Journal *of* Singularities
Volume 2 (2010), 128-142

Proc. of Singularities in Aarhus
received 1 March 2010
DOI: 10.5427/jsing.2010.2h

A Short Note on Hauser's Kangaroo Phenomena and Weak Maximal Contact in Higher Dimensions

Anne Frühbis–Krüger

Abstract

Currently there are several approaches to resolution of singularities in positive characteristic all of which have hit some obstruction. One natural idea is to try to construct new meaningful examples at this point to gain a wider range of experience. To produce such examples we mimic the characteristic zero approach and focus on cases where it fails. In particular, this short note deals with an example-driven study of failure of maximal contact and the search for an appropriate replacement.

1 Introduction

Hypersurfaces of maximal contact are one of the key concepts in Hironaka's inductive proof of desingularization in characteristic zero, but unfortunately they need not even exist locally in positive characteristic as e.g. Narasimhan's example [13] shows. In [5] and [9] Hauser replaces hypersurfaces of maximal contact by the characteristic-free notion of hypersurfaces of weak maximal contact, i.e. hypersurfaces which maximize the order of the subsequent coefficient ideal, but which do not necessarily contain the equiconstant points after all sequences of blowing ups in permissible centers. In the corresponding approach ([8], [10]) to resolution of surface singularities in positive characteristic, this modification of the concept of maximal contact turns out to be sufficient to enter into an approach in the flavour of Hironaka's original induction on the dimension of the ambient space. To obtain desingularisation of surfaces along those lines, this is, of course, not the only change to the characteristic zero arguments; important further modifications to certain components of the desingularisation invariant are required. Considering higher dimensions, however, the first step toward a construction of a desingularization similar to the characteristic zero approach or even toward new meaningful examples illustrating the obstructions against it again needs to be a reconsideration of the right generalization of maximal contact.

For readers convenience, we briefly recall some key concepts in section 2. Here one focus will be on the question of recognition of a potential kangaroo. In section 3, we start by considering an example where the original definition of weak maximal contact does not suffice for the description of a kangaroo phenomenon and then suggest a slightly modified version which is suitable for any dimension and not just surfaces. Using this new notion of a flag of weak maximal contact, section 4 is then devoted to examples of the different roles which the hypersurfaces originating from the flag can play in the course of a sequence of permissible blowing ups.

The author would like to thank Herwig Hauser, Vincent Cossart, Dominique Wagner and Santiago Encinas for fruitful discussions which originally started her interest in desingularisation in positive characteristic. The author is also indebted to the referee of the article whose comments were a great help in revising a previous version of this article. All examples appearing in this article originated from structured experiments using the computer algebra system Singular.

2 Basic facts and definitions

A section of just a few pages is obviously not sufficient to even give a brief overview of the tools and general philosophy of algorithmic desingularization, let alone all the delicacies of the case of positive characteristic. On the other hand, more than just 5 pages would be by far too long compared to the following two sections. Hence we do not attempt this here, but only very briefly sketch the idea of the characteristic zero resolution process to give a context, subsequently recalling the notions of hypersurfaces of weak maximal contact and of kangaroo points in positive characteristic. For additional background information on the characteristic zero case, we would like to point to more thorough discussions in section 4.2 of [6] from the practical point of view and in [5] embedded in a detailed treatement of the resolution process. For a detailed introduction to characteristic p phenomena and kangaroo points see [8].

2.1 The philosophy of the characteristic zero approach

In Hironaka's original work [11] and in all algorithmic approaches based on it, e.g. [3],[1],[5], the general approach is that of a finite sequence of blow-ups at appropriate non-singular centers. The very heart of these proofs is the choice of center which is controlled by a tuple of invariants assigned to each point; it is of a structure similar[1] to the following one

$$(ord, n; ord, n; \dots)$$

with lexicographic comparison, the upcoming center being the set of maximal value of the invariant. Here ord stands for an order of an appropriate (auxiliary) ideal (see below), n for a counting of certain exceptional divisors. At each ';' a new auxiliary ideal of smaller ambient dimension, a coefficient ideal, is created by means of a hypersurface of maximal contact.

To fix notation, let W be a smooth equidimensional scheme over an algebraically closed field K of characteristic zero and $X \subset W$ a subscheme thereof. We now immediately focus on one affine chart U with coordinate ring R and denote the maximal ideal at $x \in U$ by $\mathfrak{m}_x$. The order of the ideal $I_X = \langle g_1, \dots, g_r \rangle \subset R$ at a point $x \in U$ is defined as

$$ord_x(I) := \max\{m \in \mathbb{N} | \; I \subset \mathfrak{m}_x^m\}.$$

In characteristic zero, the order of the non-monomial part of an ideal can never increase under blow-ups which makes it a good ingredient for the controlling invariant of the resolution

[1] In the case of Bierstone and Milman, the very first entry is a finer invariant, the Hilbert-Samuel function.

process whose decrease marks the improvement of the singularities.

For the descent in ambient dimension, hypersurfaces of maximal contact are required; these locally contain all points of maximal order, satisfy certain normal crossing conditions and continue to contain all points at which the maximal order did not yet drop after any permissible sequence of blow-ups. In characteristic zero, they always exist locally and can be computed in a rather straight-forward way. The construction of the coefficient ideal for I at X w.r.t. a hypersurface of maximal contact $Z = V(z)$ is then performed in the following way:

$$Coeff_Z(I) = \sum_{k=0}^{ord_x(I)-1} I_k^{\frac{k!}{k-i}}$$

where I_k is the ideal generated by all polynomials which appear as coefficients of z^k in some element of I. Given this notion of coefficient ideal, it is possible to rephrase the condition on a hypersurface of maximal contact from 'containing all points of maximal order' to 'maximizing the order of the non-monomial part of the arising coefficient ideal under all choices of hypersurfaces'.

2.2 Weak maximal contact and kangaroos

In positive characteristic, there are well known examples of failure of maximal contact in the sense that eventually the equiconstant points will leave the strict transform of any chosen smooth hypersurface (see [13]). Using the characteristic free formulation of the first condition for maximal contact, i.e. that it should maximize the order of the non-monomial part of the subsequent coefficient ideal, and dropping the condition that this should hold after any permissible sequence of blow-ups, we obtain Hauser's definition of weak maximal contact. In this way, Hauser and Wagner [10] then allow passage to a new hypersurface of weak maximal contact, if the previously chosen one happens to fail to have the maximizing property at some moment in the resolution process.

Additionally there are examples (see [12]) in which the order of the non-monomial part of the first coefficient ideal can increase under a sequence of blow-ups in positive characteristic. In [8] Hauser shows that these two phenomena are closely related in the sense that both arise in the same rather rare settings and gives an explicit criterion for the possibility of such a phenomenon, which he calls a kangaroo point focusing on the point where this occurs. In this article, we often choose to refer to this as a kangaroo phenomenon, emphasizing the fact that not the point itself is in the center of interest, but the deviation from the characteristic zero case. Using the same notation for W, X etc. as in the previous section, we now recall Hauser's definition:

Definition 1 ([8]) *Let $\pi : W' \longrightarrow W$ be a blow-up at a permissible center Z, and $x \in Z$ a point of maximal order c for I_X. Denoting the weak transform of X under π by X', let $x' \in X' \cap \pi^{-1}(x)$ be a point at which $ord_{x'}(I_{X'}) = c$. Then x' is called a kangaroo point, if the order of the non-monomial part of the coefficient ideal of I_X at x w.r.t. a hypersurface of weak maximal contact is less than the order of the non-monomial part of the coefficient ideal of $I_{X'}$ w.r.t. a (possibly newly chosen) hypersurface of weak maximal contact.*

Definition 2 *Generalizing Hauser's notion of a kangaroo point, we shall call a blowing up, at which such an increase in order occurs for one of the coefficient ideals at some level in the descent of ambient dimension, a kangaroo phenomenon.*

Remark 3 ([8]) *A kangaroo point can only occur, if the following conditions are satisfied:*

(a) the order c of the ideal I_X at x does not exceed the order of $I_{X'}$ at x' and is divisible by the charateristic of the ground field.

(b) The order of the non-monomial part of the coefficient ideal is a multiple of c. [2]

(c) The exceptional multiplicities of the coefficient ideal need to satisfy a certain numerical inequality (whose specification would need to much room here).

This remark does not yield a sufficient criterion of detection of kangaroos. However, if a kangaroo phenomenon occurs, then its effect is an increase of order of the non-monomial part of the coefficient ideal by means of leaving at least two exceptional divisors at the same time and a suitable change of hypersurface of weak maximal contact (see examples in sections 3 and 4 for details).

Combining the above observations of Hauser with well-known observations by Hironaka and Giraud, condition (a) can be made a bit more precise. To this end, we need to recall another singularity invariant, the ridge (french: la faîte). Following the exposition of [14], let us consider the tangent cone $C_{X,x}$ of I_X at x and the largest subgroup scheme $A_{X,x}$ of the tangent space $T_{W,x}$ satisfying the conditions that it is homogeneous and leaves the tangent cone stable w.r.t. the translation action. $A_{X,x}$ is called the ridge of the tangent cone of I_X at x.

It is a well-known, important fact that the ridge can be generated by additive polynomials, i.e. by polynomials of the form

$$\sum_{i=1}^{n} a_i x_i^{p^e}$$

where p is the characteristic of the underlying field. In characteristic zero the ridge is always generated by polynomials of degree one; in positive characteristic the occurrence of a ridge not generated by polynomials of degree one marks a point for which the reasoning of characteristic zero might break down. Following the exposition of [2] the ridge can also be phrased as the smallest set of additive polynomials $\{p_1, \dots, p_r\}$ generating the smallest algebra $k[p_1, \dots, p_r]$ such that

$$I_X = (I_X \cap k[p_1, \dots, p_r])k[\underline{x}].$$

Combining this with Hauser's condition (a), we obtain a refined version for hypersurfaces, which, of course, still requires $ord_{x'}(I_{X'}) = c = ord_x(I_X)$ and, additionally, that the ridge must at least have one generator in higher degree, i.e. in some degree p^e. This sharpens the

[2]For kangaroo phenomena, this condition should analogously read 'one of the coefficient ideals occurring in the descent of ambient dimension'.

condition of divisibility of the order by a p-th power to the fact that some variable actually only occurs as p-th powers in the tangent cone and is implicitly already present in [8]. According to Hauser's condition (b), the degree of the non-monomial part of the first coefficient ideal is required to be a multiple of the degree c. In contrast to condition (a), this can not be made more precise by simply adding the condition that the ridge of the non-monomial part of this coefficient ideal is not generated in degree 1, because higher order generators of the coefficient ideal might introduce lower degree polynomials into the ridge which allow dropping of certain contributions arising from the lowest order generators of the ideal. To illustrate the role of the ridge, we give 3 examples:

Example 1 Over a field K of characteristic 3, consider an affine chart $U = \mathbb{A}^4_K$ (with variables named x, y, z, w) which already results from a sequence of 2 blow-ups and contains exceptional divisors $E_1 = V(w)$ and $E_2 = V(z)$, born from the first and second blow-up respectively. (These two blow-ups are indeed necessary for the possibility of an occurrence of a kangaroo point after the subsequent blowing up, according to Hauser's technical condition (c) which was not formulated explicitly in the previously stated remark.)
Locally at the coordinate origin of this chart, consider the three subvarieties of $\mathbb{A}^4_K$ defined by the following ideals:

- $I_{X_1} = \langle x^3 + z^{14}w^{10}(z^6 - w^6)\rangle$
  This is the strict transform[3] of $\langle x^3 + z^{13} - zw^{18}\rangle$ under the two blow-ups. The ridge of I_{X_1} can obviously be described by $\{x^3\}$, the non-monomial part of its first coefficient ideal is
  $$\langle z^{12} + z^6w^6 + w^{12}\rangle,$$
  with ridge $\{z^3, w^3\}$.
  After blowing up again at the origin, we obtain (in the $E_3 = V(w)$-chart) the strict transform
  $$I_{X_1'} = \langle x^3 + z^{14}w^{27}(z^6 - 1)\rangle$$
  which after a coordinate change $z_{new} = z - 1$ and a passage to a new hypersurface of weak maximal contact $V(x + z_{new}^2 w^9) = V(x_{new})$ reads as $I_{transf.} = \langle x_{new}^3 + z_{new}^6 w^{27}(-z_{new} + h.o.t.)\rangle$. Since z_{new} does not correspond to an exceptional divisor, this has a non-monomial part of the first coefficient ideal of the form
  $$\langle z_{new}^{14} + h.o.t.\rangle.$$
  This ideal is of order 14 as compared to the corresponding order 12 before the last blowing up which clearly indicates the occurrence of a kangaroo point.

- $I_{X_2} = \langle x^2y + z^{14}w^{10}(z^6 - w^6)\rangle$
  This is the strict transform of $\langle x^2y + z^{13} - zw^{18}\rangle$ under the two blow-ups.[4] The ridge of

[3]Actually this is the weak transform of I_{X_1} which in the principal ideal case happens to coincide with the strict transform.

[4]Here we are actually already deviating a bit from Hauser's original definition, because we consider an initial part involving 2 variables and then descend in ambient dimension in one step of 2 to $V(x, y)$ seen as a hypersurface in $V(x)$ which is in turn a hypersurface in $\mathbb{A}^4$. This is possible by collecting all coefficients of monomials of the form $x^a y^b$ with $a + b = k$ into the ideal I_k; for more details on this see e.g. [7], where this has been used in a very explicit way.

I_{X_2} is obviously $\{x, y\}$, the non-monomial part of its coefficient ideal w.r.t. the descent in ambient dimension to $V(x, y)$ is

$$\langle z^{12} + z^6w^6 + w^{12}\rangle$$

as before with ridge $\{z^3, w^3\}$.
After blowing up again at the origin, we obtain (in the $E_3 = V(w)$-chart) the strict transform

$$I_{X_2'} = \langle x^2y + z^{14}w^{27}(z^6 - 1)\rangle$$

for which even a coordinate change $z_{new} = z - 1$ cannot lead to a kangaroo point, because no suitable passage to new hypersurfaces of weak maximal contact killing the term $z_{new}^6 w^{27}$ is available. This could already be expected at the beginning due to the fact that the ridge of I_{X_2} is generated in degree 1.

The third example is of a different flavor and only serves to illustrate, how higher order generators of the ideal might influence the ridge in a way which is not desirable for the consideration of coefficient ideals:

- $I_{X_3} = \langle x^3 + z^{14}w^{10}(z^6 - w^6), z^{30}w^{17}(y^{19} + y^5z^7w^3)\rangle$
This is the weak transform of $\langle x^3 + z^{13} - zw^{18}, y^5z^{18} + y^{19}w\rangle$ under the two blow-ups. The ridge of I_{X_3} is obviously $\{x^3, y, z, w\}$, whereas only the hypersurface $V(x)$ can be chosen as hypersurface of weak maximal contact. The non-monomial part of the first coefficient ideal is

$$\langle z^{12} + z^6w^6 + w^{12}, (z^6 - w^6)(z^{16}w^7(y^{19} + y^5z^7w^3)),\\ z^{32}w^{14}(y^{38} - y^{24}z^7w^3 + y^{10}z^{14}w^6)\rangle.$$

The ridge can be computed to be $\{y, z, w\}$, e.g. by the algorithm of [2].
After blowing up again at the origin, we obtain (in the $E_3 = V(w)$-chart) the weak transform

$$I_{X_3'} = \langle x^3 + z^{14}w^{27}(z^6 - 1), \ldots\rangle$$

which after a coordinate change $z_{new} = z - 1$ and a passage to a new hypersurface of weak maximal contact $V(x + z_{new}^2 w^9) = V(x_{new})$ has the same first generator of order 14 as in example 1, the second generator does not have effect on the order of the non-monomial part of the first coefficient ideal as can be checked by explicit computation. Comparing this to the first example, we see that the higher order generator, which does not actually influence the order of the non-monomial part of the coefficient ideal, masked the situation in the computation of the ridge.

From these three examples, we see the usefulness of the ridge for anticipating kangaroo points in the case of hypersurfaces, whereas in the case of ideals this may be hidden by contributions of higher order generators. However, if we only consider the ridge of the ideal which is generated precisely by the lowest-order generators of the original ideal (instead of the ridge of the whole ideal), then there is hope to use this new ridge for ideals and maybe even to slightly sharpen item (b) in Hauser's condition for kangaroo points.

Remark 4 *These considerations already suggest a strategy for finding interesting examples by constructing hypersurfaces for which the ridge is not generated in degree 1 and, additionally, at least once during the iterated descents in ambient dimension the ridge of the ideal generated by the lowest order generators (denoted from now on as n-ridge for short) of the non-monomial part of the respective coefficient ideal is also not generated in degree one. In the experiments, which lead to the examples of the subsequent sections, an additional heuristic in the choice of hypersurfaces of weak maximal contact was used: When given the choice between different hypersurfaces, more precisely between linearly independent initial parts of possible hypersurfaces, we try to minimize the degree of the generator of the ridge/n-ridge corresponding to the chosen hypersurface. The reasoning behind this heuristic is to force the unpleasant, but interesting behaviour into the lowest possible ambient dimension and hence keep a clearer view of the occurring phenomena.*

Remark 5 *Similar examples to those of the subsequent sections can easily be constructed in any positive characteristic. For section 3 this is straight forward, for section 4 it is best achieved by starting in the middle, i.e. precisely where the first kangaroo has just occurred and construct from there by blowing down and blowing up.*

3 In higher dimension not all hypersurfaces of weak maximal contact are suitable

The following example shows that the property of maximizing the order of the non-monomial part of the upcoming coefficient ideal is not sufficient to properly cover all kangaroo phenomena in higher dimensions. It is stated in characteristic 2 to allow considerations in rather low degrees, but similar examples can be constructed for any positive characteristic.

Example 2 We consider a sequence of three blow ups of the hypersurface $V(x^2 + w^3 + y^{25} + yz^{16}) \subset \mathbb{A}^4_K$, $char(K) = 2$, $K = \overline{K}$. At each step the respective maximal orders, chosen hypersurfaces of weak maximal contact and coefficient ideals are specified. In the presence of exceptional divisors, we make use of Bodnar's trick [4], which allows skipping the intersection with exceptional divisors in intermediate levels of the descent in ambient dimension, if we have normal crossing between the upcoming hypersurface of weak maximal contact and the exceptional divisors.

To keep the whole rather lengthy sequence of blowing ups more readable, we only give rather scarce comments. A more commented version of a single blowing up step was already stated at the end of the previous section.

original hypersurface:
$I = \langle f \rangle = \langle x^2 + w^3 + y^{25} + yz^{16} \rangle$

- in ambient space $\mathbb{A}^4_K$
  $I = \langle x^2 + w^3 + y^{25} + yz^{16} \rangle$
  The maximal order 2 is attained at $V(x, y, z, w)$.
  The ridge of this ideal corresponds to $\{x^2\}$.
  As hypersurface of weak maximal contact we may use $H_1 = V(x) \subset \mathbb{A}^4_K$.

- in ambient space H_1
  $I_{H_1} = \langle w^3 + y^{25} + yz^{16} \rangle$.
  The maximal order of 3 is then again attained at the origin of H_1.
  The n-ridge (in the short-hand notation introduced in section 2) is $\{w\}$
  As hypersurface of weak maximal contact we now use
  $H_2 = V(x, w) \subset H_1 \subset \mathbb{A}^4_K$.

- in ambient space H_2
  $I_{H_2} = \langle y^{50} + y^2 z^{32} \rangle = \langle (y^{25} + yz^{16})^2 \rangle$.
  The maximal order of 34 is again attained at the origin of H_2 and the n-ridge is $\{y^2, z^{32}\}$.

- The only possible choice of center is $V(x, y, z, w)$.

As a sideremark to the coefficient ideal in ambient space H_2: Here it becomes evident that there are 2 mechanisms which can cause the n-ridge to have generators in higher degree: on one hand, it may be an honest generator in higher degree, on the other hand, it might have arisen from taking powers of contributing ideals I_k when forming the coefficient ideal (see section 2). However, taking powers can not accidentally cause the degree of a generator of the ridge to drop. Hence the degree of the generators of the ridge can still be used as a rather weak indicator for the possibility of new phenomena in characteristic p. Moreover, a higher degree generator of the n-ridge arising from mechanism 2 is only likely to occur, if the contributing ideals I_k are principal, because otherwise mixed products of generators would exist in the set of generators of the power of I_k.

after first blowing up, chart $E_1 = V(y)$**:**
$I_{strict} = \langle x^2 + yw^3 + y^{23} + y^{15} z^{16} \rangle$

- in ambient space $\mathbb{A}^4_K$
  $I_{strict} = \langle x^2 + y(w^3 + y^{22} + y^{14} z^{16}) \rangle$
  The maximal order is again 2, attained at the origin and the ridge is again $\{x^2\}$. We can keep the strict transform of H_1 as our hypersurface of weak maximal contact. (As $\{E_1, H_{1strict}\}$ has normal crossings, we may use Bodnàr's trick [4] and hand the exceptional divisor down to the lower dimension instead of intersecting with it at this point.)

- in ambient space $H_{1strict}$
  The non-monomial part of the coefficient ideal[5] is $\langle w^3 + y^{22} + y^{14} z^{16} \rangle$.
  The maximal order of 3 is again attained at the origin and the n-ridge is $\{w\}$ as before. We may also use the strict transform of H_2 again for the descent in ambient dimension. (Here we have normal crossing of $\{E_1, H_{1strict}, H_{2strict}\}$ and can again use Bodnàr's trick.)

- in ambient space $H_{2strict}$
  non-monomial part of coefficient ideal: $\langle y^{16} + z^{32} \rangle = \langle (y^8 + z^{16})^2 \rangle$
  maximal order 16 attained at the origin
  n-ridge: $\{y^{16}\}$

[5] As taking the coefficient ideal and subsequently calculating the controlled transform under the blowing up on one hand and calculating the weak transform of the ideal followed by computing the new coefficient ideal on the other hand are known (e.g. [5]) to lead to the same ideal, we won't go into details on this point.

- It is easy to check that here again the choice of center has to be the origin.

after second blowing up, chart $E_2 = V(z)$:
$I_{strict} = \langle x^2 + yz^2w^3 + y^{23}z^{21} + y^{15}z^{29} \rangle$

- in ambient space $\mathbb{A}^4_K$
  $I_{strict} = \langle x^2 + yz^2(w^3 + y^{22}z^{19} + y^{14}z^{27}) \rangle$
  maximal order: 2 at $V(x, zw, yz)$
  ridge: $\{x^2\}$
  hypersurface of weak maximal contact: strict transform of H_1
  ($\{E_{1strict}, E_2, H_{1strict}\}$ n.cr.)
- in ambient space $H_{1strict}$
  non-monomial part of coefficient ideal: $\langle w^3 + y^{14}z^{19}(y^8 + z^{16}) \rangle$
  maximal order: 3 at $V(w, yz)$
  n-ridge: $\{w\}$
  hypersurface of weak maximal contact: strict transform of H_2
  ($\{E_{1strict}, E_2, H_{2strict}\}$ n.cr.)
- in ambient space $H_{2strict}$
  non-monomial part of coefficient ideal: $\langle y^{16} + z^{16} \rangle = \langle (y^8 + z^8)^2 \rangle$
  maximal order: 16 at $V(y + z)$
  n-ridge: $\{y^{16} + z^{16}\}$
- center needs to be $V(x, y, z, w)$ as the locus of maximal order after the second descent in ambient dimension is not normal crossing with the exceptional divisors

after third blowing up, chart $E_3 = V(z)$:
$I_{strict} = \langle x^2 + yz^4w^3 + y^{23}z^{42} + y^{15}z^{42} \rangle$

- in ambient space $\mathbb{A}^4_K$
  $I_{strict} = \langle x^2 + yz^4(w^3 + y^{22}z^{38} + y^{14}z^{38}) \rangle$
  maximal order: 2 at $V(x, zw)$
  ridge: $\{x^2\}$
  hypersurface of weak maximal contact: strict transform of H_1
  (E_1 does not meet this chart, $\{E_{2strict}, E_3, H_{1strict}\}$ n.cr.)
- in ambient space $H_{1strict}$
  non-monomial part of coefficient ideal: $\langle w^3 + y^{12}z^{42}(y^8 + 1) \rangle$
  maximal order: 3 at $V(w, yz(y + 1))$ n-ridge: $\{w\}$
  hypersurface of weak maximal contact: strict transform of H_2
  ($\{E_{2strict}, E_3, H_{2strict}\}$ n.cr.)
- in ambient space $H_{2strict}$
  non-monomial part of coefficient ideal: $\langle y^{16} + 1 \rangle = \langle (y^8 + 1)^2 \rangle$
  maximal order: 16 at $V(y + 1)$

Changing the hypersurface for the first descent in ambient dimension from $H_{1strict}$ to $V(x + (y+1)^4z^{21})$, however, we may increase the order of the coefficient ideal in ambient dimension 2. For simplicity of notation, we first make a coordinate change which translates the point of maximal order to the coordinate origin:

- in ambient space $\mathbb{A}^4_K$
  $\langle x^2 + z^4(w^3 + y_{new}w^3 + y_{new}^8 z^{38} + y_{new}^9 z^{38} + h.o.t.)\rangle$
  maximal order 2 at $V(x, zw)$
  ridge: $\{x^2\}$
  new hypersurface of weak maximal contact: $H_1' = V(x + y^4 z^{21})$
  (E_1 does not meet this chart, $\{E_{2strict}, E_3, H_1'\}$ n.cr.)

- in ambient space H_1'
  non-monomial part of coefficient ideal: $\langle w^3 + y_{new}^9 z^{38} + h.o.t.\rangle$
  maximal order 3 at $V(w, y_{new}z)$
  n-ridge: $\{w\}$
  hypersurface of weak maximal contact: $H_2' = V(w)$
  ($\{E_{2strict}, E_3, H_2'\}$ n.cr.)

- in ambient space H_2'
  non-monomial part of the coefficient ideal: $\langle y_{new}^{18} + h.o.t.\rangle$
  maximal order: 18 **exceeds previous order** 16
  kangaroo phenomenon

Here the new phenomenon is that the change of the hypersurface of weak maximal contact was not forced by the first coefficient ideal, but by one of the later ones which would not be covered by the standard definition of weak maximal contact.

In the light of the previous example, we suggest a slightly modified version of weak maximal contact:

Definition 6 *Consider a given point x of a scheme X (possibly in the presence of an exceptional divisor E) and pass to an affine chart U containing this point. We call a flag*

$$\mathcal{H} = H_1 \supset H_2 \supset \cdots \supset H_s$$

admissible at x, if the following properties hold:

(a) H_1 is a smooth hypersurface in the ambient space U. H_{i+1} is a smooth hypersurface in H_i.

(b) H_i is a hypersurface of weak maximal contact for the coefficient ideal obtained by descent of the ambient space through $H_1, \ldots, H_{i-1}$.

(c) $x \in H_s$.

$\mathcal{H}$ is called a flag of weak maximal contact for I_X at x if it maximizes the resolution invariant lexicographically among all choices of admissible flags at x.

This definition obviously behaves well under passage to a coefficient ideal w.r.t. H_1 by omitting the first entry H_1 from $\mathcal{H}$ to obtain the new flag $\mathcal{H}_{H_1}$. This is again a flag of maximal contact, since conditions (a)-(c) and maximality follow trivially from the respective conditions on $\mathcal{H}$. Hence considering a flag of weak maximal contact instead of a hypersurface of weak maximal contact does not change any of the key properties, but allows more flexibility for dealing with lower level kangaroos.

4 Two different kinds of double kangaroos

It is a well known fact that the situation in positive characteristic can only differ from the one in characteristic zero in rather special situations. Hauser studied such phenomena in great detail in [8] by considering precisely the two levels involved in a kangaroo point. For surfaces, he and Wagner extended these considerations to a general treatment of the purely inseparable case in [10]. The situation in higher dimension differs from this easiest case in the sense that there might be more than just two levels at which the ridge is not generated in degree 1 at some time during the process of blowing ups. The following two examples illustrate three different roles of the different levels of the flag of weak maximal contact in such a setting.

Definition 7 *Let $\mathcal{H}$ be a flag of weak maximal contact for an ideal $I_X \subset W$ at the point x which we assume for simplicity to be the origin of our coordinate chart. We denote the i-th coefficient ideal, which arises when descending to H_i, by $J_i \subset \mathcal{O}_{H_i}$. If the ideal generated by the lowest order generators of J_{i-1} is not a principal ideal, H_i is called*

- **neutral**, *if the degree 1 part of the generator of the principal ideal $I_{H_i} \subset \mathcal{O}_{H_{i-1},0}$ is in the $\mathbb{C}$-span of the degree 1 elements of the ridge/n-ridge of J_{i-1}.*
- **active**, *if it is the H_i of lowest index i which is not neutral.*
- **dormant**, *if it is neither active nor neutral.*

If, on the other hand, the ideal generated by the lowest order generators of J_{i-1} is principal, it is of the form $g^{\frac{b!}{b-k}}$ for some $k < b$ and we change the notions of neutral, active and dormant by replacing the ridge/n-ridge of J_{i-1} by the one of $\langle g \rangle$.

Remark 8

1. *According to Hauser's description of the process leading to kangaroo points, at least one active H_i and one dormant H_j are necessary to produce a kangaroo phenomenon.*
2. *If the ideal generated by the lowest order generators of J_{i-1} is not principal, there is at least one ideal among the contributing I_k, of which the ideal generated by its lowest order generators is itself not principal, e.g. generated by f_1 and f_2. Hence taking the $\frac{b!}{b-k}$-th power of of this I_k upon forming the coefficient ideal, we obtain all mixed products of the form $f_1^a f_2^b$, $a + b = \frac{b!}{b-k}$. This implies that higher degree generators of the n-ridge can only occur if they would also occur for $\langle f_1, f_2 \rangle$.*
   If on the other hand, the ideal generated by the lowest order generators of J_{i-1} is principal, the generator is of the form $g^{\frac{b!}{b-k}}$ for some k and hence masks the true situation of the (n-)ridge of g. This is the reason for the special treatment of this case in the above definition.

Both of the following examples were constructed in a straight forward way, combining two occurrences of kangaroos at two different levels. Similar examples can be constructed in any positive characteristic and for any ambient dimension exceeding 4. However, these examples involve several blow-ups between the first and the second occurrence, basically making a fresh start after the first. Here no effort is made to reduce this number of blow-ups, since the context of this article is the study of the roles of the hypersurfaces of weak maximal contact.

To keep these rather lengthy examples more readable, we only state the blow-ups, the weak transform at each step and the flag of weak maximal contact, whenever the latter changes, but omit all data which is related to coefficient ideals, since these can easily be computed for these examples.

Example 3 In this example, a hypersurface in $\mathbb{A}^5_K$, $char K = 3$, we shall see 2 occurrences of kangaroo points on two different levels of coefficient ideals. For both occurrences, the active hypersurface of weak maximal contact is the first one in the flag. Note that the two blowing ups with chart $E = V(y)$ after the first kangaroo are only used for setting up the degrees for the following kangaroo.[6]

- before 1st blowing up
  $I = \langle w^3 + y^6 z^3 v^2 + x^9 y^8 + x^{18} y^2 + x^{18} v^2 \rangle$
  Flag:
  $V(w)$ active, $V(w,v)$ neutral, $V(w,v,z)$ dormant, $V(w,v,z,y)$ neutral
- after blowing up at the origin, chart $E_{new} = V(x)$
  $I = \langle w^3 + x^8(y^6 z^3 v^2 + x^6 y^8 + x^9 y^2 + x^9 v^2) \rangle$
- after blowing up at the origin, chart $E_{new} = V(y)$
  $I = \langle w^3 + x^8 y^{16}(z^3 v^2 + x^6(x^3 + y^3) + x^9 v^2) \rangle$
  Flag: $V(w)$ active, $V(w,v)$ neutral, $V(w,v,z)$ dormant, $V(w,v,z,y)$ dormant
- after blowing up at the origin, chart $E_{new} = V(x)$
  $I = \langle w^3 + x^{26} y^{16}(z^3 v^2 + x^4 + x^4 y^3 + x^6 v^2) \rangle$
  coordinate change: $y_{new} = y_{old} + 1$, $w_{new} = w_{old} + x^{10} y$
  $I = \langle w^3 + x^{26}((y-1)^{16}(z^3 v^2 + x^6 v^2) - x^4 y^4 + h.o.t.) \rangle$
  Flag in new coordinates:
  $V(w)$ active, $V(w,v)$ neutral, $V(w,v,z)$ dormant, $V(w,v,z,y)$ neutral
  Kangaroo at 3rd coefficient ideal
- after blowing up at the origin, chart $E_{new} = V(x)$
  $I = \langle w^3 + x^{28}((xy-1)^{16}(z^3 v^2 + x^3 v^2) - x^3 y^4 + h.o.t.) \rangle$
- after blowing up at the origin, chart $E_{new} = V(v)$
  $I = \langle w^3 + x^{28} v^{30}(z^3 + x^3 - x^4 y v^2 + h.o.t.) \rangle$
- after blowing up at the origin, chart $E_{new} = V(y)$
  $I = \langle w^3 + x^{28} y^{58} v^{30}(x^3 + z^3 - x^4 y^4 v^2 + h.o.t.) \rangle$
- after blowing up at the origin, chart $E_{new} = V(y)$
  $I = \langle w^3 + x^{28} y^{116} v^{30}(x^3 + z^3 - x^4 y^7 v^2 + h.o.t.) \rangle$

[6] Whenever we write 'h.o.t.' we want to indicate that there are further terms of higher degree, which are irrelevant for the further considerations. In this case only the first non-relevant term is stated, even if this does not happen to be the term originating from the previous first non-relevant term

- after blowing up at the origin, chart $E_{new} = V(z)$
  $I = \langle w^3 + x^{28}y^{116}z^{174}v^{30}(1 + x^3 - xy^7z^{10}v^2 + h.o.t.)\rangle$
  coord. change: $x_{new} = x_{old} + 1$, $y_{new} = y_{old} + 1$, $w_{new} = w_{old} + z^{58}v^{10}x^3$
  $I = \langle w^3 + z^{174}v^{30}(x^4 + x^3y + h.o.t)\rangle$
  Flag: $V(w)$, $V(w,x)$, ...
  Kangaroo at 1st coefficient ideal

Example 4 In this example, again in the same affine space as before, we shall see 2 occurrences of kangaroo points on two different levels of coefficient ideals. For the first occurrence, a dormant hypersurface of weak maximal contact acts as the active one, for the second it is the top-level active hypersurface of weak maximal contact. This example again basically consists of two regular kangaroo phenomena in a row, occurring on two different levels, but in a different flavor than example 4.

- before first blowing up
  $I = \langle w^3 + xy^9z^9v + x^7y^{20}v + x^{34}y^2v + x^{46}v\rangle$
  Flag:
  $V(w)$ active, $V(w,v)$ neutral, $V(w,v,z)$ dormant, $V(w,v,y,z)$ neutral

- after blowing up at the origin, chart $E_{new} = V(x)$
  $I = \langle w^3 + x^{17}(y^9z^9v + x^8y^{20}v + x^{17}y^2v + x^{27}v)\rangle$

- after blowing up at the origin, chart $E_{new} = V(y)$
  $I = \langle w^3 + x^{17}y^{33}(z^9v + x^8y^{10}v + x^{17}yv + x^{27}y^9v)\rangle$
  Flag: $V(w)$ active, $V(w,v)$ neutral, $V(w,v,z)$ dormant, $V(w,v,y,z)$ dormant

- after blowing up at the origin, chart $E_{new} = V(x)$
  $I = \langle w^3 + x^{57}y^{33}(z^9v + x^9y^{10}v + x^9yv + x^{27}y^9v)\rangle$
  coord. change: $y_{new} = y_{old} + 1$, $z_{new} = z_{old} + xy$
  $I = \langle w^3 + x^{57}(y-1)^{33}(z^9v + x^9y^{10}v - x^{27}v + x^{27}y^9v)\rangle$
  Flag in new coordinates:
  $V(w)$ active, $V(w,v)$ neutral, $V(w,v,z)$ dormant, $V(w,v,y,z)$ neutral
  Kangaroo at 3rd coefficient ideal

- after blowing up at the origin, chart $E_{new} = V(x)$
  $I = \langle w^3 + x^{64}(xy-1)^{33}(z^9v + x^{10}y^{10}v - x^{18}v + x^{27}y^9v)\rangle$

- after blowing up at the origin, chart $E_{new} = V(x)$
  $I = \langle w^3 + x^{71}(x^2y-1)^{33}(z^9v + x^{11}y^{10}v - x^9v + x^{27}y^9v)\rangle$

- after blowing up at the origin, chart $E_{new} = V(v)$
  $I = \langle w^3 + x^{71}(x^2yv^3-1)^{33}v^{78}(z^9 - x^9 + h.o.t)\rangle$
  Flag: $V(w)$ active, $V(w,z)$ dormant, $V(w,x,z)$ dormant, ...

- after blowing up at the origin, chart $E_{new} = V(y)$
  $I = \langle w^3 + x^{71}y^{155}v^{78}(x^2y^6v^3-1)^{33}(z^9 - x^9 + h.o.t)\rangle$

- after blowing up at the origin, chart $E_{new} = V(y)$
  $I = \langle w^3 + x^{71}y^{310}v^{78}(x^2y^6v^3 - 1)^{33}(z^9 - x^9 + h.o.t)\rangle$

- after blowing up at the origin, chart $E_{new} = V(z)$
  $I = \langle w^3 + x^{71}y^{310}v^{78}z^{465}(x^2y^6v^3z^{16} - 1)^{33}(1 - x^9 + h.o.t.)\rangle$
  coord. change: $x_{new} = x_{old} - 1$, $y_{new} = y_{old} + 1$, $w_{new} = w_{old} - v^{26}z^{155}x^3$
  Kangaroo at 1st coefficient ideal

In both examples the relevant order of the first respectively second coefficient ideal dropped significantly after the first kangaroo phenomenon, but before the occurrence of the kangaroo on this level. The examples have been constructed to illustrate roles of hypersurfaces of maximal contact in multiple kangaroos and not to specifically illustrate the increase in order. Nevertheless the observed behaviour raises several questions, which seem to be natural starting points for further experiments in the search for new meaningful examples:

- Is it possible to find an occurrence of two kangaroo phenomena whose 'distance' is less than 3 blow ups?
- Is it possible to find an occurrence of two kangaroo phenomena for which the drop of order between the first and the second kangaroo does not outweigh the increase of order?
- If one of the previous question has an affirmative answer, what is the smallest dimension in which this occurs?

References

[1] Bravo,A., Encinas,S., Villamayor,O.:*A Simplified Proof of Desingularisation and Applications*, Rev. Math. Iberoamericana 21 (2005), 349–458.

[2] Berthomieu,J., Hivert,P., Mourtada,H.:*Computing Hironaka's invariants: Ridge and Directrix*, arXiv hal-00492824 (2010)

[3] Bierstone,E., Milman,P.:*Canonical Desingularization in Characteristic Zero by Blowing up the Maximum Strata of a Local Invariant*, Invent.Math. **128** (1997), pp. 207–302. DOI: 10.1007/s002220050141

[4] Bodnàr,G.: *Algorithmic Resolution of Singularities*, PhD-Thesis, RISC Linz (2001)

[5] Encinas,S., Hauser,H.: *Strong resolution of singularities in characteristic zero*, Comment. Math. Helv. 77 (2002), 821–845. DOI: 10.1007/PL00012443

[6] Frühbis-Krüger,A.: *Computational Aspects of Singularities*, in J.-P. Brasselet, J.Damon et al.: Singularities in Geometry and Topology, World Scientific Publishing, 253–327 (2007)

[7] Frühbis-Krüger,A.: A modified coefficient ideal for use with the strict transform, to appear in J. Symb. Comp. (2010)

[8] Hauser,H.: *Why Hironaka's proof of resolution of singularities fails in positive characteristic*, preprint, http://homepage.univie.ac.at/herwig.hauser/

[9] Hauser,H.: *Wild Singularities and Kangaroo Points for the Resolution in Positive Characteristic*, preprint, http://homepage.univie.ac.at/herwig.hauser/

[10] Hauser,H., Wagner,D.: *Two new Invariants for the Resolution of Surfaces in Positive Characteristic*, preprint, http://homepage.univie.ac.at/herwig.hauser/

[11] Hironaka,H.:*Resolution of Singularities of an Algebraic Variety over a Field of Characteristic Zero*, Annals of Math. **79** (1964), pp. 109–326. DOI: 10.2307/1970486

[12] Moh,T.-T.: *On a stability theorem for local uniformization in characteristic p*, Publ.Res.Inst.Math.Sci. **23** (1987), pp. 965–973. DOI: 10.2977/prims/1195175867

[13] Narasimhan,R.:*Hyperplanarity of the equimultiple locus*, Proc.Amer.Math.Soc. **87** (1983), pp.403–406

[14] Oda,T.: *Hironaka's additive group scheme*, in Number Theory, Algebraic Geometry and Commutative Algebra, in honor of Y. Akizuki, Kinokuniya, Tokyo (1973), pp.181–219

[15] Greuel,G., Pfister,G., Schönemann,H.: *Singular version 3.1.0 – A Computer Algebra System for Polynomial Computations*, (2009) http://www.singular.uni-kl.de