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EXTRINSIC GEOMETRY AND HIGHER ORDER CONTACTS OF SURFACES IN $\mathbb{R}^5$

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Abstract. We study the extrinsic geometry of a surface in $\mathbb{R}^5$ in relation to contact theory. We first completely determine the numerical invariants of the second fundamental form and describe the corresponding curvature ellipse. We then introduce and study a new quadratic map closely related to the degenerate directions of the surface, we characterize inflection and umbilic points of the surface in terms of the invariants, and we obtain an intrinsic equation of the asymptotic lines. Finally, we give a simple condition which guarantees the existence of an isometric reduction of codimension of the surface into $\mathbb{R}^4$.

Introduction

The purpose of the paper is to study the extrinsic geometry of a surface immersed in $\mathbb{R}^5$, in relation to the contact theory. Since the second fundamental form determines the second order properties of the surface, it is important to have a systematic method to study the numerical invariants of a quadratic map from a plane into a 3-dimensional space: this is the aim of the first part of the paper. We also completely describe the associated curvature ellipse in terms of the invariants. Using the Gauss map of the surface, we interpret these invariants as angular velocities of rotations of the tangent planes of the surface. Some notions then appear naturally: we introduce the *axial directions* of a surface M in $\mathbb{R}^5$, we relate invariants with the pull-back by the Gauss map G of the Lie bracket in $\Lambda^2\mathbb{R}^5$, and we introduce a new quadratic map $\delta : TM \to NM$ from the tangent to the normal bundle of M which measures the complexity of dG. This object is a natural generalization to the dimension 5 of a classical quadratic differential used in the contact theory of surfaces in $\mathbb{R}^4$ [9], and also appears to be fundamental for the study of surfaces in $\mathbb{R}^5$ from the contact viewpoint. We then study the invariants and the ellipse attached to δ and apply these constructions to study contact properties of a surface in $\mathbb{R}^5$: we first relate δ and its ellipse to the degenerate directions of the surface and especially characterize inflection and umbilic points of the surface by the vanishing of some of the invariants, we then use δ to give an intrinsic equation for the asymptotic lines on the surface (a quintic form introduced and studied in local coordinates in [12] and [13]), we characterize the points where the Gauss map is not regular in terms of contact in the Grassmannian of 2-planes in $\mathbb{R}^5$, and we finally state a result of reduction of codimension: if u is a tangent vector field such that $\delta(u)/|\delta(u)|$ is parallel in the direction u, then there exists an isometric reduction of codimension of M into $\mathbb{R}^4$, and the vector field u determines asymptotic directions on M in both ambient spaces. We end the paper with an explicit example, for which we compute the invariants of the second fundamental form and apply the reduction of codimension.

Surfaces in $\mathbb{R}^4$ and $\mathbb{R}^5$ from the contact point of view have been studied by many authors; see e.g. [7] and the references therein.

The outline of the paper is as follows. In the first section we describe the invariants of a quadratic map $\mathbb{R}^2 \to \mathbb{R}^3$ and study the properties of its associated ellipse: this description naturally leads to a classification of the quadratic maps from $\mathbb{R}^2$ to $\mathbb{R}^3$, up to the actions of the linear isometries on $\mathbb{R}^2$ and $\mathbb{R}^3$. In Section 2 we study the invariants of a surface in $\mathbb{R}^5$, we interpret these invariants in terms of the Gauss map and we define the axial directions of the surface. We then introduce in Section 3 the quadratic map δ and study its invariants together with its associated ellipse. In Section 4, we study degenerate directions, inflection and umbilic points on a surface in $\mathbb{R}^5$, give an intrinsic equation for the asymptotic lines and characterize the points where the Gauss map is not regular. We finally study in Section 5 a reduction of codimension of a surface in $\mathbb{R}^5$.

1. Quadratic maps $\mathbb{R}^2 \to \mathbb{R}^3$

We study here the quadratic maps $\mathbb{R}^2 \to \mathbb{R}^3$: we first determine their numerical invariants and obtain their classification up to the linear isometries of $\mathbb{R}^2$ and $\mathbb{R}^3$; we then study their associated ellipses.

1.1. **Forms associated to a quadratic map.** Consider $q : \mathbb{R}^2 \to \mathbb{R}^3$ a quadratic map. If ν belongs to $\mathbb{R}^3$, let us denote by S_ν the symmetric endomorphism of $\mathbb{R}^2$ associated to the quadratic form $q_\nu := \langle q, \nu\rangle$, which is such that

$$\langle q(u), \nu\rangle = \langle S_\nu(u), u\rangle, \ \forall u \in \mathbb{R}^2.$$

We define, for $\nu, \nu_1, \nu_2 \in \mathbb{R}^3$,

$$L_q(\nu) := \frac{1}{2}\mathrm{tr} S_\nu, \quad Q_q(\nu) := \det S_\nu \quad \text{and} \quad A_q(\nu_1, \nu_2) := \frac{1}{2}[S_{\nu_1}, S_{\nu_2}],$$

where $[S_{\nu_1}, S_{\nu_2}]$ denotes the skew-symmetric operator $S_{\nu_1} \circ S_{\nu_2} - S_{\nu_2} \circ S_{\nu_1}$ which identifies with the real number

$$\langle [S_{\nu_1}, S_{\nu_2}](e_2), e_1\rangle,$$

where (e_1, e_2) stands here for the canonical basis of $\mathbb{R}^2$. Thus L_q is a linear form, Q_q is a quadratic form and A_q is a bilinear skew-symmetric form defined on $\mathbb{R}^3$. These forms are connected by the following lemma, which may be obtained by a straightforward computation.

Lemma 1.1. *The quadratic form $\Phi_q := L_q^2 - Q_q$ is non-negative and the following equation holds: for all $\nu_1, \nu_2 \in \mathbb{R}^3$*

$$\Phi_q(\nu_1)\Phi_q(\nu_2) = \widetilde{\Phi}_q(\nu_1, \nu_2)^2 + A_q(\nu_1, \nu_2)^2, \tag{1.1}$$

where $\widetilde{\Phi}_q$ denotes the polar form of Φ_q.

The forms $L_q, \ \Phi_q, \ A_q$ are invariant by the right-action of SO_2 on q : for all $g \in SO_2$,

$$L_{q\circ g} = L_q, \ \Phi_{q\circ g} = \Phi_q \quad \text{and} \quad A_{q\circ g} = A_q.$$

Conversely, $L_q, \ \Phi_q$ and A_q determine q modulo this action:

Lemma 1.2. *There is a positively oriented orthonormal basis (e_1, e_2) of $\mathbb{R}^2$ and a vector $\nu_0 \in \mathbb{R}^3$ such that, for all $\nu \in \mathbb{R}^3$,*

$$S_\nu = L(\nu)I + \tilde{\Phi}(\nu_o, \nu)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + A(\nu_o, \nu)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1.2}$$

in the basis (e_1, e_2).

The proof of this lemma is analogous to that presented in [3] for the case of quadratic maps from $\mathbb{R}^2$ into $\mathbb{R}^2$, so we omit it.

Setting

$$P = \{(L, \Phi, A) : \ \Phi \text{ is non-negative, and (1.1) holds}\},$$

where L, Φ and A are respectively linear, bilinear symmetric and bilinear skew-symmetric forms on $\mathbb{R}^2$, the following result holds:

Lemma 1.3. *Let us denote by $Q(\mathbb{R}^2,\mathbb{R}^3)$ the set of quadratic maps $\mathbb{R}^2 \to \mathbb{R}^3$. The map*

$$\begin{array}{rccl} \Theta: & Q(\mathbb{R}^2,\mathbb{R}^3)/SO_2 & \to & P \\ & [q] & \mapsto & \big(L_{[q]},\ \Phi_{[q]},\ A_{[q]}\big) \end{array}$$

is bijective.

Proof. The representation of S_ν provided in (1.2) directly implies that Θ is injective. For the surjectivity, if L, Φ and A are given, choose first ν_o such that $\Phi(\nu_o) = 1$ (if $\Phi \neq 0$) and then define S_ν in the canonical basis of $\mathbb{R}^2$ by (1.2) (if $\Phi = 0$, A has to vanish and $[q]$ is determined by L). □

By the natural left-action of SO_3 on $Q(\mathbb{R}^2,\mathbb{R}^3)/SO_2$, the forms $L_{[q]}, \Phi_{[q]}$ transform as

$$L_{g.[q]} = L_{[q]} \circ g^{-1}, \qquad \Phi_{g.[q]} = \Phi_{[q]} \circ g^{-1},$$

whereas the form $A_{[q]}$ is invariant. Thus, if SO_3 acts on P by

$$g.(L, \Phi, A) := (L \circ g^{-1}, \Phi \circ g^{-1}, A), \tag{1.3}$$

the map Θ is SO_3-equivariant and thus induces a bijective map

$$\overline{\Theta}: \quad SO_3\backslash Q(\mathbb{R}^2,\mathbb{R}^3)/SO_2 \to SO_3 \setminus P. \tag{1.4}$$

Since the formula (1.1) permits to recover A (up to sign) from Φ, the description of the quotient set $SO_3\backslash Q(\mathbb{R}^2,\mathbb{R}^3)/SO_2$ will be achieved with the simultaneous reduction of the forms $L_{[q]}$ and $\Phi_{[q]}$ in $\mathbb{R}^3$.

1.2. **Invariants on the quotient set.** We introduce the classical invariants on the quotient space $SO_3\backslash Q(\mathbb{R}^2,\mathbb{R}^3)/SO_2$ associated to the forms $L_q,\ \Phi_q$, and A_q.

Definition 1.4. We consider:

(1) the vector $\vec{H} \in \mathbb{R}^3$ such that $\langle \vec{H}, \nu\rangle = L_{[q]}(\nu)$ for all $\nu \in \mathbb{R}^3$, and the square of its norm
$$|\vec{H}|^2 := \langle \vec{H}, \vec{H}\rangle;$$
(2) the skew-symmetric operator $U_A : \mathbb{R}^3 \to \mathbb{R}^3$ associated to $A_{[q]}$ and the number
$$|U_A|^2 := \frac{1}{2}\mathrm{tr}(U_A U_A^*),$$
where $U_A^* : \mathbb{R}^3 \to \mathbb{R}^3$ is its adjoint map;
(3) the symmetric operator $U_Q : \mathbb{R}^3 \to \mathbb{R}^3$ associated to Q and the three elementary symmetric functions of its eigenvalues
$$K := \sigma_1(U_Q), \quad \Delta := \sigma_2(U_Q) \quad \text{and} \quad \Gamma^2 := \sigma_3(U_Q).$$

We will see below that the last invariant Γ^2 is always non-negative.

The numbers $|\vec{H}|^2, |U_A|^2, K, \Delta$ and Γ^2 are invariant by the left action of SO_3 on $Q(\mathbb{R}^2,\mathbb{R}^3)/SO_2$, and thus define invariants on $SO_3\backslash Q(\mathbb{R}^2,\mathbb{R}^3)/SO_2$.

Remark 1.5. i) Since U_A is skew-symmetric, there exists a unique vector $\vec{N} \in \mathbb{R}^3$ such that

$$U_A(\nu) = \vec{N} \times \nu, \qquad \forall \nu \in \mathbb{R}^3, \tag{1.5}$$

where $\times$ stands for the usual cross product in $\mathbb{R}^3$. Observe that $\vec{N} = 0$ if and only if $U_A = 0$, i.e. if and only if $A_q = 0$. On the other hand, if $\vec{N} \neq 0$ then $A_q \neq 0$ and (1.1) shows that Ker A_q = Ker Φ_q, which in turn implies that $\vec{N}$ belongs to the kernel of U_Φ, the symmetric operator on $\mathbb{R}^3$ associated to $\Phi_{[q]}$. Therefore, there exists a positively oriented orthonormal basis

$(\vec{N}_1, \vec{N}_2, \vec{N}_3)$ of eigenvectors of U_Φ such that $\vec{N}_3 = \lambda \vec{N} \in \text{Ker } U_\Phi, \quad \lambda > 0$. In this basis the matrix of $\Phi_{[q]}$ which coincides with that of U_Φ is of the form

$$U_\Phi = \begin{pmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.6}$$

for some numbers $a, b \geq 0$. We will moreover consider below the real numbers α, β and γ such that

$$\vec{H} = \alpha \vec{N}_1 + \beta \vec{N}_2 + \gamma \vec{N}_3. \tag{1.7}$$

ii) The expression (1.6) implies that the dimension of $\text{Ker}(U_\Phi)$ is larger than or equal to 1, with equality if and only if $A_q \neq 0$.

We provide a geometric interpretation of the invariants $|U_A|^2$ and Γ as follows. We first compute

$$\begin{aligned} |U_A|^2 &= A_q(\vec{N}_1, \vec{N}_2)^2 + A_q(\vec{N}_2, \vec{N}_3)^2 + A_q(\vec{N}_3, \vec{N}_1)^2 \\ &= \langle U_A(\vec{N}_1), \vec{N}_2 \rangle^2 + \langle U_A(\vec{N}_2), \vec{N}_3 \rangle^2 + \langle U_A(\vec{N}_3), \vec{N}_1 \rangle^2 \\ &= \langle \vec{N} \times \vec{N}_1, \vec{N}_2 \rangle^2 + \langle \vec{N} \times \vec{N}_2, \vec{N}_3 \rangle^2 + \langle \vec{N} \times \vec{N}_3, \vec{N}_1 \rangle^2 \\ &= |\vec{N}|^2. \end{aligned} \tag{1.8}$$

On the other hand, by the very definition of $L_{[q]}$ and $Q_{[q]}$ have the following expressions in $(\vec{N}_1, \vec{N}_2, \vec{N}_3)$:

$$U_{L^2} = \begin{pmatrix} \alpha^2 & \alpha\beta & \alpha\gamma \\ \alpha\beta & \beta^2 & \beta\gamma \\ \alpha\gamma & \beta\gamma & \gamma^2 \end{pmatrix}, \quad U_Q = \begin{pmatrix} \alpha^2 - a^2 & \alpha\beta & \alpha\gamma \\ \alpha\beta & \beta^2 - b^2 & \beta\gamma \\ \alpha\gamma & \beta\gamma & \gamma^2 \end{pmatrix}. \tag{1.9}$$

A straightforward computation using (1.6) and (1.9) then provides the invariants of the quadratic form $\Phi_{[q]}$ in terms of those of $[q]$:

Lemma 1.6. *Let U_Φ be the symmetric operator on $\mathbb{R}^3$ associated to the quadratic form $\Phi_{[q]}$. Then*

$$\text{tr}\, U_\Phi = |\vec{H}|^2 - K \qquad \textit{and} \qquad \det U_{\Phi_{|\vec{N}^\perp}} = \frac{1}{4}|\vec{N}|^2. \tag{1.10}$$

So, by (1.6) we get that $|\vec{N}| = 2ab$. We compute moreover from (1.9) the determinant of U_Q and get

$$\Gamma^2 = \gamma^2 a^2 b^2. \tag{1.11}$$

Now, since (1.7) and the second equality in (1.10) imply that

$$\gamma^2 = \langle \vec{H}, \frac{\vec{N}}{|\vec{N}|} \rangle^2 \qquad \text{and} \qquad a^2 b^2 = \frac{1}{4}|\vec{N}|^2,$$

we deduce that

$$\Gamma^2 = \frac{1}{4}\langle \vec{H}, \vec{N} \rangle^2,$$

and thus naturally set

$$\Gamma := \frac{1}{2}\langle \vec{H}, \vec{N} \rangle. \tag{1.12}$$

1.3. **Classification on the quotient set.** By substituting expressions (1.10) in the characteristic polynomial of the endomorphism U_Φ we obtain the following:

Lemma 1.7. *There is an orthogonal basis of unit eigenvectors* $(\vec{N}_1, \vec{N}_2, \vec{N}_3)$ *of* U_Φ *in which the matrix of* Φ *is diagonal with diagonal entries* $(a^2, b^2, 0)$*, where*

$$a^2 = \frac{1}{2}\left(\left(|\vec{H}|^2 - K\right) + \sqrt{\left(|\vec{H}|^2 - K\right)^2 - |\vec{N}|^2}\right) \tag{1.13}$$

and

$$b^2 = \frac{1}{2}\left(\left(|\vec{H}|^2 - K\right) - \sqrt{\left(|\vec{H}|^2 - K\right)^2 - |\vec{N}|^2}\right). \tag{1.14}$$

Moreover, assuming that $\vec{N} \neq 0$ *and* $|\vec{H}|^2 - K \neq |\vec{N}|$*, the components* α, β, γ *of* $\vec{H}$ *in* $(\vec{N}_1, \vec{N}_2, \vec{N}_3)$ *are given by the following formulas:*

$$\alpha^2 = \frac{1}{\sqrt{\left(|\vec{H}|^2 - K\right)^2 - |\vec{N}|^2}}\left(\Delta + a^2|\vec{H}|^2 + b^2\frac{4\Gamma^2}{|\vec{N}|^2} - \frac{1}{4}|\vec{N}|^2\right), \tag{1.15}$$

$$\beta^2 = \frac{1}{\sqrt{\left(|\vec{H}|^2 - K\right)^2 - |\vec{N}|^2}}\left(\Delta + b^2|\vec{H}|^2 + a^2\frac{4\Gamma^2}{|\vec{N}|^2} - \frac{1}{4}|\vec{N}|^2\right) \tag{1.16}$$

and

$$\gamma = \frac{2\Gamma}{|\vec{N}|}. \tag{1.17}$$

We point out that the basis $(\vec{N}_1, \vec{N}_2, \vec{N}_3)$ of the lemma is not uniquely determined.

Instead of the set of invariants $|\vec{H}|^2, |\vec{N}|^2, K, \Delta, \Gamma$ we may consider the (almost) equivalent set of invariants $a,\ b,\ \alpha,\ \beta,\ \gamma$ (α and β are defined up to sign) to prove the following:

Theorem 1.8. *Let* $[q] \in SO_3\backslash Q(\mathbb{R}^2, \mathbb{R}^3)/SO_2$*. The invariants* $a, b, \alpha, \beta, \gamma$ *determine* $[q]$ *up to the natural left action of the subgroup of reflections with respect to the lines* $\mathbb{R}\vec{N}_1$ *and* $\mathbb{R}\vec{N}_2$ *in* $\mathbb{R}^3$*.*

We finally describe the first set of invariants in terms of the second one: we have

$$|\vec{H}|^2 = \alpha^2 + \beta^2 + \gamma^2, \tag{1.18}$$

$$K = \alpha^2 + \beta^2 + \gamma^2 - a^2 - b^2, \tag{1.19}$$

$$|\vec{N}|^2 = 4a^2b^2, \tag{1.20}$$

$$\Delta = a^2b^2 - \alpha^2b^2 - \gamma^2b^2 - a^2\beta^2 - a^2\gamma^2 \tag{1.21}$$

and

$$\Gamma = ab\gamma. \tag{1.22}$$

Details are given in [11].

1.4. **The ellipse of a quadratic map.** Let $q \in Q(\mathbb{R}^2, \mathbb{R}^3)$. We define the ellipse of q as the set

$$\mathcal{E} := \{q(u) \in \mathbb{R}^3 : u \in \mathbb{R}^2,\ |u| = 1\} \ \subset \mathbb{R}^3.$$

This is an ellipse in $\mathbb{R}^3$. We consider $q^o \in Q(\mathbb{R}^2, \mathbb{R}^3)$ such that

$$q(u) = \vec{H} + q^o(u) \tag{1.23}$$

for all $u \in \mathbb{R}^2,\ |u| = 1$: the ellipse $\mathcal{E}$ of q is the translation defined by the vector $\vec{H}$ of the ellipse $\mathcal{E}^o$ of q^o. Let us fix an arbitrary orthonormal and positively oriented basis $(u, u^\perp)$ of $\mathbb{R}^2$. By a direct computation relying on the equation $q^o(u, u^\perp) = q(u, u^\perp)$, we easily get

$$\Phi_q(\nu) = \langle q^o(u), \nu\rangle^2 + \langle q^o(u, u^\perp), \nu\rangle^2, \quad \forall \nu \in \mathbb{R}^3. \tag{1.24}$$

Proposition 1.9. *The ellipse $\mathcal{E}$ is degenerate if and only if $\vec{N} = 0$.*

Proof. If $\vec{N} = 0$, Remark 1.5 i) and ii) imply that $A_q \equiv 0$, and $\dim(\mathrm{Ker}\ U_\Phi) \geq 2$, respectively. So, there exist two linearly independent vectors $\nu_1, \nu_2\ \in \mathrm{Ker}\ \Phi_q$, and, for any orthonormal basis $(u, u^\perp)$ of $\mathbb{R}^2$ the following equations hold

$$0 = \Phi_q(\nu_i) = \langle q^o(u), \nu_i\rangle^2 + \langle q^o(u, u^\perp), \nu_i\rangle^2, \quad i = 1, 2.$$

On the other hand, if $w = xu + yu^\perp \in \mathbb{R}^2$ is unitary, then

$$q^o(w) = (x^2 - y^2)q^o(u) + 2xyq^o(u, u^\perp),$$

which implies that $q^o(w) = \lambda(w)\vec{v}$ for some $\vec{v} \in \{\nu_1, \nu_2\}^\perp$ and $\lambda(w) \in \mathbb{R}$. This implies that $\mathcal{E}$ is degenerate.

Assume now that $\mathcal{E}$ is degenerate, that is that $q(w) = \vec{H} + \lambda(w)\vec{v}$ for all unit $w \in \mathbb{R}^2$. Then, $\langle q^o(w), \nu\rangle = 0$ for all $\nu \in \vec{v}^\perp$. Since $q^o(w, w^\perp) = q^o(\frac{w+w^\perp}{2})$ is also colinear to $\vec{v}$, we get that

$$\Phi_q(\nu) = \langle q^o(w), \nu\rangle^2 + \langle q^o(w, w^\perp), \nu\rangle^2 = 0$$

for all $\nu \in \vec{v}^\perp$. Because $\dim\ \vec{v}^\perp = 2$ we get that $\dim\ \mathrm{Ker}\ U_\Phi = 2$, and Remark 1.5 ii) implies that $A_q = 0$, and Remark 1.5 i) then guarantees that $\vec{N} = 0$. □

If $\vec{N} \neq 0$ we can choose an orthonormal and positively oriented basis (e_1, e_2) of $\mathbb{R}^2$ such that $q^o(e_1)$ and $q^o(e_1, e_2)$ are tangent to the semi-axes of $\mathcal{E}^o$. By defining

$$\vec{N}_1 = \frac{q^o(e_1)}{|q^o(e_1)|} \quad \text{and} \quad \vec{N}_2 = \frac{q^o(e_1, e_2)}{|q^o(e_1, e_2)|},$$

Equation (1.24) implies that

$$\Phi_q(\nu) = |q^o(e_1)|^2\langle \vec{N}_1, \nu\rangle^2 + |q^o(e_1, e_2)|^2\langle \vec{N}_2, \nu\rangle^2 \tag{1.25}$$

for all $\nu \in \mathbb{R}^3$. We then complete the set $(\vec{N}_1, \vec{N}_2)$ to a positively oriented and orthonormal basis $(\vec{N}_1, \vec{N}_2, \vec{N}_3)$ of $\mathbb{R}^3$: for $\nu = x\vec{N}_1 + y\vec{N}_2 + z\vec{N}_3$, we get

$$\Phi_q(x, y, z) = |q^o(e_1)|^2x^2 + |q^o(e_1, e_2)|^2y^2, \tag{1.26}$$

and the matrix of U_Φ in $(\vec{N}_1, \vec{N}_2, \vec{N}_3)$ is

$$U_\Phi = \begin{pmatrix} |q^o(e_1)|^2 & 0 & 0 \\ 0 & |q^o(e_1, e_2)|^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{1.27}$$

which implies that $|q^o(e_1)|^2$ and $|q^o(e_1, e_2)|^2$ are eigenvalues of U_Φ. We may assume that $|q^o(e_1)|^2 = a^2$ and $|q^o(e_1, e_2)|^2 = b^2$, where a^2 and b^2 are respectively given by (1.13) and (1.14). Let us note that the basis constructed here is in fact the basis mentioned in Remark 1.5, especially, that $\vec{N}_3$ is colinear to $\vec{N}$. We then consider the restriction $\Phi_{|_{\vec{N}^\perp}}$ of Φ_q to the plane

$\vec{N}^\perp$. The symmetric operator associated to $\Phi_{|_{\vec{N}^\perp}}$ is $U_{\Phi_{|_{\vec{N}^\perp}}}$. Since $\det U_{\Phi_{|_{\vec{N}^\perp}}} = a^2b^2 \neq 0$, we may define

$$\Phi_q^*(\nu) := \langle U^{-1}_{\Phi_{|_{\vec{N}^\perp}}}(\nu), \nu\rangle \tag{1.28}$$

for all $\nu \in \vec{N}^\perp$. This quadratic form gives an intrinsic equation for the ellipse $\mathcal{E}^o$:

Proposition 1.10. *Let $q \in Q(\mathbb{R}^2, \mathbb{R}^3)$ such that $\vec{N} \neq 0$. Then the ellipse $\mathcal{E}^o$ belongs to $\vec{N}^\perp$ and we have: $\nu \in \mathcal{E}^o$ if and only if $\Phi_q^*(\nu) = 1$.*

Proof. Let us consider the basis (e_1, e_2) of $\mathbb{R}^2$ introduced above and, for $w = w_1e_1 + w_2e_2 \in \mathbb{R}^2$, compute

$$\begin{aligned} q^o(w) &= (w_1^2 - w_2^2)q^o(e_1) + 2w_1w_2q^o(e_1, e_2) \\ &= (w_1^2 - w_2^2)a\vec{N}_1 + 2w_1w_2b\vec{N}_2. \end{aligned} \tag{1.29}$$

The ellipse $\mathcal{E}^o$ thus belongs to $\vec{N}_3^\perp = \vec{N}^\perp$, and we have

$$\begin{aligned} \Phi_q^*(q^o(w)) &= \langle U^{-1}_{\Phi_{|_{\vec{N}^\perp}}}(q^o(w)), q^o(w)\rangle \\ &= \left\langle \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{pmatrix} \begin{pmatrix} (w_1^2 - w_2^2)a \\ 2w_1w_2b \end{pmatrix}, ((w_1^2 - w_2^2)a, 2w_1w_2b) \right\rangle \\ &= (w_1^2 + w_2^2)^2. \end{aligned} \tag{1.30}$$

Consequently, $\Phi_q^*(q^o(w)) = 1$ if and only if $|w|^2 = w_1^2 + w_2^2 = 1$. □

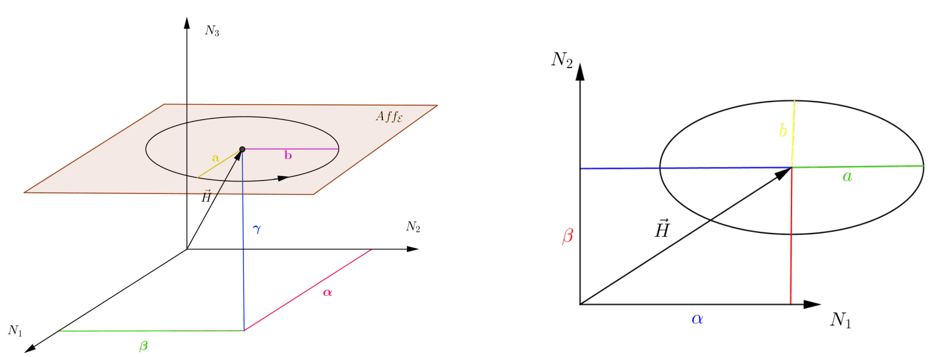
By writing $q^o(w) = x\vec{N}_1 + y\vec{N}_2$ we have that $\mathcal{E}^o$ is defined by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

in the basis $(\vec{N}_1, \vec{N}_2)$ of $\vec{N}^\perp$. This implies that the positive square roots a and b of the non-zero eigenvalues of U_Φ are the lengths of the semi-axes of $\mathcal{E}^o$, and thus also those of $\mathcal{E}$. Therefore, using (1.20), the area of $\mathcal{E}$ is

$$A(\mathcal{E}) = \pi ab = \frac{1}{2}\pi|\vec{N}|. \tag{1.31}$$

Moreover, since $\mathcal{E}^o$ lies in the plane $\vec{N}^\perp$, the ellipse $\mathcal{E}$ lies in the affine plane $\mathrm{Aff}_\mathcal{E} := \vec{H} + \vec{N}^\perp$. Consequently, the invariant $\Gamma = \frac{1}{2}\langle \vec{H}, \vec{N}\rangle$ is zero if and only if the plane $\mathrm{Aff}_\mathcal{E}$ goes through the origin of $\mathbb{R}^3$.



(a) If $\Gamma \neq 0$ then $O \notin \mathrm{Aff}_\mathcal{E}$ (b) If $\Gamma = 0$ then $O \in \mathrm{Aff}_\mathcal{E}$

2. Invariants and Gauss map of a surface in $\mathbb{R}^5$

The method of reduction developed in Section 1 can be applied to an arbitrary quadratic map defined on the tangent plane of a surface in $\mathbb{R}^5$ and with values in the corresponding normal space. We first apply it below to the second fundamental form II of the surface.

2.1. **An adapted basis at a point such that $II^\circ \neq 0$.** For sake of simplicity we keep the notation used in the previous section for the invariants of a general quadratic map to denote the corresponding invariants of the second fundamental form. Denote by

$$II^\circ := II - \vec{H}\langle .,.\rangle,$$

the traceless part of II, and assume that $II^\circ \neq 0$ (equivalently, $a \neq 0$). We consider in this setting the basis (e_1, e_2) of T_pM introduced in Section 1.4: we choose first a unit vector e_1 of T_pM such that $II^\circ(e_1) = a\vec{N}_1$; since the map II° increases the angle two times, the unit vector e_2 such that (e_1, e_2) is a positively oriented orthogonal basis satisfies $II^\circ(e_2) = -a\vec{N}_1$. Therefore,

$$II^\circ(e_1, e_2) = II^\circ\left(\frac{\sqrt{2}}{2}(e_1 + e_2)\right) = b\vec{N}_2,$$

where the last equality holds because the unit vector $\frac{\sqrt{2}}{2}(e_1 + e_2)$ makes an angle of $\frac{\pi}{4}$ with e_1. Thus, setting $u = x_1e_1 + x_2e_2$, we get

$$II^\circ(u) = a(x_1^2 - x_2^2)\vec{N}_1 + 2bx_1x_2\vec{N}_2, \tag{2.1}$$

and the matrix of II in (e_1, e_2) reads

$$II = \begin{pmatrix} a+\alpha & 0 \\ 0 & -a+\alpha \end{pmatrix}\vec{N}_1 + \begin{pmatrix} \beta & b \\ b & \beta \end{pmatrix}\vec{N}_2 + \begin{pmatrix} \gamma & 0 \\ 0 & \gamma \end{pmatrix}\vec{N}_3. \tag{2.2}$$

The ellipse of the second fundamental form at a point $p \in M$ is known as *the curvature ellipse* and has been studied in [6]; it will be denoted by $\mathcal{E}$.

2.2. **An interpretation of the invariants a, b, α, β and γ by means of the Gauss map.** Consider $\Lambda^2\mathbb{R}^5$ the vector space of bivectors of $\mathbb{R}^5$ endowed with its natural metric. The Grassmannian of the oriented 2-planes in $\mathbb{R}^5$ identifies with the submanifold of unit and simple bivectors

$$\mathcal{Q} = \{\eta \in \Lambda^2\mathbb{R}^5 : \ |\eta| = 1, \ \eta \wedge \eta = 0\},$$

and the oriented Gauss map with the map

$$G : \ M \to \mathcal{Q}, \ p \mapsto G(p) = u \wedge u^\perp,$$

where $(u, u^\perp)$ is a positively oriented orthonormal basis of T_pM.

For all unit vector $u \in T_pM$, the bivector $dG_p(u)$ is interpreted as an infinitesimal rotation in $\mathbb{R}^5$ of the tangent plane T_pM in the direction u, and, in fact, may be written in the form

$$dG_p(u) = m_u + t_u + r_u,$$

where

$$m_u = -u^\perp \wedge \vec{H}, \ t_u = -u^\perp \wedge II^o(u,u) \quad \text{and} \quad r_u = u \wedge II^o(u, u^\perp)$$

(see (3.2) below). Here and below $u^\perp$ stands for the image of u by the rotation of angle $+\pi/2$ in T_pM. Each term of this decomposition represents an infinitesimal rotation of the tangent plane in the direction u, in some 3-space: the term m_u represents an infinitesimal rotation of angular velocity $|\vec{H}|$ around the line $\mathbb{R}.u^\perp$ in the mean 3-space $T_pM \oplus \mathbb{R}\vec{H}$ (*the infinitesimal mean rotation*), the term t_u an infinitesimal rotation around the line $\mathbb{R}.u^\perp$ in $T_pM \oplus \mathbb{R}.II^o(u,u)$, and the term r_u an infinitesimal rotation around the line $\mathbb{R}.u$ in $T_pM \oplus \mathbb{R}.II^o(u, u^\perp)$; see [10], Chapter 2. Note that $t_u + r_u$ represents the difference between the infinitesimal rotation $dG_p(u)$ and the infinitesimal mean rotation.

Interpretation of a ***and*** b. The angular velocities $|II^o(u,u)|$ of t_u and $|II^o(u,u^\perp)|$ of r_u take maximal and minimal values, respectively, at the two orthogonal directions e_1, $e_2 \in T_pM$ defined in Section 2.1. These directions are called *the axial principal directions* at p. Since

$$t_{e_1} = -ae_2 \wedge \vec{N}_1 \quad \text{and} \quad r_{e_1} = be_1 \wedge \vec{N}_2,$$

the infinitesimal rotations t_{e_1} and r_{e_1} are rotations in the *perpendicular 3-spaces* $T_pM \oplus \mathbb{R}.\vec{N}_1$ and $T_pM \oplus \mathbb{R}.\vec{N}_2$, and the numbers a and b are their respective angular velocities. Note that the same interpretation holds for the tangent direction e_2. Because these directions are the pull-back by the second fundamental form of the vertices of the curvature ellipse, they are the natural generalizations of the axial principal directions for surfaces immersed in $\mathbb{R}^4$ [5]. Moreover, this approach allows us to interpret the axial principal directions of a surface in $\mathbb{R}^4$ or in $\mathbb{R}^5$ as those directions where the angular velocities considered above reach their extreme values.

Interpretation of γ***.*** Resting the mean rotation m_u to the infinitesimal rotation $dG_p(u)$ of the tangent plane in the direction u, we get a rotation in the hyperplane $T_pM \oplus II^o(u,u) \oplus II^o(u,u^\perp)$; in fact, this hyperplane does not depend on the direction $u \in T_pM$: this is the hyperplane orthogonal to the vector $\vec{N}_3$. The invariant γ is then interpreted as the velocity of the infinitesimal mean rotation which takes place in the 3-space $T_pM \oplus \mathbb{R}.\vec{N}_3$

Interpretation of α ***and*** β***.*** First, we rest the rotation $-u^\perp \wedge \gamma \vec{N}_3$ to the rotation $dG_p(u)$, and obtain the infinitesimal rotation in the hyperplane $\vec{N}_3^\perp$. Then, the numbers α and β determine the angular velocity $|\vec{H} - \gamma \vec{N}_3|$ of the infinitesimal mean rotation in this hyperplane, and, if $\vec{H} \neq \gamma \vec{N}_3$, also determine the position of the 3-spaces $T_pM \oplus \mathbb{R}.\vec{N}_1$ and $T_pM \oplus \mathbb{R}.\vec{N}_2$ relatively to the mean hyperplane $T_pM \oplus \mathbb{R}(\vec{H} - \gamma \vec{N}_3)$ in $\vec{N}_3^\perp$, since

$$\vec{N}_1 = \frac{\alpha}{\sqrt{\alpha^2+\beta^2}}\vec{n}_1 - \frac{\beta}{\sqrt{\alpha^2+\beta^2}}\vec{n}_2, \;\; \vec{N}_2 = \frac{\beta}{\sqrt{\alpha^2+\beta^2}}\vec{n}_1 + \frac{\alpha}{\sqrt{\alpha^2+\beta^2}}\vec{n}_2,$$

where $(\vec{n}_1, \vec{n}_2)$ is the positively oriented orthonormal basis of $\vec{N}_3^\perp \subset N_pM$ such that

$$\vec{H} - \gamma \vec{N}_3 = |\vec{H} - \gamma \vec{N}_3|.\vec{n}_1.$$

2.3. **The pull-back of the Lie bracket by the Gauss map.** We consider the Lie bracket

$$[.,.] : \Lambda^2\mathbb{R}^5 \times \Lambda^2\mathbb{R}^5 \to \Lambda^2\mathbb{R}^5.$$

Its restriction to $\mathcal{Q}$ is a 2-form with values in $\Lambda^2\mathbb{R}^5$. Its pull-back by the Gauss map gives the Gauss and the normal curvatures of the surface:

Proposition 2.1. *If* ∇ *denotes the Levi-Civita connection on* TM *and* $\nabla^\perp$ *the normal connection on the normal bundle* NM*, we have*

$$G^*[.,.] = R^{\nabla \oplus \nabla^\perp}, \tag{2.3}$$

where $R^{\nabla \oplus \nabla^\perp}$ *is the curvature of the connection* $\nabla \oplus \nabla^\perp$ *on* $TM \oplus NM$*, considered as a 2-form on* M *with values in* $\Lambda^2 TM \oplus \Lambda^2 NM \subset M \times \Lambda^2\mathbb{R}^5$.

This is essentially because the bracket $[.,.]$ may be interpreted as the curvature of the tautological bundles on $\mathcal{Q}$. We refer to [1] for the proof of a much more general result.

Corollary 2.2. *Let us consider the two 2-forms on $\mathcal{Q}$ defined by*

$$\omega_{\mathcal{Q}_p}(\eta,\eta') := \langle p, [\eta,\eta']\rangle \in \mathbb{R} \quad \text{and} \quad \omega'_{\mathcal{Q}_p}(\eta,\eta') := *\,(p\wedge[\eta,\eta']) \in \mathbb{R}^5 \tag{2.4}$$

for all $p \in \mathcal{Q}$, $\eta,\eta' \in T_p\mathcal{Q}$. Then

$$G^*\omega_{\mathcal{Q}} = K\ \omega_M \quad \text{and} \quad G^*\omega'_{\mathcal{Q}} = \vec{N}\ \omega_M \tag{2.5}$$

where ω_M is the area form of M.

Proof. This is an easy consequence of (2.3) and the fact that, if (e_1,e_2) is a positively oriented and orthonormal basis of TM,

$$R^{\nabla}(e_1,e_2) = Ke_1\wedge e_2 \quad \text{and} \quad R^{\nabla^\perp}(e_1,e_2)(Z) = \vec{N}\wedge Z$$

for all $Z \in NM$ (the last identity follows from the definition of $\vec{N}$ in Section 1 together with the fundamental Ricci equation). ☐

3. The quadratic map δ

We introduce here a new quadratic map $\delta:\ T_pM \to N_pM$ which is strongly related to the notion of asymptotic directions of M in $\mathbb{R}^5$. We apply to this map the method of reduction exposed in Section 1, especially, we determine its invariants and the geometry of its associated ellipse.

3.1. **Definition.** Let $p \in M$, and $u \in T_pM$. We define

$$\delta(u) := -\frac{1}{2}dG_p(u)\wedge dG_p(u)\ \in \Lambda^4\mathbb{R}^5. \tag{3.1}$$

Let us first recall the decomposition

$$dG_p(u) = -u^\perp \wedge II(u,u) + u\wedge II(u,u^\perp), \tag{3.2}$$

which is valid for all $u \in T_pM$ such that $|u| = 1$. It is obtained as follows: if $\alpha:\ (-\varepsilon,\varepsilon)\to M$ is the geodesic such that $\alpha(0) = p$ and $\alpha'(0) = u$, then $G(\alpha(t)) = \alpha'(t)\wedge\alpha'(t)^\perp$, and the derivative of that expression at $t = 0$ yields (3.2). This formula gives, if $|u| = 1$,

$$\delta(u) = u\wedge u^\perp \wedge II(u,u)\wedge II(u,u^\perp).$$

By applying the Hodge operator

$$*: \Lambda^4\mathbb{R}^5 \to \mathbb{R}^5,$$

and keeping the notation δ to denote $*\delta$, we regard δ as a map with values in $\mathbb{R}^5$. Since it is normal to $T_pM \oplus \mathbb{R}.II(u,u)\oplus\mathbb{R}.II(u,u^\perp)$ we get

$$\delta(u) = II(u,u)\times II(u,u^\perp)\ \in N_pM, \tag{3.3}$$

where $\times$ denotes the standard cross product in the normal space N_pM. Formula (3.1) (for all $u \in T_pM$), or equivalently (3.3) (if $|u| = 1$), thus defines a quadratic map

$$\delta:\quad T_pM \to N_pM.$$

Interpretations of the map δ: we see that $\delta(u)$ measures the independence of the infinitesimal rotations $-u^\perp\wedge II(u,u)$ and $u\wedge II(u,u^\perp)$ of the tangent plane around the axes $\mathbb{R}.u^\perp$ and $\mathbb{R}.u$. Let us give an interpretation of $\delta(u)$ in terms of a natural ellipse attached to the tangent direction u : consider the linear transformation

$$\begin{array}{rccc} II(u,.): & T_pM & \to & N_pM \\ & w & \mapsto & II(u,w). \end{array} \tag{3.4}$$

The image by $II(u,.)$ of the unit vectors in T_pM is an ellipse $\mathcal{E}_u$ in N_pM, with center at $0 \in N_pM$, and contained in the plane orthogonal to $\delta(u)$. This plane will be called *the osculating plane in the direction* u. Moreover, the area of $\mathcal{E}_u$ is

$$\pi|II(u,u) \times II(u,u^\perp)| = \pi|\delta(u)|.$$

Finally, we note that $\delta(u) = 0$ if and only if $II(u,u)$ and $II(u,u^\perp)$ are linearly dependent vectors, i.e. the image of $II(u,.)$ belongs to a line $\mathcal{L} \subset N_pM$; this equivalently means that $dG_p(u)$ is an infinitesimal rotation of the tangent plane in the 3-dimensional space $T_pM \oplus \mathcal{L}$, or alternatively that the ellipse $\mathcal{E}_u$ degenerate to a segment (supported by $\mathcal{L}$).

3.2. **Invariants of δ.** We describe here the numerical invariants of the quadratic map δ in terms of the invariants of the second fundamental form.

Proposition 3.1. *Let us denote by L_δ, Q_δ and A_δ the forms associated to the quadratic map δ as in Section 1. Their invariants (see Definition 1.4) are given as follows:*
i) the invariant associated to L_δ is $\frac{1}{4}|\vec{N}|^2$;
ii) the invariants associated to Q_δ are

$$\sigma_1 = \Delta, \quad \sigma_2 = \Gamma^2 K \quad \textit{and} \quad \sigma_3 = \Gamma^4; \tag{3.5}$$

iii) the invariant associated to A_δ is $\Gamma^2|\vec{H}|$.
Here $|\vec{H}|$, K, Δ, Γ and $|\vec{N}|$ still denote the invariants of II.

The quadratic map δ and the invariant Δ are thus natural generalizations of objects introduced in [9] for surfaces in $\mathbb{R}^4$.

Proof. First, using (2.2) and (3.3) and the notation introduced there, we easily obtain that the matrix of δ in the bases (e_1, e_2) and $(\vec{N}_1, \vec{N}_2, \vec{N}_3)$ is

$$\delta = \begin{pmatrix} -b\gamma & 0 \\ 0 & b\gamma \end{pmatrix} \vec{N}_1 + \begin{pmatrix} 0 & a\gamma \\ a\gamma & 0 \end{pmatrix} \vec{N}_2 + \begin{pmatrix} b(a+\alpha) & a\beta \\ a\beta & b(a-\alpha) \end{pmatrix} \vec{N}_3. \tag{3.6}$$

Consider $\nu \in N_pM$ and S^δ_ν the symmetric operator associated to the quadratic form

$$\delta_\nu(u) = \langle \delta(u), \nu \rangle.$$

We easily get

$$L_\delta(\nu) = \frac{1}{2}\operatorname{tr}\ S^\delta_\nu = \left\langle \frac{1}{2}\left(\delta(e_1) + \delta(e_2)\right), \nu \right\rangle = \left\langle \frac{1}{2}\vec{N}, \nu \right\rangle,$$

since $\vec{N} = 2ab\vec{N}_3$ (see (1.20)), and obtain *i*). Now, we have by definition

$$Q_\delta(\nu) = \langle \delta(e_1), \nu \rangle \langle \delta(e_2), \nu \rangle - \langle \delta(e_1, e_2), \nu \rangle^2,$$

and thus

$$U_{Q_\delta} = \begin{pmatrix} -b^2\gamma^2 & 0 & b^2\gamma\alpha \\ 0 & -a^2\gamma^2 & a^2\beta\gamma \\ b^2\gamma\alpha & a^2\beta\gamma & b^2a^2 - b^2\alpha^2 - a^2\beta^2 \end{pmatrix}$$

in the basis $(\vec{N}_1, \vec{N}_2, \vec{N}_3)$. Formulas in (3.5) then follow from the very definition of the invariants. Analogously, *iii*) may be proved with the explicit expression of A_δ in $(\vec{N}_1, \vec{N}_2, \vec{N}_3)$. $\square$

3.3. **The ellipse defined by δ.** We denote by $\mathcal{E}_\delta \subset N_pM$ the ellipse defined by the quadratic map $\delta : T_pM \to N_pM$. We describe it as follows.

Proposition 3.2. *$\mathcal{E}_\delta$ is an ellipse with center $\frac{1}{2}\vec{N}$, contained in the plane orthogonal to $\vec{H}$. Indeed, for all unit vector $u \in T_pM$, we have*

$$\delta(u) = \frac{1}{2}\vec{N} + \vec{H} \times II(u, u^\perp), \tag{3.7}$$

which in particular yields

$$tr_g\ \delta = \vec{N} \quad \text{and} \quad \delta^o(u) = \vec{H} \times II(u, u^\perp)$$

where $\delta^o := \delta - \frac{1}{2}tr_g\delta$ is the traceless part of δ, and g the standard inner product in $\mathbb{R}^3$. The square of the lengths of the semi-axes of $\mathcal{E}_\delta$ are

$$a_\delta^2 = \frac{1}{8}\left(|\vec{N}|^2 - 4\Delta + \sqrt{\left(|\vec{N}|^2 - 4\Delta\right)^2 - 64\Gamma^2|\vec{H}|^2}\right), \tag{3.8}$$

$$b_\delta^2 = \frac{1}{8}\left(|\vec{N}|^2 - 4\Delta - \sqrt{\left(|\vec{N}|^2 - 4\Delta\right)^2 - 64\Gamma^2|\vec{H}|^2}\right). \tag{3.9}$$

Proof. (3.7) is obtained by a straightforward computation using (3.6) together with the formulas

$$\vec{H} = \alpha\vec{N}_1 + \beta\vec{N}_2 + \gamma\vec{N}_3 \quad \text{and} \quad \vec{N} = 2ab\vec{N}_3.$$

Now, we compute the invariants of $\Phi_\delta = L_\delta^2 - Q_\delta$ using those of δ, and easily get

$$\sigma_1(U_{\Phi_\delta}) = \frac{1}{4}|\vec{N}|^2 - \Delta, \quad \sigma_2(U_{\Phi_\delta}) = \Gamma^2|\vec{H}|^2 \quad \text{and} \quad \sigma_3(U_{\Phi_\delta}) = 0.$$

Since $a^2(\delta), b^2(\delta)$ are the eigenvalues of U_{Φ_δ}, their expression follows from these invariants. □

Remark 3.3. We consider the quadratic map defined by

$$\delta'(u) := II^o(u,u) \times II^o(u, u^\perp)$$

for all unit vector u tangent to M. We have

$$\delta' = \vec{N}\ g = \frac{1}{2}\operatorname{tr}\delta_g.$$

We have the following interpretation of this formula: for all direction $u \in TM$, the infinitesimal rotations t_u and r_u are rotations in the 4-space $\vec{N}^\perp$; the independence of these rotations is measured by $|\vec{N}|$.

Remark 3.4. Assume that the curvature ellipse $\mathcal{E}$ at $p \in M$ is not degenerate. Then, it lies in a plane Π containing the origin of N_pM if and only if $\Gamma = 0$ at p. In that case, $\mathcal{E}_\delta$ becomes a segment (possibly reduced to a point) perpendicular to Π: indeed, if $\Gamma = 0$ and by Proposition 3.1, A_δ vanishes ($\mathcal{E}_\delta$ is degenerate) and $\sigma_2(U_{Q_\delta}) = \sigma_3(U_{Q_\delta}) = 0$ (the line containing $\mathcal{E}_\delta$ goes through $0 \in N_pM$).

3.4. **A normal curvature associated to the mean curvature directions.** If $u \in T_pM$, $|u| = 1$, we define *the normal curvature in the direction* u by the expression

$$\kappa_p(u) := \left\langle \frac{\vec{H} \times \vec{N}}{|\vec{H} \times \vec{N}|}, II^0(u, u^\perp) \right\rangle. \tag{3.10}$$

This is the component along the direction $\vec{H} \times \vec{N}$ of the vector tangent to the curvature ellipse $II : S^1 \subset T_pM \to N_pM$ at u. It is quadratic in $u \in T_pM$. In the bases (e_1, e_2) and $(\vec{N}_1, \vec{N}_2, \vec{N}_3)$ considered above, we have the following expressions:

$$\vec{H} \times \vec{N} = 2ab(\beta\vec{N}_1 - \alpha\vec{N}_2) \quad \text{and} \quad II^0(u, u^\perp) = -2ax_1x_2\vec{N}_1 + b(x_1^2 - x_2^2)\vec{N}_2,$$

and therefore

$$\kappa_p(u) = \frac{-1}{\sqrt{\alpha^2 + \beta^2}}(b\alpha x_1^2 + 2a\beta x_1x_2 - b\alpha x_2^2).$$

This curvature has the following relation with δ and $\vec{N}$: in view of (3.6), we have, if $u = x_1e_1 + x_2e_2$ in T_pM is a unit vector,

$$\begin{aligned} \langle \delta(u), \vec{N} \rangle &= \frac{1}{2}|\vec{N}| - \sqrt{\alpha^2 + \beta^2}\ \kappa_p(u) \\ &= b(a + \alpha)x_1^2 + 2a\beta x_1x_2 + b(a - \alpha)x_2^2. \end{aligned} \tag{3.11}$$

The zeroes of this quadratic expression are reached at the directions u along which the osculating plane is orthogonal to the plane of the curvature ellipse. This condition holds if and only if

$$\kappa_p(u) = \frac{1}{2\sqrt{\alpha^2 + \beta^2}}|\vec{N}|.$$

Moreover, the extreme values of the quadratic form (3.11) (or equivalently of κ_p) are reached at the directions along which the osculating planes are respectively the best and the worst approximations of the plane of the ellipse: these directions are in fact the mean curvature directions introduced in [6], and appear here associated to the quadratic differential

$$\langle \delta, \vec{N} \rangle = b(a + \alpha)dx_1^2 + 2a\beta dx_1dx_2 + b(a - \alpha)dx_2^2$$

(or equivalently to κ_p).

4. The contacts of higher order of M in terms of δ

4.1. **The degenerate directions and δ.** The *family of height functions* of the surface M immersed in $\mathbb{R}^5$ is

$$H : M \times \mathbb{S}^4 \to \mathbb{R}, \quad (p, \nu) \mapsto h_\nu(p) = \langle p, \nu \rangle,$$

where $\mathbb{S}^4$ is the unit sphere in $\mathbb{R}^5$ centered at the origin. Let us recall that a direction ν in N_pM *is degenerate* if the height function h_ν has a singularity at p which is not of Morse type. In this case the kernel of the Hessian of h_ν contains non zero vectors. Any direction u in this kernel is called a *contact direction* associated to ν. We have the following relation between the curvature ellipse and the cone determined by $\mathcal{E}_\delta$ in N_pM.

Proposition 4.1. *If $u \in T_pM$ is such that $\delta(u) \neq 0$, the direction*

$$\tilde{\delta}(u) := \frac{\delta(u)}{|\delta(u)|}$$

is a degenerate direction. Thus, the cone determined by the ellipse $\mathcal{E}_\delta$ is the cone of degenerate directions of M at p. Moreover, if $\Gamma \neq 0$, for all unit $u \in T_pM$ the direction $\tilde{\delta}(u)$ determines a height function with only one contact direction.

Proof. Let us consider a unit vector $u \in T_pM$, and the vector $u^\perp \in T_pM$ such that $(u, u^\perp)$ is a positively oriented and orthonormal basis of T_pM. We have

$$\begin{aligned} S_{\tilde{\delta}(u)}(u) &= \langle S_{\tilde{\delta}(u)}(u), u\rangle u + \langle S_{\tilde{\delta}(u)}(u), u^\perp\rangle u^\perp \\ &= \langle II(u), \tilde{\delta}(u)\rangle u + \langle II(u, u^\perp), \tilde{\delta}(u)\rangle u^\perp \\ &= 0 \end{aligned}$$

since $\delta(u) = II(u) \times II(u, u^\perp)$ is orthogonal to both $II(u)$ and $II(u, u^\perp)$. Then $u \in \mathrm{Ker}\ S_{\tilde{\delta}(u)}$ and

$$Q_{II}(\tilde{\delta}(u)) = \det\ S_{\tilde{\delta}(u)} = 0.$$

Moreover, recalling Proposition 3.2,

$$\begin{aligned} S_{\tilde{\delta}(u)}(u^\perp) &= \langle S_{\tilde{\delta}(u)}(u^\perp), u^\perp\rangle u^\perp \\ &= \langle II(u^\perp), \tilde{\delta}(u)\rangle u^\perp \\ &= \frac{1}{|\delta(u)|}\left(\frac{1}{2}\langle \vec{H}, \vec{N}\rangle - \langle II^o(u), \delta^o(u)\rangle\right) u^\perp \\ &= \frac{1}{|\delta(u)|}\left(\Gamma + \langle II^o(u) \times II^o(u, u^\perp), \vec{H}\rangle\right) u^\perp \\ &= \frac{2\Gamma}{|\delta(u)|} u^\perp, \end{aligned} \tag{4.1}$$

which finally shows that u is the only contact direction associated to the degenerate direction $\tilde{\delta}(u)$.

□

Remark 4.2. For surfaces in $\mathbb{R}^4$ the fact that a shape operator S_ν for some $\nu \in N_pM$ vanishes implies that the rank of the second fundamental form at p decreases, namely, it has rank less than or equal to 1. In the case analyzed here we conclude that there exists a vector $u \in T_pM,\ |u| = 1$ such that $S_{\tilde{\delta}(u)} = 0$ if and only if $S_{\tilde{\delta}(v)} = 0$ for all $v \in T_pM,\ |v| = 1$ (since then $\Gamma = 0$ by (4.1) applied to u, and then by (4.1) applied to v), or equivalently, the second fundamental form has rank less than or equal to two. We say in this case that p is an *inflection point*. On the other hand, the second fundamental form has rank less than or equal to 1 if and only if $\Gamma = \Delta = |\vec{N}| = 0$ at p. See [11] for details. In [12] the authors prove that, generically, such a point is an *umbilic* singularity for the height functions corresponding to all the degenerate directions of M at p. We will refer to this kind of point as an *umbilic point*.

4.2. **Intrinsic equation of asymptotic directions at a non-umbilic point.** From now on we consider third order geometric properties of the surface. Let p be a point of a surface M immersed in $\mathbb{R}^5$. Among the degenerate directions, the *binormal directions* are the directions $\nu \in N_pM$ such that the height function h_ν has, at p, a singularity of cusp type, or worse; they were introduced and studied in [12]. Any binormal direction ν defines a hyperplane in $\mathbb{R}^5$ having a higher order of contact with the surface at p. The corresponding contact direction is called an *asymptotic direction* of M at p. Let us recall that asymptotic lines are projectively invariant lines of the surface. In fact, there is an old reference for their equation different from that we obtain here, see [8], p. 415. If M is generically embedded, it admits at least one and at most five asymptotic directions. In [13] is presented a local analysis of the foliations defined by the asymptotic directions. The main tool for this description is a quintic form whose coefficients depend on the second fundamental form. We provide here an intrinsic expression for this quintic form, which thus gives an intrinsic equation for the asymptotic directions. Let us first introduce the covariant derivative of the second fundamental form: this is the $(1,3)$-tensor $\widetilde{\nabla} II$ defined by

$$\widetilde{\nabla}_X II(Y,Z) := \nabla_X^{\perp}(II(Y,Z)) - II(\nabla_X Y, Z) - II(Y, \nabla_X Z) \tag{4.2}$$

for all smooth vector fields X, Y, Z on M.

Proposition 4.3. *Let M be a surface immersed in $\mathbb{R}^5$, $p \in M$ and $u \in T_pM$ a unit vector such that $\delta(u) \neq 0$. Then M has a contact of order greater than or equal to 3 with the hyperplane*

$$\delta(u)^{\perp} := \{v \in \mathbb{R}^5 : \langle v, \delta(u)\rangle = 0\},$$

with contact direction u at p, if and only if

$$\langle \widetilde{\nabla}_u II(u,u), \delta(u)\rangle = 0. \tag{4.3}$$

Proof. Consider the height function with respect to $\delta(u)$, defined by

$$\begin{array}{rccc} h: & \mathbb{R}^5 & \to & \mathbb{R} \\ & p & \mapsto & \langle p, \delta(u)\rangle. \end{array}$$

Observe that $h^{-1}(0) = \delta(u)^{\perp}$ and that h is a submersion. If $\alpha : (-\varepsilon, \varepsilon) \to M \subset \mathbb{R}^5$ is the geodesic of M such that $\alpha(0) = p$ and $\alpha'(0) = u$, we have

$$(h \circ \alpha)^{(m)}(0) = \langle \alpha^{(m)}(0), \delta(u)\rangle$$

for all $m \in \mathbb{N}$, with

$$\begin{aligned} \alpha^{(1)}(0) &= u, \\ \alpha^{(2)}(0) &= II(u,u), \\ \alpha^{(3)}(0) &= \frac{d}{dt}_{|t=0} II(\alpha', \alpha') \\ &= -S_{II(u,u)}(u) + \widetilde{\nabla}_u II(u,u). \end{aligned}$$

Since $\delta(u)$ is a vector normal to the surface and orthogonal to $II(u,u)$, we get that

$$(h \circ \alpha)^{(1)}(0) = (h \circ \alpha)^{(2)}(0) = 0$$

and that $(h \circ \alpha)^{(3)}(0) = 0$ if and only if (4.3) holds. □

We thus obtain the following:

Corollary 4.4. *The equation of the asymptotic lines introduced in* [12] *may be intrinsically written as (4.3).*

We now analyze the points of M where the Gauss map is not regular, i.e. is not an immersion. Equation (3.2) implies that p is such a point if and only if there exists a unit vector $u \in T_pM$ such that

$$II(u) = II(u, u^{\perp}) = 0;$$

this equivalently means that the curvature ellipse $\mathcal{E}_p$ is a segment $\overline{OA} \subset N_pM$ for some point $A \in N_pM$. Namely, it is an umbilic singularity for all the height functions corresponding to all the degenerate directions for M at p, see Corollary 4 in [12]; such a point is called a *parabolic umbilic.*

We state the following characterization of these points:

Proposition 4.5. *The Gauss map $G : M \to \mathcal{Q}$ is not regular at $p \in M$ if and only if there exists a regular curve $\alpha : t \in (-\varepsilon, \varepsilon) \to \alpha(t)$ of M such that $\alpha(0) = p$ and $G \circ \alpha$ has a order of contact with $T_{G(p)}\mathcal{Q}$ at $t = 0$ greater than or equal to 2.*

Proof. Let us fix a unit vector $u \in T_pM$. If (e_3, e_4, e_5) is a given orthonormal basis of N_pM, we consider the function

$$H : \quad \Lambda^2\mathbb{R}^5 \to \mathbb{R}^4$$

defined by

$$H(\eta) = (\langle \eta, u \wedge u^\perp \rangle, \langle \eta, e_4 \wedge e_5 \rangle, \langle \eta, e_5 \wedge e_3 \rangle, \langle \eta, e_3 \wedge e_4 \rangle).$$

Observe that this map is a submersion since it is linear and its kernel has dimension 6. Moreover, $H^{-1}(0) = T_{u\wedge u^\perp}\mathcal{Q}$.

Without loss of generality we assume that $\alpha : t \in (-\varepsilon, \varepsilon) \to \alpha(t) \in M$ is parameterized by arc length, and $\alpha(0) = p$, $\alpha'(0) = u$, $|u| = 1$. The order of contact of $G \circ \alpha$ with $T_{G(p)}\mathcal{Q}$ at $t = 0$ is greater than or equal to 2 if and only if

$$H((G \circ \alpha)'(0)) = H((G \circ \alpha)''(0)) = 0. \tag{4.4}$$

We compute

$$\begin{aligned}(G \circ \alpha)'(t) &= dG_{\alpha(t)}(\alpha'(t)) \\ &= \alpha'(t) \wedge II(\alpha'(t), \alpha'(t)^\perp) - \alpha'(t)^\perp \wedge II(\alpha'(t), \alpha'(t)).\end{aligned}$$

If $\overline{\nabla}$ stands for the natural covariant derivative in $\mathbb{R}^5$, and omitting the parameter t,

$$\begin{aligned}(G \circ \alpha)'' &= \overline{\nabla}_{\alpha'}\alpha' \wedge II(\alpha', \alpha'^\perp) + \alpha' \wedge \overline{\nabla}_{\alpha'} II(\alpha', \alpha'^\perp) \\ &\quad -\overline{\nabla}_{\alpha'}\alpha'^\perp \wedge II(\alpha', \alpha') - \alpha'^\perp \wedge \overline{\nabla}_{\alpha'} II(\alpha', \alpha') \\ &= \nabla_{\alpha'}\alpha' \wedge II(\alpha', \alpha'^\perp) + II(\alpha', \alpha') \wedge II(\alpha', \alpha'^\perp) \\ &\quad -\alpha' \wedge S_{II(\alpha',\alpha'^\perp)}(\alpha') + \alpha' \wedge \nabla^\perp_{\alpha'} II(\alpha', \alpha'^\perp) \\ &\quad -\nabla_{\alpha'}\alpha'^\perp \wedge II(\alpha', \alpha') - II(\alpha', \alpha'^\perp) \wedge II(\alpha', \alpha') \\ &\quad -\alpha'^\perp \wedge \nabla^\perp_{\alpha'} II(\alpha', \alpha') + \alpha'^\perp \wedge S_{II(\alpha',\alpha')}(\alpha') \\ &= 2II(\alpha', \alpha') \wedge II(\alpha', \alpha'^\perp) \\ &\quad +\nabla_{\alpha'}\alpha' \wedge II(\alpha', \alpha'^\perp) - \nabla_{\alpha'}\alpha'^\perp \wedge II(\alpha', \alpha') \\ &\quad -(|II(\alpha', \alpha')|^2 + |II(\alpha', \alpha'^\perp)|^2)\alpha' \wedge \alpha'^\perp \\ &\quad +\alpha' \wedge \nabla^\perp_{\alpha'} II(\alpha', \alpha'^\perp) - \alpha'^\perp \wedge \nabla^\perp_{\alpha'} II(\alpha', \alpha'^\perp).\end{aligned}$$

At $t = 0$, we obtain

$$(G \circ \alpha)'(0) = u \wedge II(u, u^\perp) - u^\perp \wedge II(u, u)$$

and

$$\begin{aligned}(G \circ \alpha)''(0) &= 2II(u, u) \wedge II(u, u^\perp) - (|II(u, u)|)^2 + |II(u, u^\perp)|^2)u \wedge u^\perp \\ &\quad +\nabla_u\alpha'|_{t=0} \wedge II(u, u^\perp) - \nabla_u\alpha'^\perp|_{t=0} \wedge II(u, u) \\ &\quad +u \wedge \nabla^\perp_u II(\alpha', \alpha')|_{t=0} - u^\perp \wedge \nabla^\perp_u II(\alpha, \alpha'^\perp)|_{t=0}.\end{aligned}$$

Consequently, $H((G \circ \alpha)''(0)) = 0$, if and only if

$$\begin{aligned}\langle (G \circ \alpha)''(0), u \wedge u^\perp \rangle &= -(|II(u, u)|^2 + |II(u, u^\perp)|^2) = 0, \\ \langle (G \circ \alpha)''(0), e_4 \wedge e_5 \rangle &= 2\langle II(u, u) \wedge II(u, u^\perp), e_4 \wedge e_5 \rangle = 0, \\ \langle (G \circ \alpha)''(0), e_5 \wedge e_3 \rangle &= 2\langle II(u, u) \wedge II(u, u^\perp), e_5 \wedge e_3 \rangle = 0, \\ \langle (G \circ \alpha)''(0), e_3 \wedge e_4 \rangle &= 2\langle II(u, u) \wedge II(u, u^\perp), e_3 \wedge e_4 \rangle = 0,\end{aligned} \tag{4.5}$$

that is, if and only if $II(u,u) = II(u,u^\perp) = 0$. We observe that the first contact relation in (4.4) holds because $(G \circ \alpha)'(0) = 0$. □

Remark 4.6. If $u \in T_pM$ is in the kernel of dG then $\delta(u) = 0$, and $\Delta \le 0$. If the equality holds the origin lies in the curvature ellipse otherwise, it does not lies in such ellipse. If $\Gamma = 0$ then, for all $u \in T_pM$, $\delta(u)$ and $\vec{N}$ are colinear.

5. A reduction of codimension of a parallel osculating bundle

We consider a Riemannian surface M isometrically immersed in $\mathbb{R}^5$ and a differentiable vector field ν normal to M. Denote by B_ν the rank 2 fiber bundle on M with fiber the subspace $\nu_p^\perp \subset N_pM$ normal to ν at p. We say that M *admits an isometric reduction of codimension into* $\mathbb{R}^4$ *preserving the projection of the second fundamental form on* B_ν if there exists an isometric immersion

$$\varphi : M \to \mathbb{R}^4$$

that extends to a fiber bundle isomorphism

$$\bar{\varphi} : B_\nu \to NM \subset \mathbb{R}^4$$

such that for every $X, Y \in TM$ and μ, μ' sections of B_ν the following equations hold

$$\langle \bar{\varphi}(\mu), \bar{\varphi}(\mu') \rangle = \langle \mu, \mu' \rangle, \ \bar{\varphi}(II_\nu(X,Y)) = II(X,Y), \ \bar{\varphi}(\nabla^B_X \mu) = \nabla^\perp_X \bar{\varphi}(\mu),$$

where II_ν and ∇^B_X are the orthogonal projections of the second fundamental form and the normal connection of $M \subset \mathbb{R}^5$ on B_ν, respectively, and II and $\nabla^\perp$ are the second fundamental form and the normal connection of $\varphi(M) \subset \mathbb{R}^4$, respectively. We call an isometric immersion with these properties *a ν-isometric reduction of codimension of M*. Conditions for the existence of such immersions are provided in [14]. We recall the following:

Definition 5.1. We say that a unit vector field ν normal to M is a Codazzi field if the dimension of Ker S_ν is equal to 1 and ν is parallel along all the vectors in Ker S_ν, namely,

$$\nabla^\perp_u \nu = 0, \ \forall u \in \text{Ker } S_\nu.$$

The following result is then a straightforward consequence of Theorem 4.11 in [14]:

Theorem 5.2. *Let M be a simply connected surface immersed in $\mathbb{R}^5$ and ν a unit normal vector field such that the dimension of* Ker S_ν *is 1. Then, there exists a ν-isometric reduction of codimension of M if and only if ν is a Codazzi field.*

We now determine conditions on a tangent vector $u \in T_pM$ expressing that δ is parallel in the direction u; it appears that u is necessarily an asymptotic direction of M :

Proposition 5.3. *Let $\alpha : (-\varepsilon, \varepsilon) \to M$ be a geodesic at p tangent to $u \in T_pM$, $|u| = 1$. Suppose that $\delta(u) \neq 0$ and define along α the vector field*

$$\mu(t) = \tilde{\delta}(\alpha'(t)).$$

Then, $\mu(t)$ is parallel at $t = 0$ in the direction $u \in T_pM$ if and only if

$$\langle \widetilde{\nabla}_u II(u,u), \delta(u) \rangle = \langle \widetilde{\nabla}_u II(u,u^\perp), \delta(u) \rangle = 0.$$

Proof. We consider the covariant derivative of the $(1,2)$-tensor δ defined by

$$\widetilde{\nabla}_X \delta(Y,Z) := \nabla^\perp_X(\delta(Y,Z)) - \delta(\nabla_X Y, Z) - \delta(Y, \nabla_X Z) \tag{5.1}$$

for all smooth vector fields X, Y, Z tangent to M, and denote the value of this $(1,3)$-tensor at $u \in T_pM$ by

$$\widetilde{\nabla}_u \delta(u) := \widetilde{\nabla}_u \delta(u,u).$$

Then

$$\nabla^{\perp}_{\alpha'(t)}\mu(t)_{|t=0} = \frac{\widetilde{\nabla}_u\delta(u)}{|\delta(u)|} - \frac{\langle\widetilde{\nabla}_u\delta(u),\delta(u)\rangle}{|\delta(u)|^3}\delta(u).$$

So, if $\nabla^{\perp}_{\alpha'(t)}\mu(t)_{|t=0} = 0$,

$$\widetilde{\nabla}_u\delta(u) = \frac{\langle\widetilde{\nabla}_u\delta(u),\delta(u)\rangle}{|\delta(u)|^2}\delta(u).$$

Therefore,

$$\langle\widetilde{\nabla}_u\delta(u), II(u,u)\rangle = 0 \quad \text{and} \quad \langle\widetilde{\nabla}_u\delta(u), II(u,u^{\perp})\rangle = 0.$$

Since

$$\widetilde{\nabla}_u\delta(u) = \widetilde{\nabla}_u II(u,u)\times II(u,u^{\perp}) + II(u,u)\times\widetilde{\nabla}_u II(u,u^{\perp})$$

we deduce that

$$\langle\widetilde{\nabla}_u II(u,u)\times II(u,u^{\perp}), II(u,u)\rangle = 0$$

and

$$\langle II(u,u)\times\widetilde{\nabla}_u II(u,u^{\perp}), II(u,u^{\perp})\rangle = 0,$$

which implies

$$\langle\widetilde{\nabla}_u II(u,u),\delta(u)\rangle = \langle\widetilde{\nabla}_u II(u,u^{\perp}),\delta(u)\rangle = 0.$$

Conversely, if

$$\langle\widetilde{\nabla}_u II(u,u),\delta(u)\rangle = \langle\widetilde{\nabla}_u II(u,u^{\perp}),\delta(u)\rangle = 0$$

then $\widetilde{\nabla}_u\delta(u) = \lambda\delta(u)$ with $\lambda = \frac{\langle\widetilde{\nabla}_u\delta(u),\delta(u)\rangle}{|\delta(u)|^2}$, which implies that

$$\nabla^{\perp}_{\alpha'(t)}\mu(t)_{|t=0} = 0.$$

□

Assume that a differentiable vector field u tangent to M is given and consider the rank 2 fiber bundle on M whose fiber at each point p is the osculating plane in the direction $u(p)$. We recall that this plane is orthogonal to $\delta(u)$, see (3.4). We call this fiber bundle *the osculating fiber bundle defined by* u and denote it by $\mathcal{O}_u$. Moreover, suppose that the unit normal field $\tilde{\delta}(u)$ is parallel along u. Then, the fibre bundle $\mathcal{O}_u$ is parallel, and according to Proposition 5.3, the field u satisfies Equation (4.3). That is, u is a field of asymptotic directions. Since this vector field belongs to the kernel of $S_{\tilde{\delta}(u)}$ and

$$\langle S_{\tilde{\delta}(u)}(u^{\perp}), u^{\perp}\rangle = \frac{2\Gamma}{|\delta(u)|},$$

if $\Gamma \neq 0$ along M the vector field $\tilde{\delta}(u)$ is a Codazzi field. We thus obtain the following:

Corollary 5.4. *Let M be a simply connected surface M immersed in $\mathbb{R}^5$ such that $\Gamma(p) \neq 0$ for all $p \in M$. Assume that u is a vector field tangent to M such that $\tilde{\delta}(u)$ is parallel along u. Then, u is a field of asymptotic directions, and there exists a $\tilde{\delta}(u)$-isometric reduction of codimension of M whose normal bundle is isomorphic to $\mathcal{O}_u$. That is, the isometric immersion $\varphi : M \to \mathbb{R}^4$ extends to a fiber bundle isomorphism $\tilde{\varphi} : \mathcal{O}_u \to NM \subset \mathbb{R}^4$.*

Consider $\pi_u : N_pM \to O_u$, the orthogonal projection on the osculating plane in the direction u at p, and the ellipse $\mathcal{E}(u) = \pi_u(\mathcal{E}_p)$. If $\mathcal{E}(u)$ is not degenerate, there are two unit vectors v, w in T_pM whose corresponding osculating planes are orthogonal to O_u. Moreover, the intersections of these planes with O_u are tangent to $\mathcal{E}(u)$. It can be shown that this condition implies that $\delta(v)$ and $\delta(w)$ are orthogonal to these intersection lines.

If a unit vector $e \in T_pM$ is given we consider the normal section $\Sigma_p(e)$ of M at p, which is the curve obtained by the local intersection of the subspace $\mathbb{R}.e \oplus N_pM \subset \mathbb{R}^5$ with M in a neighborhood of p. Under these conditions we have

Proposition 5.5. *The order of contact of the hyperplane $\delta(u)^\perp \subset \mathbb{R}^5$ with the normal section $\Sigma_p(u)$ coincides with the order of contact of this hyperplane with the orthogonal projection of $\Sigma_p(u)$ on any of the osculating planes O_v or O_w orthogonal to O_u.*

Therefore, if we have an isometric reduction of codimension as in Corollary 5.4, $\delta(v)$ and $\delta(w)$ identify with the binormal directions defined by the curvature ellipse $\tilde{\varphi}(\mathcal{E}_u)$ of $\varphi(M) \subset \mathbb{R}^4$ at p. Moreover, this proposition implies that the order of contact along asymptotic directions is preserved by the isometric reduction of codimension. For instance for the binormal direction $\delta(v)$, the order of contact of the normal section of $\varphi(M)$ at $\varphi(p)$ along the asymptotic direction with the hyperplane $\delta(v)^\perp \subset \mathbb{R}^4$ coincides with the order of contact of the normal section of M at p along the same contact direction with the hyperplane $\delta(v)^\perp \subset \mathbb{R}^5$. Moreover, the mean curvature $\vec{H}_\varphi$ and the normal curvature $K_\varphi^\perp$ of $\varphi(M) \subset \mathbb{R}^4$ at p are such that

$$|\vec{H}_\varphi|^2 = \left|\vec{H} \times \frac{\delta(u)}{|\delta(u)|}\right|^2 \quad \text{and} \quad K_\varphi^\perp = 2\left\langle \vec{N}, \frac{\delta(u)}{|\delta(u)|}\right\rangle,$$

where $\vec{H}$ and $|\vec{N}|^2$ are respectively defined in Definition 1.4 and in (1.5).

In the following example, we calculate the invariants of an explicit surface immersed in $\mathbb{R}^5$. This surface is not contained in any affine 4-dimensional subspace of $\mathbb{R}^5$. Nevertheless, it admits isometric reductions of codimension satisfying the hypothesis of Corollary 5.4. We show one of them.

Example 5.6. Consider the surface M immersed in $\mathbb{R}^5$ parametrized by

$$\mathbf{x}(u,v) = (u, v, u^2 - v^2, 4uv, u^2 + v^2).$$

We define the moving frame

$$(e_1(u,v), e_2(u,v), \vec{N}_1(u,v), \vec{N}_2(u,v), \vec{N}_3(u,v))$$

as follows: for $(u,v) \neq (0,0)$, $i = 1,2$ and $j = 1,2,3$,

$$e_i(u,v) = \frac{\widetilde{e}_i(u,v)}{|\widetilde{e}_i(u,v)|} \quad \text{and} \quad \vec{N}_j(u,v) = \frac{\widetilde{N}_j(u,v)}{|\widetilde{N}_j(u,v)|}$$

with

$$\begin{aligned}
\widetilde{e}_1(u,v) &= (1, 0, 2u, 4v, 2u)\,, \\
\widetilde{e}_2(u,v) &= (-16uv, 16v^2 + 8u^2 + 1, -2v(6v^2 + 24u^2 + 1), 4u(8u^2 + 1), \\
&\quad\ 2v(16v^2 - 8u^2 + 1)), \\
\widetilde{N}_1(u,v) &= \left(0, 0, v^2 - u^2, -uv, v^2 + u^2\right), \\
\widetilde{N}_2(u,v) &= -\widetilde{e}_1 \times \widetilde{e}_2 \times \widetilde{N}_1 \times \widetilde{N}_3, \\
\widetilde{N}_3(u,v) &= \left(2v(2v^2 - u^2), 2u(v^2 - 2u^2), -uv, u^2 - v^2, 0\right),
\end{aligned}$$

and, if $(u,v)=(0,0)$,

$$\begin{aligned}
e_1(0,0) &= (1,0,0,0,0)\,, \\
e_2(0,0) &= (0,1,0,0,0)\,, \\
\vec{N}_1(0,0) &= \left(0,0,\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right), \\
\vec{N}_2(0,0) &= \left(0,0,-\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right), \\
\vec{N}_3(0,0) &= (0,0,0,-1,0)\,.
\end{aligned}$$

Using this moving frame to express the second fundamental form of the surface, we easily obtain the following expressions of the invariants:

$$\begin{aligned}
K(u,v) &= -16(8v^2+8u^2+1)C(u,v)^2, \\
|\vec{H}(u,v)|^2 &= 4(4096(u^8+v^8)+3840(u^6+v^6)+(672+65536u^2v^2)(u^4+v^4)+ \\
&\quad (44+38400u^2v^2)(u^2+v^2)+270336u^4v^4+3456u^2v^2+1)C(u,v)^3, \\
\Delta(u,v) &= -16(16v^2+16u^2+1)C(u,v)^3, \\
|\vec{N}(u,v)|^2 &= 64(256(u^6+v^6)+224(u^4+v^4)+(28+6144u^2v^2)(u^2+v^2)+ \\
&\quad 832u^2v^2+1)C(u,v)^4, \\
\Gamma(u,v) &= 8C(u,v)^2,
\end{aligned}$$

where $C(u,v)=(128v^4+64u^2v^2+24v^2+128u^4+24u^2+1)^{-1}$.

Note that the condition $\Gamma>0$ at each point of M, means that the curvature ellipse is not degenerate and does not lie in any plane passing through $O\in N_{\mathbf{x}(u,v)}M$.

Now, a direct calculation shows that the vector $w\in T_{\mathbf{x}(u,v)}M$ given in (e_1,e_2) by

$$w(u,v)=\left(u(32v^2+8u^2+1)\sqrt{C(u,v)},v\right)$$

satisfies $S_1(w)=0$ and $\nabla^{\perp}_{w}\vec{N}_1=0$, where S_1 denotes the shape operator associated to $\vec{N}_1$ and given by

$$S_1=\frac{1}{B_1}\begin{pmatrix} 4v^2(C(u,v))^{-1/2} & -4uv(32v^2+8u^2+1) \\ -4uv(32v^2+8u^2+1) & 4u^2(32v^2+8u^2+1)^2\sqrt{C(u,v)} \end{pmatrix}$$

with

$$B_1=(16v^2+8u^2+1)\sqrt{2v^4+u^2v^2+2u^4}(C(u,v))^{-1/2}.$$

Since $\widetilde{w}=\frac{w}{|w|}$ is in the kernel of S_1 we have

$$II(\widetilde{w},\widetilde{w}) = \langle S_{\vec{N}_2}(\widetilde{w}),\widetilde{w}\rangle\vec{N}_2+\langle S_{\vec{N}_3}(\widetilde{w}),\widetilde{w}\rangle\vec{N}_3,$$

and

$$II(\widetilde{w},\widetilde{w}^{\perp}) = \langle S_{\vec{N}_2}(\widetilde{w}),\widetilde{w}^{\perp}\rangle\vec{N}_2+\langle S_{\vec{N}_3}(\widetilde{w}),\widetilde{w}^{\perp}\rangle\vec{N}_3,$$

which implies that

$$\widetilde{\delta}(\widetilde{w}) = \frac{1}{|II(\widetilde{w},\widetilde{w}) \times II(\widetilde{w},\widetilde{w}^{\perp})|} II(\widetilde{w},\widetilde{w}) \times II(\widetilde{w},\widetilde{w}^{\perp}) = \pm \vec{N}_1.$$

Thus, $\widetilde{\delta}(\widetilde{w})$ is parallel along $\widetilde{w}$. Consequently, Corollary 5.4 implies that $\widetilde{w}$ is a field of asymptotic directions, and there exists a $\tilde{\delta}(\widetilde{w})$-isometric reduction of codimension of M whose normal bundle is isomorphic to $\mathcal{O}_{\widetilde{w}}$.

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