SINGULARITIES AND STABLE HOMOTOPY GROUPS OF SPHERES. I

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ABSTRACT. We establish an interesting connection between Morin singularities and stable homotopy groups of spheres. We apply this connection to computations of cobordism groups of certain singular maps. The differentials of the spectral sequence computing these cobordism groups are given by the composition multiplication in the ring of stable homotopy groups of spheres.

§1. INTRODUCTION

We are considering locally stable smooth maps of n-dimensional manifolds into $(n + 1)$-dimensional manifolds with the simplest, corank 1 singularities (those where the rank of the differential map is at most 1 less than the maximum possible). These singularities, called Morin singularities, form a single infinite family, with members denoted by symbols Σ^0 (nonsingular points), $\Sigma^{1,0}$ (fold points), $\Sigma^{1,1,0} = \Sigma^{1_2}$ (cusp points), ..., $\Sigma^{1,..,1,0}_j = \Sigma^{1_j}$, ... (see [M]). A map that only has singularities Σ^{1_j} with $j \leq r$ is called a Σ^{1_r}-map, and we are interested in calculating the cobordism groups of such maps. Two Σ^{1_r}-maps with the same target manifold P are (Σ^{1_r}-)cobordant if there exists a Σ^{1_r}-map into $P \times [0, 1]$ from a manifold with boundary whose boundary is the disjoint union of those two maps. Unless specified otherwise maps between manifolds will be assumed to be cooriented (equipped with an orientation of the virtual normal bundle), Morin and of codimension 1 (that is, the dimension of the target is 1 greater than the dimension of the source).

The cobordism group of fold maps of oriented n-manifolds into $\mathbb{R}^{n+1}$, which is denoted by $\mathrm{Cob}^{SO}\Sigma^{1,0}(n+1)$, was computed in [Sz5] completely, while that of cusp maps, $\mathrm{Cob}^{SO}\Sigma^{1,1,0}(n+1)$, only modulo 2- and 3-torsion.[1] Here we compute the 3-torsion part (up to a group extension). We shall also consider a subclass of such maps, the so-called prim (projection of immersion) cusp maps. These are the cusp maps with trivial and trivialized kernel bundle of the differential over the set of singular points. The cobordism group of prim fold and cusp maps of oriented n-manifolds to $\mathbb{R}^{n+1}$ will be denoted by $\mathrm{Prim}^{SO}\Sigma^{1,0}(n+1)$ and $\mathrm{Prim}^{SO}\Sigma^{1,1,0}(n+1)$ respectively. We shall compute these groups up to a group extension, and in the case of cusp maps up to 2-primary torsion.

§2. NOTATION AND FORMULATION OF THE RESULT

We shall denote by $\pi^s(n)$ the nth stem, that is,

$$\pi^s(n) = \lim_{q \to \infty} \pi_{n+q}(S^q).$$

[1]These groups were denoted in [Sz5] by $\mathrm{Cob}\Sigma^{1,0}(n)$ and $\mathrm{Cob}\Sigma^{1,1}(n)$, respectively; note the shifted dimensions.

Denote by $\mathcal{G}$ the direct sum $\bigoplus_n \pi^s(n)$. Recall that $\mathcal{G}$ is a ring, with multiplication $\circ$ defined by the composition (see [To]). This product is skew commutative: for homogeneous elements x and y of $\mathcal{G}$

$$x \circ y = (-1)^{\dim x \cdot \dim y} y \circ x.$$

Recall that $\pi^s(3) \cong \mathbb{Z}_3 \oplus \mathbb{Z}_8$; the standard notation for the generator of the $\mathbb{Z}_3$ part is α_1. By a slight abuse of notation we shall also denote by α_1 the group homomorphism of $\mathcal{G}$ defined by left multiplication by α_1, i.e. $\alpha_1(g) = \alpha_1 \circ g$ for any $g \in \mathcal{G}$.

To formulate our results more compactly, we use the language of Serre classes of abelian groups [Se]. In particular, we will denote by $\mathcal{C}_2$ the class of finite 2-torsion groups and $\mathcal{C}_{\{2,3\}}$ will denote the class of finite groups of order a product of powers of 2 and 3. Given a Serre class $\mathfrak{C}$, we call a homomorphism $f : A \to B$ a $\mathfrak{C}$-isomorphism if both $\ker f \in \mathfrak{C}$ and $\mathrm{Coker}\, f \in \mathfrak{C}$. Two groups are considered to be isomorphic modulo $\mathfrak{C}$ (denoted by $\underset{\mathfrak{C}}{\cong}$) if there exists a chain of $\mathfrak{C}$-isomorphisms that connects them. For example, isomorphism modulo $\mathcal{C}_{\{2,3\}}$ is isomorphism modulo the 2-primary and 3-primary torsions. A complex $\ldots \overset{d_{m+1}}{\to} C_m \overset{d_m}{\to} \ldots$ of maps is called $\mathfrak{C}$-exact if both $\mathrm{im}(d_m \circ d_{m+1})$ and $\ker d_m/(\ker d_m \cap \mathrm{im}\, d_{m+1})$ belong to $\mathfrak{C}$ for all m.

Theorem 1. *There is a $\mathcal{C}_2$-exact sequence*

$$0 \to \mathrm{Coker}\left(\alpha_1 : \pi^s(n-3) \to \pi^s(n)\right) \to \mathrm{Cob}^{SO}\Sigma^{1,1,0}(n+1) \to \ker\left(\alpha_1 : \pi^s(n-4) \to \pi^s(n-1)\right) \to 0.$$

Remark. Recall that in [Sz5] it was shown that

$$\mathrm{Cob}^{SO}\Sigma^{1,1,0}(n+1) \underset{\mathcal{C}_{\{2,3\}}}{\cong} \pi^s(n) \oplus \pi^s(n-4)$$

(in particular $\mathrm{Cob}^{SO}\Sigma^{1,1,0}(n+1)$ is finite unless $n = 0$ or $n = 4$, when its rank is 1). Since α_1 is a homomorphism of order 3, Theorem 1 is compatible with this result and determines the 3-primary part of $\mathrm{Cob}^{SO}\Sigma^{1,1}(n+1)$ up to a group extension.

Remark. Recall ([To, Chapter XIV]) the first few groups $\pi^s(n)$ and the standard names of the generators of the 3-primary components (with the exception of α_3, which is denoted by α_3' in [To]):

n	0	1	2	3	4	5	6	7	8	9	10	11
$\pi^s(n)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_{240}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_6$	$\mathbb{Z}_{56} \times \mathbb{Z}_9$
$(\pi^s(n))_3$	$\mathbb{Z}$			$\mathbb{Z}_3\langle\alpha_1\rangle$				$\mathbb{Z}_3\langle\alpha_2\rangle$			$\mathbb{Z}_3\langle\beta_1\rangle$	$\mathbb{Z}_9\langle\alpha_3\rangle$

Here and later we denote by $(G)_3$ the 3-primary part[2] of the abelian group G, omitting the parentheses when this does not cause confusion.

For $n \leq 11$ the only occasion when the homomorphism α_1 is non-trivial on the 3-primary part is the epimorphism $\pi^s(0) \to \pi^s(3)$ (the homomorphism $\pi^s(7) \to \pi^s(10)$ is trivial modulo 3 as a consequence of [To, Lemma 13.8.] and the fact that $(\pi^s(10))_3 \cong \mathbb{Z}_3$). Hence $\left(\mathrm{Cob}^{SO}\Sigma^{1,1,0}(n+1)\right)_3$ fits into the short exact sequence

$$0 \to (\pi^s(n))_3 \to \left(\mathrm{Cob}^{SO}\Sigma^{1,1,0}(n+1)\right)_3 \to (\pi^s(n-4))_3 \to 0$$

for $n \leq 11$, $n \neq 3$.

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[2]the quotient by the subgroup of torsion elements with orders coprime to 3

§3. ELEMENTS OF STABLE HOMOTOPY GROUPS OF SPHERES ARISING FROM SINGULARITIES

§3.1. **Representing elements of $\mathcal{G}$.** The following is a well-known corollary of the combination of the Pontryagin-Thom isomorphism and the Smale-Hirsch immersion theory:

Fact 1. *The cobordism group of framed immersions of n-manifolds into $\mathbb{R}^{n+k}$ is isomorphic to $\pi^s(n)$ for any $k \geq 1$.*

An equivalent formulation is the following:

Fact 2. *The cobordism group of pairs (M^n, F), where M^n is a stably parallelizable n-manifold and F is a trivialization of its stable normal bundle is isomorphic to $\pi^s(n)$.*

These Facts follow from the so-called (Multi-)Compression Theorem (Theorem 2 below).

Definition 1. Let K be a compact subcomplex of a smooth manifold M. We say that M *retracts nicely* onto K if there is a homotopy ρ_t, $t \in [0,1]$ (that we call a *nice retraction* of M onto K) such that

- $\rho_0 = id_M$,
- $\rho_t|_K = id_K$ for all $t \in [0,1]$,
- every ρ_t for $t < 1$ is an embedding of M into itself,
- the image of ρ_1 is K.

Before formulating the theorem, note that given natural numbers k, m and $n \geq m$ as well as an immersion $i : M^m \looparrowright \mathbb{R}^n \times \mathbb{R}^k$ of a compact m-dimensional manifold M^m with k independent normal vector fields $v_1, \ldots, v_k$, one can consider an extension of i to $M \times D_\varepsilon^k$ for a small k-disc $D_\varepsilon^k = \{(y_1, \ldots, y_k) : \sum y_j^2 < \varepsilon\}$ for any sufficiently small positive ε:

$$\hat{i} : M \times D_\varepsilon^k \looparrowright \mathbb{R}^n \times \mathbb{R}^k$$

$$(p, y_1, \ldots, y_k) \mapsto i(p) + \sum_{j=1}^k y_j v_j(p)$$

Theorem 2 ([RS])**.** *a) Given an embedding $i : M^m \hookrightarrow \mathbb{R}^n \times \mathbb{R}^k$ of a compact manifold M (possibly with boundary) equipped with k linearly independent normal vector fields $v_1, \ldots, v_k$, such that $n > m$, there is an isotopy Φ_t, $t \in [0, \infty)$, of $\mathbb{R}^n \times \mathbb{R}^k$ and a time $t^* > 0$ such that Φ_0 is the identity and $d\Phi_t(v_j)$ is parallel to the jth coordinate vector $\mathbf{e}_j$ of $\mathbb{R}^k$ for all j with $1 \leq j \leq k$ and $t \geq t^*$.*

The same statement remains true if $n = m$ and each component of M is a compact manifold with non-empty boundary.

b) If the vector fields v_j, $j \in I \subseteq \{1, \ldots, k\}$ were already parallel to the corresponding vectors $\mathbf{e}_j$, $j \in I$, the vector fields $v_{\hat{j}}$, $\hat{j} \notin I$ were orthogonal to all $\mathbf{e}_j$, $j \in I$, and either $n > m$ or $n = m$ and there exists a compact $(n-1)$-dimensional cell complex $K \subset M$ onto which M retracts nicely via a homotopy ρ_t for which the composition homotopy $i \circ \rho_t$ keeps the coordinates of $\mathbb{R}^k$ that belong to I fixed, then for any $\hat{j} \notin I$ the isotopy Φ_t can be chosen so that

- *the coordinates of $\mathbb{R}^k$ that belong to I are kept fixed;*
- *the map $(p, x, t) \mapsto \hat{\Phi}_t(p, x) := \Phi_t(\hat{i}(p, x))$ with $p \in M$, $x \in D_\varepsilon^{|I|}$ and $t \in [0, \infty)$ is an immersion of $M \times D_\varepsilon^{|I|} \times [0, \infty)$, where $D_\varepsilon^{|I|} = \{(y_j)_{j \in I} : \sum y_j^2 < \varepsilon\}$ for some sufficiently small but positive ε;*
- *for some time t^* and all $t \geq t^*$ we have $\Phi_t(p) = \Phi_{t^*}(p) + (t - t^*) \cdot \mathbf{e}_{\hat{j}}$.*

c) The isotopy Φ_t can be chosen in such a way that for any point $p \in M$ and any time $\tau \in [0, \infty)$ the tangent vector of the curve $t \mapsto \Phi_t(i(p))$ at the point $\Phi_\tau(i(p))$ will not be tangent to

the manifold $\Phi_\tau(i(M))$. That is, the map $M \times [0, \infty) \to \mathbb{R}^n \times \mathbb{R}^k$ defined by the formula $(p, t) \mapsto \Phi_t(i(p))$ is an immersion.

d) **(relative version)** Let L be a compact subset of $\mathbb{R}^n$ such that the vector fields $v_1, \ldots, v_k$ are already parallel to $\mathbf{e}_1, \ldots, \mathbf{e}_k$ in an open neighbourhood of $i^{-1}(L \times \mathbb{R}^k)$; if $n = m$, assume additionally that there exists a compact $(n-1)$-dimensional cell complex $K \subset M$ such that $i^{-1}(L \times \mathbb{R}^k) \subset K$ and M retracts nicely onto K. Then the isotopy Φ_t can be chosen to keep the vector fields $v_1, \ldots, v_k$ on $i^{-1}(L \times \mathbb{R}^k)$ parallel to $\mathbf{e}_1, \ldots, \mathbf{e}_k$ at all times.

The part c) of the theorem is not stated explicitly in [RS]. In Appendix 3 we give a proof of the theorem with emphasis on part c) and some simplifications in the case $k = 1$.

We shall mostly use the following consequence of Hirsch-Smale theory:

Theorem 3. *Given an* immersion $i : M^m \looparrowright \mathbb{R}^n \times \mathbb{R}^k$ *of a compact manifold M with k independent normal vector fields $v_1, \ldots, v_k$ for which $m < n$, there is a positive ε, a regular homotopy $\Phi_t : M \times D_\varepsilon^k \looparrowright \mathbb{R}^n \times \mathbb{R}^k$, $t \in [0, \infty)$ and a time t^* such that*

- $\Phi_0 = \hat{i}$;
- $d\Phi_t|_{M \times \mathbf{0}}(v_j)$ *is parallel to the jth coordinate vector $\mathbf{e}_j$ of $\mathbb{R}^k$ for all j with $1 \leq j \leq k$ and $t \geq t^*$;*
- *the map $(p, t) \mapsto \Phi_t(p, \mathbf{0})$ is an immersion of the cylinder $M \times [0, \infty)$ into $\mathbb{R}^n \times \mathbb{R}^k$.*

If $m = n$ and M has no closed components, then for any $(m-1)$-dimensional compact subcomplex $K \subset M$ onto which M retracts nicely there exists a positive ε and a neighbourhood U of K in M for which there is a regular homotopy $\Phi_t : U \times D_\varepsilon^k \looparrowright \mathbb{R}^n \times \mathbb{R}^k$ with the same properties.

Proof. We address the case $m < n$ first. Applying [RS, Multi-compression Theorem 4.5][3] gives us a regular homotopy $\widetilde{\Phi}_t^M : M \looparrowright \mathbb{R}^n \times \mathbb{R}^k$ that comes from an isotopy of M within its induced neighbourhood $\mathcal{V}$ pulled back by i from $\mathbb{R}^n \times \mathbb{R}^k$; for an induced neighbourhood $\mathcal{U} \subset \mathcal{V}$ of M this defines a regular homotopy $\widetilde{\Phi}_t : \mathcal{U} \looparrowright \mathbb{R}^n \times \mathbb{R}^k$ whose restriction to the disc bundle spanned by the normal vector fields v_j satisfies the required properties except the last one. We may also assume that for some time t^* and all $t > t^*$ and $q \in \mathcal{U}$ we have $\widetilde{\Phi}_t(q) = \widetilde{\Phi}_{t^*}(q) + (t - t^*)\mathbf{e}_1$.

Let us define the map

$$\widetilde{\mathbf{\Phi}} : M \times [0, \infty) \times D_\delta^{k-1} \to \mathbb{R}^n \times \mathbb{R}^k$$

$$\widetilde{\mathbf{\Phi}}(p, t_1, \ldots, t_k) = \widetilde{\Phi}_{t_1}(p) + (d\widetilde{\Phi}_{t_1})_p(t_2 v_2(p) + \cdots + t_k v_k(p))$$

for a small $\delta > 0$, and define the bundle map

$$D : T(M \times [0, \infty) \times D_\delta^{k-1}) \to T(\mathbb{R}^n \times \mathbb{R}^k)$$

$$(w, s_1, s_2, \ldots, s_k)_{(p, t_1, t_2, \ldots, t_k)} \mapsto (d\widetilde{\Phi}_{t_1})_p(w + s_1 v_1(p) + \cdots + s_k v_k(p))$$

$$\text{taken at the basepoint } \widetilde{\mathbf{\Phi}}(p, t_1, \ldots, t_k)$$

covering $\widetilde{\mathbf{\Phi}}$. This map is fibrewise injective since the linear span of $v_1, \ldots, v_k$ is normal to TM. In addition, we construct the immersion

$$\alpha : M \times [0, 2\delta) \times D_\delta^{k-1} \looparrowright \mathbb{R}^n \times \mathbb{R}^k$$

$$(p, t_1, \ldots, t_k) \mapsto i(p) + t_1 v_1(p) + \cdots + t_k v_k(p)$$

that has the property

$$d\alpha|_{T(M \times [0, 2\delta) \times D_\delta^{k-1})|_{M \times \{0\} \times \mathbf{0}}} = D|_{T(M \times [0, \infty) \times D_\delta^{k-1})|_{M \times \{0\} \times \mathbf{0}}}$$

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[3]Note that although the cited theorem is about embeddings, [RS, Addendum (vi) to Multi-compression Theorem 4.5] ensures that embeddings can be replaced by immersions.

and note that $\widetilde{\boldsymbol{\Phi}}$ is an immersion of $M \times [t^*, \infty) \times D_\delta^{k-1}$ for which

$$d\widetilde{\boldsymbol{\Phi}}|_{T(M \times [0,\infty) \times D_\delta^{k-1})|_{M \times [t^*, \infty) \times \mathbf{0}}} = D|_{T(M \times [0,\infty) \times D_\delta^{k-1})|_{M \times [t^*, \infty) \times \mathbf{0}}}$$

also holds. The differentials of these immersions can therefore be blended into D by using appropriate bump functions

$$\lambda : [0, \infty) \to [0, 1], \quad \lambda|_{[0,\delta]} = 1, \ \lambda|_{[2\delta,\infty)} = 0$$

and

$$\mu : [0, \infty) \to [0, 1], \quad \mu|_{[0,t^*-\delta)} = 0, \ \mu|_{[t^*,\infty)} = 1$$

as follows (we may assume that λ and μ have disjoint supports, i.e. $t^* > 3\delta$):

$$\widetilde{D}|_{T_{(p,t_1,\mathbf{x})}(M \times [0,\infty) \times D_\delta^{k-1})} =$$

$$\lambda(t_1)d\alpha_{(p,t_1,\mathbf{x})} + (1 - \lambda(t_1) - \mu(t_1))D|_{T_{(p,t_1,\mathbf{x})}(M \times [0,\infty) \times D_\delta^{k-1})} + \mu(t_1)d\widetilde{\boldsymbol{\Phi}}_{(p,t_1,\mathbf{x})}.$$

Here we shift the tangent vector values of $d\alpha$ to the basepoint $\widetilde{\boldsymbol{\Phi}}(p, t_1, \mathbf{x})$ of the other two summands. If δ is sufficiently small, then $\widetilde{D}$ is fibrewise injective on $M \times [0, \infty) \times \mathbf{0}$ and consequently also on $M \times [0, \infty) \times D_\varepsilon^{k-1}$ as well for some $\varepsilon > 0$. By applying [Hi, Theorem 5.7] to the pair of complexes $M \times (\{0\} \cup [t^*, \infty)) \times D_\varepsilon^{k-1} \subset M \times [0, \infty) \times D_\varepsilon^{k-1}$ we obtain an immersion Φ_t of $M \times [0, \infty) \times D_\varepsilon^{k-1}$ into $\mathbb{R}^n \times \mathbb{R}^k$ that coincides with α on $M \times \{0\} \times D_\varepsilon^{k-1}$ and with $\widetilde{\boldsymbol{\Phi}}$ on $M \times [t^*, \infty) \times D_\varepsilon^{k-1}$. Reinterpreting Φ_t as a regular homotopy of $\Phi|_{M \times \{0\} \times D_\varepsilon^{k-1}} = \hat{i}$ finishes the proof when $m < n$.

In the case $m = n$ the proof is similar, we use [RS, Addendum (v) to Multi-compression Theorem 4.5] to construct the initial regular homotopy $\widetilde{\Phi}_t$, restrict it to a neighbourhood U of K in M that has compact closure and such that M retracts nicely onto U, and repeat the rest of the proof with U substituted for M. $\qquad\square$

§3.2. **Framed immersions on the boundary of a singularity.** We outline a connection between singularities and stable homotopy groups of spheres. Namely, we define elements of $\mathcal{G}$ that describe incidences of the images of singularity strata.

We use the notation ∂f to denote the *link* of a map $f : \mathbb{R}^a \to \mathbb{R}^b$, that is, the restriction of f to the preimage $f^{-1}(S^{b-1})$ of some sufficiently small sphere $S^{b-1} \subset \mathbb{R}^b$ centered at the origin. Later, we denote the same way the restriction of a map $g : (M, \partial M) \to (P, \partial P)$ to the boundary: $\partial g = g|_{\partial M} : \partial M \to \partial P$. This notation is consistent with the notion of a link in the sense that the link of $f : \mathbb{R}^a \to \mathbb{R}^b$ is the map ∂g, where $g : D^a \to D^b$ is the restriction of f to the preimage D^a of a sufficiently small ball D^b in the target.

Example 1: Let us consider the Whitney umbrella map

$$\sigma_1 : \mathbb{R}^2 \to \mathbb{R}^3, \quad \sigma_1(t, x) = (t, tx, x^2).$$

(the normal form of an isolated $\Sigma^{1,0}$-point). The preimage of the unit 2-sphere $S^2 \subset \mathbb{R}^3$ is a closed curve $\sigma_1^{-1}(S^2) \subset \mathbb{R}^2$. The restriction of σ_1 to this curve – the link of the map σ_1 – is an immersion. The image $\sigma_1(\sigma_1^{-1}(S^2))$ is an immersed curve in S^2 with a single double point. The orientations of $\mathbb{R}^2$ and $\mathbb{R}^3$ give a coorientation on this curve. Hence this curve can be equipped with a normal vector in S^2 and with an additional normal vector to S^2 in $\mathbb{R}^3$, resulting in an immersed framed curve in $\mathbb{R}^3$; this represents an element of $\pi^s(1)$ that we shall denote by $d^1\sigma_1$. It is easy to see that $d^1\sigma_1 \neq 0$ (since this immersed curve in S^2 has a single double point). Using the standard notation η for the generator of $\pi^s(1) = \mathbb{Z}_2$, we get that $d^1\sigma_1 = \eta$.

Example 2: Let us consider the normal form of a map with an isolated cusp-point at the origin

$$\sigma_2 : \mathbb{R}^4 \to \mathbb{R}^5$$

$$(t_1, t_2, t_3, x) \mapsto (t_1, t_2, t_3, z_1, z_2)$$
$$z_1 = t_1 x + t_2 x^2$$
$$z_2 = t_3 x + x^3$$

The link of this map is its restriction to $\sigma_2^{-1}(S^4)$, where S^4 is the unit sphere in $\mathbb{R}^5$. Note that the 3-manifold $\sigma_2^{-1}(S^4)$, which we shall denote by L^3, is diffeomorphic to S^3. The link map $\partial \sigma_2 = \sigma_2|_{L^3} : L^3 \to S^4$ is a fold map, it has only $\Sigma^{1,0}$ singularities along a closed curve γ. The image of this curve γ in S^4 has a canonical framing. Indeed, the map σ_2 can be lifted to an embedding $\hat{\sigma}_2 : \mathbb{R}^4 \hookrightarrow \mathbb{R}^6$, $\hat{\sigma}_2(t_1, t_2, t_3, x) = (\sigma_2(t_1, t_2, t_3, x), x)$ such that the composition of $\hat{\sigma}_2$ with the projection $\mathbb{R}^6 \to \mathbb{R}^5$ is σ_2. Hence the two preimages of any double point of σ_2 near the singularity curve $\sigma_2(\gamma)$ have an ordering and so one gets two of the normal framing vectors on the singularity curve $\sigma_2(\gamma)$. In order to get the third framing vector we note that $\sigma_2(\gamma)$ is the boundary of the surface formed by the double points of $\partial \sigma_2$ in S^4. The inward-pointing normal vector along $\sigma_2(\gamma)$ of this surface will be the third framing vector. (In Appendix 2 we shall describe the framing that arises naturally on the image of the top singularity stratum of a map obtained as a generic projection of an immersion.) The curve $\sigma_2(\gamma)$ with this framing represents an element in $\pi^s(1)$ that we denote by $d^1(\sigma_2)$. We shall show that $d^1(\sigma_2) = 0$ (see Lemma 4 and Appendix 1).

In the present situation we can construct one more element associated to σ_2 in a quotient group of $\mathcal{G}$, which we shall denote by $d^2(\sigma_2)$. We construct this element (after making some choices) in $\pi^s(3) \cong \mathbb{Z}_{24}$ but it will be well-defined only in the quotient group $\mathbb{Z}_{24}/\mathbb{Z}_2$. The definition of $d^2(\sigma_2)$ is the following. By a result of [Sz4] (that we shall recall in §5, see also [Te]) a framed cobordism of the embedded singularity curve $\sigma_2(\gamma)$ can be extended to a cobordism of the link map $\partial \sigma_2 : L^3 \to S^4$. In particular, since the (framed) curve $\sigma_2(\gamma)$ is (framed) null-cobordant, the link map $\partial \sigma_2$ is fold-cobordant to an (oriented) immersion. That is, there is a compact oriented 4-manifold W^4 such that $\partial W^4 = L^3 \sqcup V^3$ and there is a $\Sigma^{1,0}$-map

$$C : (W^4, L^3, V^3) \to (S^4 \times [0,1], S^4 \times \{0\}, S^4 \times \{1\})$$

such that the restriction $C|_{L^3} : L^3 \to S^4 \times \{0\}$ is $\partial \sigma_2$ and the restriction $C|_{V^3} : V^3 \to S^4 \times \{1\}$ is an immersion, which we denote by $\partial' \sigma_2$. It represents an element in $\pi^s(3)$ and its image in $\pi^s(3)/\mathbb{Z}_2$ is independent of the choice of C (as detailed in the description of d^2 in subsection §5.2). The obtained element in $\pi^s(3)/\mathbb{Z}_2$ is $d^2(\sigma_2)$.

For future reference we introduce the notation σ_2^* for the map

$$\sigma_2^* = \sigma_2 \cup C : D^4 \underset{L^3}{\cup} W^4 \to D_2^5 = D^5 \underset{S^4 \times \{0\}}{\cup} S^4 \times [0,1].$$

This definition makes sense since the maps C and σ_2 coincide on the common part L^3 of their source manifolds, and it is easy to see that the gluing can be performed to make σ_2^* smooth. Note that $\partial \sigma_2^*$ is the immersion $V^3 \looparrowright S^4 \times \{1\}$.

We shall show that the stable homotopy group elements $d^1(\sigma_1)$, $d^1(\sigma_2)$, the coset $d^2(\sigma_2)$ and other analogously defined objects can be computed from a spectral sequence associated to the classifying spaces of singularities (d^1 and d^2 are in fact differentials of this spectral sequence). Next we shall describe these classifying spaces and the spectral sequence, first for the simpler case of prim maps.

§4. Classifying spaces of cobordisms of singular maps

Definition. A smooth map $f : M^n \to P^{n+k}$ is called a prim map (prim stands for the abbreviation of *pr*ojected *im*mersion) if

1) it is the composition of an immersion $g : M \looparrowright P \times \mathbb{R}^1$ and the projection $p_P : P \times \mathbb{R}^1 \to P$, and

2) an orientation is given on the kernels of the differential of f that agrees with the orientation pulled back from $\mathbb{R}^1$ via the composition $p_{\mathbb{R}} \circ g$, where $p_{\mathbb{R}} : P \times \mathbb{R}^1 \to \mathbb{R}^1$ is the projection.

For maps between manifolds with boundary $f : (M, \partial M) \to (P, \partial P)$, we shall always require that they should be *regular*, that is, $f^{-1}(\partial P) = \partial M$ and the map f in a neighbourhood of ∂M can be identified with the direct product $f|_{\partial M} \times id_{[0, \varepsilon)}$ for a suitable positive ε.

Remark. Note that for a Morin map the kernels of df form a line bundle $\ker df \to \Sigma(f)$, where $\Sigma(f)$ is the set of singular points of f. The conditions 1) and 2) ensure that for a prim map this bundle is orientable (trivial) and an orientation (trivialization) is chosen (the same map f with a different choice of orientation on the kernels is considered to be a different prim map). The converse also holds: if a Morin map $f : M^n \to P^{n+k}$ is equipped with a trivialization of its kernel bundle, then there exists a unique (up to regular homotopy) immersion $g : M \looparrowright P \times \mathbb{R}^1$ such that $f = p_P \circ g$, where $p_P : P \times \mathbb{R}^1 \to P$ is the projection and the trivialization of the kernel bundle is the same as the one defined by the projection to $\mathbb{R}^1$.

Notation. The cobordism group of all prim maps of n-dimensional oriented manifolds into $\mathbb{R}^{n+1}$ will be denoted by $\mathrm{Prim}^{SO}(n+1)$. The analogous cobordism group of prim maps having only (at most) $\Sigma^{1,0}$-singular points (i.e. both the maps and the cobordisms between them are prim fold maps) will be denoted by $\mathrm{Prim}^{SO}\Sigma^{1,0}(n+1)$; that of prim cusp maps will be denoted by $\mathrm{Prim}^{SO}\Sigma^{1,1,0}(n+1)$; and that of prim Σ^{1_i}-maps will be denoted by $\mathrm{Prim}^{SO}\Sigma^{1_i}(n+1)$.

One can define cobordism sets of prim $\Sigma^{1,0}$ and $\Sigma^{1,1,0}$ (cooriented) maps of n-manifolds in arbitrary fixed $(n+1)$-dimensional manifold P^{n+1} (instead of $\mathbb{R}^{n+1}$). The obtained sets we denote by $\mathrm{Prim}^{SO}\Sigma^{1,0}(P)$ and $\mathrm{Prim}^{SO}\Sigma^{1,1,0}(P)$, respectively.[4]

Definition. If $f_0 : M_0^n \to P^{n+k}$ and $f_1 : M_1^n \to P^{n+k}$ are two regular prim $\Sigma^{1,(1,)0}$-maps of the oriented n-manifolds M_0^n and M_1^n to the oriented manifold P^{n+k}, then a cobordism between them is a regular prim $\Sigma^{1,(1,)0}$ map $F : W^{n+1} \to P \times [0, 1]$, where W is a compact oriented $(n+1)$-manifold such that $\partial W = M_0 \underset{\partial M_0}{\cup} F^{-1}(\partial P \times [0, 1]) \underset{\partial M_1}{\cup} (-M_1)$, $F|_{M_0} = f_0$ and $F|_{M_1} = f_1$. Note that both the domain W and the target $P \times [0, 1]$ of F may have "corners": $\partial M_0 \sqcup \partial M_1$ and $\partial P \times \{0\} \sqcup \partial P \times \{1\}$, respectively. Regularity of F shall mean that near the corners ∂M_0 and ∂M_1 the map F has to be the direct product of the maps $f_0|_{\partial M_0} : \partial M_0 \to \partial P \times \{0\}$ and $f_1|_{\partial M_1} : \partial M_1 \to \partial P \times \{1\}$ with the identity map of $[0, \varepsilon) \times [0, \varepsilon)$.

The classifying spaces. There exist (homotopically unique) spaces $\overline{X}\Sigma^{1,0}$ and $\overline{X}\Sigma^{1,1,0}$ that represent the functors

$$P \longrightarrow \mathrm{Prim}^{SO}\Sigma^{1,0}(P) \text{ and}$$

$$P \longrightarrow \mathrm{Prim}^{SO}\Sigma^{1,1,0}(P)$$

in the sense of Brown representability theorem[5] (see [Sw]), in particular

$$\mathrm{Prim}^{SO}\Sigma^{1,0}(P) = [P, \overline{X}\Sigma^{1,0}] \text{ and}$$

$$\mathrm{Prim}^{SO}\Sigma^{1,1,0}(P) = [P, \overline{X}\Sigma^{1,1,0}]$$

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[4]These sets carry a natural abelian group structure, but we do not use this fact here. In [Sz4, Remark 8] the analogous statement for not necessarily prim maps with $k = 1$ is shown and used, with the proof easily adaptable to prim maps.

[5]In order to apply Brown's theorem directly, one has to extend these functors to arbitrary simplicial complexes (not only manifolds). This is done in [Sz4].

for any compact manifold P (note that we have $\text{Prim}^{SO}\Sigma^{1,0}(n+1) \cong \text{Prim}^{SO}\Sigma^{1,0}(S^{n+1})$ and $\text{Prim}^{SO}\Sigma^{1,1,0}(n+1) \cong \text{Prim}^{SO}\Sigma^{1,1,0}(S^{n+1})$). We call the spaces $\overline{X}\Sigma^{1,0}$ and $\overline{X}\Sigma^{1,1,0}$ the *classifying spaces* for prim fold and prim cusp maps respectively. This type of classifying spaces in a more general setup has been explicitly constructed and investigated earlier, see [Sz2], [Sz1], [RSz], [Sz4], [Te].

Key fibrations. For any space Y we shall denote by ΓY the space

$$\Omega^{\infty}S^{\infty}Y = \lim_{q \to \infty} \Omega^q S^q Y.$$

A crucial observation in the investigation of these classifying spaces is the existence of the so-called *key fibrations* (see [Sz4]), which in the present cases states that there exist Serre fibrations

$$\overline{p}_j : \overline{X}\Sigma^{1_j} \to \Gamma S^{2j+1}$$

of $\overline{X}\Sigma^{1_j}$ over ΓS^{2j+1} with fibre $\overline{X}\Sigma^{1_{j-1}}$. In particular, we have fibrations

- $\overline{p}_1 : \overline{X}\Sigma^{1,0} \to \Gamma S^3$ of $\overline{X}\Sigma^{1,0}$ over ΓS^3 with fibre ΓS^1; and
- $\overline{p}_2 : \overline{X}\Sigma^{1,1,0} \to \Gamma S^5$ of $\overline{X}\Sigma^{1,1,0}$ over ΓS^5 with fibre $\overline{X}\Sigma^{1,0}$.

§5. The spectral sequences

§5.1. **The first page.** Let us denote by $\overline{X}_i$ for $i = -1, 0, 1, 2$ the following spaces:

$$\overline{X}_{-1} = \text{point}; \quad \overline{X}_0 = \Gamma S^1; \quad \overline{X}_1 = \overline{X}\Sigma^{1,0}; \quad \overline{X}_2 = \overline{X}\Sigma^{1,1,0}.$$

One can define a spectral sequence with starting page

$$\overline{E}^1_{i,j} = \pi_{i+j+1}\big(\overline{X}_i, \overline{X}_{i-1}\big), \quad i = 0, 1, 2, \quad j = 0, 1, \dots$$

and differentials $d^r_{i,j} : \overline{E}^r_{i,j} \to \overline{E}^r_{i-r,j+r-1}$ that converges to $\pi_{n+1}(\overline{X}_2) = \text{Prim}^{SO}\Sigma^{1,1,0}(n+1)$.

The existence of Serre fibrations described above implies that

$$\overline{E}^1_{i,j} \cong \pi^S_{i+j+1}(\Gamma S^{2i+1}) = \pi^s(j-i).$$

§5.2. **The geometric meaning of the groups and differentials of the spectral sequence.** Let $\overline{X}_i$ denote the classifying space of prim Σ^{1_i}-maps, so that $\text{Prim}^{SO}\Sigma^{1_i}(n+1) = \pi_{n+1}(\overline{X}_i)$. The relative versions of the analogous cobordism groups of maps into the halfspace

$$\mathbb{R}^{n+1}_+ = \big\{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} : x_0 \geq 0\big\}$$

can also be introduced and they will be isomorphic to the corresponding relative homotopy groups:

Definition. Let $(M^n, \partial M)$ be a compact n-manifold with (possibly empty) boundary. Let $f : (M, \partial M) \to (\mathbb{R}^{n+1}_+, \mathbb{R}^n)$ be a prim Σ^{1_i}-map for which $f|_{\partial M} : \partial M \to \mathbb{R}^n$ is a (necessarily prim) Σ^{1_j}-map for some $j \leq i$. Such a map will be called a *prim $(\Sigma^{1_i}, \Sigma^{1_j})$-map* (recall that we always assume f to be regular in the sense of the definition in §4).

If f_0 and f_1 are two prim $(\Sigma^{1_i}, \Sigma^{1_j})$-maps of the oriented n-manifolds $(M_0^n, \partial M_0)$ and $(M_1^n, \partial M_1)$ to $(\mathbb{R}^{n+1}_+, \mathbb{R}^n)$, then a cobordism between them is a map

$$F : (W^{n+1}, \partial W) \to (\mathbb{R}^{n+1}_+ \times [0,1], \mathbb{R}^n \times [0,1])$$

(where W is a compact oriented $(n+1)$-manifold) such that $\partial W = M_0 \underset{\partial M_0}{\cup} Q^n \underset{\partial M_1}{\cup} (-M_1)$, $F|_{M_0} = f_0$ and $F|_{M_1} = f_1$, for which

a) Q is a cobordism between ∂M_0 and $-\partial M_1$;

b) $F|_Q : Q \to \mathbb{R}^n \times [0,1]$ is a prim Σ^{1_j}-cobordism between $f_0|_{\partial M_0} : \partial M_0 \to \mathbb{R}^n \times \{0\}$ and $f_1|_{\partial M_1} : \partial M_1 \to \mathbb{R}^n \times \{1\}$;

c) F is a prim Σ^{1_i}-map.

Note that both the domain W and the target $\mathbb{R}_+^{n+1} \times [0,1]$ of F have "corners": $\partial M_0 \sqcup \partial M_1$ and $\mathbb{R}^n \times \{0\} \sqcup \mathbb{R}^n \times \{1\}$ respectively. Near the corners ∂M_0 and ∂M_1 the map F has to be the direct product of the maps $f_0|_{\partial M_0}$ and $f_1|_{\partial M_1}$ with the identity map of $[0,\varepsilon) \times [0,\varepsilon)$.

The cobordism group of prim $(\Sigma^{1_i}, \Sigma^{1_j})$-maps of oriented n-manifolds to $(\mathbb{R}_+^{n+1}, \mathbb{R}^n)$ will be denoted by $\mathrm{Prim}^{SO}(\Sigma^{1_i}, \Sigma^{1_j})(n+1)$.

Analogously to the isomorphism $\mathrm{Prim}^{SO}\Sigma^{1_i}(n+1) \cong \pi_{n+1}(\overline{X}_i)$ one obtains the isomorphism

$$(\dagger) \qquad\qquad \mathrm{Prim}^{SO}(\Sigma^{1_i}, \Sigma^{1_j})(n+1) \cong \pi_{n+1}(\overline{X}_i, \overline{X}_j).$$

Remark. Note that $\overline{X}_0 = \Gamma S^1$. Indeed, $\Sigma^{1_0} = \Sigma^0$-maps are non-singular, i.e. immersions, and the classifying space for codimension 1 oriented immersions is known to be ΓS^1 (see Fact 1).

The fibration $\overline{p}_i : \overline{X}_i \to \Gamma S^{2i+1}$ with fiber $\overline{X}_{i-1}$ induces an isomorphism of homotopy groups $(\overline{p}_i)_* : \pi_{n+1}(\overline{X}_i, \overline{X}_{i-1}) \to \pi_{n+1}(\Gamma S^{2i+1}) = \pi^s(n-2i)$. The geometric interpretation of this isomorphism is the following: to the cobordism class of a prim $(\Sigma^{1_i}, \Sigma^{1_{i-1}})$-map

$$f : (M, \partial M) \to (\mathbb{R}_+^{n+1}, \mathbb{R}^n)$$

the mapping $(\overline{p}_i)_*$ associates the cobordism class of the framed immersion $f|_{\Sigma^{1_i}(f)}$ (the framing is described in Appendix 2). Note that in particular this description implies that whenever two prim $(\Sigma^{1_i}, \Sigma^{1_{i-1}})$-maps of n-manifolds have framed cobordant images of their Σ^{1_i}-points, then they represent the same element in $\mathrm{Prim}^{SO}(\Sigma^{1_i}, \Sigma^{1_{i-1}})(n+1)$.

Geometric descriptions of d^1. The differential

$$d^1 : \overline{E}_{i,j}^1 \cong \pi_{i+j+1}(\overline{X}_i, \overline{X}_{i-1}) \to \overline{E}_{i-1,j}^1 \cong \pi_{i+j}(\overline{X}_{i-1}, \overline{X}_{i-2})$$

is simply the boundary homomorphism ∂ in the homotopy exact sequence of the triple

$$(\overline{X}_i, \overline{X}_{i-1}, \overline{X}_{i-2}).$$

Composing ∂ with the isomorphism $(\overline{p}_{i-1})_*$ one can see that if $f : (M^n, \partial M) \to (\mathbb{R}_+^{n+1}, \mathbb{R}^n)$ is a prim $(\Sigma^{1_i}, \Sigma^{1_{i-1}})$-map that represents the cobordism class

$$[f] = u \in \pi_{n+1}(\overline{X}_i, \overline{X}_{i-1}) = \overline{E}_{i,n-i}^1 \cong \pi^s(n-2i),$$

then $d^1(u) \in \pi^s((n-1)-2(i-1)) = \pi^s(n-2i+1)$ is represented by the framed immersion $f|_{\Sigma^{1_{i-1}}(f|_{\partial M})}$ in $\mathbb{R}^n$.

There is an alternative description of d^1 that we shall use later as well. Let $u \in \pi_{n+1}(\overline{X}_i, \overline{X}_{i-1})$ and f be a representative of u as above, and let T and $\widetilde{T}$ be the (immersed) tubular neighbourhoods of the top singularity strata $\Sigma^{1_i}(f)$ and $f\left(\Sigma^{1_i}(f)\right)$ in M and $\mathbb{R}_+^{n+1}$, respectively, with the property that $f|_T$ maps $(T, \partial T)$ to $(\widetilde{T}, \partial \widetilde{T})$. Now $f|_{\Sigma^{1_{i-1}}(f|_{\partial T})} : \Sigma^{1_{i-1}}(f|_{\partial T}) \to \partial \widetilde{T}$ is a framed immersion into $\partial \widetilde{T}$. Note that the normal framing of $f\left(\Sigma^{1_{i-1}}(f|_{\partial T})\right)$ inside $\partial \widetilde{T}$ defines a framing of the stable normal bundle of $f\left(\Sigma^{1_{i-1}}(f|_{\partial T})\right)$ because adding the unique outward-pointing normal vector of $\partial \widetilde{T}$ in $\mathbb{R}_+^{n+1}$ one obtains a normal framing in $\mathbb{R}_+^{n+1}$. Hence $f\left(\Sigma^{1_{i-1}}(f|_{\partial T})\right)$ with the given framing represents an element of $\pi^s(n-2i+1)$; this element is $d^1(u)$. The fact that these two descriptions of $d^1(u)$ yield the same element in $\pi^s(n-2i+1)$ follows from the fact that $f\left(\Sigma^{1_{i-1}}\left(f|_{M\backslash \mathring{T}}\right)\right)$ is a framed immersed cobordism between the two representatives (here we use Theorem 3).

This alternative description of d^1 actually generalizes to the higher differentials as well (even though here we only consider d^1 and d^2; in [ST] we utilize this fact for d^r with greater r as well).

Geometric description of d^2. Turning to the differential d^2, we first give a homotopic description (an expansion of the definition, in fact). Let $u \in \pi^s(n-4) \cong \pi_{n+1}(\overline{X}_2, \overline{X}_1) = \overline{E}^1_{2,n-2}$ be an element such that $d^1(u) = 0$. Then u represents an element of the page $\overline{E}^2$ as well (no differential is going into the groups $\overline{E}^1_{2,*}$). The class $d^2(u) \in \overline{E}^2_{0,n-1}$ is defined utilizing the boundary homomorphism $\partial : \pi_{n+1}(\overline{X}_2, \overline{X}_1) \to \pi_n(\overline{X}_1)$ as follows: since $d^1(u) = 0$, the class $\partial u \in \pi_n(\overline{X}_1)$ vanishes when considered in $\pi_n(\overline{X}_1, \overline{X}_0)$. Hence there is a class y in $\pi_n(\overline{X}_0)$ whose image in $\pi_n(\overline{X}_1)$ is ∂u. The class y is not unique, but the coset

$$[y] \in \pi_n(\overline{X}_0)/\operatorname{im}\left(\partial' : \pi_{n+1}(\overline{X}_1, \overline{X}_0) \to \pi_n(\overline{X}_0)\right) = \overline{E}^2_{0,n-1}$$

is unique. By definition $d^2(u) = [y]$.

Geometrically, if $f : (M, \partial M) \to (\mathbb{R}^{n+1}_+, \mathbb{R}^n)$ represents the class $u \in \pi_{n+1}(\overline{X}_2, \overline{X}_1)$, then $d^1(u) = 0$ means that $f|_{\Sigma^{1,0}(\partial f)} : \Sigma^{1,0}(\partial f) \to \mathbb{R}^n$ is a null-cobordant framed immersion (recall that here ∂f is the restriction $f|_{\partial M}$ and is a prim fold map). This means that the classifying map $S^n \to \overline{X}_1$ of ∂f becomes null-homotopic after being composed with $\overline{p}_1$, hence the classifying map itself can be deformed into the fiber $\overline{X}_0$. Since homotopy classes of maps into $\overline{X}_0$ correspond to cobordism classes of immersions, this deformation gives a (prim $\Sigma^{1,0}$-)cobordism between the prim $\Sigma^{1,0}$-map $\partial f : \partial M \to \mathbb{R}^n$ and an immersion that we will denote by g. The immersion g is not unique, not even its framed cobordism class $[g] \in \pi^s(n-1)$ is, but its coset in $\pi^s(n-1)/\operatorname{im} d^1$ is well-defined. This coset is $d^2(u)$.

Claim 1. *For any $u \in \pi^s(n)$*
a) $d^1_{2,n+2}(u) = d^1_{2,2}(\sigma_2) \circ u$ and $d^1_{1,n+1}(u) = d^1_{1,1}(\sigma_1) \circ u$
b) $d^2_{2,n+2}(u)$ is represented by $d^2_{2,2}(\sigma_2) \circ u$ whenever $u \in \ker d^1_{2,n+2}$.

Note that in order for statement *b)* to make sense, we need to show that:

- the ambiguity of $d^2_{2,2}(\sigma_2) \circ u$, which is $\left(\operatorname{im} d^1_{1,3}\right) \circ \pi^s(n)$, is contained in the ambiguity of $d^2_{2,n+2}(u)$, which is $\operatorname{im} d^1_{1,n+3}$. This holds by the second part of statement *a)*: we have

$$\operatorname{im} d^1_{1,3} = \operatorname{im} d^1_{1,1} \circ \pi^s(2)$$

  and

$$\operatorname{im} d^1_{1,3} \circ \pi^s(n) = \operatorname{im} d^1_{1,1} \circ \pi^s(2) \circ \pi^s(n) \subset \operatorname{im} d^1_{1,1} \circ \pi^s(n+2) = \operatorname{im} d^1_{1,n+3}.$$

- $d^2_{2,2}(\sigma_2)$ is meaningful, that is, $d^1_{2,2}(\sigma_2) = 0$. This is shown later in Lemma 4.

Proof. First we need a description of the composition product in the language of Pontryagin's framed embedded manifolds.

Given $\alpha \in \pi^s(m)$, $\beta \in \pi^s(n)$ let (M^m, U^p) and (N^n, V^{m+p}) be representatives of α and β, where M, N are manifolds of dimensions m and n immersed to $\mathbb{R}^{m+p}$ and $\mathbb{R}^{n+m+p}$, respectively, U^p and V^{m+p} are their framings: $U^p = (u_1, \dots, u_p)$; $V^{m+p} = (v_1, \dots, v_{m+p})$, where u_i, v_j are linearly independent normal vector fields to M and N. These framings identify open tubular neighbourhoods of M and N with $M \times \mathbb{R}^p$ and $N \times \mathbb{R}^{m+p}$.

Put the framed immersed submanifold (M, U) of $\mathbb{R}^{m+p}$ into each fiber of the tubular neighbourhood $N \times \mathbb{R}^{m+p}$. We obtain $N \times M$ as a framed immersed manifold in $\mathbb{R}^{n+m+p}$. This is the representative of $\alpha \circ \beta$.

Now we come to the proof of the first part of Claim 1*a)*; the proof of the second part is completely analogous. Let u be an element in $\pi^s(n) = \pi_{n+5}\left(\overline{X}_2, \overline{X}_1\right)$ and let

$$f : (M^{n+4}, \partial M) \to (\mathbb{R}^{n+5}_+, \mathbb{R}^{n+4})$$

be a prim $(\Sigma^{1_2}, \Sigma^{1_1})$-map that represents a cobordism class corresponding to u. The boundary of f is the prim fold map $\partial f = f|_{\partial M} : \partial M \to \mathbb{R}^{n+4}$. Let $\Sigma^{1,0}(\partial f)$ be the manifold of fold points of ∂f, and $\partial f\left(\Sigma^{1,0}(\partial f)\right)$ be its image. Each normal fiber of $\Sigma^{1,0}(\partial f)$ in ∂M is $\mathbb{R}^2$, and it is mapped by ∂f to the corresponding normal fiber $\mathbb{R}^3$ of $\partial f\left(\Sigma^{1,0}(\partial f)\right)$ in $\mathbb{R}^{n+4}$ by the Whitney umbrella map σ_1, hence $\partial f\left(\Sigma^{1,0}(\partial f)\right)$ has a natural framing (see Appendix 2). Then $d^1(u)$ is represented by this framed manifold $\partial f\left(\Sigma^{1,0}(\partial f)\right)$.

Next we construct another representative of $d^1(u)$ using the alternative description of d^1. Choose small tubular neighbourhoods T and $\widetilde{T}$ of $\Sigma^{1,1,0}(f)$ in M and of $f\left(\Sigma^{1,1,0}(f)\right)$ in $\mathbb{R}^{n+5}_+$, respectively. $\widetilde{T}$ is *immersed* into $\mathbb{R}^{n+5}_+$, it is a D^5-bundle over $\Sigma^{1,1,0}(f)$. Recall that f restricted to $\Sigma^{1,1,0}(f)$ is an immersion. For simplicity of the description we suppose that it is an embedding. We choose these tubular neighbourhoods T and $\widetilde{T}$ so that f maps T to $\widetilde{T}$ and ∂T to $\partial \widetilde{T}$. Now the map $f|_{\partial T} : \partial T \to \partial \widetilde{T}$ is a fold map. Its singularity is a framed manifold clearly representing $d^1(\sigma_2) \circ u$. By the two alternative descriptions of d^1 the framed manifold $\partial f(\Sigma^{1,0}(\partial f))$ (a representative of $d^1(u)$) represents the same framed cobordism class as the singularity of the fold map $f|_{\partial T} : \partial T \to \partial \widetilde{T}$. Hence $d^1(u) = d^1(\sigma_2) \circ u$.

The proof of $b)$ is very similar. As before, for simplicity of the description of d^2 we suppose that $f|_{\Sigma^{1,1,0}(f)}$ is an embedding rather than an immersion. Let T and $\widetilde{T}$ as above be the tubular neighbourhoods of $\Sigma^{1,1,0}(f)$ and $f\left(\Sigma^{1,1,0}(f)\right)$ respectively. Note that (as shown in Appendix 2) $T = \Sigma^{1,1,0}(f) \times D^4$, $\widetilde{T} = f\left(\Sigma^{1,1,0}(f)\right) \times D^5$ and the map $f|_T : (T, \partial T) \to (\widetilde{T}, \partial \widetilde{T})$ is the product map $\left(f|_{\Sigma^{1,1,0}(f)}\right) \times \sigma_2$.

Recall that in §3.2 we defined a map $\sigma_2^* : D^4 \underset{L}{\cup} W^4 \to D_2^5$ such that $\partial \sigma_2^*$ is an immersion $V^3 \to S^4 = \partial D_2^5$ that represents $d^2(\sigma_2)$. Now we define a map

$$f^* = f|_{\Sigma^{1,1,0}(f)} \times \sigma_2^* : \Sigma^{1,1,0}(f) \times \left(D^4 \underset{L}{\cup} W^4\right) \to f\left(\Sigma^{1,1,0}(f)\right) \times D_2^5.$$

We will denote by T_2 the source manifold of f^* and by $\widetilde{T}_2$ the target manifold, which is an enlarged tubular neighbourhood of $f\left(\Sigma^{1,1,0}(f)\right)$ in $\mathbb{R}^{n+5}_+$. Then

$$\partial f^* : \partial T_2 = \Sigma^{1,1,0}(f) \times V^3 \to \partial \widetilde{T}_2 = f\left(\Sigma^{1,1,0}(f)\right) \times S^4 \times \{1\}$$

is an immersion that represents $d^2(\sigma_2) \circ u$.

We claim that this immersion can be extended to a proper, regular cusp map $\widehat{f^*}$ (of some compact $(n+4)$-manifold with boundary) into the entire $\mathbb{R}^{n+5}_+$ without changing the singular set. Indeed, the source manifold of ∂f^* has dimension $n+3$, hence the image of ∂f^* is an $(n+3)$-dimensional compact complex (denote it by K), and it can be covered by a small neighbourhood U in $\partial \widetilde{T}_2$ of K whose closure $\overline{U}$ is a compact manifold with non-empty boundary. By Theorem 2, there exists a deformation of $\overline{U}$ (equipped with the outward-pointing normal vector field) within $\mathbb{R}^{n+5}_+$ with time derivative nowhere tangent to the image of $\overline{U}$ that takes $\overline{U}$ into $\mathbb{R}^{n+4} = \partial \mathbb{R}^{n+5}_+$ (and the normal vector field into the outward-pointing vector field of $\mathbb{R}^{n+4}$). The trace of this deformation glued along ∂f^* to f^* gives an extension $\widehat{f^*}$ whose set of cusp points is the same as that of f^* and in particular represents u in $\mathcal{G}$.

This construction combined with Theorem 3 shows that ∂f^* and $\partial \widehat{f^*}$ are cobordant as framed immersions and therefore represent the same element in $\mathcal{G}$; the statement $b)$ follows since ∂f^* represents $d^2(\sigma_2) \circ u$ and $\partial \widehat{f^*}$ represents $d^2(u)$. $\qquad \square$

§5.3. **Calculation of the first page of the spectral sequence for prim maps.** Recall that

$$\overline{E}^1_{i,j} = \pi_{i+j+1}(\overline{X}_i, \overline{X}_{i-1})$$
$$= \pi_{i+j+1}(\Gamma S^{2i+1}) = \pi^s(j - i).$$

Hence on the diagonal $j = i$ we have $\pi^s(0) = \mathbb{Z}$ with generator ι_i in $\overline{E}^1_{i,i}$ represented by the map $\sigma_i : (D^{2i}, S^{2i-1}) \to (D^{2i+1}, S^{2i})$ that has an isolated Σ^{1_i} singularity at the origin. On the line $j = i + t$ we have $\pi^s(t)$.

	$i=0$	1	2
3	$\mathbb{Z}_{24} \xleftarrow{d^1_{1,3}} \mathbb{Z}_2$	$\mathbb{Z}_2$	
2	$\mathbb{Z}_2 \xleftarrow[\cong]{d^1_{1,2}} \mathbb{Z}_2 \xleftarrow[0]{d^1_{2,2}} \mathbb{Z}$		
1	$\mathbb{Z}_2 \xleftarrow{d^1_{1,1}} \mathbb{Z}$		
$j=0$	$\mathbb{Z}$		

The value of $d^1_{1,1}(\iota_1)$ is nothing else but $[\partial\sigma_1] = \eta \in \pi^s(1) = \mathbb{Z}_2$.

By Claim 1 we have $d^1_{1,2}(\eta) = d^1_{1,1}(\iota_1) \circ \eta = \eta \circ \eta \neq 0$ in $\pi^s(2)$ (here and later we refer the reader to [To, Chapter XIV] for the information that we need about the composition product). Hence $d^1_{1,2}$ is an isomorphism and it follows that $d^1_{2,2}$ is zero (since $d^1_{1,2} \circ d^1_{2,2} = 0$). In particular, we have obtained the following lemma:

Lemma 4. *The class $d^1(\iota_2)$, represented by the image $\sigma_2(\gamma)$ of the fold singularity curve on the boundary of the isolated cusp $\sigma_2 : \mathbb{R}^4 \to \mathbb{R}^5$, vanishes.*

In Appendix 1 we give an independent, elementary proof for this statement.

§5.4. **The second page $(\overline{E}^2_{i,j}, \overline{d}^2_{i,j})$ for prim maps.** The differential $d^1_{1,3} : \mathbb{Z}_2 \to \mathbb{Z}_{24}$ maps the generator $\eta \circ \eta$ of $\pi^s(2)$ to $d^1_{1,1}(\iota_1) \circ \eta \circ \eta = \eta \circ \eta \circ \eta$ and that is not zero ([To, Theorem 14.1]). Hence the group $\overline{E}^2_{0,3}$ is $\mathbb{Z}_{24}/\mathbb{Z}_2 = \mathbb{Z}_{12}$.

	$i=0$	1	2
3	$\mathbb{Z}_{12}$	0	$?$
2	0	0	$\mathbb{Z}$
1	0	$\mathbb{Z}$	
$j=0$	$\mathbb{Z}$		

(with the differential $d^2_{2,2}$ from position $(2,2)$ to $(0,3)$)

Now we compute the differential $d^2_{2,2} : \mathbb{Z} \to \mathbb{Z}_{12}$. Note that this is precisely the computation of the cobordism class of the framed immersion $\partial'\sigma_2$ of the 3-manifold V^3 considered in Example 2 in §3.2.

Lemma 5. $d^2_{2,2} : \mathbb{Z} \to \mathbb{Z}_{12}$ *maps the generator* ι_2 *of* $\overline{E}^2_{2,2} \cong \mathbb{Z}$ *into an element of order* 6.

Proof. $\overline{E}^1_{2,2} \cong \pi_5(\overline{X}_2, \overline{X}_1) = \pi_5(\Gamma S^5) = \pi^s_5(S^5) = \mathbb{Z}$. Since $d^1_{2,2} : E^1_{2.2} \to E^1_{1,2}$ is identically zero, $\overline{E}^2_{2,2} = \overline{E}^1_{2,2}$.

Consider the following commutative diagram with exact row and column:

$$\pi_4(\overline{X}_0)/\operatorname{im}\left(\pi_5(\overline{X}_1, \overline{X}_0) \xrightarrow{\partial} \pi_4(\overline{X}_0)\right)$$

$$\pi_5(\overline{X}_2) \xrightarrow{\varphi} \pi_5(\overline{X}_2, \overline{X}_1) \cong \mathbb{Z} \xrightarrow{} \pi_4(\overline{X}_1)$$

$$\pi_4(\overline{X}_1, \overline{X}_0)$$

with maps d^2 and d^1.

The generator ι_2 of $\pi_5(\overline{X}_2, \overline{X}_1) \cong \mathbb{Z}$ is represented by the cusp map $\sigma_2 : (D^4, L^3) \to (D^5, S^4)$ (using the notation of Example 2). Simple diagram chasing shows that the order of $d^2(\iota_2)$ is equal to the order of Coker φ. The latter is the minimal positive algebraic number of cusp points of prim cusp maps of oriented closed 4-manifolds into $\mathbb{R}^5$. Indeed, φ assigns to the class of a map $f : M^4 \to \mathbb{R}^5$ the algebraic number of its cusp points. This minimal number of cusps is known to be 6, see [Sz3, Theorem 4]. □

Applying Claim 1 *b*) we immediately get:

Corollary. *On the* 3*-torsion part, the differential* $d^2_{2,n+2}$ *acts as the homomorphism* α_1 *(as defined in* §2*) up to sign.*

§5.5. Computation of the cobordism group of prim fold maps of oriented n-manifolds to $\mathbb{R}^{n+1}$.

Theorem 6. *a)* $\operatorname{Prim}^{SO}\Sigma^{1,0}(n+1) \underset{\mathcal{C}_2}{\cong} \pi^s(n) \oplus \pi^s(n-2)$.

b) $\operatorname{Prim}^{SO}\Sigma^{1,1,0}(n+1) \underset{\mathcal{C}_{\{2,3\}}}{\cong} \pi^s(n) \oplus \pi^s(n-2) \oplus \pi^s(n-4)$.

Proof. We have seen that the spectral sequences computing

$$\operatorname{Prim}^{SO}\Sigma^{1,0}(n+1) \quad \text{and} \quad \operatorname{Prim}^{SO}\Sigma^{1,1,0}(n+1)$$

degenerate modulo $\mathcal{C}_2$ and modulo $\mathcal{C}_{\{2,3\}}$ respectively, because $d^1_{1,n+1}$ is multiplication by the order 2 element η and $d^2_{2,n+2}$ is multiplication by an element of order 6.

The fact that the cobordism groups $\operatorname{Prim}^{SO}\Sigma^{1,0}(n+1)$ and $\operatorname{Prim}^{SO}\Sigma^{1,1,0}(n+1)$ are direct sums (modulo 2- and 3-primary torsion parts) can be shown in the same way as in [Sz5, Theorem B]. Namely, the homotopy exact sequence of the fibration $\overline{p}_1 : (\overline{X}_1, \overline{X}_0) \to \Gamma S^3$

$$\pi_{n+1}(\overline{X}_0) \to \pi_{n+1}(\overline{X}_1) \xrightarrow{(\overline{p}_1)_*} \pi_{n+1}(\Gamma S^3)$$

has a 2-splitting s, that is, there is a homomorphism $s : \pi_{n+1}(\Gamma S^3) \to \pi_{n+1}(\overline{X}_1)$ such that $(\overline{p}_1)_* \circ s$ is the multiplication by 2.

The construction of s goes as follows: choose an immersion $S^2 \looparrowright \mathbb{R}^4$ with normal Euler number 2. Then its generic projection to $\mathbb{R}^3$ will be a map $\psi : S^2 \to \mathbb{R}^3$ with finitely many Whitney umbrella points that inherit a sign from the orientation of the kernel bundle, and the algebraic number of these points will be 2 (see [SS, Proposition 2.5.]). Now choosing any framed immersion $q : Q^{n-2} \looparrowright \mathbb{R}^{n+1}$ that represents an element $[q]$ in $\pi_{n+1}(\Gamma S^3) \cong \pi^s(n-2)$, the framing of Q defines a prim fold map $Q \times S^2 \overset{id \times \psi}{\to} Q \times \mathbb{R}^3 \looparrowright \mathbb{R}^{n+1}$. Its class will be $s([q])$. The existence of the 2-splitting map s implies part a).

The existence of an analogous 6-splitting of the homotopy exact sequence of the fibration $\overline{p}_2 : (\overline{X}_2, \overline{X}_1) \to \Gamma S^5$ is shown in [Sz5, Lemma 4]. It shows that

$$\mathrm{Prim}^{SO}\Sigma^{1,1,0}(n+1) \underset{\mathcal{C}_{\{2,3\}}}{\cong} \mathrm{Prim}^{SO}\Sigma^{1,0}(n+1) \oplus \pi^s(n-4)$$

and together with part a) proves part b). $\qquad\square$

Theorem 7. *Let $\eta_n : \pi^s(n) \to \pi^s(n+1)$ be the homomorphism $x \mapsto \eta \circ x$. Then the following sequence is exact:*

$$0 \to \mathrm{Coker}\,\eta_{n-1} \to \mathrm{Prim}^{SO}\Sigma^{1,0}(n+1) \to \ker\eta_{n-2} \to 0$$

Proof. In the homotopy exact sequence of the pair $(\overline{X}_1, \overline{X}_0)$ the boundary homomorphism is the differential $d^1_{1,n+1}$, which by Claim 1 a) is just the corresponding homomorphism η_n. The statement follows immediately. $\qquad\square$

Recall that for any abelian group G, we denote by $(G)_3$ its 3-primary part. While

$$\mathrm{Prim}^{SO}\Sigma^{1,1,0}(n+1)$$

is computed by Theorem 6 modulo its 2- and 3-primary parts, we can also compute the 3-primary part (up to a group extension).

Theorem 8. *The 3-primary part of $\mathrm{Prim}^{SO}\Sigma^{1,1,0}(n+1)$ fits into the short exact sequence*

$$0 \to (\mathrm{Coker}\,(\alpha_1 : \pi^s(n-3) \to \pi^s(n)))_3 \oplus (\pi^s(n-2))_3 \to$$
$$\to \left(\mathrm{Prim}^{SO}\Sigma^{1,1,0}(n+1)\right)_3 \to (\ker(\alpha_1 : \pi^s(n-4) \to \pi^s(n-1)))_3 \to 0$$

Proof. The spectral sequence $\overline{E}^r_{i,j}$ converges to $\mathrm{Prim}^{SO}\Sigma^{1,1,0}(n+1)$ and stabilizes at page 3. Recall that on the 3-primary part d^2 can be identified up to sign with the homomorphism α_1 (Corollary of Lemma 5). Hence the 3-primary parts $\left(\overline{E}^3_{i,j}\right)_3$ of the groups $\overline{E}^3_{i,j} \cong \overline{E}^\infty_{i,j}$ are the following:

(*)
$$\left(\overline{E}^3_{0,j}\right)_3 = (\mathrm{Coker}\,(\alpha_1 : \pi^s(j-3) \to \pi^s(j)))_3$$
$$\left(\overline{E}^3_{1,j}\right)_3 = (\pi^s(j-1))_3$$
$$\left(\overline{E}^3_{2,j}\right)_3 = (\ker(\alpha_1 : \pi^s(j-2) \to \pi^s(j+1)))_3$$

By general properties of spectral sequences it holds that if we define the groups

$$F_{2,n} = \mathrm{Prim}^{SO}\Sigma^{1,1,0}(n+1) = \pi_{n+1}(\overline{X}_2)$$
$$F_{1,n} = \mathrm{im}\left(\mathrm{Prim}^{SO}\Sigma^{1,0}(n+1) \to \mathrm{Prim}^{SO}\Sigma^{1,1,0}(n+1)\right) = \mathrm{im}\left(\pi_{n+1}(\overline{X}_1) \to \pi_{n+1}(\overline{X}_2)\right)$$
$$F_{0,n} = \mathrm{im}\left(\pi^s(n) \to \mathrm{Prim}^{SO}\Sigma^{1,1,0}(n+1)\right) = \mathrm{im}\left(\pi_{n+1}(\overline{X}_0) \to \pi_{n+1}(\overline{X}_2)\right)$$

then

$$F_{2,n}/F_{1,n} = \overline{E}_{2,n-2}^{\infty}$$
$$F_{1,n}/F_{0,n} = \overline{E}_{1,n-1}^{\infty}$$
$$F_{0,n} = \overline{E}_{0,n}^{\infty}$$

We will show that the exact sequence

($\ddagger$) $$0 \to (F_{0,n})_3 \to (F_{1,n})_3 \to (F_{1,n}/F_{0,n})_3 \to 0$$

splits and hence $(F_{1,n})_3 \cong (F_{0,n})_3 \oplus (F_{1,n}/F_{0,n})_3$. Then the exact sequence

$$0 \to (F_{1,n})_3 \to (F_{2,n})_3 \to (F_{2,n}/F_{1,n})_3 \to 0$$

can be written as

$$0 \to (F_{0,n})_3 \oplus (F_{1,n}/F_{0,n})_3 \to (F_{2,n})_3 \to (F_{2,n}/F_{1,n})_3 \to 0,$$

and substituting (*) gives us the statement of Theorem 8.

It remains to show that ($\ddagger$) splits. Consider the following commutative diagram:

Consider the composition map $F_{1,n} \rightarrowtail \pi_{n+1}(\overline{X}_2) \xrightarrow{r} \pi_{n+1}(\overline{X}_2, \overline{X}_0)$. Its kernel is the intersection $\ker r \cap F_{1,n}$; but $\ker r$ is the image of $\pi_{n+1}(\overline{X}_0)$, which is $F_{0,n}$. Hence the map r defines uniquely a map $j : F_{1,n}/F_{0,n} \to \pi_{n+1}(\overline{X}_2, \overline{X}_0)$. Its image $\mathrm{im}\, j$ is a subset of $\mathrm{im}\, i$ due to the commutativity of the right-hand square.

By Claim 1, in the (exact) rightmost column the map ∂ (which can be identified with $d_{2,n-1}^1$) acts on $\pi_{n+2}(\overline{X}_2, \overline{X}_1) \cong \pi_{n+2}(\Gamma S^5)$ by composition from the left by $\partial[\sigma_2]$, which is zero. Hence the map i is injective. Consequently, the map j can be lifted to a map $\hat{j} : F_{1,n}/F_{0,n} \to \pi_{n+1}(\overline{X}_1, \overline{X}_0)$ and composing it with the 2-splitting s gives us a map

$$\hat{s} = s \circ \hat{j} : F_{1,n}/F_{0,n} \to \pi_{n+1}(\overline{X}_1)$$

such that $pr \circ \hat{s}$ is a 2-splitting of the short exact sequence $0 \to F_{0,n} \to F_{1,n} \to F_{1,n}/F_{0,n} \to 0$. This proves that on the level of 3-primary parts this extension is trivial, as claimed.

$\square$

§5.6. **The spectral sequence for arbitrary (not necessarily prim) cusp maps.** There are classifying spaces for the cobordisms of codimension 1 cooriented arbitrary (not necessarily prim) Σ^{1_i}-maps as well. We denote by $X_i = X\Sigma^{1_i}$ the classifying space of such Σ^{1_i}-maps with the convention that $X_{-1} = *$. Here we will mostly be interested in X_0, X_1 and X_2. The filtration $X_{-1} \subset X_0 \subset X_1 \subset X_2$ gives again a spectral sequence with $E_{i,j}^1 = \pi_{i+j+1}(X_i, X_{i-1})$ for $i = 0, 1, 2$, $j = 0, 1, \ldots$. Analogously to the fibrations $\overline{p}_i$ we have fibrations

- $p_1 : X_1 \to \Gamma T(2\varepsilon^1 \oplus \gamma^1) = \Gamma S^2 \mathbb{R}P^\infty$ with fibre X_0
- $p_2 : X_2 \to \Gamma T(3\varepsilon^1 \oplus 2\gamma^1) = \Gamma S^3(\mathbb{R}P^\infty/\mathbb{R}P^1)$ with fibre X_1 (for the identification of $T(2\gamma^1)$ and $\mathbb{R}P^\infty/\mathbb{R}P^1$ see e.g. [Hu, Example 1.7, ch. 15]).

Here γ^1 and ε^1 are the canonical and the trivial line bundles over $\mathbb{R}P^\infty$, respectively, and T stands for Thom space (recall that $\Gamma = \Omega^\infty S^\infty$). Note that $X_0 = \overline{X}_0 = \Gamma S^1$.

Observe that the base spaces of p_i are different from those of $\overline{p}_i$. This change is due to the fact that while the normal bundles of the singularity strata for a prim map are trivial and even canonically trivialized (see Appendix 2), for arbitrary cooriented codimension 1 Morin maps they are direct sums of not necessarily trivial line bundles (see [RSz, Theorem 6] and the definition of G^{SO} that precedes it). The bundles $2\varepsilon^1 \oplus \gamma^1$ and $3\varepsilon^1 \oplus 2\gamma^1$ are the universal normal bundles in the target of the fold and cusp strata respectively.

Proposition 9. *a)* $E^1_{1,j} \in \mathcal{C}_2$.

b) $E^1_{i,j} \cong \overline{E}^1_{i,j}$ *modulo* $\mathcal{C}_2$ *for* $i = 0, 2$.

Proof. a) Since $H_*(\mathbb{R}P^\infty; \mathbb{Z}_p) = 0$ for p odd, the Serre-Hurewicz theorem implies that $\pi^s_*(\mathbb{R}P^\infty) \in \mathcal{C}_2$ and therefore

$$E^1_{1,j} \cong \pi_{j+2}(\Gamma S^2 \mathbb{R}P^\infty) = \pi^s_j(\mathbb{R}P^\infty) \in \mathcal{C}_2.$$

b) Since the inclusion $S^2 = \mathbb{R}P^2/\mathbb{R}P^1 \hookrightarrow \mathbb{R}P^\infty/\mathbb{R}P^1$ induces isomorphism of $\mathbb{Z}_p$-homologies (the groups $H_*(\mathbb{R}P^\infty/\mathbb{R}P^2; \mathbb{Z}_p)$ all vanish) for p odd, we have

$$E^1_{2,j} \cong \pi_{j+3}(\Gamma S^3(\mathbb{R}P^\infty/\mathbb{R}P^1)) \underset{\mathcal{C}_2}{\cong} \pi_{j+3}(\Gamma S^5) \cong \overline{E}^1_{2,j}.$$

We also have $X_0 = \overline{X}_0$ and consequently

$$E^1_{0,j} \cong \overline{E}^1_{0,j}.$$

$\square$

§5.7. Computation of the cobordism group of (arbitrary) cusp maps.

Proof of Theorem 1. The natural forgetting map $\overline{E}^1_{i,j} \to E^1_{i,j}$ induces a $\mathcal{C}_2$-isomorphism for $i = 0, 2$, and $E^1_{1,j} \in \mathcal{C}_2$. Since the d^1 differential is trivial modulo $\mathcal{C}_2$ for both spectral sequences, the map $\overline{E}^2_{i,j} \to E^2_{i,j}$ is a $\mathcal{C}_2$-isomorphism for $i = 0, 2$.

Hence the differential d^2 restricted to the 3-primary part can be identified in the two spectral sequences, and we obtain that $\left(E^\infty_{i,j}\right)_3 = \left(E^3_{i,j}\right)_3 \cong \left(\overline{E}^3_{i,j}\right)_3$ for $i = 0, 2$ (but not for $i = 1$). The statement of the theorem follows analogously to Theorem 8. $\square$

APPENDIX 1: AN ELEMENTARY PROOF OF LEMMA 4

In this Appendix we give an elementary and independent proof of the fact that the curve of folds on the boundary sphere of an isolated cusp map $\sigma_2 : \mathbb{R}^4 \to \mathbb{R}^5$ with the natural framing represents the trivial element in $\pi^s(1) = \mathbb{Z}_2$.

$\sigma_2 : \mathbb{R}^4 \to \mathbb{R}^5$, $\sigma_2(t_1, t_2, t_3, x) = (t_1, t_2, t_3, t_1 x + t_2 x^2, t_3 x + x^3)$

$$\mathrm{d}\sigma_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ x & x^2 & 0 & t_1 + 2xt_2 \\ 0 & 0 & x & t_3 + 3x^2 \end{bmatrix}$$

The set of singular points of σ_2 is $\Sigma = \{(-2xt_2, t_2, -3x^2, x) \mid t_2, x \in \mathbb{R}\}$, its image is

$$\tilde{\Sigma} = \sigma_2(\Sigma) = \{(-2xt_2, t_2, -3x^2, -t_2x^2, -2x^3) \mid t_2, x \in \mathbb{R}\}.$$

For a point $p \in \mathbb{R}^4 \setminus \Sigma$ the vector

$$n(p) = (-x(t_3 + 3x^2), -x^2(t_3 + 3x^2), x(t_1 + 2xt_2), t_3 + 3x^2, -(t_1 + 2xt_2))$$

is non-zero and orthogonal to the columns of $d\sigma_2$, so it is a normal vector of the immersed hypersurface $\sigma_2(\mathbb{R}^4 \setminus \Sigma) \subset \mathbb{R}^5$ at $\sigma_2(p)$.

$\tilde{\Sigma} \setminus \{0\}$ is an embedded surface in $\mathbb{R}^5$, and it has a canonical framing:

Through each point $p = (-2xt_2, t_2, -3x^2, x) \in \Sigma \setminus \{0\}$ we can define a curve

$$p_\varepsilon = (-2xt_2, t_2, -3x^2 - \varepsilon^2, x + \varepsilon)$$

such that $p_0 = p$, $\frac{\partial p_\varepsilon}{\partial \varepsilon}(0) = (0,0,0,1) \in \ker d\sigma_2$, and $\sigma_2(p_\varepsilon) = \sigma_2(p_{-\varepsilon}) = q_{\varepsilon^2}$, where

$$q_\delta = (-2xt_2, t_2, -3x^2 - \delta, -t_2(x^2 - \delta), -2x^3 + 2\delta x).$$

(Note that by taking this curve for each p we have defined an orientation of the kernel line bundle of $d\sigma_2$.)

The first vector of the framing is the tangent vector of the image curve q_δ:

$$v_1 = \frac{\partial q_\delta}{\partial \delta}(0) = (0, 0, -1, t_2, 2x).$$

Since $\sigma_2(p_\varepsilon) = \sigma_2(p_{-\varepsilon}) = q_{\varepsilon^2}$, in this point we have defined two normal vectors of $\sigma_2(\mathbb{R}^4 \setminus \Sigma)$, namely

$$n(p_{\pm\varepsilon}) = \pm\varepsilon(-6x^2 - 2\varepsilon^2, -6x^3 - 10x\varepsilon^2, 2xt_2, 6x, -2t_2) + \varepsilon^2(-8x, -14x^2 - 2\varepsilon^2, 2t_2, 2, 0).$$

The sum and the difference of these vectors are

$$n(p_\varepsilon) + n(p_{-\varepsilon}) = 2\varepsilon^2(-8x, -14x^2 - 2\varepsilon^2, 2t_2, 2, 0) \text{ and}$$
$$n(p_\varepsilon) - n(p_{-\varepsilon}) = 2\varepsilon(-6x^2 - 2\varepsilon^2, -6x^3 - 10x\varepsilon^2, 2xt_2, 6x, -2t_2).$$

The last two vectors of the framing are the limits of (the directions of) these vectors:

$$v_2 = (-3x^2, -3x^3, xt_2, 3x, -t_2)$$
$$v_3 = (-4x, -7x^2, t_2, 1, 0)$$

The following Claim implies that the framed curve $\sigma_2(\gamma)$ is null-cobordant (recall that $\gamma = \sigma_2^{-1}(S^4) \cap \Sigma$).

Claim 2. *There is a smooth embedding $F : D^2 = \{(t_2, x) \mid t_2^2 + x^2 \leq 1\} \to \mathbb{R}^5$ and a framing of $F(D^2)$ that extends $\sigma_2 \circ i$ and the canonical framing of $\tilde{\Sigma} \setminus \{0\}$ restricted to $\sigma_2 \circ i(S^1)$, where $i : S^1 = \{(t_2, x) \mid t_2^2 + x^2 = 1\} \to \Sigma$, $i(t_2, x) = (-2xt_2, t_2, -3x^2, x)$.*

Proof. We define such an F and a framing:

$$F(t_2, x) = (-2xt_2, t_2, -3x^2, -t_2x^2, 2x(t_2^2 - 1))$$
$$v_1 = (0, 0, -1, t_2, 2x)$$
$$v_2 = (3 - 3t_2^2 - 6x^2, -3x^3, xt_2, 3x, -t_2)$$
$$v_3 = (-4x, -7x^2, t_2, 1, 0)$$

These are smooth, and in the case $t_2^2 + x^2 = 1$ they coincide with the previously defined map and framing. It is easy to check that F is injective. We need to prove that the differential of

F is injective, and the vectors really form a framing, i.e. that the partial derivatives of F and v_1, v_2, v_3 are linearly independent. Equivalently, the following matrix should be non-singular:

$$M = \begin{bmatrix} -2x & 1 & 0 & -x^2 & 4xt_2 \\ t_2 & 0 & 3x & xt_2 & 1-t_2^2 \\ 0 & 0 & -1 & t_2 & 2x \\ 3-3t_2^2-6x^2 & -3x^3 & xt_2 & 3x & -t_2 \\ -4x & -7x^2 & t_2 & 1 & 0 \end{bmatrix}$$

$$\begin{aligned} \det M = {} & 180x^8 + 568x^6t_2^2 + 323x^4t_2^4 + 120x^6 - 197x^4t_2^2 + 8x^2t_2^4 + 3t_2^6 + \\ & 51x^4 - 12x^2t_2^2 - 2t_2^4 + 24x^2 - 2t_2^2 + 3 \\ = {} & 50(2x^2t_2^2 - x^2)^2 + 6(xt_2^2 - x)^2 + 2(t_2^3 - t_2)^2 + 2(t_2^2 - 1)^2 + \\ & 180x^8 + 568x^6t_2^2 + 123x^4t_2^4 + 120x^6 + 3x^4t_2^2 + 2x^2t_2^4 + \\ & t_2^6 + x^4 + 18x^2 + 1 \\ > {} & 0. \end{aligned}$$

Therefore M is always non-singular, and the proof is complete. $\qquad\square$

APPENDIX 2: THE NATURAL FRAMING ON THE IMAGE OF THE MANIFOLD FORMED BY THE Σ^{1r}-POINTS OF A COORIENTED PRIM MAP

Let us consider the map

$$\sigma_r : (\mathbb{R}^{2r}, 0) \to (\mathbb{R}^{2r+1}, 0),$$
$$(t_1, \ldots, t_{2r-1}, x) \mapsto (t_1, \ldots, t_{2r-1}, z_1, z_2)$$
$$z_1 = t_1 x + \cdots + t_r x^r$$
$$z_2 = t_{r+1} x + \cdots + t_{2r-1} x^{r-1} + x^{r+1}$$

the Morin normal form of a map with an isolated Σ^{1r}-point at 0. Denote by Δ_{r+1} the set of $(r+1)$-tuple points, i.e. the points $\{q \in \mathbb{R}^{2r+1} : \sigma_r^{-1}(q)$ consists of $r+1$ different points$\}$, and let $\overline{\Delta}$ be the closure of Δ_{r+1}.

Lemma 10. *a) The set $\overline{\Delta}$ is contained in the linear subspace $\mathcal{P}$ of $\mathbb{R}^{2r+1}$ of dimension r defined by the equations $z_1 = t_1 = \cdots = t_r = 0$.*
b) Identifying the r-tuple $(t_{r+1}, \ldots, t_{2r-1}, z_2)$ with the polynomial

$$-z_2 + t_{r+1}x + \cdots + t_{2r-1}x^{r-1} + x^{r+1},$$

the points of $\overline{\Delta}$ correspond precisely to the polynomials whose roots are all real.

Proof. a) Assume that σ_r maps the $r+1$ different points

$$p_j = (t_1^{(j)}, \ldots, t_{2r-1}^{(j)}, x_j), \quad j = 1, \ldots, r+1,$$

to the same point

$$q = (t_1, \ldots, t_{2r-1}, z_1, z_2).$$

Then necessarily $t_i^{(j)} = t_i$ for all $i = 1, \ldots, 2r-1$ and $j = 1, \ldots, r+1$. Since the points p_j are pairwise different, the values $x_1, \ldots, x_{r+1}$ are also pairwise different. But they are roots of the polynomial

$$-z_1 + t_1 x + \cdots + t_r x^r,$$

which has degree at most r, hence this polynomial must identically vanish. Consequently we have $z_1 = t_1 = \cdots = t_r = 0$ on Δ_{r+1} and therefore also on $\overline{\Delta}$ as claimed.

b) As in part a), in the preimage of Δ_{r+1} the coordinates $x_1, \ldots, x_{r+1}$ are the roots of the polynomial

$$-z_2 + t_{r+1}x + \cdots + t_{2r-1}x^{r-1} + x^{r+1}.$$

This immediately implies part b) since for any polynomial that corresponds to a point in $\overline{\Delta}$ any sum-preserving perturbation of its (real) roots that makes them distinct gives a polynomial that corresponds to a point in Δ_{r+1}. $\qquad \square$

In the next lemma we describe $\overline{\Delta}$ by identifying it with the orthant

$$\mathbb{R}^r_{\angle} = \{(u_1, \ldots, u_r) : u_j \geq 0 \text{ for all } j = 1, \ldots, r\}.$$

Moreover, we relate the natural stratification on $\mathcal{P}$ – where strata are sets of polynomials with the same multiplicities of the ordered roots – to that on $\mathbb{R}^r_{\angle}$, where the strata are the faces.

Lemma 11. $\overline{\Delta}$ *is homeomorphic to the orthant* $\mathbb{R}^r_{\angle}$*, and a homeomorphism* $\varphi : \mathbb{R}^r_{\angle} \to \overline{\Delta}$ *can be chosen to map the natural stratification of* $\mathbb{R}^r_{\angle}$ *bijectively onto that of* $\overline{\Delta}$ *inherited from* $\mathcal{P}$*, and to be a diffeomorphism from each stratum onto the corresponding stratum.*

Proof. For any point $(u_1, \ldots, u_r) \in \mathbb{R}^r$ define real numbers $x_1, \ldots, x_{r+1}$ in such a way that $x_{j+1} - x_j = u_j$ for all $j = 1, \ldots, r$ and $\sum_{j=1}^{r+1} x_j = 0$; denote the resulting (injective linear) map $\mathbf{u} = (u_1, \ldots, u_r) \mapsto \mathbf{x} = (x_1, \ldots, x_{r+1})$ by A. Using the identification of Lemma 10 b), define $\varphi : \mathbb{R}^r_{\angle} \to \overline{\Delta}$ by sending $(u_1, \ldots, u_r)$ to the polynomial that has roots $x_1, \ldots, x_{r+1}$. This map is clearly a homeomorphism, we only need to show that φ maps strata onto strata diffeomorphically.

For computational reasons, we extend φ to a map $\Phi : \mathbb{R}^{r+1} \to \mathbb{R}^{r+1}$ by mapping the point $(u_1, \ldots, u_r, \delta)$ to the coefficients of the monic polynomial with roots $x_1 + \frac{1}{r+1}\delta, \ldots, x_{r+1} + \frac{1}{r+1}\delta$. Then $\varphi(\mathbf{u})$ can be identified with $\Phi(\mathbf{u}, 0)$ and to prove our claim it is enough to show that Φ is a diffeomorphism from the strata of $\mathbb{R}^r_{\angle} \times \mathbb{R}$ onto their images. It is clear that Φ maps the strata of $\mathbb{R}^r_{\angle} \times \mathbb{R}$ homeomorphically onto their images, we only need to show that the rank of derivative at any point of an s-dimensional stratum is s.

To do this, we write Φ as the composition

$$\Phi = E \circ \tilde{A},$$

where the linear map $\tilde{A}$ is defined as

$$\tilde{A}(\mathbf{u}, \delta) = -A\mathbf{u} - \frac{1}{r+1}(\delta, \ldots, \delta)$$

and $E : \mathbb{R}^{r+1} \to \mathbb{R}^{r+1}$ is the map whose jth coordinate function is the jth elementary symmetric function:
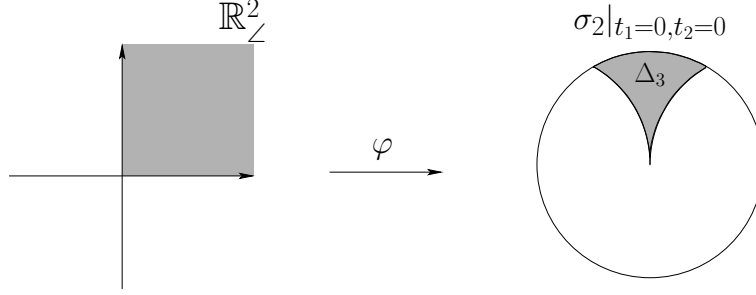
$$E(x_1, \ldots, x_{r+1}) = \big(e_1(x_1, \ldots, x_{r+1}), \ldots, e_{r+1}(x_1, \ldots, x_{r+1})\big).$$

The map $\tilde{A}$ is a linear isomorphism, composing with it does not change the rank of the differential, therefore it is enough to show that the differential of E has maximal rank when restricted to the strata of $\tilde{A}(\mathbb{R}^r_{\angle} \times \mathbb{R})$. Note that these strata have the form

$$S(a_1, \ldots, a_{s-1}) = \{(x_1, \ldots, x_{r+1}) : \quad x_1 = \cdots = x_{a_1-1} <$$
$$< x_{a_1} = \cdots = x_{a_2-1} <$$
$$< \cdots < x_{a_{s-1}} = \cdots = x_{r+1}\}$$

for some $s \geq 1$ and an index set $1 < a_1 < a_2 < \cdots < a_{s-1} \leq r+1$ (so that in particular $\dim S(a_1, \ldots, a_{s-1}) = s$).

The following claim is easily proved by induction on r:

FIGURE 1. The homeomorphism φ for $r = 2$.

Claim 3. *The Jacobi matrix of E at the point $\mathbf{x} = (x_1, \ldots, x_{r+1})$ is*

$$
J(\mathbf{x}) = \begin{bmatrix}
1 & \ldots & 1 & \ldots & 1 \\
e_1(x_2, \ldots, x_{r+1}) & \ldots & e_1(\ldots, x_{j-1}, x_{j+1}, \ldots) & \ldots & e_1(x_1, \ldots, x_r) \\
e_2(x_2, \ldots, x_{r+1}) & \ldots & e_2(\ldots, x_{j-1}, x_{j+1}, \ldots) & \ldots & e_2(x_1, \ldots, x_r) \\
\vdots & & \vdots & & \vdots \\
e_r(x_2, \ldots, x_{r+1}) & \ldots & e_r(\ldots, x_{j-1}, x_{j+1}, \ldots) & \ldots & e_r(x_1, \ldots, x_r)
\end{bmatrix}.
$$

Its determinant is $\prod_{1 \le i < j \le r+1}(x_i - x_j)$.

In particular, this matrix is nondegenerate if the x_j are pairwise different. To estimate the rank of the differential of E restricted to $S(a_1, \ldots, a_{s-1})$ at a point $\mathbf{x} \in S(a_1, \ldots, a_{s-1})$, notice that the columns of $J(\mathbf{x})$ with indices i and j coincide if $x_i = x_j$, therefore $\operatorname{rank} dE(\mathbf{x}) \le s$. On the other hand, the columns with indices $1, a_1, \ldots, a_{s-1}$ are linearly independent – the minor formed by the first s rows of these columns can be calculated to be nonzero in the same way as in Claim 3. Hence $\operatorname{rank} dE(\mathbf{x}) = s$ and we get that the kernel of $dE(\mathbf{x})$ is spanned by the vectors $\mathbf{u}_i - \mathbf{u}_j$ for those i and j for which $x_i = x_j$, where $\mathbf{u}_l$ denotes the lth unit coordinate vector. All these vectors are orthogonal to $S(a_1, \ldots, a_{s-1})$, consequently their span is also transverse to $S(a_1, \ldots, a_{s-1})$ and hence the restriction of E to $S(a_1, \ldots, a_{s-1})$ has full rank because its differential is the restriction of the differential dE to the tangent space of $S(a_1, \ldots, a_{s-1})$.   □

We want to show that $\overline{\Delta}$ has a (homotopically) canonical parallelization. Clearly any homeomorphism φ that is a diffeomorphism on the strata does give a parallelization induced from $\mathbb{R}^r_{\angle}$. But there are many possible choices of φ (even up to isotopy), and to obtain a parallelization along the whole $\overline{\Delta}$ that does not depend on the choice of local coordinates we need to show that there is a canonical (unique up to isotopy) such choice. For this purpose it is enough to show that there is a canonical ordering of the 1-dimensional edges of $\overline{\Delta}$, because if φ respects the ordering of the edges (and it can be chosen to do so), then it is isotopically unique. We shall consider σ_r as a prim map, namely the projection of the immersion $(t_1, \ldots, t_{2r-1}, x) \mapsto (\sigma_r(t_1, \ldots, t_{2r-1}, x), x) \in \mathbb{R}^{2r+2}$. This gives an ordering of the preimages of multiple points: for an $(r+1)$-tuple point with preimages $a_1, \ldots, a_{r+1}$ we assume that the indexing is such that $i < j$ if and only if $x(a_i) < x(a_j)$. If we choose a sequence $(q_n) \in \Delta$ converging to a point $\overline{q} \in \overline{\Delta} \setminus 0$ that lies on a 1-dimensional edge, then the σ_r-preimages of the $(r+1)$-tuple points $\sigma_r^{-1}(q_n) = \{a_1^{(n)}, \ldots, a_{r+1}^{(n)}\}$ degenerate in the limit in the sense that there is an integer s, $1 \le s \le r$ and there are two different points $\underline{a}$ and $\overline{a}$ with $x(\underline{a}) < x(\overline{a})$ such that $\lim_{n \to \infty} a_i^{(n)} = \underline{a}$ for $i = 1, \ldots, s$ while $\lim_{n \to \infty} a_i^{(n)} = \overline{a}$ for $i = s+1, \ldots, r+1$. Hence to each edge

we can associate an integer s, $1 \leq s \leq r$, and thus we obtain an ordering of the edges of $\overline{\Delta}$. We choose the map φ in such a way that it respects the ordering of the edges.

Note that any automorphism of σ_r as a prim map keeps the ordering of the edges of $\overline{\Delta}$. We call a pair (α, β) of germs of diffeomorphisms $\alpha : (\mathbb{R}^{2r}, 0) \to (\mathbb{R}^{2r}, 0)$ and $\beta : (\mathbb{R}^{2r+1}, 0) \to (\mathbb{R}^{2r+1}, 0)$ a prim automorphism of the prim germ $\sigma_r = \pi \circ \hat{\sigma}_r : (\mathbb{R}^{2r}, 0) \leftrightarrow (\mathbb{R}^{2r+2}, 0) \to (\mathbb{R}^{2r+1}, 0)$ if in addition to being an automorphism of σ_r (that is, $\sigma_r \circ \alpha = \beta \circ \sigma_r$) it preserves the selected orientation of $\ker d\sigma_r(0)$, i.e. the partial derivative of the $2r$th coordinate function of α with respect to x is positive (recall that $\frac{\partial \sigma_r}{\partial x}(0) = 0$).

Let now $g : M^n \leftrightarrow \mathbb{R}^{n+2}$ be an immersion, where M is a compact oriented n-dimensional manifold. Let f be the prim map $f = \pi \circ g$, with $\pi : \mathbb{R}^{n+2} \to \mathbb{R}^{n+1}$ the projection that omits the last coordinate x_{n+2} in $\mathbb{R}^{n+2}$. Suppose that f is a Σ^{1_r}-map, that is, it has no Σ^{1_j} points for $j > r$. The set of Σ^{1_r}-points of f will be denoted by $\Sigma^{1_r}(f)$, or Σ for brevity. It is a submanifold of M of dimension $n - 2r$. Its image $f(\Sigma^{1_r}(f))$ (denoted by $\tilde{\Sigma}$) is an immersed submanifold in $\mathbb{R}^{n+1}$ and has codimension $2r + 1$. We claim that $\tilde{\Sigma}$ has a natural normal framing, unique up to homotopy. To simplify the presentation of the proof, we will suppose that $\tilde{\Sigma}$ is embedded into $\mathbb{R}^{n+1}$.

There is an embedding $\theta : \Sigma \times (-\varepsilon, \varepsilon) \hookrightarrow \mathbb{R}^{n+1}$ (for some small positive number ε), unique up to isotopy, such that

- $\theta(x, 0) = f(x)$ for all $x \in \Sigma$,
- $\theta(x, t)$ is an $(r + 1)$-tuple point of f for all $0 < t < \varepsilon$.

Choose any $t^* \in (0, \varepsilon)$ and denote by Σ^* the image set $\theta(\Sigma \times \{t^*\})$. Clearly it is enough to give a canonical normal framing of Σ^*. Let $\Delta_{r+1}(f)$ denote the set of $(r + 1)$-tuple points of f that lie in a tubular neighbourhood of $\tilde{\Sigma}$ and let $\overline{\Delta_{r+1}(f)}$ be its relative closure. There is a fibration $\overline{\Delta_{r+1}(f)} \to \Sigma^*$ with fiber $\overline{\Delta}$.

The normal bundle of Σ^* in $\mathbb{R}^{n+1}$ is the sum of its normal bundle in $\Delta_{r+1}(f)$ and the restriction of the normal bundle of $\Delta_{r+1}(f)$ in $\mathbb{R}^{n+1}$ to Σ^*. The latter bundle is trivial, because it is the sum of the trivial normal line bundles of the $(r + 1)$ non-singular branches of f that intersect at the points of $\Delta_{r+1}(f)$ (and these branches have a canonical ordering by the last coordinate of g). The former bundle is trivial because there exists a canonical parallelization of $\overline{\Delta}$ that is (homotopically) invariant under the prim automorphisms of σ_r.

In general, when $\tilde{\Sigma}$ is not embedded, we need to consider only the preimages of multiple points that are close to the singular points in the source manifold to make the same argument work.

Alternatively, one can notice that the structure group of the bundle $\overline{\Delta_{r+1}(f)} \overset{\overline{\Delta}}{\to} \Sigma^*$ can be reduced to the maximal compact subgroup of the group of automorphisms of σ_r that also respect the orientation of the kernel of $d\sigma_r$, and that group is trivial [RSz]. Consequently the normal bundle of Σ^* in $\Delta_{r+1}(f)$ has a homotopically unique trivialization. We have seen above that the normal bundle of $\Delta_{r+1}(f)$ admits a canonical trivialization as well. Therefore we get a (homotopically unique) normal framing of Σ^*.

APPENDIX 3: PROOF OF THE COMPRESSION THEOREM

Throughout the proof we will assume that all the vector fields v_j have unit length and are pairwise orthogonal. To see that this can be achieved, assume that in the process of the proof an isotopy $\widetilde{\Phi}_t$ has already been constructed but it does not satisfy our assumption; we will now correct it to make it preserve the lengths and the pairwise orthogonality of the fields v_j. Let $A(t, p)$ for all $t \in \mathbb{R}$ and $p \in i(M)$ be the (unique) linear transformation of the linear space $N(t, p) = \langle d\widetilde{\Phi}_t(v_1(p)), \ldots, d\widetilde{\Phi}_t(v_k(p)) \rangle$ that sends the base $d\widetilde{\Phi}_t(v_1(p))$, ..., $d\widetilde{\Phi}_t(v_k(p))$ to the orthonormed base obtained from it by the Gram-Schmidt orthonormalization process. There

exists an orthonorming isotopy $\mathscr{O}_t$ of $\mathbb{R}^n \times \mathbb{R}^k$ that fixes $i(M)$ and the complement of a tubular neighbourhood of $i(M)$ pointwise such that for all $p \in i(M)$ it maps a small disk of the affine space $p + N(0, p)$ into the same affine space by the affine transformation $d\widetilde{\Phi}_t^{-1}|_{\Phi_t(p)} \circ A(t, p) \circ d\widetilde{\Phi}_t|_p$. Defining $\Phi_t = \widetilde{\Phi}_t \circ \mathscr{O}_t$ yields the claimed isotopy. It is easy to check that this construction can be performed in such a way as to respect the extra requirements of statements $b)$ and $d)$ of Theorem 2 – the details are mentioned at the corresponding points of the proof. Repeating this process every time a new isotopy is introduced ensures that the fields v_j stay pairwise orthogonal throughout the construction.

We first address the **case** $k = 1$, when we have an embedding $i : M \hookrightarrow \mathbb{R}^n \times \mathbb{R}^1$. For brevity the unit normal vector field v_1 will be denoted by v, the vector $\mathbf{e}_1$ will be denoted by $\uparrow$ and $\downarrow$ will be the vector $-\mathbf{e}_1$. We think of the direction of $\mathbf{0} \times \mathbb{R}^1$ as vertical, and those in $\mathbb{R}^n \times 0$ as horizontal. We may and will assume that v is perpendicular to $i(M)$ (but it may not remain so during the constructed isotopy).

Case $k = 1$, $n > m$.

Step 1: construct a diffeomorphism $\Psi \in \text{Diff}(\mathbb{R}^n \times \mathbb{R}^1)$ in order to obtain the reparametrization $i' = \Psi \circ i$ such that

- the image $d\Psi(v)$ of v under the differential of Ψ – we denote it by w – is orthogonal to $i'(M)$ and *grounded*, that is, w is never directed parallel to $\downarrow$;
- Ψ is the identity outside a compact subset of $\mathbb{R}^n \times \mathbb{R}^1$.

We perform the construction by taking the flow Ψ_t of a vector field $\boldsymbol{\theta}$ on $\mathbb{R}^n \times \mathbb{R}^1$ and setting $\Psi = \Psi_\varepsilon$ for a suitably chosen $\varepsilon > 0$; we start by constructing $\boldsymbol{\theta}$. Since $m < n$, the image of v in the unit sphere S^n has measure zero. Consequently, for almost every horizontal direction $\mathbf{h} \in S^{n-1} \hookrightarrow S^n$ the arc $\{\cos \lambda \cdot \downarrow + \sin \lambda \cdot \mathbf{h} : \lambda \in [0, \pi]\}$ intersects the image of v in a null set according to the 1-dimensional Lebesgue measure. Taking such an $\mathbf{h}$, let R denote the rotation of $\mathbb{R}^n \times \mathbb{R}^1$ that turns $\mathbf{h}$ into $\downarrow$ and is the identity on the orthogonal complement of the 2-dimensional linear subspace spanned by $\downarrow$ and $\mathbf{h}$. Let the vector field $\boldsymbol{\theta}$ be equal to the infinitesimal generator of R in a tubular neighbourhood V of $i(M)$ and extend it arbitrarily to a smooth vector field on $\mathbb{R}^n \times \mathbb{R}^1$ that vanishes outside a compact set. For all sufficiently small $\varepsilon > 0$ the time-ε flow of the vector field $\boldsymbol{\theta}$ will keep the points of $i(M)$ within the tubular neighbourhood V, hence the vector field v will be moved to $R^\varepsilon v$ and for almost all such ε the rotated vector field $R^\varepsilon v$ will miss $\downarrow$. Note that the diffeomorphism Ψ preserves the orthogonality of v to the tangent space of M (i.e. $w \perp i'(M)$).

Step 2: denote by b the inner bisector vector field (along $i'(M)$) of $\uparrow$ and w. Note that

a) b nowhere belongs to the tangent bundle $T(i'(M))$;

b) the scalar product $\langle b, \uparrow \rangle$ is everywhere positive.

In this step we construct an isotopy $\Xi_t \in \text{Diff}(\mathbb{R}^n \times \mathbb{R}^1)$, $t \in \mathbb{R}$ such that

(i) Ξ_0 is the identity;

(ii) for some $t^* > 0$ the image $d\Xi_{t^*}(b)$ of b under $d\Xi_{t^*}$ is $\uparrow$;

(iii) $\Xi_{t^*+t}(x) = \Xi_{t^*}(x) + t \cdot \uparrow$ for all $t > 0$ and for all $x \in \mathbb{R}^n \times \mathbb{R}^1$.

We proceed like in [RS], by extending the vector field b given on $i'(M)$ to a vector field $\mathbf{b}$ on $\mathbb{R}^n \times \mathbb{R}^1$ in such a way that $\mathbf{b}$ also satisfies $\langle \mathbf{b}, \uparrow \rangle > \delta > 0$ everywhere for some $\delta > 0$ and $\mathbf{b} = \uparrow$ holds outside a tubular neighbourhood V' of $i'(M)$. Let $\widetilde{\Xi}_t$ be the 1-parameter family of diffeomorphisms generated by the vector field $\mathbf{b}$; note that $\widetilde{\Xi}_t$ satisfies properties (i) and (ii), while property (iii) holds for $x \in i'(M)$ but not everywhere else. In order to define Ξ_t that also satisfies property (iii) we choose a function $\varphi : [0, \infty) \to [0, 1]$ such that

- $\varphi(t) = 1$ for all t for which $\widetilde{\Xi}_t(i'(M)) \cap V' \neq \emptyset$, and

- $\varphi(t) = 0$ for all $t \geq t^*$ for some $t^* > 0$.

We define Ξ_t to be the 1-parameter family of diffeomorphisms generated by the vector field $\hat{\mathbf{b}}_t = \varphi(t) \cdot \mathbf{b} + (1 - \varphi(t)) \cdot \uparrow$.

Since in a neighbourhood of $\widetilde{\widetilde{\Xi}}_t(i'(M))$ the vector fields $\mathbf{b}$ and $\hat{\mathbf{b}}_t$ coincide, so do the flows $\widetilde{\widetilde{\Xi}}_t$ and Ξ_t in a neighbourhood of $i'(M)$. This ensures that Ξ_t still satisfies properties (i) and (ii), and property (iii) holds by the definition of $\hat{\mathbf{b}}_t$.

Step 3: construct another isotopy $\Theta_t \in \mathrm{Diff}(\mathbb{R}^n \times \mathbb{R}^1)$, $t \in [0, 1]$ (with Θ_0 the identity) that keeps $i'(M)$ fixed pointwise and turns the vector w into b. First, we set w_τ, $\tau \in [0, 1]$ to be the vector field on $i'(M)$ obtained by smoothly rotating $w = w_0$ into $b = w_1$ along the shortest arc connecting them, staying constant in neighbourhoods of $\tau = 0$ and of $\tau = 1$, respectively. Since w is orthogonal to $i'(M)$ and b forms an acute angle with w, the vector field w_τ is not tangent to $i'(M)$ for any $\tau \in [0, 1]$. Consequently the map

$$I_\tau : M \times [0, 1] \to \mathbb{R}^{n+1}$$

$$(p, s) \mapsto i'(p) + \varepsilon_\tau \cdot s \cdot w_\tau \left(i'(p)\right)$$

is an embedding for every τ for some sufficiently small $\varepsilon_\tau > 0$.

By compactness of the interval $[0, 1]$ there is a common $\varepsilon > 0$ such that I_τ is an embedding for all τ with the choice $\varepsilon_\tau = \varepsilon$. Then I_τ is an isotopy of embeddings of the cylinder $M \times [0, 1]$ as τ changes from 0 to 1. This isotopy can be extended to an isotopy of $\mathbb{R}^n \times \mathbb{R}^1$, which we denote by Θ_t; since w_τ stayed constant in a neighbourhood of $\tau = 1$, we may assume that Θ_t also stays constant in a neighbourhood of $t = 1$. The isotopy Θ_t can then be extended to all times $t \in [0, \infty)$ by putting $\Theta_t = \Theta_1$ for $t > 1$.

Step 4: we compose the isotopies constructed in the previous steps:

$$\Phi_t = \Psi^{-1} \circ \Xi_t \circ \Theta_t \circ \Psi.$$

We claim that for t sufficiently big this isotopy will turn the vector field v into the constant vector field $\uparrow$. Indeed, Ψ sends v to w; for t sufficiently big Θ_t sends w into b; and Ξ_t sends b into $\uparrow$. Additionally, $\Xi_t \circ \Theta_t|_{i'(M)} = \Xi_t|_{i'(M)}$ moves the image of M at a constant nonzero velocity for $t > t^*$. In finite time it leaves the compact set where Ψ differs from the identity map. Therefore Ψ^{-1} is the identity on the image of $\Xi_t \circ \Theta_t \circ \Psi \circ i$ for t big enough. That is, the image of v under Φ_t is eventually $\uparrow$.

We need to show that the map $(p, t) \mapsto \Phi_t(i(p))$ is an immersion of $M \times [0, \infty)$ into $\mathbb{R}^n \times \mathbb{R}^1$. Let $p \in M$ be an arbitrary point and consider the image of the ray $p \times [0, \infty)$: it is the curve $\Psi^{-1}\left(\Xi_t\left(\Theta_t\left(\Psi\left(i(p)\right)\right)\right)\right) = \Psi^{-1}\left(\Xi_t\left(\Theta_t\left(i'(p)\right)\right)\right)$. The isotopy Θ_t keeps $i'(p)$ fixed, and the curve $\Xi_t(i'(p))$ $(= \widetilde{\widetilde{\Xi}}_t(i'(p)))$ is the trajectory of the point $i'(p)$ under the (time-independent) flow of the vector field $\mathbf{b}$. The derivative of this trajectory at the starting point $i'(p)$ is $b(p)$, a vector linearly independent of $T_{i'(p)}i'(M)$, hence the same independence condition holds for the images under Ξ_t at any time $t > 0$. A final composition with the diffeomorphism Ψ^{-1} keeps the condition intact.

Case $k = 1$, $n = m$. Since M has no closed components, there is an $(n-1)$-dimensional compact subcomplex K in M such that for any neighbourhood U of K there is an isotopy α_τ^U, $\tau \in [0, 1]$ of M that deforms M into U, i.e. $\alpha_0^U = id_M$ and $\alpha_1^U(M) \subset U$. The neighbourhood U will be chosen later and α_τ will denote the isotopy corresponding to that choice of U.

We consider the compressing procedure (turning v to $\uparrow$) of the case $n > m$ on the subcomplex K. The result is an isotopy $\hat{\Phi}_t$ such that for some t^* and for all $t \geq t^*$

- $d\hat{\Phi}_t(v) = \uparrow$ holds on K;
- $\hat{\Phi}_t(p) = \hat{\Phi}_{t^*}(p) + (t - t^*) \cdot \uparrow$ holds for all $p \in K$;

- for a neighbourhood $\mathcal{N}$ of K in M the restriction of the isotopy to the image $i(\mathcal{N})$ is an immersion of the infinite cylinder $\mathcal{N} \times [0, \infty)$.

Since on K the image $d\hat{\Phi}_{t^*}(v) = \uparrow$ does not belong to the tangent space $T\hat{\Phi}_{t^*}(i(M))$, there is neighbourhood $U \subset \mathcal{N}$ of K in M such that in the closure of U the shortest arc connecting the image $d\hat{\Phi}_{t^*}(v|_{i(\overline{U})})$ with $\uparrow$ does not intersect $T\hat{\Phi}_{t^*}(i(M))$. This neighbourhood U will be the one defining α_τ. We also consider the following extension $\overline{\alpha}_\tau$ of α_τ to the cylinder $M \times [0, 1]$ in the ambient space $\mathbb{R}^n \times \mathbb{R}^1$:

$$\overline{\alpha}_\tau(p, s) = i(\alpha_\tau(p)) + \varepsilon \cdot s \cdot v(i(\alpha_\tau(p))) \quad \text{for } p \in M, \ s \in [0, 1]$$

for an $\varepsilon > 0$ small enough so that $\overline{\alpha}_0$ (and consequently every $\overline{\alpha}_\tau$) is an embedding, and extend it to an isotopy $\widetilde{\alpha}_\tau$ of $\mathbb{R}^n \times \mathbb{R}^1$. It can be supposed that $\widetilde{\alpha}_\tau$ is the identity outside a compact set $B \subset \mathbb{R}^n \times \mathbb{R}^1$. By increasing t^* if needed we may also assume that for all $t \geq t^*$ we have $\hat{\Phi}_t(i(M)) \cap B = \emptyset$.

In the neighbourhood U we can again repeat the argument of Step 3 to obtain an isotopy Λ_t, $t \geq 0$ of the identity of $\mathbb{R}^n \times \mathbb{R}^1$ that fixes $i(M)$ pointwise, turns v into $d\hat{\Phi}_{t^*}^{-1}(\uparrow)|_{i(\overline{U})}$ and stays constant $\Lambda_t = \Lambda_{t^*}$ for $t \geq t^*$. We can now define the isotopy

$$\Phi_t = \widetilde{\alpha}_1^{-1} \circ \hat{\Phi}_t \circ \Lambda_t \circ \widetilde{\alpha}_1.$$

This isotopy will send v to $\uparrow$ for $t \geq t^*$ (for $t \geq t^*$ the image $\hat{\Phi}_t(i(M))$ is disjoint from B, hence $\widetilde{\alpha}_1^{-1}$ is the identity near $\hat{\Phi}_t(i(M))$), and on $i(M)$ we have that $\Phi_t = \widetilde{\alpha}_1^{-1} \circ \hat{\Phi}_t \circ \Lambda_t \widetilde{\alpha}_1 = \widetilde{\alpha}_1^{-1} \circ \hat{\Phi}_t \widetilde{\alpha}_1$ is a reparametrization of $\hat{\Phi}_t|_{i(U)}$, hence the restriction of the obtained isotopy to $i(M)$ is an immersion of the cylinder $M \times [0, \infty)$ as required.

Case $k > 1$. Here we perform the construction in two steps, first straightening out the vector field v_1 as detailed above both when $n = m$ and when $n > m$, and then utilizing different corollaries of the original Rourke-Sanderson compression theorem appropriate for the cases $n = m$ and $n > m$ to straighten the rest of the normal vector fields. In the first step we obtain an isotopy $\Phi_t^{(0)}$ such that for some t^* and all $t > t^*$ we have $\Phi_t^{(0)}(p) = \Phi_{t^*}^{(0)}(p) + (t - t^*)\mathbf{e}_1$ for all $p \in \mathbb{R}^n \times \mathbb{R}^k$ and $d\Phi_t^{(0)}(v_1) = \mathbf{e}_1$, while the restriction of the isotopy to $i(M)$ is an immersion of the cylinder $M \times [0, \infty)$. Then we apply [RS, Multi-compression Theorem 4.5] when $n > m$ and [RS, Addendum (v) to Multi-compression Theorem 4.5] when $n = m$ to the composition $\Phi_{t^*}^{(0)} \circ i$ and obtain an isotopy $\Phi_t^{(1)}$, $t \in [0, t_1]$, such that it straightens the images of all the vector fields $v_1, \ldots, v_k$. By construction the isotopy $\Phi_t^{(1)}|_{\Phi_{t^*}^{(0)}(i(M))}$ is a lift of a C^0-small regular homotopy $\check{\Phi}_t$ of the projection of $\Phi_{t^*}^{(0)}(i(M))$ parallel to $\mathbf{e}_1$, hence we can choose this lift to be locally of the form $\check{\Phi}_t \times id_{\langle \mathbf{e}_1 \rangle}$ and consequently preserve the first coordinate function as well as the v_1 direction.

By reparametrizing time we may assume that $\Phi_t^{(1)}$ can be extended smoothly to all $t \in \mathbb{R}$ as the identity for all $t < 0$ and as a time-independent diffeomorphism for all $t > t^*$, and by multiplying the time-dependent generating vector field $\frac{\partial \Phi_t^{(1)}}{\partial t}$ by an appropriate bump function we may also assume that $\Phi_t^{(1)}$ only moves points within a compact subset of $\mathbb{R}^n \times \mathbb{R}^k$.

We define

$$(\diamondsuit) \qquad \Phi_t(p) = \begin{cases} \Phi_t^{(0)}(p) & \text{if } t \leq t^*, \\ \Phi_{t-t^*}^{(1)}(\Phi_{t^*}^{(0)}(p)) + (t - t^*)\mathbf{e}_1 & \text{if } t \geq t^*. \end{cases}$$

This is an isotopy that straightens all the vector fields $v_1, \ldots, v_k$, we only need to check that its restriction to $i(M)$ is an immersion of the cylinder $M \times [0, \infty)$. For times $t \leq t^*$ this is already

proven above in the $k = 1$ case. For $t \geq t^*$ and all $p \in i(M)$ we have

$$\frac{\partial \Phi_t(p)}{\partial t} = \frac{\partial \Phi^{(1)}_{t-t^*}(\Phi^{(0)}_{t^*}(p))}{\partial t} + \mathbf{e}_1,$$

and this is not in the tangent space of $\Phi_t(i(M))$ exactly if the image of $\frac{\partial \Phi^{(1)}_{t-t^*}(\Phi^{(0)}_{t^*}(p))}{\partial t}$ avoids $-\mathbf{e}_1 + d\Phi_t(di(TM))$. This latter condition can be achieved by a linear reparametrization of time in $\Phi^{(1)}_t$: we can slow down $\Phi^{(1)}_t$ by a sufficiently small factor $\varepsilon > 0$ so that the length of the derivative $\frac{\partial \Phi^{(1)}_{\varepsilon t}}{\partial t}$ becomes always less than the minimum of the distance between $\mathbf{e}_1$ and $\operatorname{im} d\Phi_t$ on $i(M)$ – this latter is positive since the projection of $\Phi^{(1)}_t$ parallel to the $\mathbf{e}_1$ direction is a regular homotopy of the projection of $\Phi^{(0)}_{t^*}(i(M))$.

For part b) of Theorem 2 we proceed in the same way: first we straighten the vector field $v_{\hat{\jmath}}$ for some $\hat{\jmath} \notin I$ (if $I = \{1, \dots, k\}$, then we are already done). Next we apply [RS, Addendum (vi) to Multi-compression Theorem 4.5] to the projection of $\Phi^{(0)}_{t^*}(i(M))$ parallel to the linear span $\langle \mathbf{e}_h : h \in I \cup \{\hat{\jmath}\} \rangle$ and obtain a regular homotopy of this projection. We choose an isotopy lift $\Phi^{(1)}_t$ of the obtained regular homotopy that keeps coordinates with indices in $I \cup \{\hat{\jmath}\}$ fixed and combine it with the constant speed shift in the $\mathbf{e}_{\hat{\jmath}}$ direction as given by ($\diamondsuit$) (with $\mathbf{e}_{\hat{\jmath}}$ substituted for $\mathbf{e}_1$). This yields an isotopy Φ_t that straightens all the vector fields $v_1, \dots, v_k$ and keeps coordinates that belong to I fixed. Therefore we only need to perform the straightening of the vector field $v_{\hat{\jmath}}$ in a manner that satisfies the requirements of part b) of the theorem. In order to keep unchanged the coordinates that belong to I, all the vector fields that generate the isotopies forming the final isotopy Φ_t will be tangent to the orthogonal complement of the linear span of $\mathbf{e}_j$, the linear space $W = W_I = \langle \mathbf{e}_j : j \in I \rangle^{\perp}$:

- In Step 1, the vector field $v_{\hat{\jmath}}$ lies in the $(n + k - |I| - 1)$-dimensional unit sphere of the linear space W, and we can make it miss the $\downarrow$ direction by a small move parallel to W.
- Step 2 can be performed so that the isotopy Ξ_t preserves parallel translates of W since $v_{\hat{\jmath}}$ and its target state $\mathbf{e}_{\hat{\jmath}}$ are both parallel to W.
- Step 3 can be performed so that the isotopy Θ_t preserves parallel translates of W since both $v_{\hat{\jmath}}$ and b are parallel to W.
- The orthonorming isotopy $\mathscr{O}_t$ mentioned in the beginning of Appendix 3 is constructed as an extension of an isotopy that acts affinely on normal disks of $i(M)$. Each of these affine transformations fixes a point in $i(M)$ and has a linear part that is a composition of the differential of Φ_t, its inverse map and the maps $A(t, p)$; all of these linear maps preserve W if the Gram-Schmidt orthogonalization process is run on the v_j with $j \in I$ first. Hence $\mathscr{O}_t$ is an extension of an isotopy that preserves translates of W and can therefore be chosen to preserve translates of W itself. Consequently the corrected isotopy $\Phi_t \circ \mathscr{O}_t$ will preserve translates of W as well.

If, in the course of the proof, instead of $di(T_p M)$ we use its orthogonal projection onto W, $(di(T_p M) + W^{\perp}) \cap (W + i(p))$, then not only will the map $(p, t) \mapsto \Phi_t(i(p))$ be an immersion of $M \times [0, \infty)$, but the map

$$(p, (x_j)_{j \in I}, t) \mapsto \Phi_t(i(p)) + \sum_{j \in I} x_j \cdot v_j$$

will be an immersion of $M \times D^{|I|}_{\varepsilon} \times [0, \infty)$ for some sufficiently small positive ε.

The **relative version** of Theorem 2 only requires substantial adaption of Steps 1 and 2 of the proof – in Step 3, we only have to additionally require that Θ_t is the identity on $L \times \mathbb{R}^k$ at all times, and in the construction of $\mathscr{O}_t$ (see the beginning of Appendix 3) we have to require

that $\mathcal{O}_t$ should be the identity on a neighbourhood of $L \times \mathbb{R}^k$. We shall again consider only the field v_1, which we denote by v. If the vector field v is grounded ($v \neq \downarrow$ everywhere) then Step 1 can be skipped (setting $\Psi = id_{\mathbb{R}^n \times \mathbb{R}^1}$) and no further change in the proof is needed. If the subset $\{p \in M : v(i(p)) = \downarrow\}$ is non-empty, then we need to modify the definition of the vector field $\boldsymbol{\theta}$ in Step 1 since its flow does not necessarily preserve v on $L \times \mathbb{R}^1$. We choose $\boldsymbol{\theta}$ to satisfy the following for some positive $\delta < \pi/6$ ($\angle$ stands for angle):

- $\boldsymbol{\theta}$ is the infinitesimal generator of the rotation R (chosen as in the $m < n$ case) on a neighbourhood of the set

$$\mathcal{U} = \{p \in M : \angle(v(i(p)), \uparrow) \geq \delta\};$$

- $\boldsymbol{\theta} = \mathbf{0}$ outside a compact set;
- $\boldsymbol{\theta} = \mathbf{0}$ in a neighbourhood of $L \times \mathbb{R}^1$.

While this $\boldsymbol{\theta}$ no longer preserves orthogonality, taking a sufficiently small ε in the construction of $\Psi = \Psi_\varepsilon$ will make Ψ

- preserve orthogonality on $i(\mathcal{U})$;
- change direction of all vectors by less than δ;
- act as the identity in a neighbourhood of $L \times \mathbb{R}^1$.

Hence the image of v under Ψ (again denoted by w) will still miss $\downarrow$ – at points in $\mathcal{U}$ the diffeomorphism Ψ is the fixed rotation R^ε and at points outside $\mathcal{U}$ a rotation of an angle less than δ cannot turn them into the lower hemisphere $\{z \in S^n : \langle z, \uparrow \rangle < 0\}$. On $L \times \mathbb{R}^1$ the vector fields v and w coincide. We now define the vector field b to be the inner bisector of $\uparrow$ and w as in Step 2; once its property a) is verified, the rest of the proof will proceed without change (property b) holds trivially).

To check condition a) of Step 2, first note that for any vector $w \neq \downarrow$ the new vector b forms an acute angle with w, hence the condition is satisfied on $\mathcal{U}$, where w is orthogonal to M. Outside $\mathcal{U}$ the angle $\angle(w, \uparrow)$ is less than 2δ since Ψ changed all angles by at most δ. At these points the angle $\angle(b, w)$ is at most δ, hence $b(i'(p))$ and $v(i(p))$ form an angle of at most $\delta + \delta = 2\delta$ for all $p \in M \setminus \mathcal{U}$. Since the tangent space of $i(M)$ was also rotated by Ψ by an angle less than δ, the vector field b and $Ti'(M)$ form a strictly positive angle as required by condition a) since we chose $\delta < \pi/6$.

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