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GEOMETRY OF IRREDUCIBLE PLANE QUARTICS AND THEIR QUADRATIC RESIDUE CONICS

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Dedicated to Professor Du Plessis on his sixtieth birthday.

Abstract. Let D be an irreducible plane curve in $\mathbb{P}^2$. In this article, we first introduce a notion of a quadratic residue curve mod$D$, and study quadratic residue conics C mod an irreducible quartic curve Q. As an application, we study a dihedral cover of $\mathbb{P}^2$ with branch locus $C + Q$ and give two examples of Zariski pairs as by-products.

Introduction

In this article, we study the geometry of irreducible plane quartic Q and a smooth conic C which is tangent to Q with even order at each point in $C \cap Q$. The geometry of a smooth plane quartic and its bitangent lines is a classical object and well studied by many mathematicians from various points of view. We hope that this article adds another interesting topic to geometry of plane quartics. All varieties throughout this paper are defined over the field of complex numbers, $\mathbb{C}$. In order to explain our motivation and results on the above subject, let us start with introducing some notions and definitions.

Let Σ be a smooth projective surface. Let $f' : Z' \to \Sigma$ be a double cover of Σ, i.e., Z' is a normal surface and f' is a finite surjective morphism of degree 2. We denote its canonical resolution by $\mu : Z \to Z'$ (see [7] for the canonical resolution). Note that μ is the identity if Z' is smooth. We put $f := f' \circ \mu$. We denote the involution on Z induced by the covering transformation of f' by σ_f. The branch locus $\Delta_{f'}$ of f' is the subset of Σ consisting of points x such that f' is not locally isomorphic over x. Similarly we define the branch locus Δ_f of f. Note that $\Delta_{f'} = \Delta_f$.

Definition 0.1. Let D be an irreducible curve on Σ. We call D a splitting curve with respect to f if f^*D is of the form

$$f^*D = D^+ + D^- + E,$$

where $D^+ \neq D^-$, $\sigma_f^* D^+ = D^-$, $f(D^+) = f(D^-) = D$ and $\mathrm{Supp}(E)$ is contained in the exceptional set of μ. If the double cover $f : Z \to \Sigma$ is determined by its branch locus Δ_f, i.e., any double cover with branch locus Δ_f is isomorphic to Z' over Σ, and D is a splitting curve with respect to f, we say that "Δ_f *is a quadratic residue curve mod* D".

Remark 0.1.

- Note that if Σ is simply connected, then any double cover of Σ is determined by its branch locus.
- In our previous results on dihedral covers and their application to the study of the topology of the complements of plane curves, we see that splitting curves play important roles and that it is indispensable to know their properties of them. (see [2], [17], [18], for example). This is our first motivation to study splitting curves.
- Our terminology comes from elementary number theory. Let m be a square free positive integer, let p be an odd prime with $p \not| m$ and let $\mathcal{O}_{\mathbb{Q}(\sqrt{m})}$ be the integer ring of $\mathbb{Q}(\sqrt{m})$. It

is well known that the ideal (p) generated by p in $\mathcal{O}_{\mathbb{Q}(\sqrt{m})}$ satisfies the following properties (See [8, Proposition 13.1.3], p.190, for example):

- If m is a quadratic residue mod p, then $(p) = \mathfrak{p}_1\mathfrak{p}_2$, where $\mathfrak{p}_i$ $(i = 1, 2)$ are distinct prime ideals.
- If m is not a quadratic residue mod p, then (p) is a prime ideal.

Suppose that $f : Z \to \Sigma$ is uniquely determined by Δ_f. Likewise the Legendre symbol in elementary number theory, we here introduce a notation to describe if Δ_f is a quadratic residue mod D or not. For an irreducible curve D on Σ, we put

$$(\Delta_f/D) = \begin{cases} 1 & \text{if } \Delta_f \text{ is a quadratic residue curve mod } D \\ -1 & \text{if } \Delta_f \text{ is not a quadratic residue curve mod } D \end{cases}$$

As $\mathbb{P}^2$ is simply connected, any double cover of $\mathbb{P}^2$ is just determined by its branch locus. On the other hand, any reduced plane curve B of even degree can be the branch locus of a double cover. Hence for any irreducible plane curve D, one can consider (B/D).

In this article, we consider the case when *any point $x \in B \cap D$ is a smooth point of both B and D*. For such a case, if the intersection multiplicity at some point in $B \cap D$ is odd, then we infer that $(B/D) = -1$. This leads us to introduce a notion of *even tangential curve*.

Definition 0.2. Let D_1 and D_2 are reduced divisors on a smooth projective surface without any common irreducible component. We say that D_1 and D_2 are even tangential or D_1 (resp. D_2) is even tangential to D_2 (resp. D_1) if

(i) For $\forall P \in D_1 \cap D_2$, $P \notin \mathrm{Sing}(D_1) \cup \mathrm{Sing}(D_2)$, and
(ii) the intersection multiplicity of D_1 and D_2 at P, $I_P(D_1, D_2)$, is even for $\forall P \in D_1 \cap D_2$.

Note that we do not pay attention to $\sharp(D_1 \cap D_2)$ to define even tangential curves.

Now our basic problem can be formulated as follows:

Problem 0.1. Let B be a reduced plane curve of even degree.

(i) Find an even tangential curve D to B and determine the value of (B/D).
(ii) What can we say about the topology of $\mathbb{P}^2 \setminus (B + D)$ from the value of (B/D)?

As a first step, we consider the case when B is a smooth conic C. Suppose that D is an irreducible plane curve which is even tangential to C. We easily see the following:

- If $\deg D = 1, 2$, we have $(C/D) = 1$.
- If $\deg D = 3$, we have
  (i) $(C/D) = -1$ if D is smooth, and
  (ii) $(C/D) = 1$ if D is a nodal cubic.

  Note that there is no even tangential cuspidal cubic to C.

Hence the case of $\deg D = 4$ seems to be the first interesting case. Now let us restate our exact problems which we consider in this article:

Problem 0.2. Fix an irreducible quartic Q.

(i) Find even tangential conics C to Q and determine the value (C/Q).
(ii) Does the value (C/Q) affect the topology of $\mathbb{P}^2 \setminus (C + Q)$?

In this article, we first consider Problem 0.2 (i) and give a formula to determine (C/Q) (see Theorem 2.1). We next count the number of even tangential conics passing through a smooth point x on Q. Now our result is as follows:

Theorem 0.1. *Choose a smooth point x of Q and let l_x be the tangent line to Q at x. There exist finitely many (possibly no) even tangential conics C to Q through x and we have the following table:*

- Ξ_Q: *the set of types of singularities of* Q. *Note that* Q *has at worst simple singularities and we use the notation in* [3] *in order to describe the type of a singularity.*
- $l_x \cap Q$: *This shows how* l_x *meets* Q. *We use the following notation to describe it.*
  - s: $I_x(l_x, Q) = 2$ *or* 3, *and* l_x *meets* Q *transversely at other point(s).*
  - b: l_x *is either bitangent line through* x *or* $I_x(l_x, Q) = 4$.
  - sb: $I_x(l_x, Q) = 2$ *and* l_x *passes through a double point of* Q.
- ETC: *the set of even tangential conics passing through* x *and* $\sharp$ETC *denotes its cardinality.*
- QRETC: *the set of even tangential conics passing through* x *with* $(C/Q) = 1$ *and* $\sharp$QRETC *denotes its cardinality.*
- *We omit cases of* $(\Xi_Q,\, l_x \cap Q)$ *which do not occur. For example, the case of* $(\Xi_Q, l_x \cap Q) = (A_6, b)$ *is omitted, as such a case does not occur.*

No.	Ξ_Q	$l_x \cap Q$	$\sharp$ETC	$\sharp$QRETC
1	A_6	s	0	0
2	A_6	sb	0	0
3	E_6	s	0	0
4	E_6	b	0	0
5	A_5	s	1	1
6	A_5	b	1	1
7	A_5	sb	0	0
8	D_5	s	1	1
9	D_5	b	0	0
10	D_4	s	3	3
11	D_4	b	0	0
12	$A_4 + A_2$	s	0	0
13	$A_4 + A_2$	sb	0	0
14	$A_4 + A_1$	s	0	0
15	$A_4 + A_1$	b	0	0
16	$A_4 + A_1$	sb	0	0
17	$A_4 + A_1$	sb	0	0
18	$A_3 + A_2$	s	1	1
19	$A_3 + A_2$	sb	0	0
20	$A_3 + A_2$	sb	1	1
21	$A_3 + A_1$	s	2	2
22	$A_3 + A_1$	b	1	1
23	$A_3 + A_1$	sb	1	1
24	$A_3 + A_1$	sb	0	0
25	$3A_2$	s	0	0
26	$3A_2$	b	0	0
27	$2A_2 + A_1$	s	0	0
28	$2A_2 + A_1$	b	0	0
29	$2A_2 + A_1$	sb	0	0
30	$A_2 + 2A_1$	s	1	1
31	$A_2 + 2A_1$	b	0	0
32	$A_2 + 2A_1$	sb	0	0
33	$A_2 + 2A_1$	sb	1	1

No.	Ξ_Q	$l_x \cap Q$	$\sharp$ETC	$\sharp$QRETC
34	$3A_1$	s	4	4
35	$3A_1$	b	1	1
36	$3A_1$	sb	2	2
37	A_4	s	3	0
38	A_4	b	1	0
39	A_4	sb	1	0
40	A_3	s	7	1
41	A_3	b	2	0
42	A_3	sb	4	1
43	$2A_2$	s	3	0
44	$2A_2$	b	3	0
45	$2A_2$	sb	1	0
46	$A_2 + A_1$	s	6	0
47	$A_2 + A_1$	b	3	0
48	$A_2 + A_1$	sb	3	0
49	$A_2 + A_1$	sb	2	0
50	$2A_1$	s	13	1
51	$2A_1$	b	6	0
52	$2A_1$	sb	7	1
53	A_2	s	15	0
54	A_2	b	6	0
55	A_2	sb	10	0
56	A_1	s	30	0
57	A_1	b	15	0
58	A_1	sb	20	0
59	$\emptyset$	s	63	0
60	$\emptyset$	b	36	0

Note that there exist both quadratic and non-quadratic residue even tangential conics to Q for the cases 40, 42, 50 and 52. These cases are interesting when we consider Problem 0.2 (ii). In fact, we study dihedral covers of $\mathbb{P}^2$ whose branch loci are $C + Q$, and have the following result (see §3 for the notations on dihedral covers):

Theorem 0.2. *Let Q be an irreducible quartic, let C be an even tangential conic to Q and let $f_C : Z_C \to \mathbb{P}^2$ be a double cover with $\Delta_{f_C} = C$. If there exists a $\mathcal{D}_{2p}$-cover $\pi : S \to \mathbb{P}^2$ with $\Delta_\pi = C + Q$ for an odd prime $p \geq 5$, then we have the following:*

(i) $D(X/\mathbb{P}^2) = Z_C \cong \mathbb{P}^1 \times \mathbb{P}^1$, i.e., π is branched at $2C + pQ$.

(ii) $(C/Q) = 1$. Moreover, if we put $f_C^* Q = Q^+ + Q^-$, then $Q^+ \sim Q^- \sim (2, 2)$.

Conversely, if the second condition holds, then there exist $\mathcal{D}_{2n}$-covers $\pi_n : S_n \to \mathbb{P}^2$ branched at $2C + nQ$ for any $n \geq 3$.

Since both of $\deg C$ and $\deg Q$ are even, we infer that there exists a $(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$-cover of $\mathbb{P}^2$ with branch locus $C + Q$. Hence, from Theorem 0.2, we have the following corollaries:

Corollary 0.1. *If there exists a $\mathcal{D}_{2p}$-cover of $\mathbb{P}^2$ with $\Delta_\pi = C + Q$ for some odd prime $p \geq 5$, then there exists a $\mathcal{D}_{2n}$-cover $\mathbb{P}^2$ with $\Delta_\pi = C + Q$ for any $n \geq 2$.*

Corollary 0.2. *(i) Let p be an odd prime ≥ 5. If there exists an epimorphism from the fundamental group $\pi_1(\mathbb{P}^2 \setminus (C+Q), *)$ to $\mathcal{D}_{2p}$, then $(C/Q) = 1$ and $Q^+ \sim Q^-$.*

(ii) If there exists an epimorphism $\pi_1(\mathbb{P}^2 \setminus (C+Q), *)$ to $\mathcal{D}_{2p}$, then there exists an epimorphism $\pi_1(\mathbb{P}^2 \setminus (C+Q), *)$ to $\mathcal{D}_{2n}$ for any $n \geq 2$.

This paper consists of 5 sections. In §1, we start with preliminaries on theory of elliptic surface. We prove Theroem 0.1 in §2. In §3, we give a summary on branched Galois covers, mainly dihedral covers. We prove Theorem 0.2 in §4. In §5, we consider an application of Theorem 0.2 and give two examples of Zariski pairs.

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1. Preliminaries on elliptic surfaces

1.1. **Elliptic surfaces.** We review some general facts on elliptic surfaces. For details, we refer to [9], [10] and [14]. Let $\varphi : \mathcal{E} \to C$ be an elliptic surface over a smooth projective curve C with a section O. Throughout this article, we always assume that

(i) φ is relatively minimal and
(ii) there exists at least one singular fiber.

Let $\mathrm{NS}(\mathcal{E})$ be the Néron-Severi group of $\mathcal{E}$ and let T_φ be the subgroup of $\mathrm{NS}(\mathcal{E})$ generated by O and all the irreducible components of fibers of φ. T_φ has a canonical basis as follows:

O, a general fiber $\mathfrak{f}$, and $\{\Theta_{v,1}, \ldots, \Theta_{v,m_v-1}\}_{v \in R_\varphi}$, where

- $R_\varphi := \{v \in C \mid \varphi^{-1}(v) \text{ is reducible.}\}$, and
- we label the irreducible components of $\varphi^{-1}(v)$ as follows: $\Theta_{v,0}, \Theta_{v,1}, \ldots, \Theta_{v,m_v-1}$, $\Theta_{v,0}O = 1$.

Let $\mathrm{MW}(\mathcal{E})$ be the Mordell-Weil group, the group of sections, of $\mathcal{E}$, O being the zero sections. Under these circumstances, we have

Theorem 1.1. [14, Theorem 1.3] *There is a natural isomorphism*

$$\mathrm{MW}(\mathcal{E}) \cong \mathrm{NS}(\mathcal{E})/T_\varphi.$$

Also in [14], a symmetric bilinear form $\langle\ ,\ \rangle$, called the height pairing, on $\mathrm{MW}(\mathcal{E})$ is defined by using the intersection pairing as follows:

For any $s \in \mathrm{MW}(\mathcal{E})$, $\langle s, s\rangle \geq 0$ and $= 0$ if and only if s is a torsion. More explicitly, for $s_1, s_2 \in \mathrm{MW}(\mathcal{E})$, $\langle s_1, s_2\rangle$ is given by

$$\langle s_1, s_2\rangle = \chi(\mathcal{O}_\mathcal{E}) + s_1O + s_2O - s_1s_2 - \sum_{v \in R_\varphi} \mathrm{Corr}_v(s_1, s_2),$$

where $\mathrm{Corr}_v(s_1, s_2)$ is given by

$$\mathrm{Corr}_v(s_1, s_2) = (s_2\Theta_{v,1}, \ldots, s_2\Theta_{v,m_{v-1}})(-A_v^{-1}) \begin{pmatrix} s_1\Theta_{v,1} \\ \cdot \\ s_1\Theta_{v,m_v-1} \end{pmatrix},$$

and A_v is the intersection matrix $(\Theta_{v,i}\Theta_{v,j})$ $(1 \leq i, j \leq m_v - 1)$. As for explicit values of $\mathrm{Corr}_v(s_1, s_2)$, see Table 8.16 in [14].

1.2. **A "reciprocity" between sections and trisections on rational ruled surfaces.** Let Σ_d be the Hirzebruch surface of degree d (d: even positive integer). We denote its section with self-intersection number $-d$ and its fiber of the ruling by $\Delta_{0,d}$ and F_d, respectively. Let Γ_d be an irreducible curve on Σ_d such that

(1) $\Gamma_d \sim 3(\Delta_{0,d} + dF_d)$ and
(2) Γ_d has at worst simple singularities.

Let Δ be a section on Σ_d such that (i) $\Delta \sim \Delta_{0,d} + dF_d$ and (ii) Δ and Γ_d are even tangential.

Let $p'_d : S'_d \to \Sigma_d$ be the double cover with branch locus $\Delta_{0,d} + \Gamma_d$ and $\mu_d : S_d \to S'_d$ be the canonical resolution and put $p_d := p'_d \circ \mu_d$. Since $\Delta_{0,d} + \Gamma_d$ meets a general fiber $F_d \cong \mathbb{P}^1$ in 4 distinct points, one can easily see that S_d has an elliptic fibration $\varphi_d : S_d \to \mathbb{P}^1$ over $\mathbb{P}^1$. Moreover, by its construction, we infer that

(a) φ_d is relatively minimal,
(b) the preimage of $\Delta_{0,d}$ gives a section which we denote by O, and
(c) Δ gives rise to two sections s^+_Δ and s^-_Δ of φ_d.

Let $\mathrm{MW}(S_d)$ be the group of sections , the Mordell-Weil group, of φ_d, where O is the zero element. Let $q_d : W_d \to \Sigma_d$ be a double cover with branch locus $\Delta_{0,d} + \Delta$. Note that q_d is uniquely determined by $\Delta_{0,d} + \Delta$ as Σ_d is simply connected and that $W_d \cong \Sigma_{d/2}$. Then we have

Theorem 1.2.
$$((\Delta_{0,d} + \Delta)/\Gamma_d)) = (-1)^{\varepsilon(s^+_\Delta)}$$
where, for a section $s \in \mathrm{MW}(S_d)$, $\varepsilon(s)$ is defined as follows:
$$\varepsilon(s) = \begin{cases} 0 & \exists s_o \in \mathrm{MW}(S_d) \text{ such that } s = 2s_o \\ 1 & \nexists s_o \in \mathrm{MW}(S_d) \text{ such that } s = 2s_o \end{cases}$$
Note that $\varepsilon(s^+_\Delta) = \varepsilon(s^-_\Delta)$ as $s^+_\Delta = -s^-_\Delta$ on $\mathrm{MW}(S_d)$.

Proof. It is enough to show
$$((\Delta_{0,d} + \Delta)/\Gamma_d)) = 1 \Leftrightarrow s^\pm_\Delta \in 2\,\mathrm{MW}(S_d).$$
($\Rightarrow$) As we have seen, $W_d \cong \Sigma_{d/2}$. We choose affine open subsets $V \subset W_d (\cong \Sigma_{d/2})$, and $U \subset \Sigma_d$ as follows:

(i) Both U and V are $\mathbb{C}^2$.
(ii) We choose affine coordinates (t, u) and $(\tilde{t}, \zeta)$ of U and V, respectively, in such a way that q_d is given by
$$q_d : (\tilde{t}, \zeta) \mapsto (t, u) = (\tilde{t}, \zeta^2 + f(t)),$$
where $f(t)$ is a polynomial of degree $\leq d$. Note that with respect to these coordinates (t, u) and $(\tilde{t}, \zeta)$, $\Delta \cap U : u - f(t) = 0$, $\Delta_{0,d}$ corresponds to the section given by $u = \infty$ and the involution σ_{q_d} is given by $(\tilde{t}, \zeta) \mapsto (\tilde{t}, -\zeta)$.

Since $((\Delta_{0,d} + \Delta)/\Gamma_d) = 1$, $q_d^*\Gamma_d$ is of the form $\Gamma^+ + \Gamma^-$. Since $\sigma_{q_d}^*\Gamma^+ = \Gamma^-, \sigma_{q_d}^*\Delta_{0,d/2} = \Delta_{0,d/2}$ and $\sigma_{q_d}^* F_{d/2} = F_{d/2}$, $\Gamma^+ \sim \Gamma^- \sim 3(\Delta_{0,d/2} + d/2F_{d/2})$. Hence we may assume
$$\begin{aligned} \Gamma^+ &: \quad F(\tilde{t}, \zeta) = \zeta^3 + a_1(\tilde{t})\zeta^2 + a_2(\tilde{t})\zeta + a_3(\tilde{t}) = 0 \\ \Gamma^- &: \quad -F(\tilde{t}, -\zeta) = \zeta^3 - a_1(\tilde{t})\zeta^2 + a_2(\tilde{t})\zeta - a_3(\tilde{t}) = 0, \end{aligned}$$
where $\deg a_k(\tilde{t}) \leq kd/2$. Since $\zeta^2 = u - f(t), t = \tilde{t}$, we have
$$F(\tilde{t}, \zeta) = (a_1(t)u - a_1(t)f(t) + a_3(t)) + (u - f(t) + a_2(t))\zeta.$$

As $q_d^*\Gamma = \Gamma^+ + \Gamma^-$, we may assume that Γ_d is given by

$$-F(\tilde{t},\zeta)F(\tilde{t},-\zeta) = (a_1(t)u - a_1(t)f + a_3(t))^2 - (u - f(t) + a_2(t))^2(u - f(t)) = 0.$$

On the other hand, over U is S'_d is given by

$$S'_d|_{p'_d{}^{-1}} : y^2 = (a_1(t)u - a_1(t)f + a_3(t))^2 - (u - f(t) + a_2(t))^2(u - f(t)),$$

and the above equation considered as a Weierstrass equation of the generic fiber, $S_{d,\eta}$, of φ_d. By our construction, $s^{\pm}_{\Delta}$ is given by

$$s^{\pm}_{\Delta} : (f(t), \pm a_3(t)).$$

Put

$$s^{\pm}_o : (\mp(f(t) - a_2(t)), \pm(a_1(t)a_2(t) - a_3(t)).$$

Then $s^{\pm}_o \in \mathrm{MW}(S_d)$ and by the definition of the group law, we have

$$2s^{\pm}_o = s^{\pm}_{\Delta}.$$

($\Leftarrow$) We use the affine open subsets of Σ_d and W_d as before. Suppose that Γ_d is given by

$$\Gamma_d : F_{\Gamma_d}(t,u) = u^3 + c_1(t)u^2 + c_2(t)u + c_3(t) = 0$$

where $c_k(t)(i = 1,2,3)$ are polynomials of degrees $\leq kd$. Then S'_d over U is given by $y^2 = F_{\Gamma_d}(t,u)$ and, as we have seen, this equation can be regarded as a Weierstrass equation of the generic fiber $S_{d,\eta}$. Since $s^+_{\Delta}O = 0$ and $p_d(s^+_{\Delta}) = \Delta$, $s^+_{\Delta} \in \mathrm{MW}(S_d)$ is given by

$$s^+_{\Delta} : (u,y) = (f(t), g(t)),$$

where $g(t)$ is a polynomial of degree $\leq 3d/2$. Let $s_o \in \mathrm{MW}(S_d)$ such that $2s_o = s^+_{\Delta}$. Since s_o is a $\mathbb{C}(\mathbb{P}^1)(= \mathbb{C}(t))$-rational point of $S_{d,\eta}$, there exist $f_o(t), g_o(t) \in \mathbb{C}(t)$ such that

$$s_o : (u,y) = (f_o(t), g_o(t)).$$

Since $s^+_{\Delta}O = 0$, by [9, Theorem 9.1], we infer that $s_oO = 0$. Therefore $f_o(t), g_o(t) \in \mathbb{C}[t]$ and $\deg f_o \leq d, \deg g_o \leq 3d/2$. Now let

$$y = \alpha(t)u + \beta(t), \alpha(t), \beta(t) \in \mathbb{C}(t)$$

be the tangent line of the elliptic curve $S_{d,\eta}$ over $\mathbb{C}(t)$ at s_o. By the definition of the group law on $S_{d,\eta}$, we have

$$F(t,u) = (\alpha(t)u + \beta(t))^2 + (u - f_o(t))^2(u - f(t)).$$

As $F(t,u), f, f_o \in \mathbb{C}[t,u]$, we infer that $\alpha(t), \beta(t) \in \mathbb{C}[t]$. Thus we may assume that $\Gamma_d \cap U$ is given by

$$(\alpha(t)u + \beta(t))^2 + (u - f_o(t))^2(u - f(t)) = 0.$$

As $q_d^*\Gamma_d$ on V is given by

$$\begin{aligned} & (\alpha(t)u + \beta(t))^2 + (u - f_o(t))^2\zeta^2 \\ = \ & \{(\alpha(t)u + \beta(t)) + \sqrt{-1}(u - f_o(t))\zeta\} \times \{(\alpha(t)u + \beta(t)) - \sqrt{-1}(u - f_o(t))\zeta\}, \end{aligned}$$

Γ_d is splitting with respect to q_d, i.e., $((\Delta_{0,d} + \Delta)/\Gamma_d) = 1$. □

Remark 1.1. Theorem 1.2 can be generalized to the case when S_d has a hyperelliptic fibration under some restriction. See [19].

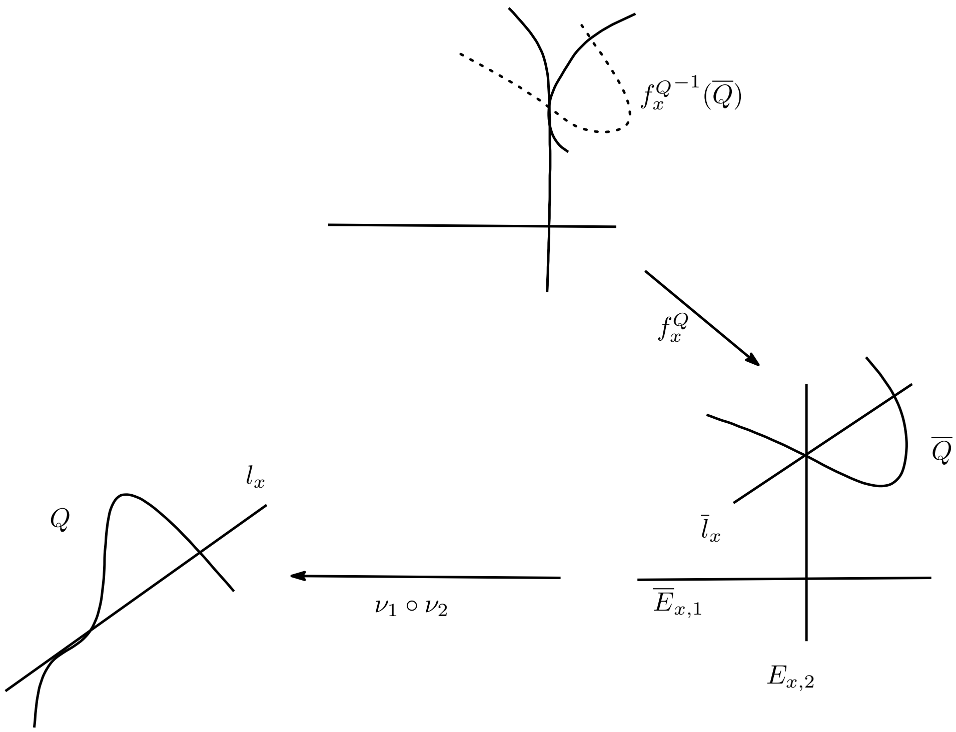
1.3. **Double covers of $\mathbb{P}^2$ branched along quartics and rational elliptic surfaces.** An elliptic surface $\mathcal{E}$ is said to be rational, if $\mathcal{E}$ is a rational surface. Hence it is an elliptic surface over $\mathbb{P}^1$. Analogously to [17], we associate a rational elliptic surface $\mathcal{E}_x^Q$ to a reduced quartic Q in $\mathbb{P}^2$ with a distinguished smooth point $x \in Q$ as follows:

Let $\nu_1 : \mathbb{P}^2_x \to \mathbb{P}^2$ be a blowing-up at x. We denote the proper transform of the tangent line l_x at x by $\bar{l}_{x,1}$, and the exceptional curve of ν_1 by $E_{x,1}$. We next consider another blowing up $\nu_2 : \widehat{\mathbb{P}}^2 \to \mathbb{P}^2_x$ at $\bar{l}_{x,1} \cap E_{x,1}$, and denote the proper transforms of $\bar{l}_{x,1}, E_{x,1}$ and the exceptional curve of ν_2 by $\bar{l}_x$, $\overline{E}_{x,1}$, and $E_{x,2}$, respectively. Let $f' : \mathcal{E}' \to \widehat{\mathbb{P}}^2$ be a double cover with branch locus $\overline{E}_{x,1}$ and $\overline{Q}$, where $\overline{Q}$ is the proper transform of Q with respect to $\nu_2 \circ \nu_1$. Let $\mu_x^Q : \mathcal{E}_x^Q \to \mathcal{E}'$ be the canonical resolution of $\mathcal{E}'$ and put $f_x^Q := f' \circ \mu_x^Q$. Then we see that $\mathcal{E}_x^Q$ satisfies the following properties:

(i) The pencil Λ_x of lines through x induces a relatively minimal elliptic fibration $\varphi_x^Q : \mathcal{E}_x^Q \to \mathbb{P}^1$.
(ii) The preimage of $\overline{E}_{x,1}$ gives rise to a section O of φ_x^Q, and the generic fiber has a group structure, O being the zero element. Moreover the covering transformation of $\mathcal{E}_x^Q$ coincides with the involution induced by the inversion of the group law.
(iii) The preimages of $E_{x,2}$ and $\bar{l}_x$ in $\mathcal{E}_x^Q$ are irreducible components of singular fibers. The types of the singular fiber cointainig the preimages of $E_{x,2}$ and $\bar{l}_x$ are as follows:

I_2	l_x meets Q at x and at another two distinct points.
III	l_x is a 3-fold tangent point.
I_3	l_x is a bitangent line.
IV	l_x is a 4-fold tangent point.
I_n $(n \geq 4)$	l_x passes through a singular point of type $A_n (n \geq 1)$.

We use here Kodaira's notation ([9]) in order to describe the types of singular fibers. The following picture describes the case that l_x is a 3-fold tangent line at x.



(iv) Other singular fibers of $\mathcal{E}_x^Q$ correspond to lines in Λ_x not meeting Q at 4 distinct points. We refer to [11, Table 6.2] for details.

Remark 1.2. Note that any rational elliptic surface $\mathcal{E}$ with at least one reducible singular fiber is obtained above. Namely $\mathcal{E} = \mathcal{E}_x^Q$ for some Q and a smooth point x on Q.

1.4. **The Mordell-Weil lattices of $\mathcal{E}_x^Q$.** In this subsection, we give a table of types of singularities of Q, the relative position of l_x and Q, and the Mordell-Weil lattices of $\mathcal{E}_x^Q$. We first note that $\mathrm{MW}(\mathcal{E}_x^Q)$ has no 2-torsion, since we assume that Q is irreducible. Also we omit cases which never occur. As for the structure of the Mordell-Weil lattices for rational elliptic surfaces, we refer to [12] and to [15] for the correction of the misprints in [12]. Let us explain notations used in the table.

- Ξ_Q and $l_x \cap Q$ are the same as those in the table Theorem 0.1
- $R_{Q,x}$: the subgroup of $\mathrm{NS}(\mathcal{E}_x^Q)$ generated by $\{\Theta_{v,1}, \ldots, \Theta_{v,m_v-1}\}_{v \in R_{\varphi_x^Q}}$. Note that $R_{Q,x}$ is isomorphic to a direct sum of root lattices of A-D-E type, and we describe $R_{Q,x}$ as a direct sum of them.
- $\mathrm{MW}(\mathcal{E}_x^Q)$: the lattice structure of $\mathrm{MW}(\mathcal{E}_x^Q)$. To describe them, we use the notation in [12]. Namely $\bullet^*$ means the dual lattice of the lattice $\bullet$ and $\langle m \rangle$ denotes a lattice of rank 1, $\mathbb{Z}x$ with $\langle x, x\rangle = m$. Also a matrix means the intersection matrix with respect to a certain basis. Note that the lattice structure is determined by $R_{Q,x}$ as $\mathrm{MW}(\mathcal{E}_x^Q)$ has no 2-torsion.
- $\mathrm{MW}^0(\mathcal{E}_x^Q)$: the narrow part of $\mathrm{MW}(\mathcal{E}_x^Q)$, i.e., the subgroup of $\mathrm{MW}(\mathcal{E}_x^Q)$ consisting of sections s with $s\Theta_{v,0} = 1$.

No.	Ξ_Q	$l_x \cap Q$	$R_{Q,x}$	$\mathrm{MW}(\mathcal{E}_x^Q)$	$\mathrm{MW}^0(\mathcal{E}_x^Q)$
1	A_6	s	$A_6 \oplus A_1$	$\langle 1/14 \rangle$	$\langle 14 \rangle$
2	A_6	sb	A_8	$\mathbb{Z}/3\mathbb{Z}$	$\{0\}$
3	E_6	s	$E_6 \oplus A_1$	$\langle 1/6 \rangle$	$\langle 6 \rangle$
4	E_6	b	$E_6 \oplus A_2$	$\mathbb{Z}/3\mathbb{Z}$	$\{0\}$
5	A_5	s	$A_5 \oplus A_1$	$A_1^* \oplus \langle 1/6 \rangle$	$A_1 \oplus \langle 6 \rangle$
6	A_5	b	$A_5 \oplus A_2$	$A_1^* \oplus \mathbb{Z}/3\mathbb{Z}$	A_1
7	A_5	sb	A_7	$\langle 1/8 \rangle$	$\langle 8 \rangle$
8	D_5	s	$D_5 \oplus A_1$	$A_1^* \oplus \langle 1/4 \rangle$	$A_1 \oplus \langle 4 \rangle$
9	D_5	b	$D_5 \oplus A_2$	$\langle 1/12 \rangle$	$\langle 12 \rangle$
10	D_4	s	$D_4 \oplus A_1$	$(A_1^*)^{\oplus 3}$	$A_1^{\oplus 3}$
11	D_4	b	$D_4 \oplus A_2$	$\frac{1}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$	$\begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix}$
12	$A_4 + A_2$	s	$A_4 \oplus A_2 \oplus A_1$	$\langle 1/30 \rangle$	$\langle 30 \rangle$
13	$A_4 + A_2$	sb	$A_4 \oplus A_4$	$\mathbb{Z}/5\mathbb{Z}$	$\{0\}$
14	$A_4 + A_1$	s	$A_4 \oplus A_1^{\oplus 2}$	$\frac{1}{10}\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$	$\begin{pmatrix} 6 & -2 \\ -2 & 4 \end{pmatrix}$
15	$A_4 + A_1$	b	$A_4 \oplus A_2 \oplus A_1$	$\langle 1/30 \rangle$	$\langle 30 \rangle$
16	$A_4 + A_1$	sb	$A_4 \oplus A_3$	$\langle 1/20 \rangle$	$\langle 20 \rangle$
17	$A_4 + A_1$	sb	$A_6 \oplus A_1$	$\langle 1/14 \rangle$	$\langle 14 \rangle$
18	$A_3 + A_2$	s	$A_3 \oplus A_2 \oplus A_1$	$A_1^* \oplus \langle 1/12 \rangle$	$A_1 \oplus \langle 12 \rangle$
19	$A_3 + A_2$	sb	$A_4 \oplus A_3$	$\langle 1/20 \rangle$	$\langle 20 \rangle$
20	$A_3 + A_2$	sb	$A_5 \oplus A_2$	$A_1^* \oplus \mathbb{Z}/3\mathbb{Z}$	A_1
21	$A_3 + A_1$	s	$A_3 \oplus A_1^{\oplus 2}$	$(A_1^*)^{\oplus 2} \oplus \langle 1/4 \rangle$	$A_1^{\oplus 2} \oplus \langle 4 \rangle$
22	$A_3 + A_1$	b	$A_3 \oplus A_2 \oplus A_1$	$A_1^* \oplus \langle 1/12 \rangle$	$A_1 \oplus \langle 12 \rangle$
23	$A_3 + A_1$	sb	$A_5 \oplus A_1$	$A_1^* \oplus \langle 1/6 \rangle$	$A_1 \oplus \langle 6 \rangle$
24	$A_3 + A_1$	sb	$A_3 \oplus A_3$	$\langle 1/4 \rangle^{\oplus 2}$	$\langle 4 \rangle^{\oplus 2}$
25	$3A_2$	s	$A_2^{\oplus 3} \oplus A_1$	$\langle 1/6 \rangle \oplus \mathbb{Z}/3\mathbb{Z}$	$\langle 6 \rangle$
26	$3A_2$	b	$A_2^{\oplus 4}$	$(\mathbb{Z}/3\mathbb{Z})^{\oplus 2}$	$\{0\}$
27	$2A_2 + A_1$	s	$A_2^{\oplus 2} \oplus A_1^{\oplus 2}$	$\langle 1/6 \rangle^{\oplus 2}$	$\langle 6 \rangle^{\oplus 2}$
28	$2A_2 + A_1$	b	$A_2^{\oplus 3} \oplus A_1$	$\langle 1/6 \rangle \oplus \mathbb{Z}/3\mathbb{Z}$	$\langle 6 \rangle$
29	$2A_2 + A_1$	sb	$A_4 \oplus A_2 \oplus A_1$	$\langle 1/30 \rangle$	$\langle 30 \rangle$

No.	Ξ_Q	$l_x \cap Q$	$R_{Q,x}$	$\mathrm{MW}(\mathcal{E}_x^Q)$	$\mathrm{MW}^0(\mathcal{E}_x^Q)$
30	A_2+2A_1	s	$A_2\oplus A_1^{\oplus 3}$	$A_1^*\oplus\frac{1}{6}\begin{pmatrix}2&1\\1&2\end{pmatrix}$	$A_1\oplus\begin{pmatrix}4&-2\\-2&4\end{pmatrix}$
31	A_2+2A_1	b	$A_2^{\oplus 2}\oplus A_1^{\oplus 2}$	$\langle 1/6\rangle^{\oplus 2}$	$\langle 6\rangle^{\oplus 2}$
32	A_2+2A_1	sb	$A_4\oplus A_1^{\oplus 2}$	$\frac{1}{10}\begin{pmatrix}2&1\\1&3\end{pmatrix}$	$\begin{pmatrix}6&-2\\-2&4\end{pmatrix}$
33	A_2+2A_1	sb	$A_3\oplus A_2\oplus A_1$	$A_1^*\oplus\langle 1/12\rangle$	$A_1\oplus\langle 12\rangle$
34	$3A_1$	s	$A_1^{\oplus 4}$	$(A_1^*)^{\oplus 4}$	$A_1^{\oplus 4}$
35	$3A_1$	s	$A_2\oplus A_1^{\oplus 3}$	$A_1^*\oplus\frac{1}{6}\begin{pmatrix}2&1\\1&2\end{pmatrix}$	$A_1\oplus\begin{pmatrix}4&-2\\-2&4\end{pmatrix}$
36	$3A_1$	sb	$A_3\oplus A_1^{\oplus 2}$	$(A_1^*)^{\oplus 2}\oplus\langle 1/4\rangle$	$A_1^{\oplus 2}\oplus\langle 4\rangle$
37	A_4	s	$A_4\oplus A_1$	$\frac{1}{10}\begin{pmatrix}3&1&-1\\1&7&3\\-1&3&7\end{pmatrix}$	$\begin{pmatrix}4&-1&1\\-1&2&-1\\1&-1&2\end{pmatrix}$
38	A_4	b	$A_4\oplus A_2$	$\frac{1}{15}\begin{pmatrix}2&1\\1&8\end{pmatrix}$	$\begin{pmatrix}8&-1\\-1&2\end{pmatrix}$
39	A_4	sb	A_6	$\frac{1}{7}\begin{pmatrix}2&1\\1&4\end{pmatrix}$	$\begin{pmatrix}4&-1\\-1&2\end{pmatrix}$
40	A_3	s	$A_3\oplus A_1$	$A_3^*\oplus A_1^*$	$A_3\oplus A_1$
41	A_3	b	$A_3\oplus A_2$	$\frac{1}{12}\begin{pmatrix}7&1&2\\1&7&2\\2&2&4\end{pmatrix}$	$\begin{pmatrix}2&0&-1\\0&2&-1\\-1&-1&4\end{pmatrix}$
42	A_3	sb	A_5	$A_2^*\oplus A_1^*$	$A_2\oplus A_1$
43	$2A_2$	s	$A_2^{\oplus 2}\oplus A_1$	$A_2^*\oplus\langle 1/6\rangle$	$A_2\oplus\langle 6\rangle$
44	$2A_2$	b	$A_2^{\oplus 3}$	$A_2^*\oplus\mathbb{Z}/3\mathbb{Z}$	A_2
45	$2A_2$	sb	$A_4\oplus A_2$	$\frac{1}{15}\begin{pmatrix}2&1\\1&8\end{pmatrix}$	$\begin{pmatrix}8&-1\\-1&2\end{pmatrix}$
46	A_2+A_1	s	$A_2\oplus A_1^{\oplus 2}$	$\frac{1}{6}\begin{pmatrix}2&1&0&-1\\1&5&3&1\\0&3&6&3\\-1&1&3&5\end{pmatrix}$	$\begin{pmatrix}4&-1&0&1\\-1&2&-1&0\\0&-1&2&-1\\1&0&-1&2\end{pmatrix}$
47	A_2+A_1	b	$A_2^{\oplus 2}\oplus A_1$	$A_2^*\oplus\langle 1/6\rangle$	$A_2\oplus\langle 6\rangle$
48	A_2+A_1	sb	$A_4\oplus A_1$	$\frac{1}{10}\begin{pmatrix}3&1&-1\\1&7&3\\-1&3&7\end{pmatrix}$	$\begin{pmatrix}4&-1&1\\-1&2&-1\\1&-1&2\end{pmatrix}$
49	A_2+A_1	sb	$A_4\oplus A_1$	$\frac{1}{12}\begin{pmatrix}7&1&2\\1&7&2\\2&2&4\end{pmatrix}$	$\begin{pmatrix}2&0&-1\\0&2&-1\\-1&-1&4\end{pmatrix}$
50	$2A_1$	s	$A_1^{\oplus 3}$	$D_4^*\oplus A_1^*$	$D_4\oplus A_1$
51	$2A_1$	b	$A_2\oplus A_1^{\oplus 2}$	$\frac{1}{6}\begin{pmatrix}2&1&0&-1\\1&5&3&1\\0&3&6&3\\-1&1&3&5\end{pmatrix}$	$\begin{pmatrix}4&-1&0&1\\-1&2&-1&0\\0&-1&2&-1\\1&0&-1&2\end{pmatrix}$
52	$2A_1$	sb	$A_3\oplus A_1$	$A_3^*\oplus A_1^*$	$A_3\oplus A_1$

No.	Ξ_Q	$l_x \cap Q$	$R_{Q,x}$	$\mathrm{MW}(\mathcal{E}_x^Q)$	$\mathrm{MW}^0(\mathcal{E}_x^Q)$
53	A_2	s	$A_2 \oplus A_1$	A_5^*	A_5
54	A_2	b	$A_2^{\oplus 2}$	$(A_2^*)^{\oplus 2}$	$A_2^{\oplus 2}$
55	A_2	sb	A_4	A_4^*	A_4
56	A_1	s	$A_1^{\oplus 2}$	D_6^*	D_6
57	A_1	b	$A_2 \oplus A_1$	A_5^*	A_5
58	A_1	sb	A_3	D_5^*	D_5
59	$\emptyset$	s	A_1	E_7^*	E_7
60	$\emptyset$	b	A_2	E_6^*	E_6

2. Proof of Theorem 0.1

We keep the same notations as before. Our result will be proved case-by-case. Let us start with the following lemma.

Lemma 2.1. *Let C be an even tangential conic to Q through x. The preimage of C in $\mathcal{E}_x^Q$ consists of two sections s_C^+ and s_C^- such that*

(i) $\langle s_C^+, s_C^+\rangle = \langle s_C^-, s_C^-\rangle = 2$
(ii) $s_C^+ O = s_C^- O = 0$
(iii) $s_C^+\Theta_{v,0} = s_C^-\Theta_{v,0} = 1$ *for all* $v \in R_{\varphi_x^Q}$, *i.e,* $s_C^\pm \in \mathrm{MW}^0(\mathcal{E}_x^Q)$.

Coversely, for any section s in $\mathrm{MW}(\mathcal{E}_x^Q)$ satisfying two of the above three properties, the image of s in $\mathbb{P}^2$ is an even tangential conic to Q.

Proof. We first note that two of the properties (i), (ii) and (iii) imply the remaining. This follows from the formula

$$\langle s, s\rangle = 2 + 2sO - \sum_{v\in R_\varphi} \mathrm{Corr}_v(s, s)$$

for the rational elliptic surface $\mathcal{E}_x^Q$ and $s \in \mathrm{MW}(\mathcal{E}_x^Q)$.

Let $\overline{C}$ be the proper transform of C in $\widehat{\mathbb{P}}^2$. Since $\overline{C}$ is tangent to $\overline{Q}$ at each intersection point and $\overline{C}\cap\overline{E}_{x,1} = \emptyset$, the preimage of $\overline{C}$ in $\mathcal{E}_x^Q$ consists of 2 irreducible components s_C^+ and s_C^- so that $s_C^\pm O = 0$. Since $\overline{C}$ meets the proper transform of a general member in Λ_x at one point, both s_C^+ and s_C^- are sections of $\varphi_x^Q : \mathcal{E}_x^Q \to \mathbb{P}^1$. The property (iii) follows from the fact that $\overline{C}$ meets $E_{x,2}$ and $\overline{C}$ does not pass through singularities of $\overline{Q}$. Now the property (i) is straightforward from the explicit formula for $\langle\, ,\, \rangle$.

Conversely, suppose that we have a section s satisfying two of the properties (i), (ii) and (iii). Let C_s be the image of s in $\mathbb{P}^2$. By our construction of $\mathcal{E}_x^Q$, we infer that C_s is a conic tangent to Q at x. Since C_s is also the image of $\sigma^*_{f_x^Q} s$, we infer that C_s is an even tangent conic to Q. □

Theorem 2.1. *Let C be an even tangential conic to Q and let s_C^+ be the section as above.*

$$(C/Q) = (-1)^{\varepsilon(s_C^+)},$$

where the symbol $\varepsilon(s_C^+)$ is the same as that defined in Theorem 1.2.

Proof. Let $\widehat{\mathbb{P}}^2$ as before. Since $\bar{l}_x$ is a (-1) curve, by blowing down $\bar{l}_x$, we obtain Σ_2 with the following properties:

(i) The image of $\overline{Q}$ is a trisection $\Gamma_Q \sim 3(\Delta_{0,d} + 2F)$.
(ii) Singularities of Γ_Q are the same as those of Q except the A_1 singularity caused by blowing down $\bar{l}_x$.
(iii) The image of $\overline{E}_{x,1} = \Delta_{0,d}$.

(iv) The image of $\overline{C}$ is a section Δ_C such that $\Delta_C \sim (\Delta_{0,d} + 2F)$ and Δ_C is even tangent to Γ_Q.

Let $f_o : Z_o \to \Sigma_2$ be the induced double cover by $f_C : Z_C \to \mathbb{P}^2$, i.e., the $\mathbb{C}(Z_C)$-normalization of Σ_2. One easily see that $\Delta_{f_o} = \Delta + \Delta_C$.

$$\begin{tikzcd} Z_C \arrow[d] & \mathcal{E}_x^Q \arrow[d] & Z_o \arrow[d] \\ \mathbb{P}^2 & \widehat{\mathbb{P}}^2 \arrow[l] \arrow[r] & \Sigma_2 \end{tikzcd}$$

Since Δ_C is the image of $\overline{C}$, it is also the image of $s_C^{\pm}$. Hence we infer that

$$(C/Q) = 1 \Leftrightarrow (\Delta_{0,d} + \Delta_C/\Gamma_Q) = 1.$$

Hence by Theorem 1.2, we infer that $(C/Q) = 1$ if and only if $s_C^{+} \in 2\,\mathrm{MW}(\mathcal{E}_x^Q)$. □

Remark 2.1. Suppose that $s_C^{+} \in 2\,\mathrm{MW}(\mathcal{E}_x^Q)$. Let s_o be an element in $\mathrm{MW}(\mathcal{E}_x^Q)$ such that $2s_o = s_C^{+}$. By Lemma 2.1 (i), we have $\langle s_o, s_o\rangle = 1/2$. Hence if $\mathrm{MW}(\mathcal{E}_x^Q)$ has no section s with $\langle s, s\rangle = 1/2$, there is no quadratic residue even tangential conic to Q through x.

Lemma 2.2. *Let $\widetilde{Q}$ be the normalization of Q and we denote the genus of $\widetilde{Q}$ by $g(\widetilde{Q})$.*

(i) *No even tangential conic to Q is quadratic residue* mod$Q$ *if $g(\widetilde{Q}) \geq 2$.*

(ii) *All even tangential conic to Q are quadratic residue* mod$Q$ *if $g(\widetilde{Q}) = 0$.*

Proof. (i) Let C be an even tangential conic to Q and suppose that $(C/Q) = 1$. Let $f_C : Z_C \to \mathbb{P}^2$ be a double cover with $\Delta_{f_C} = C$. Then f_C^*Q is of the form $Q^+ + Q^-$. Since $Z_C = \mathbb{P}^1 \times \mathbb{P}^1, \mathrm{Pic}(Z_C) \cong \mathbb{Z} \oplus \mathbb{Z}$ and the covering transformation induces an involution $(a, b) \mapsto (b, a)$ on $\mathrm{Pic}(Z_C)$, we infer that $Q^+ \sim Q^- \sim (2, 2)$. Since Q^+, Q^- and Q are birationally equivalent, we have $g(\widetilde{Q}) \leq 1$ and the result follows.

(ii) Since the induced double cover on $\widetilde{Q}$ is unramified, $(C/Q) = 1$. □

Now we easily have the following theorem:

Theorem 2.2. *Let Q be an irreducible quartic. Choose a smooth point $x \in Q$ and let $\mathcal{E}_x^Q$ be the rational elliptic surface as in §1. Then we have the following:*

(i) *Let* ETC *be the set of conics passing through x. Then*

$$\begin{aligned} \sharp\mathrm{ETC} &= \sharp\{s \in \mathrm{MW}(\mathcal{E}_x^Q) \mid \langle s, s\rangle = 2, sO = 0\}/2 \\ &= \sharp\{s \in \mathrm{MW}^0(\mathcal{E}_x^Q) \mid \langle s, s\rangle = 2\}/2 \end{aligned}$$

(ii) *Let* QRETC *be the set of even tangential conics passing through x with $(C/Q) = 1$. Then*

$$\begin{aligned} \sharp\mathrm{QRETC} &= \sharp\{s \in 2\,\mathrm{MW}(\mathcal{E}_x^Q) \mid \langle s, s\rangle = 2, sO = 0\}/2 \\ &= \sharp\{s \in 2\,\mathrm{MW}(\mathcal{E}_x^Q) \cap \mathrm{MW}^0(\mathcal{E}_x^Q) \mid \langle s, s\rangle = 2\}/2 \end{aligned}$$

Proof. Our statements (i) and (ii) are immediate from Lemma 2.1 and Theorem 2.1. □

We now prove Theorem 0.1 case-by-case. We first compute $\sharp$ETC. By Lemma 2.1, it is enough to see the number of sections s in the narrow part $\mathrm{MW}^0(\mathcal{E}_x^Q)$ of $\mathrm{MW}(\mathcal{E}_x^Q)$ with $\langle s, s\rangle = 2$.

For the lattices of A-D-E types, it is nothing but the number of roots, and the following table is well known (see [6])

A_n	D_n $(n \geq 4)$	E_6	E_7
$n(n+1)$	$2n(n-1)$	72	126

From the above table and that in §2, our statement on $\sharp$ETC is straightforward except for the cases $11, 14, 30, 32, 35, 37, 38, 39, 41, 45, 46, 48, 49, 51$. For the rank 2 cases among the exceptional cases, our statement follows easily by direct computation. For the cases of rank > 2, we make use of [12, Lemma 3.8], which is as follows:

$$\begin{pmatrix} 4 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} \cong A_1^{\perp} \text{ in } A_4, \quad \begin{pmatrix} 4 & -1 & 0 & 1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 1 & 0 & -1 & 2 \end{pmatrix} \cong A_1^{\perp} \text{ in } A_5$$

$$\begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \cong A_2^{\perp} \text{ in } D_5$$

where the terminology $\bullet^{\perp}$ in $\blacksquare$ means that we embed a lattice $\bullet$ into $\blacksquare$ and $\bullet^{\perp}$ is the orthogonal complement of $\bullet$ in $\blacksquare$. Also, by [12, Lemma 3.8], the embedding is determined up to isomorphism. Hence we just count the number of roots which are orthogonal to the embedded lattices. To be more precise, we explain the case $A_1^{\perp}$ in A_5. We first consider the realization of A_5 as follows:

$$A_5 = \{(x_1, \dots, x_6) \mid \sum_i x_i = 0, x_i \in \mathbb{Z}\} \subset \mathbb{R}^6$$

and the pairing is induced from the Euclidean metric $\sum_i x_i^2$ in $\mathbb{R}^6$. Under these circumstances, the roots are given by a vector $(1, -1, 0, 0, 0, 0)$ and those obtained by permutations of the coordinates. We fix an embedding of A_1 given by $\mathbb{Z}(1, -1, 0, 0, 0, 0) \subset A_5$. Then roots in $A_1^{\perp}$ are

$$\begin{matrix} (0,0,\pm 1,\mp 1,0,0) & (0,0,\pm 1,0,\mp 1,0) & (0,0,\pm 1,0,0,\mp 1) \\ (0,0,0,\pm 1,\mp 1,0) & (0,0,0,\pm 1,0,\mp 1) & (0,0,0,0,\pm 1,\pm 1). \end{matrix}$$

Since the remaining cases are similar, we omit them. Thus we have a list for $\sharp$ETC.

We now go on to compute $\sharp$QRETC. We first note that $\sharp$QRETC $= 0$ if $\sharp$ETC $= 0$. In the following, we only cosider the case of $\sharp$ETC $\neq 0$.

Since Q is irreducible, $\mathrm{MW}(\mathcal{E}_x^Q)$ has no 2-torsion. Hence for each $s \in 2\,\mathrm{MW}(\mathcal{E}_x^Q)$, there exists a unique $s_o \in \mathrm{MW}(\mathcal{E}_x^Q)$ such that $2s_o = s$. For distinct $C_1, C_2 \in$ QRETC, $s_{C_1}^{+}$ and $s_{C_2}^{+}$ are distinct in $\mathrm{MW}^0(\mathcal{E}_x^Q)$. Hence it is enough to compute

$$\sharp\{s_o \in \mathrm{MW}(\mathcal{E}_x^Q) \mid \langle s_o, s_o \rangle = 1/2, 2s_o \in \mathrm{MW}^0(\mathcal{E}_x^Q)\}$$

Now Theorem 0.1 follows from the following claim:

Claim. *Suppose that $\sharp$ETC $\neq 0$. If $\mathrm{MW}(\mathcal{E}_x^Q)$ has an A_1^* as a direct summand, then two generators $\pm\tilde{s}$ of A_1^* are sections such that $\langle \tilde{s}, \tilde{s} \rangle = 1/2, 2\tilde{s} \in \mathrm{MW}^0(\mathcal{E}_x^Q)$. Conversely if there exists $s_o \in \mathrm{MW}(\mathcal{E}_x^Q)$ such that $\langle s_o, s_o \rangle = 1/2, 2s_o \in \mathrm{MW}^0(\mathcal{E}_x^Q)$, then $\mathbb{Z}s_o (\cong A_1^*)$ is a direct summand of $\mathrm{MW}(\mathcal{E}_x^Q)$.*

Proof of Claim. Suppose that A_1^* is a direct summand of $\mathrm{MW}(\mathcal{E}_x^Q)$ and let $\tilde{s}$ be a section such that $\mathbb{Z}\tilde{s} = A_1^*$. Then $\langle \tilde{s}, \tilde{s} \rangle = 1/2$ and $2\tilde{s} \in \mathrm{MW}^0(\mathcal{E}_x^Q)$ by [14, Theorem 9.1].

We now go on to show the converse. Let s_o be a section with $\langle s_o, s_o \rangle = 1/2, 2s_o \in \mathrm{MW}^0(\mathcal{E}_x^Q)$. As for the dual lattices of A-D-E type, we have the following table:

Type	A_n^*	D_n^* $(n \geq 4)$	E_6^*	E_7^*
Minimum norm	$\frac{n}{(n+1)}$	1	$\frac{4}{3}$	$\frac{3}{2}$

Hence we easily see that $\mathrm{MW}(\mathcal{E}_x^Q)$ has an A_1^* direct summand except for the cases 37, 38, 39, 41, 45, 46, 48, 49 and 51. We see that there is no section s with $\langle s, s \rangle = 1/2$ for these exceptional cases.

Cases 38, 39 and 45. In these cases, the paring $\langle\, ,\, \rangle$ takes its value in $1/15\mathbb{Z}$ (Cases 38 and 45), and $1/7\mathbb{Z}$ (Case 39), where $1/m\mathbb{Z} = \{a/m \mid a \in \mathbb{Z}\}$. Hence there is no section s with $\langle s, s\rangle = 1/2$.

Cases 37 and 48. Let s be any element of $\mathrm{MW}(\mathcal{E}_x^Q)$. In these cases,

$$\langle s, s\rangle = 2(1 + sO) - \frac{k_1(5 - k_1)}{5} - \frac{1}{2}k_2,$$

where $k_1 \in \{0, 1, 2, 3, 4\}, k_2 \in \{0, 1\}$. Hence we infer that there is no s with $\langle s, s\rangle = 1/2$.

Cases 41 and 49. Let s be any element of $\mathrm{MW}(\mathcal{E}_x^Q)$. In these cases,

$$\langle s, s\rangle = 2(1 + sO) - \frac{k_1(4 - k_1)}{4} - \frac{2}{3}k_2,$$

where $k_1 \in \{0, 1, 2, 3\}, k_2 \in \{0, 1\}$. Hence we infer that there is no s with $\langle s, s\rangle = 1/2$.

Cases 46 and 51. Let s be any element of $\mathrm{MW}(\mathcal{E}_x^Q)$. In these cases,

$$\langle s, s\rangle = 2(1 + sO) - \frac{2}{3}k_1 - \frac{1}{2}k_2 - \frac{1}{2}k_3,$$

where $k_1, k_2, k_3 \in \{0, 1\}$. Hence we infer that there is no s with $\langle s, s\rangle = 1/2$.

After checking each case we see that s_o generates an A_1^* direct summand.

3. Preliminaries from theory of Galois covers

3.1. **Galois covers.** In this subsection, we summarize some facts and terminologies on Galois covers. For details, see [1, §3]. Let X and Y be normal projective varieties. We call X a cover if there exists a finite surjective morphism $\pi : X \to Y$. Let $\mathbb{C}(X)$ and $\mathbb{C}(Y)$ be rational function fields of X and Y, respectively. If X is a cover of Y, then $\mathbb{C}(X)$ is an algebraic extension of $\mathbb{C}(Y)$ with $\deg \pi = [\mathbb{C}(X) : \mathbb{C}(Y)]$. Let G be a finite group. A G-cover is a cover $\pi : X \to Y$ such that $\mathbb{C}(X)/\mathbb{C}(Y)$ is a Galois extension with $\mathrm{Gal}(\mathbb{C}(X)/\mathbb{C}(Y)) \cong G$. For a cover $\pi : X \to Y$, the branch locus Δ_π of π is a subset of Y as follows:

$$\Delta_\pi = \{y \in Y \mid \pi \text{ is not locally isomorphic over } y\}.$$

If Y is smooth, Δ_π is an algebraic subset of pure codimention 1 ([21]). Let $\pi : X \to Y$ be a G-cover of a smooth projective variety Y. Let $\Delta_\pi = \Delta_{\pi,1} + \ldots + \Delta_{\pi,r}$ denote the irreducible decomposition of Δ_π. We say that $\pi : X \to Y$ is branched at $e_1\Delta_{\pi,1} + \ldots + e_r\Delta_{\pi,r} (e_i \geq 2, i = 1, \ldots, r)$ if the ramification index along $\Delta_{\pi,i}$ is e_i for each i.

Let B be a reduced divisor on a smooth projective variety Y and $B = B_1 + \ldots + B_r$ denote its irreducible decomposition. It is known that the existence of a G-cover $\pi : X \to Y$ at $\sum_i e_iB_i$ can be characterized as follows:

Theorem 3.1. *There exists a G-cover of Y branched at $\sum_i e_iB_i$ if and only if there exists an epimorphism $\phi : \pi_1(Y \setminus B, *) \to G$ such that for each meridian γ_i of B_i, the image of its class $[\gamma_i]$, $\phi([\gamma_i])$, has order e_i.*

3.2. **Dihedral covers.** Let $\mathcal{D}_{2n}$ be the dihedral group of order $2n$ ($n \geq 3$) given by $\langle \sigma, \tau \mid \sigma^2 = \tau^n = (\sigma\tau)^2 = 1\rangle$. In [17], we developed a method to deal with $\mathcal{D}_{2n}$-covers, and some variants of the results in [17] have been studied since then. We summarize here some results which we need later. Let us start with introducing some notation in order to explain them.

Let $\pi : X \to Y$ be a $\mathcal{D}_{2n}$-cover. By its definition, $\mathbb{C}(X)$ is a D_{2n}-extension of $\mathbb{C}(Y)$. Let $\mathbb{C}(X)^\tau$ be the fixed field by τ. We denote the $\mathbb{C}(X)^\tau$- normalization by $D(X/Y)$. We denote the induced morphisms by $\beta_1(\pi) : D(X/Y) \to Y$ and $\beta_2(\pi) : X \to D(X/Y)$. Note that X is a $\mathbb{Z}/n\mathbb{Z}$-cover of $D(X/Y)$ and $D(X/Y)$ is a double cover of Y such that $\pi = \beta_1(\pi) \circ \beta_2(\pi)$:

$$\begin{tikzcd}
X \arrow[dd, "\pi"'] \arrow[dr, "\beta_2(\pi)"] & \\
& D(X/Y) \arrow[dl, "\beta_1(\pi)"] \\
Y &
\end{tikzcd}$$

Generic $\mathcal{D}_{2n}$-covers. A $\mathcal{D}_{2n}$-covers $\pi : S \to \Sigma$ is said to be *generic* if $\Delta(\pi) = \Delta(\beta_1(\pi))$. As for conditions for the existence of generic $\mathcal{D}_{2n}$-covers with prescribed branch loci, we have the following:

Let B be a reduced divisor on Σ with at worst simple singularities. Suppose that there exists a double cover $f'_B : Z'_B \to \Sigma$ with branch locus B and let $\mu_B : Z_B \to Z'_B$ be the canonical resolution. We define the subgroup R_B of $\mathrm{NS}(Z_B)$ as follows:

$$R_B := \oplus_{b\in \mathrm{Sing}(B)} R_b,$$

where R_b is the subgroup in $\mathrm{NS}(Z_B)$ generated by the exceptional divisor of the singularity ${f'}_B^{-1}(x)$. Then we have the following result:

Theorem 3.2. [1, Theorem 3.27] *Let p be an odd prime and suppose that Z_B is simply connected. There exists a generic $\mathcal{D}_{2p}$-cover $\pi : S \to \Sigma$ with branch locus B if and only if* $\mathrm{NS}(Z_B)/R_B$ *has p-torsion.*

Let $R_b^\vee = \mathrm{Hom}_{\mathbb{Z}}(R_b, \mathbb{Z})$. R_b can be regarded as a subgroup of $R_b^\vee$ by using the intersection pairing. Since the torsion subgroup of $\mathrm{NS}(Z_B)/R_B$ can be considered as a subgroup of $\oplus_{b\in\mathrm{Sing}(B)} R_b^\vee/R_b$, we have the following corollary:

Corollary 3.1. *If there exists no b such that $p|\sharp(R_b^\vee/R_b)$, then there exists no generic $\mathcal{D}_{2p}$-cover with branch locus B.*

Non-generic $\mathcal{D}_{2n}$-covers. A $\mathcal{D}_{2n}$-cover is said to be *non-generic* if $\Delta(\beta_1(\pi))$ is a proper subset of $\Delta(\pi)$. We consider a non-generic $\mathcal{D}_{2n}$-cover of Σ under the following setting:

Let $B = B_1 + B_2$ be a reduced divisor on Σ such that:

(i) there exists a double cover $f'_{B_1} : Z'_{B_1} \to \Sigma$ with $\Delta_{f'_{B_1}} = B_1$, and

(ii) B_2 is irreducible.

Let $f_{B_1} : Z_{B_1} \to \Sigma$ be the canonical resolution of Z'_{B_1}.

Proposition 3.1. [1, Proposition 3.31] *Suppose that Σ is simply connected and the preimage of the strict transform of B_2 consists of two distinct irreducible components B_2^+ and B_2^-. If there exist an effective divisor D and a line bundle $\mathcal{L}$ on Z_{B_1} satisfying conditions*

(i) $D = B_2^+ + D'$; D' and $\sigma^*_{f_{B_1}} D'$ have no common components,

(ii) $\mathrm{Supp}(D' + \sigma^*_{f_{B_1}} D')$ *is contained in the exceptional set of $\mu_{f'_{B_1}}$ and*

(iii) $D - \sigma^*_{f_{B_1}} D \sim n\mathcal{L}$ $(n \geq 3)$, where $\sim$ denotes linear equivalence,

then there exists a $\mathcal{D}_{2n}$-cover $\pi : S \to \Sigma$ branched at $2B_1 + nB_2$ such that $\Delta_{\beta_1(\pi)} = B_1$.

Corollary 3.2. *If $\sigma^*_{f_{B_1}} B_2^+ \sim B_2^-$ and there exists a $\mathcal{D}_{2n}$-cover of Σ branched at $2B_1 + nB_2$ for any $n \geq 3$.*

Proposition 3.2. [1, Proposition 3.32] *Under the notation above, if a $\mathcal{D}_{2n}$-cover $\pi : S \to \Sigma$ branched at $2B_1 + nB_2$ exists, then the following holds:*

(i) $D(S/\Sigma) = Z'_{B_1}$. *The preimage of the porper transform of* B_2 *in* Z_{B_1} *consists of two irreducible components,* $B_2^{\pm}$.

(ii) There exist effective divisors D_1 *and* D_2*, and a line bundle* $\mathcal{L}$ *on* Z_{B_1} *such that*

- $\mathrm{Supp}(D_1 + \sigma^*_{f_{B_1}} D_1 + D_2)$ *is contained in the exceptional set of* μ,
- D_1 *and* $\sigma^*_{f_{B_1}} D_1$ *have no common components,*
- *if* $D_2 \neq \emptyset$*, then* n *is even,* D_2 *is reduced, and* $D' = \sigma^*_{f_{B_1}} D'$ *for each irreducible component* D' *of* D_2*, and*
- $(B_2^+ + D_1 + \frac{n}{2} D_2) - (B_2^- + \sigma^*_{f_{B_1}} D_1) \sim n\mathcal{L}$.

Corollary 3.3. *If a* $\mathcal{D}_{2n}$*-cover* $\pi : S \to \Sigma$ *branched at* $2B_1 + nB_2$ *exists, then* B_2 *is a splitting curve with respect to* f_{B_1}.

4. Proof of Theorem 0.2

We first note that there are 3 possibilities for $\beta_1(\pi) : D(S/\mathbb{P}^2) \to \mathbb{P}^2$:

Case 1. $D(S/\mathbb{P}^2) = Z_C, \beta_1(\pi) = f_C$.

Case 2. $D(S/\mathbb{P}^2) = Z'_Q, \beta_1(\pi) = f'_Q$.

Case 3. $D(S/\mathbb{P}^2) = Z'_{C+Q}, \beta_1(\pi) = f'_{C+Q}$.

Note that $f'_\bullet : Z_\bullet \to \mathbb{P}^2$ denotes a double cover with branch locus $\bullet$. We show that our statements (i) and (ii) hold for Case 1 and neither Cases 2 nor 3 occur.

Case 1. In this case, π is branched at $2C + pQ$. Hence, by Corollary 3.3, we infer that $(C/Q) = 1$. Put $f_C^* Q = Q^+ + Q^-$. By Proposition 3.2, $Q^+ - Q^-$ is p-divisible in $\mathrm{Pic}(Z_C)$. Since $Q^+ + Q^- \sim (4,4)$, Q^+ is linearly equivalent to either $(3,1), (1,3)$ or $(2,2)$. Hence, $Q^+ \sim Q^- \sim (2,2)$ if $p \geq 3$.

Case 2. Let Σ_2, Δ_C and Γ_Q be the Hirzebruch surface of degree 2 and the divisors obtained as in §2. By considering the $\mathbb{C}(S)$-normalization of Σ_2, we have a D_{2p}-cover branched at $2(\Delta_{0,d} + \Gamma_Q) + p\Delta_C$. As in [18], we reduce our problem on the existence of $\mathcal{D}_{2p}$-covers to that on a linear equation on $\mathrm{MW}(\mathcal{E}_x^Q)$. By [18, Proposition 4.1], the following proposition is straightforward:

Proposition 4.1. *If there exists a* $\mathcal{D}_{2p}$*-cover of* $\mathbb{P}^2$ *branched at* $pC + 2Q$*, then* $s_C^+ \in p\,\mathrm{MW}(\mathcal{E}_x^Q)$.

Let s_o be an element in $\mathrm{MW}(\mathcal{E}_x^Q)$ such that $ps_o = s_C^+$. Then we have $\langle s_o, s_o\rangle = 2/p^2$. On the other hand, by the table in §1, the value of $\langle s_o, s_o\rangle \in 1/(2^3 \cdot 3 \cdot 5 \cdot 7)\mathbb{Z}$. Therefore Case 2 does not occur.

Case 3. Our statement may follow from the results in [13]. However, we prove our statement without using the fact that Z_B is a $K3$ surface. Put $B = C + D$. In this case, the canonical resolution of $D(S/\mathbb{P}^2)$ is Z_B. Hence by Theorem 3.2, $\mathrm{NS}(Z_B)/R_B$ has p-torsion. By Corollary 3.1 and Theorem 0.1, it is enough to show that there exists no $\mathcal{D}_{10}$-cover in the case when Q has one A_4 singularity and C is an even bitangential conic to Q. Let D be an element of $\mathrm{NS}(Z_B)$ such that D gives rise to 5-torsion in $\mathrm{NS}(Z_B)/R_B$. By using the intersection pairing, D can be regarded as an element of $R_B^\vee = \oplus_{b\in\mathrm{Sing}(B)} R_b^\vee$. Since $R_b^\vee$ can be embedded into $R_b \otimes \mathbb{Q}$ canonically, D can be expressed as an element in $\oplus_{b\in\mathrm{Sing}(B)} R_b \otimes \mathbb{Q}$. Let b_o be the unique A_4 singularity, and put

$$D \approx_{\mathbb{Q}} \sum_{b\in\mathrm{Sing}(Q)} D_b, \quad D_b \in R_b \otimes \mathbb{Q}$$

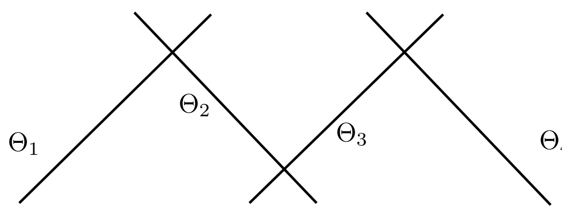
and let $\gamma(D_b)$ be the class of D_b in $R_b^\vee/R_b$. Since the type of singularity of B other than b_o is either A_3, A_7, A_{11} or A_{15}, $\gamma(D_b) = 0$ if $b \neq b_o$. As $R_{b_o}^\vee/R_{b_o}$ is generated by

$$\frac{1}{5}(4\Theta_1 + 3\Theta_2 + 2\Theta_3 + \Theta_1),$$

we have

$$D - \sum_{b \in \mathrm{Sing}(B)\setminus\{b_o\}} D_b \approx_{\mathbb{Q}} \frac{k}{5}(4\Theta_1 + 3\Theta_2 + 2\Theta_3 + \Theta_1) \bmod R_B,$$

for some $k \in \{\pm 1, \pm 2\}$. Here we label the irreducible components as follows:



By modifying D with an element in R_B suitably, we may assume $D \approx_{\mathbb{Q}} k/5(4\Theta_1+3\Theta_2+2\Theta_3+\Theta_1)$. This shows that

$$D^2 = -\frac{4k^2}{5}.$$

This leads us to a contradiction, as $D^2 \in \mathbb{Z}$. Therefore Case 3 does not occur.

The remaining part of Theorem 0.2 is immediate from Corollary 3.2. $\square$

Remark 4.1.

(1) $(C/Q) = 1$ is not enough for the existence of D_{2n}-covers. In fact, for Q with $3A_1$ singularities, there exists an even tangential conic C such that $(C/Q) = 1$ but $Q^+ \not\sim Q^-$ (see [2]).

(2) By [13], there exists an irreducible quartic Q with one A_5 singularity and an even tangential conic C to Q such that
- $C \cap Q = \{x_1, x_2\}, I_{x_1}(C, Q) = 2, I_{x_2}(C, Q) = 6$, and
- $\mathrm{NS}(Z_B)/R_B$ has 3-torsion.

By Theorem 3.2, there exists a $\mathcal{D}_6$-cover branched at $2(C + Q)$. In this case, $(C/Q) = 1$, but $Q^+ \not\sim Q^-$. In fact, if $Q^+ \sim Q^-$, then Q^+ is a rational curve with one singularity whose type is either A_1 or A_2. This singularity must give rise to another singularity of Q, which is impossible.

5. Application to the study of Zariski pairs

Let (B_1, B_2) be a pair of reduced plane curves. We call (B_1, B_2) a Zariski pair if

(1) both of B_1 and B_2 have the same combinatorial type (see [1] for the precise definition of combinatorial type), and
(2) there exists no homeomorphism $h : \mathbb{P}^2 \to \mathbb{P}^2$ such that $h(B_1) = B_2$.

In the case of an irreducible quartic Q and its even tangential conic, the combinatorial type of $C + Q$ is determined by Ξ_Q, $\sharp C \cap Q$ and $I_P(C, Q)$ for each $P \in C \cap Q$.

As an application of the previous sections, we have

Proposition 5.1. *Let Q_1 and Q_2 be irreducible quartics and let C_1 and C_2 be their even tangential conics, respectively. Suppose that $C_i + Q_i$ $(i = 1, 2)$ have the same combinatorial type.*

(i) If $(C_1/Q_1) = 1$ and $(C_2/Q_2) = -1$, then $(C_1 + Q_1, C_2 + Q_2)$ is a Zariski pair.

(ii) If $(C_i/Q_i) = 1$ $(i = 1, 2)$, $Q_1^+ \sim Q_1^-$ and $Q_2^+ \not\sim Q_2^-$, then $(C_1 + Q_1, C_2 + Q_2)$ is a Zariski pair.

Proof. (i) As $C_1 + Q_1$ and $C_2 + Q_2$ have the same combinatorial type, $\Xi_{Q_1} = \Xi_{Q_2}$. Since $(C_1/Q_1) = 1$ and $(C_2/Q_2) = -1$, by Theorem 0.1, we see that $\Xi_{Q_1} = \Xi_{Q_2} = 2A_1$ or A_3. Therefore $Q_1^+ \sim Q_1^- \sim (2, 2)$. Hence by Corollary 0.2, we infer that $\pi_1(\mathbb{P}^2 \setminus (C_1 + Q_1), *) \not\cong \pi_1(\mathbb{P}^2 \setminus (C_2 + Q_2), *)$, i.e., $(C_1 + Q_1, C_2 + Q_2)$ is a Zariski pair.

(ii) Our statement is immediate from [2, Proposition 2].

□

An example for Proposition 5.1 (ii) can be found in [2]. We end this section by giving examples for Proposition 5.1 (i). Let $\mathcal{E}_x^Q$ be the rational elliptic surface corresponding to either No. 40 or No.50 in Theorem 0.1. Choose sections s_1 and s_2 in $\mathrm{MW}(\mathcal{E}_x^Q)$ in such a way that

- $\langle s_i, s_i \rangle = 2, s_i O = 0$ $(i = 1, 2)$ and
- $s_1 \in 2\,\mathrm{MW}(\mathcal{E}_x^Q)$, while $s_2 \notin 2\,\mathrm{MW}(\mathcal{E}_x^Q)$.

By Lemma 2.1, there exist even tangential conics C_{s_1} and C_{s_2} arising from s_1 and s_2, respectively. By Theorem 2.1, we have $(C_{s_1}/Q) = 1$ and $(C_{s_2}/Q) = -1$. Hence if C_{s_1} and C_{s_2} intersects Q in the same manner, we have an example for Proposition 5.1 (i). Now we go on to give explicit examples.

Example 5.1. (*cf.* [16, Example, p.198]) Let Q be an irreducible quartic given by the affine equation

$$f(t, u) = u^3 + (271350 - 98t)u^2 + t(t - 5825)(t - 2025)u + 36t^2(t - 2025)^2 = 0.$$

By taking homogeneous coordinates, $[U, T, V]$, of $\mathbb{P}^2$ in such a way that $u = U/V, t = T/V$, we easily see that $[1, 0, 0]$ is a smooth point of Q. Choose $[1, 0, 0]$ as the distinguished point x. We easily see that the tangent line l_x is given by $V = 0$, and $I_x(l_x, Q) = 3$. The elliptic surface $\varphi_x^Q : \mathcal{E}_x^Q \to \mathbb{P}^1$ corresponding to Q and x is given by a Weierstrass equation

$$y^2 = f(t, u).$$

By [16, Example, p.198], $\mathcal{E}_x^Q$ satisfies the following properties:

(i) φ_x^Q has 3 reducible singular fibers over $t = 0, 2025, \infty$, whose types are: I_2 over $t = 0, 2025$ and III over $t = \infty$. This implies Q has $2A_1$ as its singularities.
(ii) $\mathrm{MW}(\mathcal{E}_x^Q) \cong D_4^* \oplus A_1^*$.

Choose three sections of $\mathcal{E}_x^Q$ given by [16] as follows:

$$s_o : (0, 6t^2 - 12150t),\ \tilde{s}_1 : (-32t, 2t^2 - 6930t),\ \tilde{s}_2 : (-20t, 4t^2 - 4500t).$$

For these sections, $s_o \in A_1^*$ and $\tilde{s}_i \in D_4^*$ $(i = 1, 2)$ and we have

$$\langle s_o, s_o \rangle = \frac{1}{2},\ \langle \tilde{s}_i, \tilde{s}_i \rangle = 1\,(i = 1, 2),\ \langle \tilde{s}_1, \tilde{s}_2 \rangle = 0,$$

and there is no other section s with $\langle s, s \rangle = 1/2$ other than $\pm s_o$.

The sections given by $s_1 := 2s_o$ and $s_2 := \tilde{s}_1 + \tilde{s}_2$ are

$$s_1 = \left(\frac{1}{144}t^2 + \frac{1231}{72}t - \frac{5143775}{144}, -\frac{1}{1728}t^3 - \frac{2335}{576}t^2 + \frac{13493375}{576}t - \frac{29962489375}{1728}\right)$$

$$s_2 = \left(\frac{1}{36}t^2 + \frac{435}{2}t - \frac{921375}{4}, -\frac{1}{216}t^3 - \frac{1181}{24}t^2 - \frac{41625}{8}t + \frac{373156875}{8}\right).$$

Since $s_2 \in D_4^*$, we infer that s_1 is 2-divisible, while s_2 is not 2-divisible. Also, both s_1 and s_2 do not meet the zero section O and $\langle s_1, s_1\rangle = \langle s_2, s_2\rangle = 2$. Let C_1 and C_2 be conics given by

$$\begin{aligned} C_1 : u &= \frac{1}{144}t^2 + \frac{1231}{72}t - \frac{5143775}{144} \\ C_2 : u &= \frac{1}{36}t^2 + \frac{435}{2}t - \frac{921375}{4}. \end{aligned}$$

We infer that C_1 and C_2 are the even tangent conics corresponding to s_1 and s_2, respectively. It is a straightforward computation that, for each i, C_i is tangent to Q at four distinct points. Hence $(C_1 + Q, C_2 + Q)$ is an example for Proposition 5.1 (i).

Example 5.2. (*cf.* [16, Example, p. 210]) Let Q be an irreducible quartic given by the affine equation

$$f(t, u) = u^3 + (25t + 9)u^2 + (144t^2 + t^3)u + 16t^4 = 0.$$

We take a homogeneous coordinate $[U, T, V]$ as in the previous example. With this coordinate $[1, 0, 0]$ is a smooth point and choose $[1, 0, 0]$ as the distinguished point x. The tangent line l_x is again given by $V = 0$ and $I_x(l_x, Q) = 3$. The elliptic surface $\varphi_x^Q : \mathcal{E}_x^Q \to \mathbb{P}^1$ corresponding to Q and x is given by a Weierstrass equation

$$y^2 = f(t, u).$$

Note that we change the equation slightly. The original Weierstrass equation in [16] is $y^2 - 6uy = u^3 + 25tu^2 + (144t^2 + t^3)u + 16t^4$. By [16, Example, p. 210], $\mathcal{E}_x^Q$ satisfies the following properties:

(i) φ_x^Q has 2 reducible singular fibers over $t = 0, \infty$, whose types are: I_4 over $t = 0$ and III over $t = \infty$. This implies Q has A_3 as its singularity.

(ii) $\mathrm{MW}(\mathcal{E}_x^Q) \cong A_3^* \oplus A_1*$.

By modifying the sections given [16] slightly, take three sections of $\mathcal{E}_x^Q$ as follows:

$$s_o : (0, 4t^2),\ \tilde{s}_1 : (-16t, -48t),\ \tilde{s}_2 : (-15t, t^2 + 45t).$$

For these sections, $s_o \in A_1^*$ and $\tilde{s}_i \in A_3^*$ $(i = 1, 2)$ and we have

$$\langle s_o, s_o\rangle = \frac{1}{2},\ \langle \tilde{s}_i, \tilde{s}_i\rangle = \frac{3}{4}\,(i = 1, 2),\ \langle \tilde{s}_1, \tilde{s}_2\rangle = \frac{1}{4},$$

and there is no other section s with $\langle s, s\rangle = 1/2$ other than $\pm s_o$. The sections given by $s_1 := 2s_0$ and $s_2 := \tilde{s}_1 + \tilde{s}_2$ are

$$s_1 = \left(\frac{1}{64}t^2 - \frac{41}{2}t + 315, -\frac{1}{512}t^3 - \frac{55}{32}t^2 + \frac{2637}{8}t - 5670\right)$$
$$s_2 = \left(t^2 + 192t + 8640, -t^3 - 301t^2 - 27936t - 803520\right).$$

Since $s_2 \in A_3^*$, we infer that s_1 is 2-divisible, while s_2 is not 2-divisible. Also, both $2s_o$ and $s_1 + s_2$ do not meet the zero section O and $\langle s_1, s_1\rangle = \langle s_2, s_2\rangle = 2$. Let C_1 and C_2 be conics given by

$$\begin{aligned} C_1 : u &= \frac{1}{64}t^2 - \frac{41}{2}t + 315 \\ C_2 : u &= t^2 + 192t + 8640. \end{aligned}$$

We infer that C_1 and C_2 are even tangential conics to Q corresponding to s_1 and s_2, respectivly. A straightforward computation shows that, for each i, C_i is tangent to Q at four distinct points. Hence $(C_1 + Q, C_2 + Q)$ is an example for Proposition 5.1 (i).

Remark 5.1.

(1) Zariski pairs in Examples 5.1 and 5.2 can be found in [13]. Hence our examples are not new. Our justification lies in a new point of view: quadratic residue curves.

(2) For Zariski pairs in Examples 5.1 and 5.2, there exists a Z-spitting conic for $C_1 + Q_1$, while there exists no such conic for $C_2 + Q_2$ (see [13] for the definition of Z-splitting conics). Moreover precisely, for an irreducible quartic Q with $\Xi_Q = 2A_1$ or A_3 and its even tangential conic C, one can show $(C/Q) = 1$ if and only if there exists a Z-splitting conic for $C + Q$ whose class order is 4 ([20]).

References

[1] E. Artal Bartolo, J.-I. Cogolludo and H. Tokunaga: *A survey on Zariski pairs*, Adv. Stud. Pure Math., **50**(2008), 1-100.

[2] E. Artal Bartolo and H. Tokunaga: *Zariski k-plets of rational curve arrangements and dihedral covers*, Topology Appl. **142** (2004), 227-233. DOI: 10.1016/j.topol.2004.02.003

[3] W. Barth, K. Hulek, C.A.M. Peters and A. Van de Ven: Compact complex surfaces, Ergebnisse der Mathematik und ihrer Grenzgebiete **4** 2nd Enlarged Edition, Springer-Verlag (2004).

[4] E. Brieskorn: *Über die Auflösung gewisser Singlaritäten von holomorpher Abbildungen*, Math. Ann. **166**(1966), 76-102. DOI: 10.1007/BF01361440

[5] E. Brieskorn: *Die Auflösung der rationalen Singularitäten holomorpher Abbildungne*, Math. Ann. **178**(1968), 255-270. DOI: 10.1007/BF01352140

[6] J.H. Conway and N.J.A. Sloane: *Sphere packings, lattices and groups*, Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, New York, third edition, 1999.

[7] E. Horikawa: *On deformation of quintic surfaces*, Invent. Math. **31** (1975), 43−85. DOI: 10.1007/BF01389865

[8] K. Ireland and M. Rosen: *A Classical Introduction to Modern Number Theory, Second Edition*, Graduate Text in Mathematics **84** Springer-Verlag (1990).

[9] K. Kodaira: *On compact analytic surfaces II*, Ann. of Math. **77** (1963), 563-626. DOI: 10.2307/1970131

[10] R. Miranda: *The moduli of Weierstrass fibrations over* $\mathbb{P}^1$, Math. Ann. **255**(1981), 379-394. DOI: 10.1007/BF01450711

[11] R. Miranda and U. Persson: *On extremal rational elliptic surfaces*, Math. Z. **193**(1986), 537-558. DOI: 10.1007/BF01160474

[12] K. Oguiso and T. Shioda: *The Mordell-Weil lattice of Rational Elliptic surface*, Comment. Math. Univ. St. Pauli **40**(1991), 83-99.

[13] I. Shimada: *Lattice Zariski k-ples of plane sextic curves and Z-splitting curves for double plane sextics*, arXive:0903.3308

[14] T. Shioda: *On the Mordell-Weil lattices*, Comment. Math. Univ. St. Pauli **39** (1990), 211-240.

[15] T. Shioda: *Existence of a Rational Elliptic Surface with a Given Mordell-Weil Lattice*, Proc. Japan Acad. **68**(1992), 251-255. DOI: 10.3792/pjaa.68.251

[16] T. Shioda and H. Usui: *Fundamental invariants of Weyl groups and excellent families of elliptic curves*, Comment. Math. Univ. St. Pauli **41**(1992), 169-217.

[17] H. Tokunaga: *On dihedral Galois coverings*, Canadian J. of Math. **46** (1994),1299 - 1317. DOI: 10.4153/CJM-1994-074-4

[18] H. Tokunaga: *Dihedral covers and an elemetary arithmetic on elliptic surfaces*, J. Math. Kyoto Univ. **44**(2004), 55-270.

[19] H. Tokunaga: *Splitting curves on a rational ruled surface, the Mordell-Weil groups of hyperelliptic fibrations and Zariski pairs*, arXive:0905.0047

[20] H. Tokunaga: *Quadratic residue conics for an irreducible quartic and Z-splitting conics*, in preparation.

[21] O. Zariski: *On the purity of the branch locus of algebraic functions*, Proc. Nat. Acad. USA **44** (1958), 791-796. DOI: 10.1073/pnas.44.8.791

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