4)H_{P^3} - K_\Gamma$, giving valence $(d-4)$. □

It is shown on [3, p.285] that a correspondence T with valency has the divisor class of

$$(d_1(T)+v(T))(* \times \Gamma) + (d_2(T)+v(T))(\Gamma \times *) - v(T)\Delta(\Gamma),$$

where $\Delta(\Gamma)$ denotes the diagonal. Since the diagonal has self-intersection number $2-2g$, we can now calculate all intersection numbers. In particular,

$$T.\Delta(\Gamma) = d_1(T) + d_2(T) + 2gv(T), \tag{2}$$

$$T.T' = d_1(T)d_2(T') + d_2(T)d_1(T') - 2gv(T)v(T'). \tag{3}$$

We also apply the genus formula. Since the canonical class is $(2g-2)(* \times \Gamma + \Gamma \times *)$, this gives

$$-\chi(T) = 2d_1(T)d_2(T) + (2g-2)(d_1(T)+d_2(T)) - 2gv(T)^2 - \mu(T). \tag{4}$$

We can now count the intersections of T_a, T_b and T_c with Δ in two ways: they are enumerated in Theorem 2.3 (i) and shown to have multiplicity 1, and then counted in Lemma 3.2, giving the numbers k_1, k_1 and k_2; or we can use (2), with the values given by Lemma 4.1. The first two confirm the calculation $k_1 = 4d+12g-12$ of (1); the third gives

$$k_2 = 2(d-2)(d-3) + 2g(d-6). \tag{5}$$

We can also obtain $\chi(T)$ in two ways. Projecting on the first factor gives $d_1(T)\chi(\Gamma)$, diminished by the effect of ramification, thus if the coincidence points are simple, we obtain $d_1(T)(2-2g) - I_1(T)$. Now coincidence points of T_a, T_b and T_c were enumerated in Theorem 2.3 (ii) and shown to be simple, and then counted in Lemma 3.2. Secondly, we can use (4), with the values given by Lemma 4.1. Applying this to T_a, T_a^t and T_b yields

$-\chi(T_a) = (d-3)(2g-2) + k_5$,
$-\chi(T_a) = (3d+6g-9)(2g-2) + (d-4)k_1 + k_2$,
$-\chi(T_a) = 2(d-3)(3d+6g-9) + (2g-2)(4d+6g-12) - 18g$;
$-\chi(T_b) = (2d+2g-6)(2g-2) + k_2 + k_5$,

$-\chi(T_b) = 2(2d+2g-6)^2 + (2g-2)(4d+4g-12) - 32g$.
In view of the values of k_1 and k_2 given by (1) and (5), the second and third of these equations yield the same value; comparing with the first then gives

$$k_5 = 6(d-3)(d-4) + 6g(3d-14) + 12g^2; \tag{6}$$

and now the other equations both give the same value for $\chi(T_b)$.

Similarly, the intersection numbers $T_a.T_b$, $T_a.T_c$ and $T_b.T_c$ can be computed either using Theorem 2.3 (iv) and (v), and Lemma 3.2 or using (3), with the values given by Lemma 4.1. Comparing the results gives
$k_1 + k_2 + k_5 = (4d+6g-12)(2d+2g-6) - 24g$,
$2k_2 + 2k_6 = (4d+6g-12)((d-2)(d-3)-2g) - 6g(d-4)$,
$2k_2 + 2k_3 = 2(2d+2g-6)((d-2)(d-3)-2g) - 8g(d-4)$.
Here the first is an identity in view of the known values of k_1, k_2 and k_5; the others yield values for k_6 and k_3.

Finally, applying the same procedure as for T_a and T_b to T_c, but now taking account of the fact that $\mu(T_c) = 12k_4$, gives
$-\chi(T_c) = ((d-2)(d-3)-2g)(2g-2) + k_2 + k_3 + 12k_4$,
$-\chi(T_c) = 2((d-2)(d-3)-2g)^2 + (4g-4)((d-2)(d-3)-2g) - 2g(d-4)^2 - 12k_4$.
Here substituting the known values of k_2 and k_3 allows us to solve for k_4. Collecting our results gives

Proposition 4.2. *We have*
$k_1 = 4(d-3) + 12g$,
$k_2 = 2(d-2)(d-3) + 2g(d-6)$,
$k_3 = 2(d-2)(d-3)(d-4) + 2g(d^2-10d+26) - 4g^2$,
$k_4 = \frac{1}{12}(d-2)(d-3)^2(d-4) - \frac{1}{2}g(d^2-7d+13) + \frac{1}{2}g^2$,
$k_5 = 6(d-3)(d-4) + 6g(3d-14) + 12g^2$,
$k_6 = 2(d-2)(d-3)(d-4) + 3g(d^2-8d+18) - 6g^2$.
We also have
$\chi(T_a) = -2(d-3)(3d-13) - 10g(2d-9) - 12g^2$,
$\chi(T_b) = -8(d-3)(d-4) - 8g(3d-14) - 16g^2$,
$\chi(T_c) = -(d+1)(d-2)(d-3)(d-4) + 6g(d-5) + 6g^2$.

The number k_1 of stalls comes from the Plücker relations. The number k_2 of tangents meeting Γ again was first given by Cayley [2], and the number k_4 of 4-secants was first given by Salmon 1868; with a fuller proof given by Zeuthen [11]. In [6] a formula for k_2 is given for arbitrary curves; applying this to the dual curve $\Gamma^\vee$ yields a formula for $k_5(\Gamma)$. The formulae for k_3 and k_6 appear to be new.

5. Curves in E_Γ

In this section, by studying the curves E_a, E_b and E_c, we complete the evaluation of numbers of types of special points on Γ.

At a general point of each of these curves, the projection π_E induces a submersion on Γ. The list of exceptions was given in Lemma 2.1. In particular, the projection of E_a on Γ is an isomorphism. The degrees of the projections of E_b and E_c to Γ coincide with the multiplicities along Γ of the surfaces B and C, hence are equal to m_b and m_c respectively. Applying Lemma 2.1, we obtain formulae for the Euler characteristics of E_b and E_c and for the mutual intersection numbers as follows.

$$
(7)\qquad
\begin{array}{rclcl}
E_a.E_b & = & \#(ab) + 2\#(a_2) & = & \#(ab) + 2k_1. \\
E_a.E_c & = & \#(ac) + \#(\delta) & = & \#(ac) + k_2. \\
E_b.E_c & = & \#(bc) + \#(\alpha) + 2\#(c_2) & = & \#(bc) + k_2 + 4k_3. \\
\chi(E_b) & = & m_b\chi(\Gamma) - \#(bb) - \#(b_2) - \#(\delta) & = & m_b(2-2g) - \#(bb) - k_5 - k_2. \\
\chi(E_c) & = & m_c\chi(\Gamma) - \#(cc) - \#(\beta) - 2\#(\gamma) & = & m_c(2-2g) - \#(cc) - k_3 - 8k_4.
\end{array}
$$

The group of divisors on B_Γ is free on the classes $[H]$ of a (pulled back) plane and $[E]$ of E_Γ. The strict transform $\hat{A}$ of A is obtained from the total transform by subtracting $[E]$ multiplied by the multiplicity m_a of $S(A_2)$ along Γ; similarly for B and C. Thus

$$
(8)\qquad [\hat{A}] = d_a[H] - m_a[E], \quad [\hat{B}] = d_b[H] - m_b[E], \quad [\hat{C}] = d_c[H] - m_c[E].
$$

Taking intersections with E_Γ defines a map from divisors on B_Γ to those on E_Γ. Since the blow up of a point in a surface gives a curve of self-intersection -1, the self-intersection of E_Γ in B_Γ has the class $-[D]$, where D is the class of a section of the bundle $E_\Gamma \to \Gamma$. Denote by $[F]$ the class in E_Γ of a fibre. Then since a plane meets Γ in d points, the trace of $[H]$ on E_Γ is $d[F]$.

The surface E_Γ is a P^1-bundle over Γ, associated to a plane bundle E. This situation is described in Beauville [1, III,18]: the group of divisors of E_Γ is free on $[D]$ and $[F]$, we have

$$
(9)\qquad [D].[D] = k_0, \quad [D].[F] = 1, \quad [F].[F] = 0,
$$

where $k_0 = deg\ E$, and the canonical class is $K_E = -2[D] + (deg\ E + 2g - 2)[F]$.

In fact, we have $k_0 = -4d - 2g + 2$. To see this, we can apply the adjunction formula to the blow-up $B_\Gamma \to P^3$ to see that K_E is the pullback of $K_P + 2[E] = -4[H] + 2[E]$, hence is $-4d[F] - 2[D]$.

According to [9, Corollary 7.3.1], the surface A touches E_Γ along the curve a, and also meets it in the fibres over the points of $S(\alpha)$; B meets E_Γ in b, the fibres over $S(\beta)$, and fibres over $S(\delta)$ counted twice; and C meets E_Γ in c and the fibres over $S(\gamma)$. It follows from (8) by taking traces on E (recall that $m_a = 2$) that we have divisors

$$
(10)\qquad [E_a] = [D] + c_a[F], \quad [E_b] = m_b[D] + c_b[F], \quad [E_c] = m_c[D] + c_c[F],
$$

where

$$
\begin{array}{rclcl}
c_a & = & \frac{1}{2}(dd_a - \#(\alpha)) & = & \frac{1}{2}(dd_a - k_2), \\
c_b & = & d_b d - \#(\beta) - 2\#(\delta) & = & dd_b - k_3 - 2k_2, \\
c_c & = & d_c d - \#(\gamma) & = & dd_c - 4k_4.
\end{array}
$$

Using (9) and (10) gives formulae for the mutual intersection numbers of E_a, E_b and E_c alternative to those of (7). In particular,

$\#(ab) + 2k_1 = [E_a].[E_b] = k_0 m_b + \frac{1}{2}m_b(dd_a - k_2) + d_b d - k_3 - 2k_2,$

$\#(ac) + k_2 = [E_a].[E_c] = k_0 m_c + \frac{1}{2}m_c(dd_a - k_2) + d_c d - 4k_4.$

We have already calculated m_b and m_c in Lemma 3.1, d_a in (1); by Lemma 3.2, $\#(ab) = k_5$ and $\#(ac) = k_6$, and the values of the k_i are given in Proposition 4.2. Substituting these enables us to complete the calculation of the degrees of 2-dimensional strata.

Proposition 5.1. *We have*

$d_a = 2d - 2 + 2g,$

$d_b = 2(d-1)(d-3) + 2g(d-3)$

$d_c = \frac{1}{3}(d-1)(d-2)(d-3) - g(d-2).$

Now by Lemma 2.1, E_a is isomorphic to Γ, so has genus g; E_b has $\#(bb)$ simple (A_1) nodes and $\#(b_2)$ simple (A_2) cusps; E_c has $\#(cc)$ simple nodes and $\#(\gamma)$ triple points (type D_4). Thus, first,

$-\chi(E_a) = \{(\frac{1}{2}d_a d - \#(\alpha))[F] + [D]\}.\{(\frac{1}{2}d_a d - \#(\alpha) - 4d)[F] - [D]\}$,
which indeed reduces, substituting from (9), to $2g-2$. Then we have
$-\chi(E_b) = [E_b].([E_b] + K_E) - \#(bb) - 2\#(b_2)$,
$-\chi(E_c) = [E_c].([E_c] + K_E) - \#(cc) - 4\#(\gamma)$,
and hence formulae alternative to those of (7); comparing the two and substituting known values completes the count of special points of the various types.

Proposition 5.2. *We have*
$\#(\alpha) = \#(\delta) = 2(d-2)(d-3) + 2g(d-6)$,
$\#(\beta) = 2(d-2)(d-3)(d-4) + 2g(d^2-10d+26) - 4g^2$,
$\#(\gamma) = \frac{1}{3}(d-2)(d-3)^2(d-4) - 2g(d^2-7d+13) + 2g^2$,
$\#(a_2) = 4(d-3) + 12g$,
$\#(b_2) = 6(d-3)(d-4) + 6g(3d-14) + 12g^2$,
$\#(c_2) = 4(d-2)(d-3)(d-4) + 4g(d^2-10d+26) - 8g^2$,
$\#(ab) = 6(d-3)(d-4) + 6g(3d-14) + 12g^2$,
$\#(ac) = 2(d-2)(d-3)(d-4) + 3g(d^2-8d+18) - 6g^2$,
$\#(bb) = 4(d-5)(d-3)(d-4) + 4g(3d^2-30d+77) + 12(d-6)g^2 + 4g^3$,
$\#(bc) = 3(d-5)(d-2)(d-3)(d-4) + g(5d^3-65d^2+288d-448) + (2d^2-26d+92)g^2 - 4g^3$,
$\#(cc) = \frac{1}{4}(d-2)(d-3)(d-4)(d-5)(2d-3) + \frac{1}{4}g(d^4-24d^3+177d^2-502d+468) - (d^2-10d+28)g^2 + g^3$.

We also have
$\chi(E_b) = -4(d-3)^2(d-4) - 4g(3d^2-24d+49) - 4g^2(3d-14) - 4g^3$,
$\chi(E_c) = -\frac{1}{12}(d-2)(d-3)(6d^3-55d^2+169d-192) - \frac{1}{4}g(d^4-24d^3+173d^2-490d+500) + g^2(d^2-10d+30) - g^3$.
However the Euler characteristics of the normalised curves (which give the genera) are given by $\chi(\tilde{E}_b) = \chi(E_b) + \#(bb)$ and $\chi(\tilde{E}_c) = \chi(E_c) + \#(cc) + 8k_4$, which lead to the simpler formulae

$$\begin{array}{rcl} \chi(\tilde{E}_b) & = & -8(d-3)(d-4) - 8g(3d-14) - 16g^2, \\ \chi(\tilde{E}_c) & = & -(d-2)(d-3)(2d-9) - g(3d^2-25d+60) + 6g^2. \end{array} \tag{11}$$

The degree d_c of the surface C of trisecants was first given by Cayley [2], with a full proof by Zeuthen [11]. The number of tritangent planes (equal to $\frac{1}{3}\#(bb)$) was also given by Zeuthen [11]. The formulae for d_b, $\#(bc)$ and $\#(cc)$ appear to be new.

6. Degrees of curve strata

In this section we complete the calculation of the degrees of the 1-dimensional strata. We first state the result; the formulae will be obtained in stages through the section.

Theorem 6.1. *We have*
$n(A_4) = 4(d-3) + 12g$,
$n(A_5) = 6(d-3)^3 + 12g(d-5) + 6g^2$,
$n(D_5) = 2(d-2)(d-3) + 2g(d-6)$,
$n(D_6) = 2(d-2)(d-3)(d-4) + 2g(d^2-10d+26) - 4g^2$,
$n(X_9) = \frac{1}{12}(d-2)(d-3)^2(d-4) - \frac{1}{2}g(d^2-7d+13) + \frac{1}{2}g^2$,
$n(2A_2) = 2(d-1+g)(d-3+g) = 2(d-1)(d-3) + 4g(d-2) + 2g^2$,
$n(A_2A_3) = 2d(d-3)(2d-7) + 2g(4d^2-19d+6) + 4g^2(d-3)$,
$n(A_2D_4) = \frac{1}{3}(d-2)(d-3)(d-4)(2d+1) + \frac{2}{3}g(d^3-9d^2+20d+6) - 2g^2(d-2)$,
$n(2A_3) = 2(d+1)(d-3)^2(d-4) + 4g(d^3-8d^2+13d+16) + 2g^2(d^2-7d+4)$,
$n(A_3D_4) = \frac{1}{3}(d-2)(d-3)(d-4)(2d^2-5d-9) + \frac{1}{3}g(2d^4-27d^3+103d^2-66d-204) - 2g^2(d^2-6d+2)$,
$n(2D_4) = \frac{1}{72}(d-2)(d-3)(d-4)(d-5)(4d^2-d-12) - \frac{1}{6}g(d-3)(d-5)(2d^2-3d-8) + \frac{1}{2}dg^2(d-5)$.

The values of $n(A_4)$, $n(D_5)$, $n(D_6)$ and $n(X_9)$ were denoted k_1, k_2, k_3 and k_4 and calculated in Proposition 4.2. The degrees of curves of intersection of two strata can be evaluated as follows.

Lemma 6.2. [9, Lemma 2.1] *Along $S(A_4)$, $S(A_2)$ and $S(A_3)$ are smooth, and intersect with multiplicity 2; along $S(A_5)$, $S(A_3)$ has a cuspidal edge; along $S(D_5)$, $S(A_2), S(A_3)$ and $S(D_4)$ are all smooth, and any two of them meet transversely; along $S(D_6)$, $S(A_3)$ and $S(D_4)$ are smooth, and intersect with multiplicity 2; and along $S(X_9)$, $S(D_4)$ has 4 branches, any two of which are transversal.*

It follows that intersections of cycles are given by

$$\begin{array}{rcl} [A].[B] &=& S(A_2A_3) + 2S(A_4) + S(D_5) + 2m_b[\Gamma], \\ [A].[C] &=& S(A_2D_4) + S(D_5) + 2m_c[\Gamma], \\ [B].[C] &=& S(A_3D_4) + S(D_5) + 2S(D_6) + m_bm_c[\Gamma]. \end{array} \tag{12}$$

Taking degrees, we find

$d_ad_b = n(A_2A_3) + 2k_1 + k_2 + 2dm_b$,
$d_ad_c = n(A_2D_4) + k_2 + 2dm_c$,
$d_bd_c = n(A_3D_4) + k_2 + 2k_3 + dm_bm_c$;

and substituting the values already obtained now yields the values of $n(A_2A_3)$, $n(A_2D_4)$ and $n(A_3D_4)$.

Next we consider the generic plane sections Π_a, Π_b, Π_c of the respective surfaces A, B and C. These have the same degrees as the corresponding surfaces, and have singularities given by Lemma 2.2. Applying the Plücker formula, we obtain

$-\chi(\Pi_a) = d_a(d_a-3) - 2d - n(2A_2)$,
$-\chi(\Pi_b) = d_b(d_b-3) - n(2A_3) - 2n(A_5) - d(m_b-1)^2$,
$-\chi(\Pi_c) = d_c(d_c-3) - n(2D_4) - 9n(X_9) - d(m_c-1)^2$.

However, it will be more convenient to use instead the normalisations of these curves, and here we have

$$\begin{array}{rcl} -\chi(\tilde\Pi_a) &=& d_a(d_a-3) - 2d - 2n(2A_2), \\ -\chi(\tilde\Pi_b) &=& d_b(d_b-3) - 2n(2A_3) - 2n(A_5) - dm_b(m_b-1), \\ -\chi(\tilde\Pi_c) &=& d_c(d_c-3) - 2n(2D_4) - 12n(X_9) - dm_c(m_c-1). \end{array} \tag{13}$$

To obtain alternative formulae, we first observe that $\tilde\Pi_a$ can be identified with Γ itself, so $\chi(\tilde\Pi_a) = 2-2g$.

Next we compare T_b, E_b and Π_b. A general T-secant PQ of Γ determines 2 points (P,Q), $(Q,P) \in T_b$, 2 points (P,Π), $(Q,\Pi') \in E_b$ and a single point $PQ \cap \Pi_0$ of Π_b. There are various exceptions to this, but the number of exceptions decreases if we compare instead the normalisations T_b (already normal), $\tilde E_b$ and $\tilde\Pi_b$: the only special cases now are the k_1 tangents at stalls P, which yield a single point in each of T_b, $\tilde E_b$ and $\tilde\Pi_b$. Hence $\chi(T_b) = \chi(\tilde E_b) = 2\chi(\tilde\Pi_b) - k_1$.

The equality is confirmed by our calculations, and we obtain

$$\chi(\tilde\Pi_b) = -2(d-3)(2d-9) - 2g(6d-31) - 8g^2. \tag{14}$$

Similarly for the third case, a general trisecant PQR of Γ gives rise to 6 points of T_c, 3 points of E_c and a single point of Π_c. Again using the normalisations, we find that the only exceptions are (a) each of k_2 tangents $T_P\Gamma$ meeting Γ again in Q, giving only 3 points of $\tilde T_c$ and 2 points of $\tilde E_c$, and (b) each of k_4 4-secants $PQRS$ of Γ, giving 24 points of $\tilde T_c$ (12 points of T_c), 12 points of $\tilde E_c$ (4 points of E_c) and 4 points of $\tilde\Pi_c$ (1 point of Π_c). Hence $\chi(\tilde T_c) = \chi(T_c) + 12k_4 = 2\chi(\tilde E_c) - k_2$, giving

$$\chi(\tilde T_c) = -4(d-2)(d-3)(d-4) - 6g(d^2-8d+18) + 12g^2,$$

and $\chi(\tilde E_c) = 3\chi(\tilde\Pi_c) - k_2$, giving

$$\chi(\tilde{\Pi}_c) = -\tfrac{1}{3}(d-2)(d-3)(2d-11) - g(d^2-9d+24) + 2g^2.$$

Substituting in (13) now yields the values of $n(2A_2)$ and $n(2D_4)$ and the equation

$$(15) \quad n(2A_3) + n(A_5) = 2(d-3)^2(d^2-3d-1) + 4g(d^3-8d^2+16d+1) + 2g^2(d^2-7d+7).$$

Lemma 6.3. *If $\Delta^\vee$ is the dual curve to Δ, then $S(2A_2)(\Delta^\vee)$ is the dual curve to $S(A_5)(\Delta)$.*

Proof. We can define $S(2A_2)(\Delta^\vee)$ as the set of planes through a pair of coplanar tangents of $\Delta^\vee$, or of Δ. But this is just the set of tangent planes of $S(A_3)(\Delta)$, and hence, of $S(A_5)(\Delta)$. □

We next study the curve $S(A_5)$, which from now on we denote by F. Recall that each T-secant of Γ touches F at its T-centre, thus the tangent surface to F is $S(A_3) = B$.

Theorem 6.4. *For Γ projection-generic, the curve F has no flexes, it has stalls only at $S(b_2)$, and has cusps only at $S(A_7)$ and $S(\delta)$.*

Proof. The only 0-dimensional strata lying on F are the compound singularities $S(A_2A_5)$, $S(A_3A_5)$, $S(D_4A_5)$, and $S(A_7)$, $S(D_8)$ outside Γ, and $S(b_2)$, $S(\delta)$ on Γ. By [9, §2], F is smooth at the compound singularities and at $S(D_8)$ and is cusped at $S(A_7)$. By the normal forms of [9, §6], F is smooth at $S(b_2)$ and is cusped at $S(\delta)$.

It follows in the generic case by [4] and in general by specialisation that the tangent line to a curve at a flex is singular on its tangent surface. But for Γ projection-generic, the singular locus of B is $\Gamma \cup F \cup S(2A_3)$. Now Γ cannot contain a straight line. If F or $S(2A_3)$ contained a line L, L could not itself be a T-secant (the T-trisecants form $S(D_6)$). For each T-secant meeting L, its T-plane contains the tangent line to $S(A_3)$, hence contains L. Thus the tangent lines to Γ at the end points of the T-secant meet L. We claim that this implies that Γ is planar: a contradiction. For take L as $y = z = 0$ in $\mathbb{C}^3$, and take a local parameter t on Γ. Since the tangent at (x, y, z) meets L, $(dy/dt)/(dz/dt) = y/z$. Hence $d(y/z)/dt = 0$, thus y/z is constant along Γ. Thus the singular locus of $S(A_3)$ contains no straight line, so F has no flex.

Any stall of F lies on the self-intersection curve $S(2A_3) \cup \Gamma$ of the tangent surface B of F. Now $S(2A_3)$ meets F only in $S(A_7)$ which gives cusps, not stalls on $S(A_5)$. The curve Γ itself meets F in $S(b_2) \cup S(\delta)$. By [9, Proposition 8.4], B has a cuspidal cross-cap at a point of $S(b_2)$, hence such a point is indeed a stall on F. By [9, Theorem 9.2], at a point of $S(\delta)$, B has a swallowtail singularity, and by [9, Corollary 9.2.1], the local parameters of F at such a point are $s_0 = 1$, $s_1 = s_2 = 0$, so it does not count as a stall. □

Thus we have projective characters $s_1(F) = 0$, $s_2(F) = \#(b_2) = k_5$, $r_1(F) = d_b$. Moreover, since each T-secant meets F in just one point, we can identify the normalisation $\tilde{F}$ with $\tilde{\Pi}_b$, so have $\chi(F) = \chi(\tilde{\Pi}_b)$, which was calculated in (14). The Plücker relations of Lemma 3.3 imply (since $s_1(F) = 0$)

$$r_0(F) = \tfrac{1}{2}(3r_1(F) - s_2(F) - 3\chi(F)) = \tfrac{1}{2}(3d_b - k_5 - 3\chi(\tilde{\Pi}_b)),$$

and this gives the degree $n(A_5)$ of $F = S(A_5)$. The value of $n(2A_3)$ now follows from (15).

We also have $s_0(F) = 2r_1(F) - s_2(F) - 4\chi(F) = 2d_b - k_5 - 4\chi(\tilde{\Pi}_b)$, and Theorem 6.4 gives $s_0(F) = k_2 + n(A_7)$, so we obtain

$$(16) \qquad n(A_7) = 12(d-3)(d-4) + 4g(8d-41) + 20g^2.$$

A formula for the degree of the curve $S(2A_2)$ (there called the *nodal curve*), for arbitrary space curves, can be found in [6], in the form $\frac{1}{2}\{r_1(r_1-1) - r_2 - 3(r_0+s_1)\}$. One can also calculate $n(2A_3)$ by applying this formula to F, but must then note that this nodal curve has to be interpreted as containing Γ with multiplicity $\binom{m_b}{2}$ as well as $S(2A_3)$. The other formulae in this section are new.

7. Calculations in $\Gamma \times \Gamma$

In this section we prove Theorem 2.3. We first prove most of (i) in Lemma 7.1, and all of (iv) in Lemma 7.2. The assertions in (ii) and (iii) require calculations, which we give in Lemmas 7.3, 7.4 and 7.5, for the respective curves T_a, T_b and T_c. We complete the proof of (v) in Proposition 7.7.

In these arguments we will make explicit use of the condition of projection genericity, particularly (PG6), which implies in particular that the subsets $W_1 \dots W_6$ of $\Gamma \times \Gamma$ are mutually disjoint. We also need the following consequences of (PG1):

at any stall $P \in \Gamma$ we have $s_2(P) = 1$,

there is no T-3-secant with T-centre on Γ,

the cross-ratio of the planes through a 4-secant containing the 4 tangent lines is *not* equal to the cross-ratio of the 4 points on the line.

Before starting our calculations, we note that the situation of T_b and T_c was also considered in [3, pp 290-297]. However precise conditions for counting multiplicities were passed over there, and of the hypotheses actually listed on p.291, the absence of 5-secants, T-4-secants and flexes (points with $s_1(P) > 0$) follow from (PG1), and the condition that no osculating 2-plane contain a tangent line is not generic.

The correspondences T_a and T_b are among those studied in [10], and the results concerning them are given there. The proofs below are more direct than those of [10] for this special case. Of the hypotheses of the other paper, that $s_0(P) + s_1(P) + s_2(P) \leq 1$ for each $P \in \Gamma$ follows from our hypotheses $s_0(P) = s_1(P) = 0$ and $s_2(P) \leq 1$; (PG6) implies all the other conditions (in fact we just need $S(a_2)$, $S(\alpha)$, $S(\delta)$, $S(ab)$ and $S(b_2)$ disjoint).

Lemma 7.1. *We have $T_a \cap \Delta(\Gamma) = T_b \cap \Delta(\Gamma) = W_1$, $T_c \cap \Delta(\Gamma) = W_2'$.*

Proof. If $(P, Q) \in T_a$ we have $P \in O_Q\Gamma$. Conversely, the plane $O_Q\Gamma$ has intersection number d with Γ, and the point Q accounts for 3; for the other points P, we have $(P, Q) \in T_a$. If also $P = Q$, the intersection number at Q is 4, so Q is a stall.

If $(P, P) \in T_b$, there is an intersection of $T_Y\Gamma_P$ with Γ_P at Y_P additional to that expected, i.e. Y_P is a flex of Γ_P, so again P is a stall of Γ.

If $(P, P) \in T_c$, then there is a line meeting Γ twice in P and once elsewhere. This must be the tangent at P, so $P \in S(\delta)$. □

Lemma 7.2. *We have $T_a \cap T_b = W_1 \cup W_2^t \cup W_5$, $T_a \cap T_c = W_2^t \cup W_6$, and $T_b \cap T_c = W_2 \cup W_2^t \cup W_3'$.*

Proof. Intersections on the diagonal are dealt with by Lemma 7.1, so consider pairs $P \neq Q$.

If $(P, Q) \in T_a \cap T_b$, then $P \in O_Q\Gamma$. If $P \in T_Q\Gamma$ then $(P, Q) \in W_2^t$; if not, $T_P\Gamma$ meets $T_Q\Gamma$ in a point different from P in $O_Q\Gamma$, so is contained in this plane, hence $(P, Q) \in W_5$.

If $(P, Q) \in T_a \cap T_c$, then $PQ \subset O_Q\Gamma$ and PQ is a trisecant PQR. If $R = P$, we have $Q \in T_P\Gamma \subset O_Q\Gamma$, so $Q \in S(\delta) \cap S(b_2)$, contradicting projection genericity. If $R = Q$, $P \in T_Q\Gamma$, so $(P, Q) \in W_2^t$. If P, Q, R are distinct, then Q has type ac, hence $(P, Q) \in W_6$.

If $(P, Q) \in T_b \cap T_c$, then again PQ is a trisecant PQR. If $R = P$ then $Q \in T_P\Gamma$, so $(P, Q) \in W_2$. If $R = Q$ then $P \in T_Q\Gamma$, so $(P, Q) \in W_2^t$. Otherwise, $(P, Q) \in W_3'$. □

For the next results, we need direct calculations of the low order terms in the expansions of the curves at the special points. Parts of these calculations appeared in a preliminary version of [9], where they were used to establish the local structure of the curves E_a, E_b and E_c in E_Γ with respect to each other and to the projection π_E: however in the final version of [9], the local structure is obtained from the main versality results.

We work throughout in affine 3-space, and take up the notation and calculations of [8, Proposition 6.15]: denote a typical point by $X = (x, x', x'')$; points of Γ are denoted $P = (p, p', p'')$, Q, R etc. We regard the co-ordinates p, p', p'' as functions of a local parameter t_p on Γ which vanishes at P (we omit the subscript p if there is no ambiguity). Their Taylor expansions are denoted $p = \sum_0^\infty p_r t_p^r$, $p' = \sum_0^\infty p'_r t_p^r$, etc.

Successive derivatives of the vector P with respect to t_p are denoted by suffices: $P_1, P_2, \ldots$. Thus at $t_p = 0$ we have $P_r = r!(p_r, p'_r, p''_r)$. However, we denote by P_0 the result of substituting $t_p = 0$ in P.

We take co-ordinates with P_0 at the origin, with tangent along the $x''-$axis. Since Γ is smooth at P, $p''_1 \neq 0$. We may take x'' (scaled by p''_1, which we retain to preserve homogeneity in our formulae) as local co-ordinate at P, so $p''_r = 0$ for $r \neq 1$. When $Q_0 \neq P_0$, we also suppose Q_0 in the plane $x' = 0$, so $q'_0 = 0$. We will expand an equation for the correspondence T_* as Taylor series in t_p and t_q. If the terms of degree ≤ 1 are $at_p + bt_q = 0$, then we have a coincidence point $I_1(T)$ if $b = 0$ (or $I_2(T)$ if $a = 0$, a singular point of T if $a = b = 0$), and it is simple iff the coefficient of t_q^2 is non-zero.

Lemma 7.3. *The curve T_a is smooth at all points. We have $I_1(T_a) = W_5^t$ and $I_2(T_a) = W_1'^t \cup W_2$. All coincidence points are simple. At a united point in W_1, the tangent is $3t_p + t_q = 0$.*

Proof. We have $(P, Q) \in T_a$ if $Q \in O_P\Gamma$, so $P - Q, P_1$ and P_2 are coplanar. Since $(P_0, Q_0) \in T_a$, we have $p'_2 = 0$; as the curvature does not vanish at P_0, $p_2 \neq 0$. We have

$$\Delta_0 := [P - Q, P_1, P_2] = \begin{vmatrix} -q_0 + \ldots & -q'_1 t_q + \ldots & -q''_0 + \ldots \\ 2p_2 t_p + \ldots & 3p'_3 t_p^2 + \ldots & p''_1 \\ 2p_2 + \ldots & 6p'_3 t_p + \ldots & 0 \end{vmatrix};$$

the terms of degree 1 in t_p and t_q are $6q_0 p'_3 p''_1 t_p - 2p_2 q'_1 p''_1 t_q$.

Thus for $I_1(T_a)$ we have $q'_1 = 0$, so $T_{Q_0}\Gamma \subset O_{P_0}\Gamma$ and $(P_0, Q_0) \in W_5^t$.
For $I_2(T_a)$ we have either

$q_0 = 0$:, so $Q_0 \in T_{P_0}$ and $(P_0, Q_0) \in W_2$, or

$p'_3 = 0$, so P_0 is a stall on Γ and $(P_0, Q_0) \in W_1'^t$;

we cannot have both, for this would imply $P_0 \in S(\delta) \cap S(a_2)$. This gives the coincidence points as stated, and proves smoothness.

In the W_5^t case, to obtain the coefficient of t_q^2, we set $t_p = 0$ in the determinant: the coefficient is $-2p_2 q'_2 p''_1$, which does not vanish since if $q'_2 = 0$, Q_0 would be a stall, so $Q_0 \in S(ab) \cap S(a_2)$.

In the other cases, we need the coefficient of t_p^2. If $p'_3 = 0$, the coefficient is $-12q_0 p'_4 p''_1$, which cannot vanish else we would have $s_2(P) \geq 2$; if $q_0 = 0$, the coefficient is $6p_2 p'_3 q''_0$, and this is non-zero as $q''_0 = 0$ implies $Q_0 = P_0$.

We also need to consider the case $Q_0 = P_0$. Here P and Q are given by the same parametrisation, i.e. with the same coefficients $p_r \ldots$ but with different parameters t_p, t_q. Since we have a stall, $p'_3 = 0$ and $p'_4 \neq 0$, so the determinant reduces to

$$\begin{vmatrix} p_2(t_p^2 - t_q^2) + \ldots & p'_4(t_p^4 - t_q^4) + \ldots & p''_1(t_p - t_q) \\ 2p_2 t_p + \ldots & 4p'_4 t_p^3 + \ldots & p''_1 \\ 2p_2 + \ldots & 12p'_4 t_p^2 + \ldots & 0 \end{vmatrix}.$$

A factor $(t_p - t_q)^3$ can be removed; when this is done, the terms of least degree reduce to $2p_2 p'_4 p''_1(3t_p + t_q)$; in particular, the curve is smooth. □

Lemma 7.4. *The curve T_b is smooth at all points. At a united point, the tangent is given by $t_q = -t_p$. The set $I_1(T_b) = W_2 \cup W_5$. All coincidence points are simple.*

Proof. First consider a neighbourhood of the point defined by a T-secant P_0Q_0, with $P_0 \neq Q_0$. Since $(P_0, Q_0) \in T_b$, we have $q_1' = 0$. Now $(P, Q) \in T_b$ if $P - Q$, P_1 and Q_1 are coplanar, so $\Delta_b := [P - Q, P_1, Q_1] = 0$. We have

$$\Delta_b = \begin{vmatrix} p_2t_p^2 + \ldots - q_0 - \ldots & p_2't_p^2 + \ldots - q_2't_q^2 + \ldots & p_1''t_p - q_0'' - \ldots \\ 2p_2t_p + \ldots & 2p_2't_p + \ldots & p_1'' \\ q_1 + \ldots & 2q_2't_q + \ldots & q_1'' + \ldots \end{vmatrix}.$$

The linear terms in the expansion of Δ_b are $-2p_2'(-q_1q_0'' + q_0q_1'')t_p + 2q_0q_2'p_1''t_q$. For $I_1(T_b)$, either

$q_0 = 0$, so $Q_0 \in T_{P_0}\Gamma$ and $(P_0, Q_0) \in W_2$, or

$q_2' = 0$, thus $O_{Q_0}\Gamma$ is $x' = 0$ so $T_{P_0}\Gamma \subset O_{Q_0}\Gamma$ and $(P_0, Q_0) \in W_5$.

We cannot have both, else we would have $P_0 \in S(\delta) \cap S(b_2)$. As a check, we note that for $I_2(T_b)$ we have the transposed cases:

$(-q_1q_0'' + q_0q_1'') = 0$; since $q_1' = 0$, this is the condition for $P_0 \in T_{Q_0}\Gamma$, so $P_0 \in S(\alpha)$, $(P_0, Q_0) \in W_2^t$, or

$p_2' = 0$, thus $O_{P_0}\Gamma$ is $x' = 0$, so here $T_{Q_0} \subset O_{P_0}$ and $(P_0, Q_0) \in W_5^t$.

This proves smoothness of T_b. As before, we check the second degree terms:

if $q_0 = 0$, the coefficient is $-q_1q_2'p_1''$, and $q_1 \neq 0$ for otherwise $T_{P_0}\Gamma = T_{Q_0}\Gamma$,

if $q_2' = 0$, the coefficient is $-3q_0q_3'p_1''$, and $q_3' \neq 0$, for otherwise Q_0 is a stall.

We also see that when $p_2' = 0$, the coefficient of t_p^2 is $3p_3'(q_1q_0'' - q_0q_1'')$.

Again we must also consider the case $P_0 = Q_0$. We already saw in [8, Lemma 6.11 (ii)] that in this case, (as we saw in Lemma 7.1) P_0 must be a stall, so $(P_0, P_0) \in W_1$, and also that for the two parameters we have, to first order, $t_p + t_q = 0$. □

Lemma 7.5. *At a point $(P, Q) \in T_c$, the curve is smooth and transverse to neither fibre except as follows. We have $I_1(T_c) = W_2^t \cup W_3 \cup W_4$, $I_2(T_c) = W_2 \cup W_3^t \cup W_4$. Points of $W_2^t \cup W_3$ are simple coincidence points. Points of W_4 are double points, with 2 transverse branches, each tangent to neither fibre.*

Proof. We use the same notation as before, and take the trisecant $P_0Q_0R_0$ to lie on $x' = 0$. First suppose P_0, Q_0, R_0 are distinct, so we may suppose $0 = p_0''$, q_0'' and r_0'' also distinct. Since the points are collinear, $\frac{q_0}{q_0''} = \frac{r_0}{r_0''} = \lambda$, say.

The condition for collinearity of P, Q, R is that the matrix

$$\begin{pmatrix} 1 & p & p' & p'' \\ 1 & q & q' & q'' \\ 1 & r & r' & r'' \end{pmatrix} \tag{17}$$

have rank 2. Since the first and last columns are independent for P_0, Q_0, R_0 and hence nearby, it suffices to equate to zero the determinants formed by omitting the third and second columns, which we denote respectively by Δ_c and Δ_c'.

For the terms of order at most 1 in t_p, t_q, t_r it suffices to consider

$$\begin{pmatrix} 1 & 0 & 0 & p_1''t_p \\ 1 & q_0 + q_1t_q & q_1't_q & q_0'' + q_1''t_q \\ 1 & r_0 + r_1t_r & r_1't_r & r_0'' + r_1''t_r \end{pmatrix}.$$

The terms of degree at most 1 in t_p, t_q and t_r are:

Δ_c: $p_1''(r_0 - q_0)t_p + (q_1 - \lambda q_1'')r_0''t_q + (\lambda r_1'' - r_1)q_0''t_r$,

Δ_c': $q_1'r_0''t_q - r_1'q_0''t_r$.

If $q_1' = 0$, $T_{P_0}\Gamma$ and $T_{Q_0}\Gamma$ both lie in the plane $x' = 0$, so we have a T-3-secant and $(P_0, Q_0) \in W_3'$: here $t_r = 0$ and $p_1''(r_0 - q_0)t_p + (q_1 - \lambda q_1'')r_0''t_q = 0$.

Similarly if $r_1' = 0$, we have $t_q = 0$ (we cannot have $q_1' = r_1' = 0$), $T_{P_0}\Gamma$ and $T_{R_0}\Gamma$ are coplanar: here $(P_0, Q_0) \in W_3^t \subset I_2(T_c)$.

Otherwise, we can eliminate t_r to get $0 = r_1' p_1''(r_0 - q_0)t_p + \xi r_0'' t_q$, where $\xi = (q_1 - \lambda q_1'')r_1' - (r_1 - \lambda r_1'')q_1'$.

Thus $\xi = 0$ is the condition for $T_{Q_0}\Gamma$ and $T_{R_0}\Gamma$ to be coplanar: in this case $(P_0, Q_0) \in W_3 \subset I_1(T_c)$.

Thus if $(P_0, Q_0) \in W_3$, the coefficient of t_q vanishes. We now claim that it follows from the fact that P_0 is not the T-centre of Q_0R_0 that the coefficient C in $t_p = Ct_q^2$ is non-zero. As direct calculation is messy, we proceed a little differently.

Write the co-ordinates of Q as (q, q', q''). Projecting from P_0 (the origin) to $x'' = c$ gives $(cq/q'', cq'/q'')$; similarly for R. Since the second co-ordinates have leading terms $(cq_1'/q_0'')t_q$ and $(cr_1'/r_0'')t_r$, we can solve $q'/q'' = r'/r''$ for t_r in terms of t_q. The order of contact of the projected curves is now the order of the difference of the first co-ordinates, hence the order of $qr'' - q''r$. Since P_0 is not the T-centre of Q_0R_0, this order is 2.

On the other hand, we have $\Delta_c = (qr'' - q''r) - p(r'' - q'') + p_1''t_p(r - q)$ and $\Delta_c' = (q'r'' - q''r') - p'(r'' - q'') + p_1''t_p(r' - q')$. Since p, p' each have order at least 2, we can ignore the terms involving p, p'. Substituting the above solution for t_r thus makes Δ_c' vanish to order at least 2. So to this order, the equation $\Delta_c = 0$ yields $t_p = (qr'' - q''r)/p_1''(q - r)$ which, since the denominator is non-vanishing, has order precisely 2.

Next we treat the case when P_0, Q_0 and R_0 are not all distinct. Suppose $R_0 = P_0$: then $Q_0 \in T_{P_0}\Gamma$, so $q_0 = 0$. Here $P_0 \in S(\delta)$ and $(P_0, Q_0) \in W_2$; the 3-secants near $T_{P_0}\Gamma$ give a branch of T_c. We take co-ordinates so that the plane $O_{P_0}\Gamma$ is given by $x' = 0$, and so $p_2' = 0$. Since $P_0 \notin S(a_2)$, $p_3' \neq 0$. Since $P_0 \notin S(ab)$, $T_{Q_0}\Gamma \not\subset O_{P_0}\Gamma$, so $p_2 q_1' \neq 0$.

In the matrix (17) we subtract the first row from the third, and divide the result by $t_r - t_p$ giving, to first order, $(0, p_2(t_p + t_r), 0, p_1'')$. Terms of order ≤ 2 in the matrix are:

$$\begin{pmatrix} 1 & 0 & 0 & p_1''t_p \\ 1 & q_1t_q + q_2t_q^2 & q_1't_q + q_2't_q^2 & q_0'' + q_1''t_q + q_2''t_q^2 \\ 0 & p_2(t_p + t_r) + p_3(t_p^2 + t_pt_r + t_r^2) & p_3'(t_p^2 + t_pt_r + t_r^2) & p_1'' \end{pmatrix}$$

Denote the minors corresponding to Δ_c and Δ_c' by Δ_1 and Δ_1'. These have first order terms $q_1p_1''t_q - p_2q_0''(t_p + t_r)$ and $q_1'p_1''t_q$ respectively, so to first order we have $0 = t_q = t_p + t_r$ and $(P_0, Q_0) \in I_2(T_c)$.

Assign weight 1 to t_p and t_r, 2 to t_q: then up to weight 2 we have $\Delta_1' = p_1''q_1't_q - q_0''p_3'(t_p^2 + t_pt_r + t_r^2)$, so to that order $t_q = \frac{q_0''p_3'}{p_1''q_1'}t_p^2$, with non-zero coefficient. Hence the coincidence point is simple.

We have now shown that at each point of T_c we have a smoothly immersed curve. Double points can only occur if P_0Q_0 lies in two trisecants, or more accurately, defines a 4-secant $P_0Q_0R_0S_0$, thus $(P_0, Q_0) \in W_4$. It remains to show that the two branches at such a point are not tangent. With the above notation, we had $0 = r_1'p_1''(r_0 - q_0)t_p + \xi r_0''t_q$, where $\xi = (q_1 - \lambda q_1'')r_1' - (r_1 - \lambda r_1'')q_1'$. Thus the condition for tangency of the two branches is

$$\{(q_1 - \lambda q_1'')r_1' - (r_1 - \lambda r_1'')q_1'\}r_0''s_1'p_1''(s_0 - q_0) = \{(q_1 - \lambda q_1'')s_1' - (s_1 - \lambda s_1'')q_1'\}s_0''r_1'p_1''(r_0 - q_0).$$

Substituting $q_0 = \lambda q_0''$, $r_0 = \lambda r_0''$, $s_0 = \lambda s_0''$, and dividing both sides by $\lambda p_1''q_1'r_1's_1'$, this reduces to

$$\left\{\frac{q_1 - \lambda q_1''}{q_1'} - \frac{r_1 - \lambda r_1''}{r_1'}\right\}r_0''(s_0'' - q_0'') = \left\{\frac{q_1 - \lambda q_1''}{q_1'} - \frac{s_1 - \lambda s_1''}{s_1'}\right\}s_0''(r_0'' - q_0'').$$

Now the points P_0, Q_0, R_0, S_0 lie on the line $x' = 0, x = \lambda x''$, with co-ordinates $0, q_0'', r_0'', s_0''$, and the tangents to Γ at these points lie in the planes $x' = \mu(x - \lambda x'')$, where the corresponding values of μ are $0, \frac{q_1'}{q_1-\lambda q_1''}, \frac{r_1'}{r_1-\lambda r_1''}, \frac{s_1'}{s_1-\lambda s_1''}$. The above equality requires the cross-ratios $(0, q_0'', r_0'', s_0'')$ and $(0, \frac{q_1'}{q_1-\lambda q_1''}, \frac{r_1'}{r_1-\lambda r_1''}, \frac{s_1'}{s_1-\lambda s_1''})$ to be equal. But projection genericity implies that they are not. $\square$

Before treating intersection numbers in $\Gamma\times\Gamma$ we introduce a map which, in some cases, enables us to deduce them from intersection numbers in E_Γ (which were given in Lemma 2.1). Define $\Phi : \Gamma \times \Gamma \rightsquigarrow E_\Gamma$ by $(P, Q) \mapsto (P, \Pi)$, where Π is the plane through $T_P\Gamma$ and Q. This is defined except if $Q \in T_P\Gamma$, i.e. except on the diagonal $\Delta(\Gamma)$ and W_2. We have $\Phi(T_a^t) = E_a$, $\Phi(T_b) = E_b$ and $\Phi(T_c) = E_c$; the images of points of other types are given by

$$\begin{array}{c|cccccccc} x & W_1' & W_2^t & W_3 & W_3' & W_4 & W_5 & W_5^t & W_6 \\ \Phi(x) & a_2 & \alpha & \beta & c_2 & \gamma & b_2 & ab & ac \end{array}.$$

Lemma 7.6. *The restriction of Φ to the complement of $\Delta(\Gamma) \cup T_b$ is a submersion. Along T_b, the map is a simple fold, except at points of W_5.*

Proof. With co-ordinates as above, Π is the plane through the x''−axis and (q, q', q''), which has initial position $(q_0, 0, q_0'')$. Thus for t_q small, the angle made by Π is $q_1' t_q/q_0$. Hence the map is a local submersion if $q_1' \neq 0$, i.e. if $(P_0, Q_0) \notin T_b$.

For the second assertion we may suppose $q_0 \neq 0$, $q_0' = 0$, $q_1' = 0$. Since the point P is given by the first projection, it suffices to keep $P = P_0$ fixed, and see how Π varies with Q. Here the angle is $q_2' t_q^2/q_0$ (modulo higher terms), so is a non-zero multiple of t_q^2 except if $q_2' = 0$, i.e. $(P_0, Q_0) \in W_5$. $\square$

Proposition 7.7. *The intersection number at each of the common points of Lemma 7.2 is 1, except that at W_2^t the intersection number of T_a^t and T_c is 2.*

Proof. We treat the cases in turn.

$W_1 \subset T_a \cap T_a^t \cap T_b$. We have seen that at such a point, to first order, T_a is given by $3t_p + t_q = 0$, hence T_a^t by $t_p + 3t_q = 0$, while T_b is given by $t_p + t_q = 0$.

$W_2 \subset T_a^t \cap T_b \cap T_c$. We have $W_2 \subset I_2(T_a^t) \cap I_1(T_b) \cap I_2(T_c)$, so at these points T_b is transverse to the others. From the above calculations, the least order terms at W_2 are

for T_a^t we have $t_q = \frac{3p_3' q_0''}{q_1' p_1''} t_p^2$.

for T_c we have $t_q = \frac{p_3' q_0''}{q_1' p_1''} t_p^2$.

Since $p_3' q_0'' \neq 0$, T_a^t and T_c have intersection number 2.

$W_3' \subset T_b \cap T_c$: the case of a T-3-secant $(P_0Q_0)R_0$. Here we apply Lemma 7.6. Since E_b and E_c have simple tangency at a point of $S(c_2)$, it follows since Φ is a simple fold along T_b that the pre-images T_b and T_c are transverse.

$W_5 \subset T_a \cap T_b$. Here transversality holds since $W_5 \subset I_1(T_b) \cap I_2(T_a)$.

$W_6 \subset T_a \cap T_c$. Here since E_a and E_c are transverse at ac, it follows from Lemma 7.6 that T_a and T_c are transverse at W_6. $\square$

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ISSN #1949-2006

Proceedings of Singularities in Aarhus,
17-21 August 2009,
in honor of Andrew du Plessis
on the occasion of his 60th birthday

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