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FLAT SURFACES ALONG CUSPIDAL EDGES

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ABSTRACT. We consider developable surfaces along the singular set of a cuspidal edge surface which are regarded as flat approximations of the cuspidal edge surface. For the study of singularities of such developable surfaces, we introduce the notion of Darboux frames along cuspidal edges, and introduce invariants. As a by-product, we introduce the notion of higher-order helices which are generalizations of previous notions of generalized helices (i.e., slant helices and clad helices). We use this notion to characterize special cuspidal edges.

1. Introduction

In recent decades, there have appeared several articles concerning the differential geometry of singular surfaces in Euclidean 3-space [5, 6, 19, 20, 21, 25, 27, 28, 32]. Wave fronts are particularly interesting singular surfaces which always have normal directions, even along singularities. A cuspidal edge surface is one of the generic wave fronts in Euclidean 3-space. In this paper, we consider developable surfaces along the singular curve of a cuspidal edge surface in Euclidean 3-space. Such a developable surface is called a *developable surface along* the cuspidal edge. Actually there are infinitely many developable surfaces along a cuspidal edge. Since a cuspidal edge surface has the normal direction at any point (even at a singular point), we focus on two typical developable surfaces along the cuspidal edge. One of them is a developable surface which is tangent to the cuspidal edge surface and the other is normal to the cuspidal edge surface. These two developable surfaces are considered to be flat approximations of the cuspidal edge surface along the cuspidal edge. We investigate the singularities of these developable surfaces along the cuspidal edge and introduce new invariants for the cuspidal edge.

For this purpose, we introduce the notion of Darboux frames along cuspidal edges, which is analogous to the notion of Darboux frames along curves on regular surfaces (cf. [7, 8, 14]). Since the Darboux frame along a cuspidal edge is an orthonormal frame along the cuspidal edge, we can obtain structure equations and invariants (cf. Proposition 3.1). We show that these invariants are equal to the invariants which are known as basic invariants of a cuspidal edge in [20, 21, 27], in which the normal form of the cuspidal edge was used for the study of geometric properties. The normal form of the cuspidal edge is a very strong tool from a singularity theory viewpoint. However, it is rather difficult to understand the geometric meanings intuitively. Here, we emphasize that we use the Darboux frame instead of the normal form of the cuspidal edge. By using the Darboux frame, we can directly and intuitively understand geometric properties of the cuspidal edge.

The precise definition of the cuspidal edge (surface) is given as follows: The unit cotangent bundle $T_1^*\mathbb{R}^3$ of $\mathbb{R}^3$ has a canonical contact structure and can be identified with the unit tangent bundle $T_1\mathbb{R}^3$. Let α denote that canonical contact form. Let M be a 2-dimensional manifold. A map $i : M \to T_1\mathbb{R}^3$ is said to be *isotropic* if the pull-back $i^*\alpha$ vanishes identically. We call

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the image of $\pi \circ i$ the *wave front set* of i, where $\pi : T_1\mathbb{R}^3 \to \mathbb{R}^3$ is the canonical projection and we denote it by $W(i)$. Moreover, i is called the *Legendrian lift* of $W(i)$. With this framework, we define the notion of fronts as follows: A map-germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ is called a *frontal* if there exists a unit vector field ν (called a *unit normal of* f) of $\mathbb{R}^3$ along f such that

$$L = (f, \nu) : (\mathbb{R}^2, 0) \to (T_1\mathbb{R}^3, 0)$$

is an isotropic map by an identification $T_1\mathbb{R}^3 = \mathbb{R}^3 \times S^2$, where S^2 is the unit sphere in $\mathbb{R}^3$ (cf. [1], see also [18]). A frontal f is a *front* if the above L can be taken as an immersion. A point $q \in (\mathbb{R}^2, 0)$ is a singular point if f is not an immersion at q. A map $f : M \to N$ between M and a 3-dimensional manifold N is called a frontal (respectively, a front) if for every $p \in M$, the map-germ f at p is a frontal (respectively, a front). A singular point p of a map f is called a *cuspidal edge* if the map-germ f at p is $\mathcal{A}$-equivalent to $(u, v) \mapsto (u, v^2, v^3)$ at 0. (Two map-germs $f_1, f_2 : (\mathbb{R}^n, 0) \to (\mathbb{R}^m, 0)$ are $\mathcal{A}$-*equivalent* if there exist diffeomorphisms $S : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ and $T : (\mathbb{R}^m, 0) \to (\mathbb{R}^m, 0)$ such that $f_2 \circ S = T \circ f_1$.) Therefore if the singular point p of f lies on a cuspidal edge, then f is a front at p, and furthermore, they are one of two possible types of generic singularities of fronts (the other one is a *swallowtail* which is a singular point p of f satisfying that f at p is $\mathcal{A}$-equivalent to $(u, v) \mapsto (u, u^2v + 3u^4, 2uv + 4u^3)$ at 0).

On the other hand, a developable surface is known to be a frontal, so that the normal direction is well-defined at any point. We say that a developable surface is an *osculating developable surface* along the cuspidal edge if it contains the singular set of the cuspidal edge such that the normal direction of the developable surface coincides with the normal direction of the cuspidal edge at any point of the singular set. We also say that a developable surface is a *normal developable surface* along the cuspidal edge if it contains the singular set of the cuspidal edge such that the normal direction of the developable surface belongs to the tangent plane of the cuspidal edge at any point of the singular set. In this paper, we study the geometric properties of cuspidal edges using these two developable surfaces along cuspidal edges. In particular, we show that the singular values of those developable surfaces characterize some cuspidal edges with special geometric properties. As a by-product, we introduce the notion of higher order helices which is a generalization of previous notions of generalized helices (i.e., slant helices and clad helices) in [13, 30, 31].

This paper is organized as follows: We describe basic properties of cuspidal edges in §2. The Darboux frame along a cuspidal edge is introduced in §3. Associated to the Darboux frame, we introduce three basic invariants, which are the same as those of cuspidal edges, as in [20, 21, 27]. We also introduce two vector fields along a cuspidal edge which will play critical roles in this paper. In §4, definitions and basic properties of (general) developable surfaces are described. Moreover, the notion of higher order helices is introduced and characterizations of those generalized helices by the curvature and the torsion are given (cf. Proposition 4.4, the Lancret type theorem). We also consider a tangent developable surface of a curve such that the curve is a kth-order helix. We give a characterization of such tangent developable surfaces as a corollary of Proposition 4.4 (cf. Theorem 4.6). Returning to the study of cuspidal edges, we introduce two developable surfaces along a cuspidal edge in §5. In order to classify the singularities of those two developable surfaces, we introduce four new invariants represented by the three basic invariants of a cuspidal edge. The classifications are give by those four invariants (cf. Theorems 5.1 and 5.3). Moreover, if one of the three basic invariants is identically equal to zero, we have special developable surfaces alone the cuspidal edge, whose singularities are classified in Corollaries 5.2 and 5.4. If two of these three basic invariants are identically equal to zero, the cuspidal edge is a subset of a plane (cf. §5.3). If the all three basic invariants are identically equal to zero, the cuspidal edge is a line. In §6 we investigate cuspidal edges with special properties. We compare the properties of cuspidal edges with those of curves on regular surfaces in §7. In particular, we give a geometric interpretation of the cuspidal torsion. Finally we briefly describe definitions and properties of support functions of a cuspidal edge in the appendix. By using support functions, we give geometric interpretations of singularities from the contact viewpoint.

2. Cuspidal edges

Let $f:(\mathbb{R}^2,0)\to(\mathbb{R}^3,0)$ be a frontal with a unit normal vector field ν. For a coordinate system (u,v) on $(\mathbb{R}^2,0)$, we define a function λ by $\lambda=\det(f_u,f_v,\nu)$ and call it the *signed area density of* f. We say that a singular point $0\in(\mathbb{R}^2,0)$ is a *non-degenerate singular point* if $d\lambda(0)\neq 0$. Let 0 be a non-degenerate singular point. Then there exists a vector field germ η on $(\mathbb{R}^2,0)$ such that $\langle\eta(p)\rangle_{\mathbb{R}}=\ker df_p$ for any $p\in S(f)$, where $S(f)$ is the set germ of the singular points of f. We call η a *null vector field*. We say that $0\in(\mathbb{R}^2,0)$ is a *singular point of the first kind* if it is non-degenerate and $\eta(0)$ is transversal to $S(f)$ at 0. The following lemma is well-known.

Lemma 2.1. ([28, Corollary 2.5, p.735], see also [18]) *Let* 0 *be a singular point of a front* $f:(\mathbb{R}^2,0)\to(\mathbb{R}^3,0)$. *Then* 0 *is a cuspidal edge* (*respectively, swallowtail*) *if and only if* $\eta\lambda\neq 0$ (*respectively,* $\eta\lambda=0$ $\eta\eta\lambda\neq 0$ *and* $d\lambda\neq 0$) *at* 0, *where* $\eta\lambda$ *stands for the directional derivative of* λ *by* η.

By this lemma, if f is a front, then the singular point of the first kind is a cuspidal edge. The cuspidal cross cap $((u,v)\mapsto(u,v^2,uv^3))$ is a singular point of the first kind, which is not a front. For details see [27].

On the other hand, it is known [20, 21, 27] that there exist several geometric invariants for cuspidal edges in $\mathbb{R}^3$. In [21], these invariants are defined and studied for cuspidal edges in any Riemannian 3-manifold. See [21] for details.

Let $f:(\mathbb{R}^2,0)\to(\mathbb{R}^3,0)$ be a frontal and ν the unit normal vector field. Suppose that 0 is a singular point of the first kind. Then one can easily see that there exists a coordinate system (u,v) of $(\mathbb{R}^2,0)$ with the following properties:

(1) $S(f)=\{v=0\}$,
(2) u is an arc-length parameter of the curve given by $f(u,0)$,
(3) $\ker df_{(u,0)}$ is generated by $\partial/\partial v$,
(4) (u,v) is compatible with the orientation of $\mathbb{R}^2$.

We call a coordinate system satisfying these properties an *adapted coordinate system centered at* $(u,v)=(0,0)$. On an adapted coordinate system, since $\partial/\partial u$ is tangent to $S(f)$, it holds that $\lambda_u=0$. Thus $d\lambda(0)\neq 0$ implies $\lambda_v\neq 0$. Since $f_v(0)=0$, we see that

$$\det(f_u,f_{vv},\nu)(0)=\lambda_v(0)\neq 0.$$

Hence one can choose the direction of ν such that $\det(f_u,f_{vv},\nu)(0)>0$. We always choose the unit normal vector ν of f on an adapted coordinate system centered at a singular point of the first kind so that it satisfies $\det(f_u,f_{vv},\nu)(0)>0$.

We define three invariants for f as follows on an adapted coordinate system (u,v):

$$\kappa_s(u)=\det\big(\gamma'(u),\gamma''(u),\nu(u,0)\big),\quad \kappa_\nu(u)=\langle\gamma''(u),\nu(u,0)\rangle,$$

$$\kappa_t(u)=\left[\frac{\det\big(\gamma',f_{vv},f_{uvv}\big)}{|\gamma'\times f_{vv}|^2}-\frac{\det\big(\gamma',f_{vv},f_{uu}\big)\langle\gamma',f_{vv}\rangle}{|\gamma'|^2|\gamma'\times f_{vv}|^2}\right]_{v=0},$$

where $\gamma(u)=f(u,0)$ and $\langle\ ,\ \rangle$ is the canonical inner product of $\mathbb{R}^3$. We call $\kappa_s(u)$ the *singular curvature*, $\kappa_\nu(u)$ the *normal curvature* and $\kappa_t(u)$ the *cuspidal torsion* of f at $(u,0)$, respectively. The singular curvature measures convexity or concavity of a cuspidal edge and the cuspidal torsion measures the rate of revolution of the direction of incidence of a cusp along a cuspidal edge. See [20, 27] for details. See [9, 24, 21] for other studies of geometric invariants of cuspidal edges.

3. Darboux frames along cuspidal edges

Let $f : I \times (-\varepsilon, \varepsilon) \to \mathbb{R}^3$ be a frontal with a unit normal vector ν, where I is an open interval or a circle, and $\varepsilon > 0$. Assume that $I \times \{0\}$ consists of singular points of the first kind, and we take a coordinate system (u, v) of $I \times (-\varepsilon, \varepsilon)$ satisfying that

(1) u is an arc-length parameter of the curve given by $f(u, 0)$,
(2) $\ker df_{(u,0)}$ is generated by $\partial/\partial v$,
(3) (u, v) is compatible with the orientation of $\mathbb{R}^2$.

We also call this coordinate system *adapted*. In this paper we always choose the unit normal vector ν of f on an adapted coordinate system so that it satisfies $\det(f_u, f_{vv}, \nu)(u, 0) > 0$.

We now set $\boldsymbol{\gamma}(u) = f(u, 0)$ and consider unit vector fields $\boldsymbol{e}(u) = f_u(u, 0) = \boldsymbol{\gamma}'(u)$, $\boldsymbol{\nu}(u) = \nu(u, 0)$ and $\boldsymbol{b}(u) = -\boldsymbol{e}(u) \times \boldsymbol{\nu}(u)$ along $\boldsymbol{\gamma}$. Here, $\boldsymbol{a}_1 \times \boldsymbol{a}_2$ is the exterior product of $\boldsymbol{a}_1, \boldsymbol{a}_2$ in $\mathbb{R}^3$. Then $\{\boldsymbol{e}, \boldsymbol{b}, \boldsymbol{\nu}\}$ is a orthonormal frame along $\boldsymbol{\gamma}$. We call $\{\boldsymbol{e}, \boldsymbol{b}, \boldsymbol{\nu}\}$ the *Darboux frame* along the cuspidal edge $\boldsymbol{\gamma}$. As the structure equations for the Darboux frame along the cuspidal edge, we have the following proposition.

Proposition 3.1 (Frenet-Serret type formulae)**.**

$$
\left\{
\begin{aligned}
\boldsymbol{e}'(u) &= \kappa_s(u)\boldsymbol{b}(u) + \kappa_\nu(u)\boldsymbol{\nu}(u), \\
\boldsymbol{b}'(u) &= -\kappa_s(u)\boldsymbol{e}(u) + \kappa_t(u)\boldsymbol{\nu}(u), \\
\boldsymbol{\nu}'(u) &= -\kappa_\nu(u)\boldsymbol{e}(u) - \kappa_t(u)\boldsymbol{b}(u).
\end{aligned}
\right. \tag{3.1}
$$

By using the matrix representation, we have

$$
\begin{pmatrix} \boldsymbol{e}' \\ \boldsymbol{b}' \\ \boldsymbol{\nu}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa_s & \kappa_\nu \\ -\kappa_s & 0 & \kappa_t \\ -\kappa_\nu & -\kappa_t & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{e} \\ \boldsymbol{b} \\ \boldsymbol{\nu} \end{pmatrix}.
$$

Proof. Since $\{\boldsymbol{e}, \boldsymbol{b}, \boldsymbol{\nu}\}$ is an orthonormal frame along $\boldsymbol{\gamma}$, we have

$$
\begin{pmatrix} \boldsymbol{e}' \\ \boldsymbol{b}' \\ \boldsymbol{\nu}' \end{pmatrix} = \begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \delta \\ -\beta & -\delta & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{e} \\ \boldsymbol{b} \\ \boldsymbol{\nu} \end{pmatrix},
$$

where $\alpha = \langle \boldsymbol{e}', \boldsymbol{b} \rangle$, $\beta = \langle \boldsymbol{e}', \boldsymbol{\nu} \rangle$ and $\delta = -\langle \boldsymbol{\nu}', \boldsymbol{b} \rangle$. By a straightforward calculation, we have

$$
\alpha = \langle \boldsymbol{e}', \boldsymbol{b} \rangle = -\langle \boldsymbol{e}', \boldsymbol{e} \times \boldsymbol{\nu} \rangle = \det(\boldsymbol{e}, \boldsymbol{e}', \boldsymbol{\nu}) = \det(\boldsymbol{\gamma}', \boldsymbol{\gamma}'', \boldsymbol{\nu}).
$$

Since $\det(f_u, f_{vv}, \nu) > 0$, we have $\alpha = \kappa_s$. It follows from $\beta = \langle \boldsymbol{e}', \boldsymbol{\nu} \rangle$ that $\beta = \kappa_\nu$. Since f has a singular point of the first kind at $0 \in (\mathbb{R}^2, 0)$, f_{vv}, f_u are linearly independent. We set

$$
(\tilde{u}, \tilde{v}) = \phi(u, v) = (u + a(u)v^2, v), \quad a(u) = \langle f_u(u, 0), f_{vv}(u, 0) \rangle / 2.
$$

Then we see that

$$
\begin{pmatrix} u_{\tilde{u}} & u_{\tilde{v}} \\ v_{\tilde{u}} & v_{\tilde{v}} \end{pmatrix} = \frac{1}{1 + a'(u)v^2} \begin{pmatrix} 1 & -2a(u)v \\ 0 & 1 + a'(u)v^2 \end{pmatrix} \circ \phi^{-1}(\tilde{u}, \tilde{v})
$$

$f_{\tilde{u}} = f_u$. Moreover, since

$$
f_{\tilde{v}} = f_u u_{\tilde{u}} + f_v = f_u \frac{-2a(u)v}{1 + a'(u)v^2} + f_v,
$$

it holds that

$$\begin{aligned} f_{\tilde{v}\tilde{v}}(\tilde{u},0) &= f_{\tilde{v}v}(u,0) = \Big(f_u v\frac{-2a(u)v}{1+a'(u)v^2} + f_u\frac{-2a(u)}{1+a'(u)v^2} + f_u\frac{-4a(u)a'(u)v^2}{(1+a'(u)v^2)^2} + f_{vv}\Big)(u,0) \\ &= -2a(u)f_u(u,0) + f_{vv}(u,0). \end{aligned}$$

By the definition of $a(u)$, it holds that $\langle f_{\tilde{u}}, f_{\tilde{v}\tilde{v}}\rangle(\tilde{u},0) = 0$. Therefore we can choose an adapted coordinate system (u,v) such that f_u, f_{vv} are orthogonal, namely $\boldsymbol{\nu} = f_u \times f_{vv}/|f_u \times f_{vv}|$ on the u-axis. Moreover, we have $-\boldsymbol{b} = \boldsymbol{e}\times\boldsymbol{\nu} = f_u \times (f_u \times f_{vv})/|f_u \times f_{vv}| = -f_{vv}/|f_u \times f_{vv}|$, so that

$$-\delta = \langle \boldsymbol{\nu}', \boldsymbol{b}\rangle = \frac{\langle f_u \times f_{uvv}, f_{vv}\rangle}{|f_u \times f_{vv}|^2} = \frac{\det(f_u, f_{uvv}, f_{vv})}{|f_u \times f_{vv}|^2} = -\frac{\det(f_u, f_{vv}, f_{uvv})}{|f_u \times f_{vv}|^2} = -\kappa_t,$$

on the u-axis. □

We define a vector field $D_o(u)$ along $\boldsymbol{\gamma}$ by

$$D_o(s) = \kappa_t(u)\boldsymbol{e}(u) - \kappa_\nu(u)\boldsymbol{b}(u),$$

which is called an *osculating Darboux vector field* along $\boldsymbol{\gamma}$. If $\kappa_\nu^2 + \kappa_t^2 \neq 0$, we can define the *unit osculating Darboux vector field* by

$$\overline{D_o}(u) = \frac{\kappa_t(u)\boldsymbol{e}(u) - \kappa_\nu(u)\boldsymbol{b}(u)}{\sqrt{\kappa_\nu(u)^2 + \kappa_t(u)^2}}. \tag{3.2}$$

We also define a vector field $D_r(u)$ along $\boldsymbol{\gamma}$ by

$$D_r(s) = \kappa_t(u)\boldsymbol{e}(u) + \kappa_s(u)\boldsymbol{\nu}(u),$$

which is called a *normal Darboux vector field* along $\boldsymbol{\gamma}$. If $\kappa_t^2 + \kappa_s^2 \neq 0$, we can also define the *unit normal Darboux vector field* by

$$\overline{D}_r(u) = \frac{\kappa_t(u)\boldsymbol{e}(u) + \kappa_s(u)\boldsymbol{\nu}(u)}{\sqrt{\kappa_t(u)^2 + \kappa_s(u)^2}}. \tag{3.3}$$

We now define the notion of contour edges of cuspidal edges. For a unit vector $\boldsymbol{k} \in S^2$, we say that the cuspidal edge $S(f)$ is the *tangential contour edge of the orthogonal projection* with direction $\boldsymbol{k}$ if

$$S(f) = \{(u,0) \in (\mathbb{R}^2,0) \mid \langle \boldsymbol{\nu}(u), \boldsymbol{k}\rangle = 0\}.$$

We also say that the cuspidal edge $S(f)$ is the *normal contour edge of the orthogonal projection* with direction $\boldsymbol{k}$ if

$$S(f) = \{(u,0) \in (\mathbb{R}^2,0) \mid \langle \boldsymbol{b}(u), \boldsymbol{k}\rangle = 0\}.$$

Moreover, for a point $\boldsymbol{c} \in \mathbb{R}^3$, say that the cuspidal edge $S(f)$ is the *tangential contour edge of the central projection* (respectively, *normal contour edge of the central projection*) with center $\boldsymbol{c}$ if

$$\begin{aligned} S(f) &= \{(u,0) \in (\mathbb{R}^2,0) \mid \langle f(u,0) - \boldsymbol{c}, \boldsymbol{\nu}(u)\rangle = 0\ \}. \\ \big(\text{respectively, } S(f) &= \{(u,0) \in (\mathbb{R}^2,0) \mid \langle f(u,0) - \boldsymbol{c}, \boldsymbol{b}(u)\rangle = 0\ \}.\big) \end{aligned}$$

For a regular surface, the notion of contour edges corresponds to the notion of contour generators [3].

On the other hand, there is a notion of isophotic curves on a regular surfaces. An isophotic curve of a surface is a curve consisting of points which have the same light intensity from a given light source. If the light source is infinitely far from the surface, the light rays might be considered as parallel lines. In this case, an *isophotic curve* is a curve on a regular surface such that the normal of the surface along the curve makes a constant angle with a fixed direction. Therefore, we can define the notion of isophotic curves on the cuspidal edge exactly the same way as the definition for curves on a regular surface. In particular, the cuspidal edge $S(f)$ is said to be a *normally isophotic edge* if there exists a unit vector $\boldsymbol{d}$ such that $\langle \boldsymbol{d}, \boldsymbol{\nu}(u)\rangle$ is constant. We also say that $S(f)$ is a *tangential isophotic edge* if there exists a unit vector $\boldsymbol{d}$ such that $\langle \boldsymbol{d}, \boldsymbol{b}(u)\rangle$ is constant.

We emphasize that notions of contour generators and isophotic curves on regular surfaces play important roles in the vision theory and visual psychophysics (cf. [3, 15, 16, 17]).

4. Developable surfaces and generalizations of helices

We briefly review the notions and basic properties of ruled surfaces and developable surfaces. Let $\boldsymbol{\gamma} : I \longrightarrow \mathbb{R}^3$ and $\boldsymbol{\xi} : I \longrightarrow \mathbb{R}^3 \setminus \{\boldsymbol{0}\}$ be C^∞-maps, where I is an open interval or a circle. Then we define a map $F_{(\boldsymbol{\gamma},\boldsymbol{\xi})} : I \times \mathbb{R} \longrightarrow \mathbb{R}^3$ by

$$F_{(\boldsymbol{\gamma},\boldsymbol{\xi})}(u,t) = \boldsymbol{\gamma}(u) + t\boldsymbol{\xi}(u).$$

We call the image of $F_{(\boldsymbol{\gamma},\boldsymbol{\xi})}$ a *ruled surface*, the map $\boldsymbol{\gamma}$ a *base curve* and the map $\boldsymbol{\xi}$ a *director curve*. The line defined by $\boldsymbol{\gamma}(u) + t\boldsymbol{\xi}(u)$ for a fixed $u \in I$ is called a *ruling*. If the direction of the director curve $\boldsymbol{\xi}$ is constant, we call $F_{(\boldsymbol{\gamma},\boldsymbol{\xi})}$ a (*generalized*) *cylinder*. Using the notation $\overline{\boldsymbol{\xi}}(u) = \boldsymbol{\xi}(u)/\|\boldsymbol{\xi}(u)\|$, we have $F_{(\boldsymbol{\gamma},\boldsymbol{\xi})}(I \times \mathbb{R}) = F_{(\boldsymbol{\gamma},\overline{\boldsymbol{\xi}})}(I \times \mathbb{R})$. In this case $F_{(\boldsymbol{\gamma},\boldsymbol{\xi})}$ is a cylinder if and only if $\dot{\overline{\boldsymbol{\xi}}}(u) \equiv 0$, where $\equiv$ means that equality holds identically. We say that $F_{(\boldsymbol{\gamma},\boldsymbol{\xi})}$ is *non-cylindrical* if $\dot{\overline{\boldsymbol{\xi}}}(u) \neq 0$ for any $u \in I$. Suppose that $F_{(\boldsymbol{\gamma},\boldsymbol{\xi})}$ is non-cylindrical. Then a *striction curve* is defined to be

$$\boldsymbol{s}(u) = \boldsymbol{\gamma}(u) - \frac{\langle \dot{\boldsymbol{\gamma}}(u), \dot{\overline{\boldsymbol{\xi}}}(u)\rangle}{\langle \dot{\overline{\boldsymbol{\xi}}}(u), \dot{\overline{\boldsymbol{\xi}}}(u)\rangle}\overline{\boldsymbol{\xi}}(u). \tag{4.1}$$

It is known that a singular point of the non-cylindrical ruled surface is located on the striction curve. We call the ruled surface with vanishing Gaussian curvature on the regular part a *developable surface*. It is known that a ruled surface $F_{(\boldsymbol{\gamma},\boldsymbol{\xi})}$ is a developable surface if and only if

$$\det\left(\dot{\boldsymbol{\gamma}}(u), \boldsymbol{\xi}(u), \dot{\boldsymbol{\xi}}(u)\right) = 0, \tag{4.2}$$

where $\dot{\boldsymbol{\gamma}}(u) = (d\boldsymbol{\gamma}/du)(u)$(cf., [12]). The set of singular points of a non-cylindrical developable surface coincides with the striction curve[11]. A non-cylindrical ruled surface $F_{(\boldsymbol{\gamma},\boldsymbol{\xi})}$ is a *cone* if the striction curve $\boldsymbol{s}$ is constant. It is known (cf., [12]) that a non-cylindrical developable surface $F_{(\boldsymbol{\gamma},\boldsymbol{\xi})}$ is a wave front if and only if

$$\psi(u) = \det\left(\boldsymbol{\xi}(u), \dot{\boldsymbol{\xi}}(u), \ddot{\boldsymbol{\xi}}(u)\right) \neq 0. \tag{4.3}$$

In this case we call $F_{(\boldsymbol{\gamma},\boldsymbol{\xi})}$ a (*non-cylindrical*) *developable front*. Let $F_{(\boldsymbol{\gamma},\boldsymbol{\xi})}(u,t)$ be a non-cylindrical developable surface. Then by (4.2), there exist $\alpha(u)$ and $\beta(u)$ such that $\dot{\gamma}(u) = \alpha(u)\xi(u) + \beta(u)\dot{\xi}(u)$. The striction curve of $F_{(\boldsymbol{\gamma},\boldsymbol{\xi})}$ is written as $\boldsymbol{s}(u) = \gamma(u) - \beta(u)\boldsymbol{\xi}(u)$, and we see that the signed area density of $F_{(\boldsymbol{\gamma},\boldsymbol{\xi})}$ is proportional to $\lambda = t + \beta(u)$. Thus a singular point of $F_{(\boldsymbol{\gamma},\boldsymbol{\xi})}$ is always non-degenerate. By Lemma 2.1, we have the following:

Proposition 4.1. *With the above notations, a singular point $(u, -\beta(u))$ of $F_{(\boldsymbol{\gamma},\boldsymbol{\xi})}$ is a cuspidal edge* (*respectively, a swallowtail*) *if and only if $\psi(u) \neq 0$ and $\beta'(u) - \alpha(u) \neq 0$* (*respectively, $\psi(u) \neq 0$, $\beta'(u) - \alpha(u) = 0$ and $\beta''(u) - \alpha'(u) \neq 0$*)*.*

On the other hand, by [4, Corollary 1.5], we have the following:

Proposition 4.2. *With the same notations as in Proposition 4.1, a singular point $(u, -\beta(u))$ of $F_{(\boldsymbol{\gamma},\boldsymbol{\xi})}$ is a cuspidal cross cap if and only if $\beta'(u) - \alpha(u) \neq 0$, $\psi(u) = 0$ and $\psi'(u) \neq 0$.*

See [23] for other investigations of developable surfaces with singularities.

Remarkable generalizations of helices in $\mathbb{R}^3$ were introduced and investigated in [13, 30, 31]. Let $\boldsymbol{\gamma} : I \to \mathbb{R}^3$ be a space curve with an arc-length parameter u. We call $\boldsymbol{\gamma}$ a *Frenet curve* if $\kappa(u) = \|\boldsymbol{\gamma}''(u)\| \neq 0$. For a Frenet curve $\boldsymbol{\gamma}$, let $\{\boldsymbol{t}, \boldsymbol{n}_{\boldsymbol{\gamma}}, \boldsymbol{b}_{\boldsymbol{\gamma}}\}$ be the Frenet frame along $\boldsymbol{\gamma}$, and κ, τ the curvature and torsion, respectively. Then $\boldsymbol{\gamma}$ is said to be a *cylindrical helix* (or, a *generalized helix*) if there exists a constant vector $\boldsymbol{v}$ such that $\boldsymbol{t}(u)$ makes a constant angle with $\boldsymbol{v}$. By the Frenet-Serret formulae, this condition is equivalent to the condition that $\boldsymbol{n}_{\boldsymbol{\gamma}}(u)$ is orthogonal to $\boldsymbol{v}$. Moreover, $\boldsymbol{\gamma}$ is called a *slant helix* if there exists a constant vector $\boldsymbol{v}$ such that $\boldsymbol{n}_{\boldsymbol{\gamma}}(u)$ makes a constant angle with $\boldsymbol{v}$ [13]. By definition, $\boldsymbol{\gamma}$ is a slant helix if and only if $\boldsymbol{n}_{\boldsymbol{\gamma}}(u)$ is a circle in the unit sphere. Recently, the notion of clad helices have been introduced in [30, 31]. We say that $\boldsymbol{\gamma}$ is a *clad helix* if $\boldsymbol{n}_{\boldsymbol{\gamma}}(u)$ is a cylindrical helix. Since $\boldsymbol{n}_{\boldsymbol{\gamma}}(u)$ is a curve in the unit sphere, it is a spherical cylindrical helix. It is classically known that $\boldsymbol{\gamma}$ is cylindrical helix if and only if τ/κ is constant (i.e., the Lancret theorem). If both of τ and κ are constant, $\boldsymbol{\gamma}$ is a circular helix (i.e., an ordinary helix). Therefore, a cylindrical helix is a generalization of circular helix. A curve $\boldsymbol{\gamma}$ is a slant helix if and only if

$$\theta(u) = \frac{\kappa^2}{(\kappa^2+\tau^2)^{3/2}} \left(\frac{\tau}{\kappa}\right)'(u)$$

is constant [13]. Moreover, $\boldsymbol{\gamma}$ is a clad helix if and only if

$$\eta(u) = \frac{\theta'}{(\kappa^2+\tau^2)^{1/2}(1+\theta^2)^{3/2}}(u)$$

is constant [30, 31]. See [13, 30, 31] for details. Motivated by the results in [13, 30, 31], we consider generalizations of these notions of helices. For a Frenet curve $\boldsymbol{\gamma} : I \longrightarrow \mathbb{R}^3$, we say that $\boldsymbol{\gamma}$ is a *0th-order helix* if it is a cylindrical helix, $\boldsymbol{\gamma}$ is a *1st-order helix* if it is a slant helix and $\boldsymbol{\gamma}$ is a *2nd-order helix* if it is a clad helix, respectively. For $k \geq 1$, we inductively define the notion of kth-order helices. We say that $\boldsymbol{\gamma}$ is a *kth-order helix* if $\boldsymbol{t}$ is a $(k-1)$th-order helix.

Proposition 4.3. *A Frenet curve $\boldsymbol{\gamma}$ is a kth-order helix if and only if $\boldsymbol{n}_{\boldsymbol{\gamma}}$ is a $(k-2)$th-order helix.*

Proof. For $k = 2$, $\boldsymbol{\gamma}$ is a 2nd-order helix if and only if $\boldsymbol{\gamma}$ is a clad helix. Therefore, $\boldsymbol{n}_{\boldsymbol{\gamma}}$ is a cylindrical helix. By definition, it means that $\boldsymbol{n}_{\boldsymbol{\gamma}}$ is a 0th-order helix. The assertion holds for $k = 2$. For $k > 2$, $\boldsymbol{\gamma}$ is a kth-order helix if and only if $\boldsymbol{t}$ is a $(k-1)$th-order helix. This means that $\boldsymbol{n}_{\boldsymbol{\gamma}} = \boldsymbol{t}'/\|\boldsymbol{t}'\|$ is a $(k-2)$th-order helix. This completes the proof. $\square$

We remark that a cylindrical helix is also called a *constant slope curve* because its tangent vector has a constant angle with a constant direction. We can interpret a constant slope as a 0th-order slope. In this sense, we also call a kth-order helix a *kth-order slope curve.*

On the other hand, we now give a characterization of kth-order helices by the curvature and the torsion (i.e., the Lancret-type theorem). We define $\mathscr{H}[\boldsymbol{\gamma}]_0(u) = \tau(u)/\kappa(u)$, which is called a *0th-order helical curvature* of $\boldsymbol{\gamma}$. We have

$$\theta(u) = \frac{\kappa^2}{(\kappa^2+\tau^2)^{3/2}} \left(\frac{\tau}{\kappa}\right)'(u) = \frac{1}{\kappa} \frac{\left(\frac{\tau}{\kappa}\right)'}{\left(1+\left(\frac{\tau}{\kappa}\right)^2\right)^{3/2}}(u) = \frac{1}{\kappa} \frac{\mathscr{H}[\boldsymbol{\gamma}]_0'}{\left(1+\left(\mathscr{H}[\boldsymbol{\gamma}]_0\right)^2\right)^{3/2}}(u).$$

We set $\mathscr{H}[\boldsymbol{\gamma}]_1(u) = \theta(u)$, which is called a *1st-order helical curvature.* Moreover, the *2nd-order helical curvature* of $\boldsymbol{\gamma}$ is defined to be

$$\mathscr{H}[\boldsymbol{\gamma}]_2(u) = \eta(u) = \frac{1}{\kappa(1+(\mathscr{H}[\boldsymbol{\gamma}])_0^2)^{1/2}} \frac{\mathscr{H}[\boldsymbol{\gamma}]_1'}{\left(1+\left(\mathscr{H}[\boldsymbol{\gamma}]_1\right)^2\right)^{3/2}}(u).$$

For $r \geq 2$, we inductively define

$$\mathscr{H}[\boldsymbol{\gamma}]_{2r-1}(u) = \frac{1}{(1+(\mathscr{H}[\boldsymbol{\gamma}]_{2r-3})^2)^{1/2}} \frac{\mathscr{H}[\boldsymbol{\gamma}]'_{2r-2}}{\left(1+\left(\mathscr{H}[\boldsymbol{\gamma}]_{2r-2}\right)^2\right)^{3/2}}(u),$$

which is called a $(2r-1)$*st-order helical curvature*, and

$$\mathscr{H}[\boldsymbol{\gamma}]_{2r}(u) = \frac{1}{(1+(\mathscr{H}[\boldsymbol{\gamma}]_{2r-3})^2)^{1/2}(1+(\mathscr{H}[\boldsymbol{\gamma}]_{2r-2})^2)^{1/2}} \frac{\mathscr{H}[\boldsymbol{\gamma}]'_{2r-1}}{\left(1+\left(\mathscr{H}[\boldsymbol{\gamma}]_{2r-1}\right)^2\right)^{3/2}}(u),$$

which is called a $2r$*th-order helical curvature*. On the other hand, let $\kappa_n(u)$ and $\tau_n(u)$ be the curvature and the torsion of the principal normal $\boldsymbol{n}(u)$, respectively. Then we can calculate that

$$\kappa_n(u) = \sqrt{1+(\mathscr{H}[\boldsymbol{\gamma}]_1)^2}(u),\ \tau_n(u) = \left(\frac{\mathscr{H}[\boldsymbol{\gamma}]'_1}{(1+(\mathscr{H}[\boldsymbol{\gamma}]_1)^2)(\kappa^2+\tau^2)^{1/2}}\right)(u).$$

By using these formulae, we can show that the above inductive definitions are well-defined. Then we have the following characterization of higher-order helices.

Proposition 4.4. *Let $\boldsymbol{\gamma} : I \longrightarrow \mathbb{R}^3$ be a Frenet curve. Then the following conditions are equivalent*:

(1) *$\boldsymbol{\gamma}$ is a kth-order helix,*
(2) *$\mathscr{H}[\boldsymbol{\gamma}]_k(u)$ is constant,*
(3) *$\mathscr{H}[\boldsymbol{\gamma}]_{k+1}(u)$ is identically equal to zero.*

Proof. By definition (2) and (3) are equivalent. It follows from [12, 30, 31] that conditions (1) and (2) are equivalent for $k \leq 2$. Let us write $\mathscr{H}[\boldsymbol{n}]_k(u)$ as the kth-order helical curvature of the principal normal curve $\boldsymbol{n}(u)$ of $\boldsymbol{\gamma}(u)$. By Proposition 4.3, $\boldsymbol{\gamma}(u)$ is a 3rd-order helix if and only if $\boldsymbol{n}(u)$ is a 1st-order helix. By the result in [12], this is equivalent to

$$\mathscr{H}[\boldsymbol{n}]_1(u) = \frac{1}{\kappa_n} \frac{(\mathscr{H}[\boldsymbol{n}]_0)'}{\left(1+(\mathscr{H}[\boldsymbol{n}]_0)^2\right)^{3/2}}(u)$$

being constant. If we substitute $\kappa_n(u) = \sqrt{1+(\mathscr{H}[\boldsymbol{\gamma}]_1)^2}(u)$ and $\mathscr{H}[\boldsymbol{n}]_0 = \tau_n/\kappa_n = \mathscr{H}[\boldsymbol{\gamma}]_2$, we have $\mathscr{H}[\boldsymbol{\gamma}]_3(u) = \mathscr{H}[\boldsymbol{n}]_1(u)$, so that conditions (1) and (2) are equivalent for $k = 3$. By Proposition 4.3, $\boldsymbol{\gamma}(u)$ is a 4th-order helix if and only if $\boldsymbol{n}(u)$ is a 2nd-order helix. This condition is equivalent to the condition that

$$\mathscr{H}[\boldsymbol{n}]_2(u) = \frac{1}{\kappa_n(1+(\mathscr{H}[\boldsymbol{n}]_0)^2)^{1/2}} \frac{(\mathscr{H}[\boldsymbol{n}]_1)'}{\left(1+(\mathscr{H}[\boldsymbol{n}]_1)^2\right)^{3/2}}(u)$$

is constant. If we substitute $\kappa_n(u) = \sqrt{1+(\mathscr{H}[\boldsymbol{\gamma}]_1)^2}(u)$, $\mathscr{H}[\boldsymbol{n}]_0 = \mathscr{H}[\boldsymbol{\gamma}]_2$ and $\mathscr{H}[\boldsymbol{n}]_1 = \mathscr{H}[\boldsymbol{\gamma}]_3$ into the above formulae, then the above condition is equivalent to the condition that

$$\mathscr{H}[\boldsymbol{\gamma}]_4(u) = \frac{1}{(1+(\mathscr{H}[\boldsymbol{\gamma}]_1)^2)(1+(\mathscr{H}[\boldsymbol{\gamma}]_2)^2)^{1/2}} \frac{\mathscr{H}[\boldsymbol{\gamma}]'_3}{(1+(\mathscr{H}[\boldsymbol{\gamma}]_3)^2)^{3/2}}(u)$$

is constant. Therefore, conditions (1) and (2) are equivalent for $k = 4$. We can show that condition (1) and (2) are equivalent by inductive arguments similar to the above cases. $\square$

We now consider the tangent surface $F_{(\boldsymbol{\gamma},\boldsymbol{t})}(u,t) = \boldsymbol{\gamma}(u) + t\boldsymbol{t}(u)$ for a Frenet curve $\boldsymbol{\gamma}(u)$. We remark that a tangent surface is a developable surface. Here, we consider tangent surfaces of special curves in $\mathbb{R}^3$. We also remark that $F_{(\boldsymbol{\gamma},\boldsymbol{t})}$ is non-cylindrical if and only if $\boldsymbol{\gamma}$ is a Frenet curve. We assume that $\boldsymbol{\gamma}$ is a Frenet curve and $F_{(\boldsymbol{\gamma},\boldsymbol{t})}$ is said to be a *developable surface with kth-order slope* if $\boldsymbol{\gamma}$ is a kth-order helix. In particular, a developable surface with 0th-order slope is called a *constant angle surface* [22] (or, a *developable surface of constant slope* [26, 6.3]). By Proposition 4.3, $F_{(\boldsymbol{\gamma},\boldsymbol{t})}$ is a developable surface with kth-order slope if and only if $\boldsymbol{n}_{\boldsymbol{\gamma}}(u)$ is a $(k-2)$th-order helix. By the Frenet-Serret formula $\boldsymbol{b}'_{\boldsymbol{\gamma}} = -\tau\boldsymbol{n}_{\boldsymbol{\gamma}}$, this implies that $\boldsymbol{b}_{\boldsymbol{\gamma}}$ is a $(k-1)$th-order helix. If $\tau \neq 0$, then the converse holds. Let $\boldsymbol{v} : I \longrightarrow S^2 \subset \mathbb{R}^3$ be a smooth unit vector field. For a unit constant vector $\boldsymbol{c}$, we say that $\boldsymbol{v}(u)$ has a 1st-order angle with $\boldsymbol{c}$ if $\langle \boldsymbol{v}(u), \boldsymbol{c}\rangle$ is constant. For $k \geq 2$, we say that $\boldsymbol{v}(u)$ has a kth-order angle with $\boldsymbol{c}$ if $\boldsymbol{v}'(u)/\|\boldsymbol{v}'(u)\|$ has a $(k-1)$th-order angle with $\boldsymbol{c}$. We have the following lemma.

Lemma 4.5. *Let $\boldsymbol{v} : I \longrightarrow S^2 \subset \mathbb{R}^3$ be a smooth unit vector field. For $k \geq 2$, there exists a unit constant vector $\boldsymbol{c}$ such that $\boldsymbol{v}(u)$ has a kth-order angle with $\boldsymbol{c}$ if and only if $\boldsymbol{v}(u)$ is a $(k-2)$th-order helix.*

Proof. We prove this by induction. Since a 0th-order helix is a cylindrical helix, which is equivalent to the condition that $\langle \boldsymbol{v}'(u)/\|\boldsymbol{v}'(u)\|, \boldsymbol{c}\rangle$ is constant for a unit vector $\boldsymbol{c}$. This means that $\boldsymbol{v}(u)$ has a 1st-order angle with $\boldsymbol{c}$. This completes the proof for $k = 2$. Suppose that the assertion holds for $k-1$. If $\boldsymbol{v}(u)$ has a kth-order angle with $\boldsymbol{c}$ for a unit vector $\boldsymbol{c}$. By definition, $\boldsymbol{v}'(u)/\|\boldsymbol{v}'(u)\|$ has a $(k-1)$th-order angle with $\boldsymbol{c}$ for a unit vector $\boldsymbol{c}$, by the inductive assumption, $\boldsymbol{v}'(u)/\|\boldsymbol{v}'(u)\|$ is a $(k-3)$th-order helix. By definition, $\boldsymbol{v}$ is a $(k-2)$th-order helix. The converse also holds. □

We have the following theorem.

Theorem 4.6. *Let $\boldsymbol{\gamma} : I \longrightarrow \mathbb{R}^3$ be a Frenet curve. Then the following conditions are equivalent:*
(1) *$F_{(\boldsymbol{\gamma},\boldsymbol{t})}$ is a developable surface with kth-order slope,*
(2) *$\mathscr{H}[\boldsymbol{\gamma}]_k(u)$ is constant,*
(3) *$\mathscr{H}[\boldsymbol{\gamma}]_{k+1}(u) \equiv 0$,*
(4) *$\boldsymbol{t}$ is a $(k-1)$th-order helix,*
(5) *$\boldsymbol{n}_{\boldsymbol{\gamma}}$ is a $(k-2)$th-order helix.*

If $\tau(u) \neq 0$, then the following condition is equivalent to the above:
(6) *The restriction of the unit normal vector field of $F_{(\boldsymbol{\gamma},\boldsymbol{t})}$ on the striction curve $\boldsymbol{\gamma}$ has a $(k-1)$th-order angle with a constant unit vector.*

Proof. By Propositions 4.3 and 4.4, conditions (1), (2), (3), (4) and (5) are equivalent. Suppose $\tau(u) \neq 0$. By a straightforward calculation, the restriction of the unit normal vector field of $F_{(\boldsymbol{\gamma},\boldsymbol{t})}$ on the striction curve $\boldsymbol{\gamma}(u)$ is the binormal vector field $\boldsymbol{b}_{\boldsymbol{\gamma}}(u)$ of $\boldsymbol{\gamma}(u)$. Suppose that $k = 2$. Since $\mathscr{H}[\boldsymbol{\gamma}]_2(u)$ is constant, $\boldsymbol{\gamma}(u)$ is a clad helix (i.e., 2nd-order helix), which is equivalent to the condition that $\boldsymbol{n}_{\boldsymbol{\gamma}}(u)$ is a cylindrical helix. Since $\boldsymbol{b}'_{\boldsymbol{\gamma}} = -\tau\boldsymbol{n}_{\boldsymbol{\gamma}}$, this condition is equivalent to the condition that $\boldsymbol{b}'_{\boldsymbol{\gamma}}(u)/\|\boldsymbol{b}'_{\boldsymbol{\gamma}}(u)\|$ is a cylindrical helix. By definition, $\boldsymbol{b}_{\boldsymbol{\gamma}}(u)$ has a 1st-order angle with a unit vector $\boldsymbol{c}$. For $k > 2$, by Lemma 4.5, condition (5) is equivalent to the condition that $\boldsymbol{n}_{\boldsymbol{\gamma}}(u)$ has a kth-order angle with a unit vector $\boldsymbol{c}$. By the relation $\boldsymbol{b}'_{\boldsymbol{\gamma}} = -\tau\boldsymbol{n}_{\boldsymbol{\gamma}}$ and definition, $\boldsymbol{b}_{\boldsymbol{\gamma}}(u)$ has a $(k-1)$th-order angle with $\boldsymbol{c}$. □

In the above theorem, we do not consider condition (4) for $k = 0$ and condition (5) for $k = 0, 1$ respectively.

5. Developable surfaces along cuspidal edges

In this section we introduce two kinds of flat surfaces along a cuspidal edge. Let $f : I \times (-\varepsilon, \varepsilon) \to \mathbb{R}^3$ be a frontal with a unit normal vector ν, where I is an open interval or a circle, and $\varepsilon > 0$. Assume that $I \times \{0\}$ consists of singular points of the first kind, and we take an adapted coordinate system (u, v) on $I \times (-\varepsilon, \varepsilon)$.

5.1. **Osculating developable surfaces along cuspidal edges.** If $(\kappa_\nu(u), \kappa_t(u)) \neq (0,0)$ on $u \in I$, we define a map $OD_f : I \times \mathbb{R} \longrightarrow \mathbb{R}^3$ by

$$OD_f(u,t) = f(u,0) + t\overline{D_o}(u) = f(u,0) + t\frac{\kappa_t(u)\boldsymbol{e}(u) - \kappa_\nu(u)\boldsymbol{b}(u)}{\sqrt{\kappa_t(u)^2 + \kappa_\nu(u)^2}}.$$

This is a ruled surface. Setting

$$\delta_o = \kappa_s(\kappa_\nu^2 + \kappa_t^2) - \kappa_t\kappa_\nu' + \kappa_\nu\kappa_t', \tag{5.1}$$

where $' = d/du$, by (3.1), we have

$$\overline{D_o}' = \frac{\delta_o}{(\kappa_t^2 + \kappa_\nu^2)^{3/2}}(\kappa_\nu\boldsymbol{e} + \kappa_t\boldsymbol{b}). \tag{5.2}$$

Here and in what follows, we omit "(u)" if it does not create misunderstandings. By (5.2), we have $\det\left(\boldsymbol{\gamma}', \overline{D_o}, \overline{D_o}'\right) = 0$. This means that $OD_f(I \times \mathbb{R})$ is a developable surface. We call OD_f an *osculating developable surface* of f along $S(f)$. By (5.2), OD_f is non-cylindrical if and only if $\delta_o \neq 0$. The osculating developable surface of f approximates f along $S(f)$ as a developable surface, and it has common tangent planes with f along $S(f)$ (see Figure 1). Let $\boldsymbol{s}_{OD}$ be the

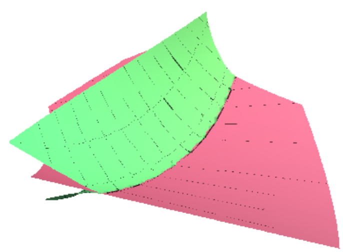

FIGURE 1. A cuspidal edge (green) with its osculating developable surface (purple)

striction curve of OD_f, which is defined by $\boldsymbol{s}_{OD}(u) = OD_f\left(u, -\sqrt{\kappa_\nu(u)^2 + \kappa_t(u)^2}\kappa_\nu(u)/\delta_o(u)\right)$. By a straightforward calculation, we see that

$$\boldsymbol{s}_{OD}' = \frac{\sigma_o}{\delta_o^2}(\kappa_t\boldsymbol{e} - \kappa_\nu\boldsymbol{b}), \tag{5.3}$$

where we set

$$\begin{aligned}\sigma_o &= \kappa_\nu\delta_o' + (\kappa_s\kappa_t - 2\kappa_\nu')\delta_o \\ &= \kappa_t(\kappa_\nu^2 + \kappa_t^2)\kappa_s^2 + 3\kappa_t(-\kappa_t\kappa_\nu' + \kappa_\nu\kappa_t')\kappa_s \\ &\qquad + \kappa_s'\kappa_\nu^3 + \kappa_t''\kappa_\nu^2 + (\kappa_t^2\kappa_s' - 2\kappa_\nu'\kappa_t' - \kappa_t\kappa_\nu'')\kappa_\nu + 2\kappa_t(\kappa_\nu')^2.\end{aligned}$$

By Propositions 4.1 and 4.2, we have the following theorem:

Theorem 5.1. *Suppose that OD_f is non-cylindrical. Then a singular point $(u, -\kappa_\nu(u)/\delta_o(u))$ of OD_f is*

(1) *a cuspidal edge if and only if $\delta_o(u) \neq 0$ and $\sigma_o(u) \neq 0$,*
(2) *a swallowtail if and only if $\delta_o(u) \neq 0$, $\sigma_o(u) = 0$ and $\sigma_o'(u) \neq 0$.*

Moreover, cuspidal cross caps never appear.

Proof. Since $D_o' = (\kappa_\nu\kappa_s + \kappa_t')\boldsymbol{e} + (\kappa_s\kappa_t - \kappa_\nu')\boldsymbol{b}$, and $D_o'' = *\boldsymbol{e} + *\boldsymbol{b} + \delta_0\boldsymbol{\nu}$, we see that $\psi = \delta_o^2$, where $*$ stands for some function. On the other hand, since

$$\boldsymbol{e} = \frac{1}{\delta}\Big((\kappa_s\kappa_t - \kappa_\nu')D_o + \kappa_\nu D_o'\Big),$$

α, β in Proposition 4.1 can be taken as $(\alpha, \beta) = (\kappa_s\kappa_t - \kappa_\nu', \kappa_\nu)/\delta_o$. Thus we see that $\beta' - \alpha = \sigma/\delta_o^2$. By Proposition 4.1, we see assertions (1) and (2). Since $\psi = \delta_o^2$, if $\psi(u) = 0$ then $\psi'(u) = 0$ for $u \in I$. This proves the last assertion. $\square$

Since OD_f is a developable surface, the striction curve $\boldsymbol{s}_{OD}$ coincides with $OD_f|_{S(OD_f)}$, and is a curve in $\mathbb{R}^3$. By (5.3), $\boldsymbol{s}_{OD}$ is regular if $\sigma_o \neq 0$. We denote by κ_{OD} (respectively, τ_{OD}) the curvature (respectively, the torsion) of $\boldsymbol{s}_{OD}$ the torsions of $OD_f|_{S(OD_f)}$ and $ND_f|_{S(OD_f)}$, respectively. By (5.3) and

$$\boldsymbol{s}_{OD}'' = \frac{1}{\delta_o^3}\bigg[\Big(\delta_o(\sigma_o'\kappa_t + \sigma_o\kappa_t' + \sigma_o\kappa_s\kappa_\nu) - 2\kappa_t\sigma_o\delta_o'\Big)\boldsymbol{e} + \Big(\delta_o(-\sigma_o'\kappa_\nu - \sigma_o\kappa_\nu' + \sigma_o\kappa_s\kappa_t) + 2\kappa_\nu\sigma_o\delta_o'\Big)\boldsymbol{b}\bigg],$$

$\boldsymbol{s}_{OD}''' = *\boldsymbol{e} + *\boldsymbol{b} + \dfrac{\sigma_o}{\delta_o^2}\Big(\kappa_s(\kappa_\nu^2 + \kappa_t^2) - \kappa_\nu'\kappa_t + \kappa_\nu\kappa_t'\Big)\boldsymbol{\nu}$, if $\sigma_o \neq 0$, then it holds that

$$\kappa_{OD} = \frac{|\delta_o|^3}{(\kappa_\nu^2 + \kappa_t^2)^{3/2}|\sigma_o|}, \quad \tau_{OD} = \frac{\delta_o^2}{\sigma_o}. \tag{5.4}$$

Therefore, $\boldsymbol{s}_{OD}$ is a Frenet curve if $\sigma_o \neq 0$ and $\delta_o \neq 0$. If $\kappa_\nu \equiv 0$, then $\boldsymbol{s}_{OD}$ is equal to $f(S(f))$. Moreover, if the cuspidal edge f is a tangent developable surface $F_{(\boldsymbol{\gamma},\boldsymbol{t})}$, then $\boldsymbol{e} = \boldsymbol{t}$, $\boldsymbol{b} = \boldsymbol{n}_{\boldsymbol{\gamma}}$ and $\boldsymbol{\nu} = \boldsymbol{b}_{\boldsymbol{\gamma}}$. By the Frenet-Serret formulae, we have $\kappa_\nu \equiv 0$, $\kappa_s = \kappa$ and $\kappa_t = \tau$. Then $\overline{D}_o(u) = \pm\boldsymbol{e}(u)$ and the image of $\boldsymbol{s}_{OD}$ coincides with $f(S(f))$. If $\kappa_t \equiv 0$ and $\kappa_\nu \neq 0$, then $\overline{D}_o(u) = \mp\boldsymbol{b}(u)$. We have the following corollary of Theorem 5.1.

Corollary 5.2. *Let f be a cuspidal edge. Then we have the following*:
(A) *Suppose that $\kappa_\nu \equiv 0$ and $\kappa_t \neq 0$. Then $\boldsymbol{s}_{OD}(I) = f(S(f))$ (i.e., OD_f is the tangent developable of $S(f)$) and a singular point $(u, 0) \in S(f)$ of OD_f is a cuspidal edge if and only if $\kappa_s(u) \neq 0$. Moreover, swallowtails never appear.*
(B) *Suppose that $\kappa_t \equiv 0$ and $\kappa_\nu \neq 0$. Then $OD_f(u, t) = f(u, 0) + t\boldsymbol{b}(u)$. If $\kappa_s(u_0) = 0$, then OD_f is cylindrical at u_0. If OD_f is non-cylindrical (i.e., $\kappa_s \neq 0$), then*

$$\boldsymbol{s}_{OD}(u) = OD_f(u, -|\kappa_\nu(u)|/\kappa_\nu(u)\kappa_s(u))$$

and a singular point $(u, -|\kappa_\nu(u)|/\kappa_\nu(u)\kappa_s(u))$ of OD_f is
(1) *a cuspidal edge if and only if $\kappa_s'(u) \neq 0$,*
(2) *a swallowtail if and only if $\kappa_s'(u) = 0$ and $\kappa_s''(u) \neq 0$.*

Proof. (A) Since $\kappa_\nu \equiv 0$, $\delta_o = \kappa_s\kappa_t^2$ and $\sigma_o = \kappa_t^3\kappa_s^2$, and then the results follow from Theorem 5.1.

(B) Since $\kappa_t \equiv 0$ and $\kappa_\nu \neq 0$, $\delta_o = \kappa_s\kappa_\nu^2$ and $\sigma_o = \kappa_\nu^3\kappa_s'$, so that $\sigma_o' = 3\kappa_\nu^2\kappa_\nu'\kappa_s' + \kappa_\nu^3\kappa_s''$, and then the results follow from Theorem 5.1. $\square$

Let f be a cuspidal edge with $\kappa_\nu \equiv 0$. Then by Corollary 5.2, $S(f) = S(OD_f)$. If $\kappa_s > 0$ (respectively, $\kappa_s < 0$), then $S(OD_f)$ locates the opposite side across the $f(S(f))$ (respectively, the same side with f with respect to $f(S(f))$). See Figure 2. For a cuspidal edge f with $\kappa_\nu \neq 0$, this is investigated in [24], and a cuspidal edge $\hat{f}$ which is isometric to f and satisfies $f(S(f)) = \hat{f}(S(\hat{f}))$. See [24] for detail.

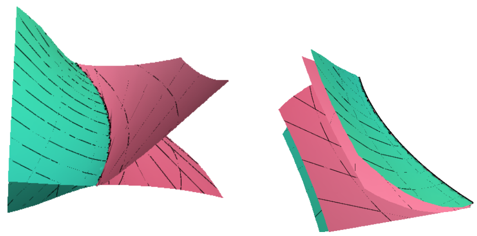

FIGURE 2. Left(respectively, right): Cuspidal edge f with $\kappa_\nu \equiv 0$ and $\kappa_s > 0$ (respectively, $\kappa_s < 0$) (green), and OD_f (purple).

5.2. **Normal developable surfaces along cuspidal edges.** If $(\kappa_t(u), \kappa_s(u)) \neq (0,0)$, we define a map $ND_f : I \times \mathbb{R} \longrightarrow \mathbb{R}^3$ by

$$ND_f(u,t) = f(u,0) + t\overline{D}_r(u) = f(u,0) + t\frac{\kappa_t(u)\boldsymbol{e}(u) + \kappa_s(u)\boldsymbol{\nu}(u)}{\sqrt{\kappa_t(u)^2 + \kappa_s(u)^2}}.$$

Since

$$\overline{D}_r' = \frac{\delta_n}{(\kappa_t^2 + \kappa_s^2)^{3/2}}(-\kappa_s \boldsymbol{e} + \kappa_t \boldsymbol{\nu}), \tag{5.5}$$

where

$$\delta_n = \kappa_\nu(\kappa_s^2 + \kappa_t^2) - \kappa_s\kappa_t' + \kappa_t\kappa_s', \tag{5.6}$$

we can also show that $ND_f(I \times \mathbb{R})$ is a developable surface (See Figure 3). By (5.5), ND_f is

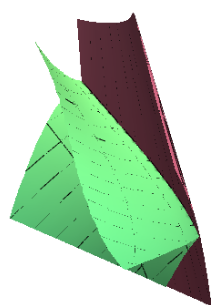

FIGURE 3. A cuspidal edge (green) with its normal developable surface (purple)

non-cylindrical if and only if $\delta_n \neq 0$. Let $\boldsymbol{s}_{ND}$ be the striction curve of ND_f, which is defined by $\boldsymbol{s}_{ND}(u) = ND_f\big(u, -\sqrt{\kappa_s(u)^2 + \kappa_t(u)^2}\kappa_s(u)/\delta_n(u)\big)$. Again by a straightforward calculation, we have

$$\boldsymbol{s}_{ND}' = \frac{\sigma_n}{\delta_n^2}(\kappa_t \boldsymbol{e} + \kappa_s \boldsymbol{\nu}), \tag{5.7}$$

where we set

$$\begin{aligned} \sigma_n &= -\kappa_s\delta_n' + (\kappa_\nu\kappa_t + 2\kappa_s')\delta_n \\ &= \kappa_t(\kappa_s^2 + \kappa_t^2)\kappa_\nu^2 + 3\kappa_t(\kappa_t\kappa_s' - \kappa_s\kappa_t')\kappa_\nu \\ &\qquad -\kappa_s\kappa_\nu'\kappa_t^2 + (2\kappa_s'^2 - \kappa_s\kappa_s'')\kappa_t + \kappa_s(-\kappa_s^2\kappa_\nu' - 2\kappa_s'\kappa_t' + \kappa_s\kappa_t''). \end{aligned}$$

Similar to Section 5.1, by Propositions 4.1 and 4.2, we have the following theorem:

Theorem 5.3. *Suppose that ND_f is non-cylindrical. Then a singular point $(u, -\kappa_s(u)/\delta_n(u))$ of ND_f is*

(1) *a cuspidal edge if and only if $\delta_n(u) \neq 0$ and $\sigma_n(u) \neq 0$,*
(2) *a swallowtail if and only if $\delta_n(u) \neq 0$, $\sigma_n(u) = 0$ and $\sigma_n'(u) \neq 0$.*

Moreover, cuspidal cross caps never appear.

If $\kappa_s \equiv 0$, then $\overline{D}_n(u) = \pm\boldsymbol{e}(u)$ and the image of $\boldsymbol{s}_{ND}$ coincides with $f(S(f))$. If $\kappa_t \equiv 0$ and $\kappa_s \neq 0$, then $\overline{D}_n(u) = \pm\boldsymbol{\nu}(u)$.

Therefore we have the following corollary of Theorem 5.3.

Corollary 5.4. *Let f be a cuspidal edge. Then we have the following*:
(A) *Suppose that $\kappa_s \equiv 0$ and $\kappa_t \neq 0$. Then $\boldsymbol{s}_{ND}(I) = f(S(f))$ (i.e., ND_f is the tangent developable of $S(f)$) and a singular point $(u, 0) \in S(f)$ of ND_f is a cuspidal edge if and only if $\kappa_\nu(u) \neq 0$. Moreover, swallowtails never appear.*
(B) *Suppose that $\kappa_t \equiv 0$ and $\kappa_s \neq 0$. Then $ND_f(u,t) = f(u,0) + t\boldsymbol{\nu}(u)$. If $\kappa_\nu(u_0) = 0$, then ND_f is cylindrical at u_0. If ND_f is non-cylindrical (i.e., $\kappa_\nu \neq 0$), then*

$$\boldsymbol{s}_{ND}(u) = ND_f(u, -|\kappa_\nu(u)|/\kappa_\nu(u)\kappa_s(u))$$

and a singular point $(u, -|\kappa_\nu(u)|/\kappa_\nu(u)\kappa_s(u))$ of ND_f is
(1) *a cuspidal edge if and only if $\kappa_\nu(u) \neq 0$,*
(2) *a swallowtail if and only if $\kappa_\nu \neq 0$, $\kappa_\nu' = 0$ and $\kappa_\nu''(u) \neq 0$.*

Proof. (A) Since $\kappa_s \equiv 0$, $\delta_n = \kappa_\nu \kappa_t^2$ and $\sigma_n = \kappa_t^3 \kappa_\nu^2$. Then the results follow from Theorem 5.1.
(B) If $\kappa_t \equiv 0$, then we have $\delta_n = \kappa_\nu \kappa_s^2$ and $\sigma_n = -\kappa_s^3 \kappa_\nu'$, so that $\sigma_n' = -3\kappa_s^2\kappa_s'\kappa_\nu' - \kappa_s^3\kappa_\nu''$. □

On the other hand, also similar to Section 5.1, if $\sigma_n \neq 0$, then the curvature κ_{ND} and the torsion τ_{ND} of $\boldsymbol{s}_{ND}$ are given by

$$\kappa_{ND} = \frac{|\delta_n|^3}{(\kappa_s^2 + \kappa_t^2)^{3/2}|\sigma_n|}, \quad \tau_{ND} = \frac{\delta_n^2}{\sigma_n}. \tag{5.8}$$

We close this subsection giving examples of OD_f and ND_f having cuspidal edges and swallowtails.

Example 5.5. Let us consider a space curve

$$\boldsymbol{\gamma}(u) = \left(\cos\frac{u}{\sqrt{2}},\ \sin\frac{u}{\sqrt{2}},\ \frac{u}{\sqrt{2}}\right). \tag{5.9}$$

Let $\boldsymbol{e}_{\boldsymbol{\gamma}}, \boldsymbol{n}_{\boldsymbol{\gamma}}, \boldsymbol{b}_{\boldsymbol{\gamma}}$ be the Frenet frame of γ. We set

$$f(u,v) = \boldsymbol{\gamma} + v^2\Big(\cos\theta(u)\boldsymbol{n}_{\boldsymbol{\gamma}} - \sin\theta(u)\boldsymbol{b}_{\boldsymbol{\gamma}}\Big) + v^3\Big(\sin\theta(u)\boldsymbol{n}_{\boldsymbol{\gamma}} - \cos\theta(u)\boldsymbol{b}_{\boldsymbol{\gamma}}\Big), \tag{5.10}$$

for a function $\theta(u)$. Then we see that $S(f) = \{v = 0\}$ and it consists of cuspidal edges. If $\theta(u) = \pi/4$, then

$$\boldsymbol{s}_{OD}(0) = OD_f(0, -2\sqrt{2/3}), \quad \boldsymbol{s}_{ND}(0) = ND_f(0, 2\sqrt{2/3}), \text{ and } \ \sigma_o(0) = \sigma_n(0) = 3/128.$$

Thus singular points of OD_f near $(0, -2\sqrt{2/3})$ and ND_f near $(0, 2\sqrt{2/3})$ consist of a cuspidal edge. See Figures 4 and 5. In these pictures, f is colored in green, and OD_f and ND_f are colored in purple.

Example 5.6. Let us consider the case $\theta = \pi/4 + u/4$ in (5.10) of Example 5.5. We see that $\boldsymbol{s}_{OD}(0) = OD_f(0, -2\sqrt{3})$, $\boldsymbol{s}_{ND}(0) = ND_f(0, 2\sqrt{3})$, $\sigma_o(0) = \sigma_n(0) = 0$, and $\sigma_o'(0) = -1/256$, $\sigma_n'(0) = 1/256$. Thus each singular point of OD_f at $(0, -2\sqrt{3})$ and ND_f at $(0, 2\sqrt{3})$ is a

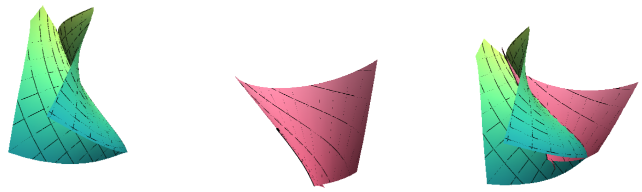

FIGURE 4. Left to right: Cuspidal edge f of $\theta = \pi/4$, OD_f and combined picture of f and OD_f

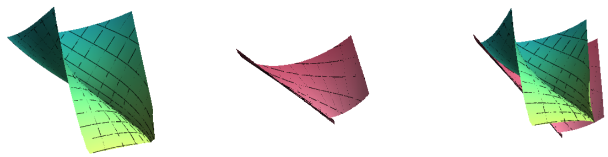

FIGURE 5. Left to right: Cuspidal edge f of $\theta = \pi/4$, ND_f and combined picture of f and ND_f

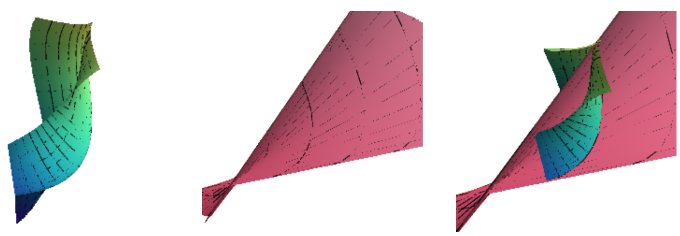

FIGURE 6. Left to right: Cuspidal edge f of $\theta = \pi/4 + u/4$, OD_f and combined picture of f and OD_f

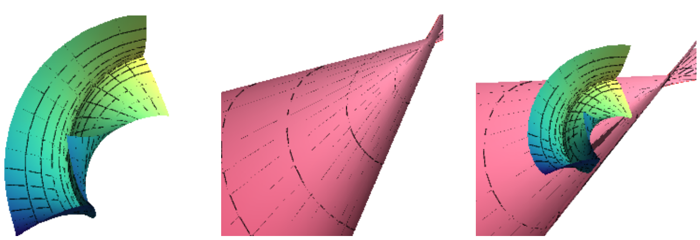

FIGURE 7. Left to right: Cuspidal edge f of $\theta = \pi/4 + u/4$, ND_f and combined picture of f and ND_f

swallowtail. See Figures 6 and 7. In these pictures, f is colored in green, and OD_f and ND_f are colored in purple.

5.3. **Planer cuspidal edges.** In the previous subsections we investigated the singularities of OD_f and ND_f with the condition $(\kappa_\nu(u), \kappa_t(u)) \neq (0,0)$ and $(\kappa_t(u), \kappa_s(u)) \neq (0,0)$ for any $u \in I$. Moreover, we also investigated the case when one of κ_s, κ_ν and κ_t is identically equal to zero as special cases (cf. Corollaries 5.2 and 5.4). Here, we study cuspidal edges with

$(\kappa_\nu(u),\kappa_t(u))=(0,0)$ and $(\kappa_t(u),\kappa_s(u))=(0,0)$ for any $u\in I$. With the same setting to the above subsections, let us assume $(\kappa_\nu(u),\kappa_t(u))=(0,0)$ and $\kappa_s\neq 0$ for any $u\in I$. Since the curvature κ and the torsion τ of the curve $f(u,0)$ as a curve in $\mathbb{R}^3$ satisfy

$$\kappa^2=\kappa_s^2+\kappa_\nu^2,\quad \tau=\frac{\kappa_s\kappa_\nu'-\kappa_\nu\kappa_s'}{\kappa_s^2+\kappa_\nu^2}+\kappa_t, \tag{5.11}$$

(see [20]) and $\boldsymbol{\nu}'(u)\equiv 0$, we see that $f(u,0)$ lies on a plane which is perpendicular to the constant vector $\boldsymbol{\nu}$. In this case, OD_f can be considered as a subset of this plane and

$$ND_f(u,t)=f(u,0)+t\boldsymbol{\nu}$$

is a cylinder. By the same argument as the above, we see that if $(\kappa_t(u),\kappa_s(u))\equiv(0,0)$ and $\kappa_\nu\neq 0$, then $f(u,0)$ lies on a plane which is perpendicular to the constant vector $\boldsymbol{b}$. In this case, ND_f can be considered as a subset of this plane and $ND_f(u,t)=f(u,0)+t\boldsymbol{b}$ is a cylinder. Moreover, if we assume $(\kappa_s(u),\kappa_\nu(u),\kappa_t(u))\equiv(0,0,0)$, then $f(u,0)$ is a straight line, and $\boldsymbol{\nu}'\equiv\boldsymbol{b}'\equiv 0$. In this case, OD_f should be defined as the plane perpendicular to $\boldsymbol{\nu}$ and ND_f as the plane perpendicular to $\boldsymbol{b}$. Since OD_f and ND_f intersect orthogonally, the cuspidal edge $S(f)$ is a line in this case.

5.4. **Normalized derivate director curves and derivate striction curves.** We set

$$\overline{\overline{D_o}'}=\frac{(\overline{D}_o)'}{|(\overline{D}_o)'|}=\frac{\kappa_\nu\boldsymbol{e}+\kappa_t\boldsymbol{b}}{\sqrt{\kappa_\nu^2+\kappa_t^2}},\quad \overline{\overline{D}_r'}=\frac{(\overline{D}_r)'}{|(\overline{D}_r)'|}=\frac{-\kappa_s\boldsymbol{e}+\kappa_t\boldsymbol{b}}{\sqrt{\kappa_s^2+\kappa_t^2}},$$

and call them the *normalized* $\overline{D_o}'$ and *normalized* $\overline{D}_r'$, respectively. They are curves in the unit sphere in $\mathbb{R}^3$. Here, we calculate their geodesic curvatures. Since

$$\begin{aligned}
\left(\overline{\overline{D_o}'}\right)' &= \frac{\delta_o}{\kappa_\nu^2+\kappa_t^2}\Big(-\kappa_t\boldsymbol{e}+\kappa_\nu\boldsymbol{b}\Big)+\frac{\boldsymbol{\nu}}{\sqrt{\kappa_\nu^2+\kappa_t^2}},\\
\left(\overline{\overline{D_o}'}\right)'' &= \frac{1}{\sqrt{\kappa_\nu^2+\kappa_t^2}^{\,5}}\Bigg\{-\bigg[\Big(\kappa_\nu^3\kappa_s+\kappa_\nu^2\kappa_t'+\kappa_\nu\kappa_t(\kappa_s\kappa_t-3\kappa_\nu')-2\kappa_t^2\kappa_t'\Big)\delta_o\\
&\qquad +(\kappa_\nu^2+\kappa_t^2)(\kappa_\nu^5+2\kappa_\nu^3\kappa_t^2+\kappa_\nu\kappa_t^4+\kappa_t\delta_o')\bigg]\boldsymbol{e}\\
&\qquad -\bigg[\Big(\kappa_t^3\kappa_s-\kappa_t^2\kappa_\nu'+\kappa_t\kappa_\nu(\kappa_\nu\kappa_s+3\kappa_t')+2\kappa_\nu^2\kappa_\nu'\Big)\delta_o\\
&\qquad +(\kappa_\nu^2+\kappa_t^2)(\kappa_\nu^4\kappa_t+2\kappa_\nu^2\kappa_t^3+\kappa_t^5-\kappa_\nu\delta_o')\bigg]\boldsymbol{b}\\
&\qquad +(\kappa_\nu^2+\kappa_t^2)^2(\kappa_\nu\kappa_\nu'+\kappa_t\kappa_t')\boldsymbol{\nu}\Bigg\},
\end{aligned}$$

we obtain the geodesic curvature of $\overline{\overline{D_o}'}$ as follows:

$$\left(\frac{\delta_o^2+1}{\kappa_\nu^2+\kappa_t^2}\right)^{3/2}\Big(-(\kappa_\nu^2+\kappa_t^2)\delta_o'+3(\kappa_\nu\kappa_\nu'+\kappa_t\kappa_t')\delta_o\Big),$$

and in a similar manner, we obtain the geodesic curvature $\overline{\overline{D}_r'}$ as follows:

$$\left(\frac{\delta_n^2+1}{\kappa_s^2+\kappa_t^2}\right)^{3/2}\Big(-(\kappa_s^2+\kappa_t^2)\delta_n'+3(\kappa_s\kappa_s'+\kappa_t\kappa_t')\delta_n\Big).$$

Next we consider normalized striction curves. By (3.2), (5.3), and (3.3), (5.7), we see that

$$\overline{\boldsymbol{s}'_{OD}} = \frac{\boldsymbol{s}'_{OD}}{|\boldsymbol{s}'_{OD}|} = \overline{D_o}, \quad \overline{\boldsymbol{s}'_{ND}} = \frac{\boldsymbol{s}'_{ND}}{|\boldsymbol{s}'_{ND}|} = \overline{D_r}.$$

Thus the normalized derivate striction curves coincide with the normalized director curves. Moreover, since $\overline{D_o}$ and $\boldsymbol{\nu}$ (respectively, $\overline{D_r}$ and $\boldsymbol{b}$) are dual to each other as curves in the unit sphere in $\mathbb{R}^3$, $\overline{\boldsymbol{s}'_{OD}}$ and $\boldsymbol{\nu}$ (respectively, $\overline{\boldsymbol{s}'_{ND}}$ and $\boldsymbol{b}$) are dual to each other.

6. Special cuspidal edges

In this section we consider the case when the singular values of OD_f and ND_f are special curves in $\mathbb{R}^3$. Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a cuspidal edge and $\{\boldsymbol{e}, \boldsymbol{b}, \boldsymbol{\nu}\}$ Darboux frame along the cuspidal edge $\boldsymbol{\gamma}$, where $\boldsymbol{\gamma} = f|_{S(f)}$.

6.1. **Contour edges.**

In this subsection we give characterizations of contour edges by using the invariants of cuspidal edges. We have the following theorem.

Theorem 6.1. *With the same notations as the previous sections, we have the following*:
(A) *Suppose that* $\kappa_\nu^2 + \kappa_t^2 \neq 0$. *Then the following properties are equivalent*:

(1) OD_f *is a cylinder,*
(2) $\delta_o \equiv 0$,
(3) $\boldsymbol{\nu}$ *is a part of a great circle in* S^2.
(4) $S(f)$ *is a tangential contour edge with respect to an orthogonal projection.*
(5) $\overline{D_o}$ *is a constant vector.*

(B) *Suppose that* $\kappa_s^2 + \kappa_t^2 \neq 0$. *Then the following properties are equivalent*:

(1) ND_f *is a cylinder,*
(2) $\delta_n(u) \equiv 0$,
(3) $\boldsymbol{b}$ *is a part of a great circle in* S^2,
(4) $S(f)$ *is a normal contour edge with respect to an orthogonal projection.*
(5) $\overline{D_r}$ *is a constant vector.*

Proof. We show the assertion (A). By (5.2), we see the equivalency of (1) and (2). The condition $\kappa_t^2 + \kappa_\nu^2 \neq 0$ means that $\boldsymbol{\nu}$ is a non-singular spherical curve. Moreover, since

$$\nu'' = (\kappa_s\kappa_t - \kappa_\nu')\boldsymbol{e} + (-\kappa_\nu\kappa_s - \kappa_t')\boldsymbol{b},$$

we see that $\det(\nu, \nu', \nu'') = \delta_o$. This implies that the geodesic curvature of $\boldsymbol{\nu}$ is $\delta_o(\kappa_t^2 + \kappa_\nu^2)^{-3/2}$, and it shows that the equivalency of (2) and (3). We assume (2). Then $\overline{D_o}(u)$ is a constant vector $\overline{D_o}$. Thus $\langle \boldsymbol{\nu}(u), \overline{D_o} \rangle = 0$ for any u. This implies that $S(f)$ is a tangential contour edge with respect to $\overline{D_o}$. This implies (4). Conversely, we assume (4). Then there exists a vector $\boldsymbol{k}$ such that $\langle \boldsymbol{\nu}(u), \boldsymbol{k} \rangle = 0$ holds for any u. This implies that $\boldsymbol{\nu}(u)$ belongs to the normal plane of $\boldsymbol{k}$ passing through the origin, and it implies (3). Since $\boldsymbol{\nu}$ and $\overline{D_o}$ are dual each other as spherical curves by (3.2) and (5.2), we see that the equivalency of (3) and (5). Thus the assertion (A) holds. One can show the assertion (B) by the same method as in the proof of (A), using (3.3) and (5.5) instead of (3.2) and (5.2). □

Theorem 6.2. *With the same notations as above, we have the following*:
(A) *Suppose that* $\kappa_t^2 + \kappa_\nu^2 \neq 0$ *and* $\delta_o \neq 0$ *for any* $u \in I$. *Then the following properties are equivalent*:

(1) OD_f *is a cone,*
(2) $\sigma_o \equiv 0$,

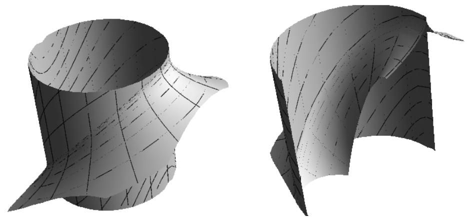

FIGURE 8. Cuspidal edge whose osculating developable surface is a cylinder

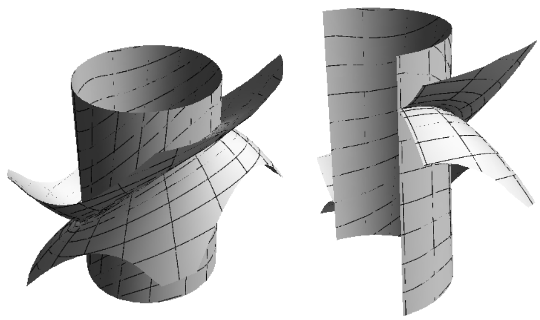

FIGURE 9. Cuspidal edge whose normal developable surface is a cylinder

(3) $S(f)$ *is a tangential contour edge with respect to a central projection.*
(4) $\boldsymbol{s}_{OD}$ *is a constant vector.*

(B) *Suppose that* $\kappa_t^2 + \kappa_s^2 \neq 0$ *and* $\delta_n \neq 0$ *for any* $u \in I$. *Then the following properties are equivalent*:

(1) ND_f *is a cone,*
(2) $\sigma_n \equiv 0$,
(3) $S(f)$ *is a normal contour edge with respect to a central projection.*
(4) $\boldsymbol{s}_{ND}$ *is a constant vector.*

Proof. By (5.3), we see that the equivalency of (1) and (2). We assume (2). Then $\boldsymbol{s}_{OD}(u)$ is a constant vector for any u. We set $\boldsymbol{c} = \boldsymbol{s}_{OD}(u)$. Then by (4.1), $f(u,0) - \boldsymbol{c}$ is parallel to $\overline{D_o}(u)$. Thus $\langle f(u,0) - \boldsymbol{c}, \boldsymbol{\nu}(u)\rangle = \langle \overline{D_o}(u), \boldsymbol{\nu}(u)\rangle = 0$ holds for any u. This implies (3). Conversely, we assume (3). Then there exists a vector $\boldsymbol{c}$ such that $\langle f(u,0) - \boldsymbol{c}, \boldsymbol{\nu}(u)\rangle \equiv 0$. By (4.1), $\boldsymbol{s}_{OD}(u) - f(u,0)$ is parallel to $\overline{D_o}(u)$, $\langle \boldsymbol{s}_{OD}(u) - \boldsymbol{c}, \boldsymbol{\nu}(u)\rangle \equiv 0$. Differentiating this equation by u, and noticing $\langle \boldsymbol{s}'_{OD}(u), \boldsymbol{\nu}(u)\rangle \equiv 0$ by (5.3), we have $\langle \boldsymbol{s}_{OD}(u), \boldsymbol{\nu}'(u)\rangle \equiv 0$. By (5.3) and (3.1), we see that $\langle \boldsymbol{s}'_{OD}(u), \boldsymbol{\nu}'(u)\rangle \equiv 0$. Thus, differentiating $\langle \boldsymbol{s}_{OD}(u), \boldsymbol{\nu}'(u)\rangle \equiv 0$ by u, we have $\langle \boldsymbol{s}_{OD}(u), \boldsymbol{\nu}''(u)\rangle \equiv 0$. On the other hand, by (3.1), the three vectors $\boldsymbol{\nu}(u), \boldsymbol{\nu}'(u), \boldsymbol{\nu}''(u)$ are linearly independent if and only if $\delta_o(u) \neq 0$. Hence

$$\langle \boldsymbol{s}_{OD}(u) - \boldsymbol{c}, \boldsymbol{\nu}(u)\rangle \equiv \langle \boldsymbol{s}_{OD}(u) - \boldsymbol{c}, \boldsymbol{\nu}'(u)\rangle \equiv \langle \boldsymbol{s}_{OD}(u) - \boldsymbol{c}, \boldsymbol{\nu}''(u)\rangle \equiv 0$$

implies $\boldsymbol{s}_{OD}(u) - \boldsymbol{c} \equiv 0$, and this implies (1). Thus the assertion (A) holds. One can show the assertion (B) by the same method as in the proof of (A) using (5.7) instead of (5.3). □

6.2. **Isophotic edges.** Recall that the curve γ is called the (normal) isophotic edge (respectively, the tangent isophotic edge) if there exists a constant vector $\boldsymbol{v}$ such that $\boldsymbol{\nu}$ (respectively, $\boldsymbol{b}$) makes a constant angle with $\boldsymbol{v}$.

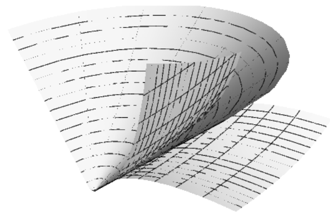

FIGURE 10. Cuspidal edge whose osculating developable surface is a cone

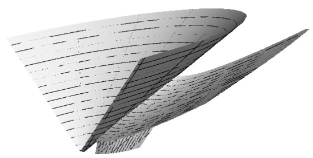

FIGURE 11. Cuspidal edge whose normal developable surface is a cone

Let us turn to our setting. With the same notations as those of Section 5, by a straightforward calculation, we have

$$\left(\frac{\tau_{OD}}{\kappa_{OD}}\right)^2 = \frac{(\kappa_\nu^2+\kappa_t^2)^3}{\delta_o^2} \quad \text{and} \quad \left(\frac{\tau_{ND}}{\kappa_{ND}}\right)^2 = \frac{(\kappa_s^2+\kappa_t^2)^3}{\delta_n^2}. \tag{6.1}$$

These are squares of the geodesic curvatures of $\boldsymbol{\nu}$ and $\boldsymbol{b}$, respectively. Thus we obtain:

Theorem 6.3. *With the same notations as those of Section 5, we have the following*:
(A) *Suppose that $\kappa_t^2+\kappa_\nu^2 \neq 0$, $\delta_o \neq 0$ and $\sigma_o \neq 0$ for any $u \in I$. Then the following properties are equivalent*:

(1) *OD_f is a constant angle surface,*
(2) *$\boldsymbol{\nu}$ is a part of a small circle,*
(3) *$S(f)$ is a normal isophotic edge,*
(4) *$\overline{D_o}$ is a part of a small circle,*
(5) *$\overline{\boldsymbol{s}'_{OD}}$ is a part of a small circle,*
(6) *$\delta_o/(\kappa_\nu^2+\kappa_t^2)^{3/2}$ is constant,*
(7) *$\boldsymbol{s}_{OD}$ is a cylindrical helix.*

(B) *Suppose that $\kappa_s^2+\kappa_\nu^2 \neq 0$, $\delta_n \neq 0$ and $\sigma_n \neq 0$ for any $u \in I$. Then the following properties are equivalent*:

(1) *ND_f is a constant angle surface,*
(2) *$\boldsymbol{b}$ is a part of a small circle,*
(3) *$\boldsymbol{\gamma}$ is a tangent isophotic edge,*
(4) *$\overline{D_r}$ is a part of a small circle,*
(5) *$\overline{\boldsymbol{s}'_{ND}}$ is a part of a small circle,*
(6) *$\delta_n/(\kappa_s^2+\kappa_t^2)^{3/2}$ is constant,*
(7) *$\boldsymbol{s}_{ND}$ is a cylindrical helix.*

Proof. By the definition and (6.1), the equivalency of (1) and (6) is obvious. By the proof of Theorem 5.3, $\delta_o/(\kappa_\nu^2+\kappa_t^2)^{3/2}$ is the geodesic curvature of $\boldsymbol{\nu}$, so that (2) and (6) are equivalent. Since $\boldsymbol{\nu}$ is a curve on the unit sphere, we see the equivalency of (2) and (3). By (5.2), $\boldsymbol{\nu}$ and $\overline{D_o}$ are spherical dual each other. Hence we see equivalency of (2) and (4). Equivalency of (2) and

(5) is obvious since $\overline{D_o}$ and $\overline{\boldsymbol{s}'_{OD}}$ are parallel. By definition, (5) and (7) are equivalent. Thus the assertion (A) holds.

One can show the assertion (B) by arguments similar to those for (A). $\square$

6.3. **General order sloped edges.** In this subsection we consider cuspidal edges such that the osculating or the normal developables of cuspidal edges are general order sloped, where we say that $S(f)$ is a *k-th order sloped edge with respect to* $\overline{D_o}$ (respectively, $\overline{D_r}$) if $\overline{D_o}$ (respectively, $\overline{D_r}$) is a $(k-1)$th-order (spherical) helix. We denote the kth-order helical curvature of $\boldsymbol{s}_{OD}(u)$ (respectively, $\boldsymbol{s}_{ND}(u)$) by $\mathscr{H}[\boldsymbol{s}_{OD}]_k(u)$ (respectively, $\mathscr{H}[\boldsymbol{s}_{ND}]_k(u)$). By (6.1), we have

$$\mathscr{H}[\boldsymbol{s}_{OD}]_0(u) = \frac{(\kappa_\nu^2+\kappa_t^2)^{3/2}}{|\delta_o|},$$

$$\mathscr{H}[\boldsymbol{s}_{ND}]_0(u) = \frac{(\kappa_s^2+\kappa_t^2)^{3/2}}{|\delta_n|},$$

$$\mathscr{H}[\boldsymbol{s}_{OD}]_1(u) = \frac{\sqrt{\kappa_\nu^2+\kappa_t^2}}{\delta_o^2+(\kappa_\nu^2+\kappa_t^2)^3}\big(3\kappa_\nu\kappa_\nu'+3\kappa_t\kappa_t'-(\kappa_\nu^2+\kappa_t^2)\delta_o'\big),$$

$$\mathscr{H}[\boldsymbol{s}_{ND}]_1(u) = \frac{\sqrt{\kappa_s^2+\kappa_t^2}}{\delta_n^2+(\kappa_s^2+\kappa_t^2)^3}\big(3\kappa_s\kappa_s'+3\kappa_t\kappa_t'-(\kappa_s^2+\kappa_t^2)\delta_n'\big)$$

and

$$\mathscr{H}[\boldsymbol{s}_{OD}]_2(u) = \frac{\sigma_o(\kappa_\nu^2+\kappa_t^2)^{3/2}\theta_{OD}'}{\delta_o\sqrt{\delta_o^2+(\kappa_\nu^2+\kappa_t^2)^3}(1+\theta_{OD}^2)^{3/2}},$$

$$\mathscr{H}[\boldsymbol{s}_{ND}]_2(u) = \frac{\sigma_n(\kappa_s^2+\kappa_t^2)^{3/2}\theta_{ND}'}{\delta_n\sqrt{\delta_n^2+(\kappa_s^2+\kappa_t^2)^3}(1+\theta_{ND}^2)^{3/2}}.$$

Higher order helical curvatures of $\boldsymbol{s}_{OD}(u)$ and $\boldsymbol{s}_{ND}(u)$ are inductively defined. However, these are very complicated, so we omit explanations by using basic invariants for the cuspidal edge. Then we have the following theorem as a simple corollary of Theorem 4.6.

Theorem 6.4. *With the same notations as those of Sections 4 and 5, we have the following*:
(A) *Suppose that $\kappa_t^2+\kappa_\nu^2\neq 0$, $\delta_o\neq 0$ and $\sigma_o\neq 0$ for any $u\in I$. Then the following properties are equivalent*:

(1) *OD_f is a developable surface with kth-order slope,*
(2) *$\boldsymbol{s}_{OD}$ is a kth-order helix,*
(3) *$\overline{\overline{D_o}'}$ is a $(k-2)$th-order (spherical) helix,*
(4) *$\overline{\boldsymbol{s}'_{OD}}$ is a $(k-1)$th-order (spherical) helix,*
(5) *$\mathscr{H}[\boldsymbol{s}_{OD}]_k$ is constant,*
(6) *$\mathscr{H}[\boldsymbol{s}_{OD}]_{k+1}\equiv 0$,*
(7) *$S(f)$ is a k-th order sloped edge with respect to $\overline{D_o}$.*

(B) *Suppose that $\kappa_t^2+\kappa_s^2\neq 0$, $\delta_n\neq 0$ and $\sigma_n\neq 0$ for any $u\in I$. Then the following properties are equivalent*:

(1) *ND_f is a developable surface with kth-order slope,*
(2) *$\boldsymbol{s}_{ND}$ is a kth-order helix,*
(3) *$\overline{\overline{D_r}'}$ is a $(k-2)$th-order (spherical) helix,*
(4) *$\overline{D_r}$ is a $(k-1)$th-order (spherical) helix,*
(5) *$\overline{\boldsymbol{s}'_{ND}}$ is a $(k-1)$th-order (spherical) helix,*
(6) *$\mathscr{H}[\boldsymbol{s}_{ND}]_k$ is constant,*
(7) *$\mathscr{H}[\boldsymbol{s}_{ND}]_{k+1}\equiv 0$.*

(8) *$S(f)$ is a k-th order sloped edge with respect to $\overline{D_r}$.*

Proof. (A) With assumptions $\kappa_t^2+\kappa_\nu^2 \neq 0$ and $\delta_o \neq 0$, $\boldsymbol{s}_{OD}$ is a Frenet curve. By definition, $\overline{D_o}'$ is the unit principal normal vector field of $\boldsymbol{s}_{OD}$. Since $\boldsymbol{s}_{OD}$ is the striation curve of OD_f, the director curve of OD_f is equal to $\boldsymbol{s}'_{OD}$, so that we can apply Theorem 4.6 to $\boldsymbol{s}_{OD}$. By definition, (4) and (8) are equivalent. For (B), we have arguments similar to the case (A) and apply Theorem 4.6 to $\boldsymbol{s}_{ND}$. □

If we consider the case when one of $\kappa_\nu,\ \kappa_t,\ \kappa_s$ is identically equal to zero, we have the following representations of helical curvatures of $\boldsymbol{s}_{OD}$ and $\boldsymbol{s}_{ND}$, respectively:
(1) Suppose that $\kappa_\nu \equiv 0$ and $\kappa_t \neq 0$. Then $\delta_o=\kappa_s\kappa_t^2$ and $\sigma_o=\kappa_t^3\kappa_s^2$. If $\kappa_s \neq 0$, then $\overline{D}_o(u)=\pm\boldsymbol{e}(u)$ and $\boldsymbol{s}_{OD}(I)=f(S(f))$. If we denote by κ and τ the curvature and the torsion of $S(f)$ respectively, then $\kappa(u)=|\kappa_s(u)|$ and $\tau(u)=\kappa_t(u)$. Therefore we have

$$\mathscr{H}[\boldsymbol{s}_{OD}]_0(u)=\mathscr{H}[S(f)]_0(u)=\kappa_t(u)/|\kappa_s(u)|.$$

Moreover, we have

$$\mathscr{H}[S(f)]_1(u)=\frac{1}{|\kappa_s(u)|}\frac{\mathscr{H}[S(f)]_0'(u)}{(1+(\mathscr{H}[S(f)]_0(u))^2)^{3/2}},$$

$$\mathscr{H}[S(f)]_2(u)=\frac{1}{|\kappa_s(u)|(1+(\mathscr{H}[S(f)]_0(u))^2)^{1/2}}\frac{\mathscr{H}[S(f)]_1'(u)}{(1+(\mathscr{H}[S(f)]_1(u))^2)^{3/2}}.$$

Higher order helical curvatures of $S(f)$ are inductively defined. Moreover, OD_f is the tangent developable of $f(S(f))$.
(2) Suppose that $\kappa_t \equiv 0$ and $\kappa_\nu \neq 0$. Then $\delta_o(u)=\kappa_s(u)\kappa_\nu(u)^2$ and $\sigma_o(u)=\kappa_\nu(u)^3\kappa_s'(u)$. If $\kappa_s \neq 0$ and $\kappa_s' \neq 0$, then $\overline{D}_o(u)=\pm\boldsymbol{b}(u)$ and $\boldsymbol{s}_{OD}(u)=OD_f(u,-|\kappa_\nu(u)|/\kappa_\nu(u)\kappa_s(u))$. Moreover, we have

$$\kappa_{OD}(u)=\frac{|\kappa_s(u)|^3|\kappa_\nu(u)|^3}{|\sigma_o(u)|} \text{ and } \tau_{OD}(u)=\frac{\kappa_s(u)^2\kappa_\nu(u)^4}{\sigma_o(u)},$$

so that $\mathscr{H}[\boldsymbol{s}_{OD}]_0(u)=\tau_{OD}(u)/\kappa_{OD}(u)=|\kappa_t(u)|/\kappa_s(u)$. We can define kth-order helical curvature $\mathscr{H}[\boldsymbol{s}_{OD}]_k(u)$ inductively. In this case $ND_f(u,t)=f(u,0)+t\boldsymbol{b}(u)$.
(3) Suppose that $\kappa_s \equiv 0$ and $\kappa_t \neq 0$. Then $\delta_n=\kappa_\nu\kappa_t^2$ and $\sigma_n=\kappa_t^3\kappa_\nu^2$. If $\kappa_s \neq 0$, then $\overline{D}_r(u)=\pm\boldsymbol{e}(u)$, $\boldsymbol{s}_{ND}(I)=f(S(f))$ and $\kappa(u)=|\kappa_\nu(u)|$ and $\tau(u)=\kappa_t(u)$. Therefore we have $\mathscr{H}[\boldsymbol{s}_{ND}]_0(u)=\mathscr{H}[S(f)]_0(u)=\kappa_t(u)/|\kappa_\nu(u)|$. We can define kth-order helical curvature $\mathscr{H}[S(f)]_k(u)$ inductively. In this case ND_f is the tangent developable of $f(S(f))$.
(4) Suppose that $\kappa_t \equiv 0$ and $\kappa_s \neq 0$. Then $\delta_n=\kappa_\nu\kappa_s^2$ and $\sigma_n=-\kappa_s^3\kappa_\nu'$. If $\kappa_\nu \neq 0$ and $\kappa_\nu' \neq 0$, then $\overline{D}_o(u)=\pm\boldsymbol{n}(u)$ and $\boldsymbol{s}_{ND}(u)=ND_f(u,-|\kappa_\nu(u)|/\kappa_\nu(u)\kappa_s(u))$. Moreover, we have

$$\kappa_{ND}(u)=\frac{|\kappa_\nu(u)|^3|\kappa_s(u)|^3}{|\sigma_n(u)|} \text{ and } \tau_{ND}(u)=\frac{\kappa_\nu(u)^2\kappa_t(u)^4}{\sigma_n(u)},$$

so that $\mathscr{H}[\boldsymbol{s}_{ND}]_0(u)=|\kappa_s(u)|/\kappa_\nu(u)$. We can define kth-order helical curvature $\mathscr{H}[\boldsymbol{s}_{ND}]_k(u)$ inductively. In this case $ND_f(u,t)=f(u,0)+t\boldsymbol{\nu}(u)$.

Corollary 6.5. *With the same notations as those in the above theorem, we have the following*:
(A) *Suppose that $\kappa_\nu \equiv 0$, $\kappa_t \neq 0$, and $\kappa_s \neq 0$. Then OD_f is the tangent developable of $S(f)$ and the following properties are equivalent*:

(1) *OD_f is a developable surface with kth-order slope,*
(2) *$S(f)$ is a kth-order helix,*
(3) *$\boldsymbol{b}$ is a $(k-2)$th-order (spherical) helix,*
(4) *$\boldsymbol{e}$ is a $(k-1)$th-order (spherical) helix,*
(5) *$\mathscr{H}[S(f)]_k$ is constant,*

(6) $\mathscr{H}[S(f)]_{k+1} \equiv 0$.

(B) *Suppose that* $\kappa_t \equiv 0$, $\kappa_\nu \neq 0$, $\kappa_s \neq 0$ *and* $\kappa_s' \neq 0$. *Then* OD_f *is the tangent developable of* $S(f)$ *and the following properties are equivalent*:

(1) OD_f *is a developable surface with kth-order slope,*
(2) $\boldsymbol{s}_{OD}$ *is a kth-order helix,*
(3) $\boldsymbol{e}$ *is a* $(k-2)$*th-order* (*spherical*) *helix,*
(4) $\boldsymbol{b}$ *is a* $(k-1)$*th-order* (*spherical*) *helix,*
(5) $\overline{\boldsymbol{s}'_{OD}}$ *is a* $(k-1)$*th-order* (*spherical*) *helix,*
(6) $\mathscr{H}[\boldsymbol{s}_{OD}]_k$ *is constant,*
(7) $\mathscr{H}[\boldsymbol{s}_{OD}]_{k+1} \equiv 0$.
(8) $S(f)$ *is a k-th order sloped edge with respect to* $\boldsymbol{b}$.

(C) *Suppose that* $\kappa_s \equiv 0$, $\kappa_t \neq 0$, *and* $\kappa_\nu \neq 0$. *Then* ND_f *is the tangent developable of* $S(f)$ *and the following properties are equivalent*:

(1) ND_f *is a developable surface with kth-order slope,*
(2) $S(f)$ *is a kth-order helix,*
(3) $\boldsymbol{\nu}$ *is a* $(k-2)$*th-order* (*spherical*) *helix,*
(4) $\boldsymbol{e}$ *is a* $(k-1)$*th-order* (*spherical*) *helix,*
(5) $\mathscr{H}[S(f)]_k$ *is constant,*
(6) $\mathscr{H}[S(f)]_{k+1} \equiv 0$.

(D) *Suppose that* $\kappa_t \equiv 0$, $\kappa_s \neq 0$, $\kappa_\nu \neq 0$ *and* $\kappa_\nu' \neq 0$. *Then* $ND_f(u,t) = f(u,0) + t\boldsymbol{\nu}(u)$ *and the following properties are equivalent*:

(1) ND_f *is a developable surface with kth-order slope,*
(2) $\boldsymbol{s}_{ND}$ *is a kth-order helix,*
(3) $\boldsymbol{e}$ *is a* $(k-2)$*th-order* (*spherical*) *helix,*
(4) $\boldsymbol{\nu}$ *is a* $(k-1)$*th-order* (*spherical*) *helix,*
(5) $\overline{\boldsymbol{s}'_{ND}}$ *is a* $(k-1)$*th-order* (*spherical*) *helix,*
(6) $\mathscr{H}[\boldsymbol{s}_{ND}]_k$ *is constant,*
(7) $\mathscr{H}[\boldsymbol{s}_{ND}]_{k+1} \equiv 0$.
(8) $S(f)$ *is a k-th order sloped edge with respect to* $\boldsymbol{\nu}$.

7. Curves on regular surfaces and relationships with cuspidal edges

In this section we consider curves on regular surfaces and investigate the relationship with the previous results on cuspidal edges. In [8, 14], developable surfaces along a curve on a regular surface are investigated. We consider a regular surface M parametrized by an embedding $\boldsymbol{X} : U \to \mathbb{R}^3$ with a unit normal vector field $\boldsymbol{n}$ (i.e., $M = \boldsymbol{X}(U)$). For a curve $c : I \to U$, we define $\boldsymbol{\gamma} = \boldsymbol{X} \circ c$ as a curve on M. We assume that $\boldsymbol{\gamma}$ is parametrized by the arc-length parameter s. The Darboux frame $\{\boldsymbol{t}, \boldsymbol{d}, \boldsymbol{n}\}$ along $\boldsymbol{\gamma}$ is defined to be the unit tangent vector $\boldsymbol{t}$ of $\boldsymbol{\gamma}$, $\boldsymbol{n} = \boldsymbol{n} \circ \boldsymbol{\gamma}$, and $\boldsymbol{d} = -\boldsymbol{t} \times \boldsymbol{n}$. Then we have

$$\left\{ \begin{array}{rcl} \boldsymbol{t}' & = & \kappa_g \boldsymbol{d} + \kappa_n \boldsymbol{n} \\ \boldsymbol{d}' & = & -\kappa_g \boldsymbol{t} + \tau_g \boldsymbol{n} \\ \boldsymbol{n}' & = & -\kappa_n \boldsymbol{t} - \tau_g \boldsymbol{d}. \end{array} \right.$$

The invariants κ_g, κ_n and τ_g are called the *geodesic curvature*, the *normal curvature* and the *geodesic torsion* respectively. It is known that $\boldsymbol{\gamma}$ is a geodesic of M if and only if $\kappa_g \equiv 0$, $\boldsymbol{\gamma}$ is an asymptotic curve of M if and only if $\kappa_n \equiv 0$ and $\boldsymbol{\gamma}$ is a principal curve of M if and only if $\tau_g \equiv 0$. Here, $\boldsymbol{\gamma}$ is said to be a *geodesic* if the curvature vector $\boldsymbol{t}'(s)$ has only a normal component of the surface M, an *asymptotic curve* if $\boldsymbol{t}'(s)$ has only a tangential component of the surface M and a *line of curvature* if $\boldsymbol{\nu}'(s)$ is parallel to $\boldsymbol{t}(s)$, respectively.

In [14], an invariant $\tilde{\delta}_o = \kappa_g + (\kappa_n\tau_g' - \kappa_n'\tau_g)(\kappa_n^2+\tau_g^2)^{-1}$ is introduced[1] and it is shown that $\tilde{\delta}_o \equiv 0$ if and only if $(\tau_g\boldsymbol{t} - \kappa_n\boldsymbol{d})(\kappa_n^2+\tau_g^2)^{-1/2}$ is a constant vector. Moreover, it is shown that $\tilde{\delta}_o \equiv 0$ if and only if $\boldsymbol{\gamma}$ is a contour generator (i.e., singular set) with respect to an orthogonal projection such that its kernel is generated by $\tau_g\boldsymbol{t} - \kappa_n\boldsymbol{d}$. Furthermore, in [7], it is shown that $\tilde{\delta}_o(\kappa_n^2+\tau_g^2)^{-1/2}$ is constant if and only if γ is an isophotic curve (i.e., $\boldsymbol{n}\circ\boldsymbol{\gamma}$ makes a constant angle with a constant vector $(\tau_g\boldsymbol{t}+\kappa_g\boldsymbol{n})(\kappa_g^2+\tau_g^2)^{-1/2}$.).

On the other hand, in [7], an invariant $\tilde{\delta}_r = \kappa_n + (\kappa_g'\tau_g - \kappa_g\tau_g')(\kappa_g^2+\tau_g^2)^{-1}$ is introduced[2] and it is shown that $\tilde{\delta}_r \equiv 0$ if and only if $(\tau_g\boldsymbol{t}+\kappa_g\boldsymbol{n})(\kappa_g^2+\tau_g^2)^{-1/2}$ is a constant.

Actually, $(\tau_g\boldsymbol{t}-\kappa_n\boldsymbol{d})(\kappa_n^2+\tau_g^2)^{-1/2}$ (respectively, $(\tau_g\boldsymbol{t}+\kappa_g\boldsymbol{n})(\kappa_g^2+\tau_g^2)^{-1/2}$) is called a *normalized osculating Darboux vector* (respectively, a *normalized rectifying Darboux vector*) along $\boldsymbol{\gamma}$ in [7, 14]. Therefore, the osculating Darboux vector and the rectifying Darboux vector along a cuspidal edge are the notions analogous to those of the case for a regular curve on a regular surface. In this section we compare their properties along regular curves on regular surfaces with those along cuspidal edges.

On the other hand, with the same setting as in Section 5, $S(f)$ is not only a curve on f but also a curve on OD_f and ND_f. In particular, if $\kappa_\nu \neq 0$, then $S(f)$ is a regular curve on the regular part of OD_f. Moreover, $S(f)$ is always a regular curve on the regular part of ND_f. Therefore, we consider the invariants of $S(f)$ as a regular curve on OD_f and ND_f, respectively. Let $\tilde{\kappa}_g$, $\tilde{\kappa}_\nu$ and $\tilde{\tau}_g$ be the geodesic curvature, normal curvature and geodesic torsion of

$$S(f) = \{f(u,0) = OD_f(u,0)\ |u \in I\}$$

as a curve on OD_f, respectively. Also let $\overline{\kappa}_g$, $\overline{\kappa}_\nu$ and $\overline{\tau}_g$ denote the geodesic curvature, normal curvature and geodesic torsion of $S(f) = \{f(u,0) = ND_f(u,0)|\ u \in I\}$ as a curve on ND_f, respectively.

Since $\boldsymbol{\nu}$ is a unit normal vector of OD_f, we see that $\tilde{\kappa}_g = \kappa_s$, $\tilde{\kappa}_n = \kappa_\nu$ and $\tilde{\tau}_g = \kappa_t$. Also, since $\boldsymbol{b}$ is a unit normal vector of ND_f, we see that $\overline{\kappa}_g = -\kappa_\nu$, $\overline{\kappa}_n = \kappa_s$ and $\overline{\tau}_g = \kappa_t$. Hence we see that the invariants $\tilde{\delta}_o$ and $\tilde{\delta}_r$ of $f(u,0) = OD_f(u,0)$ as a curve on OD_f are

$$\tilde{\delta}_o = \frac{\delta_o}{\kappa_\nu^2+\kappa_t^2}, \quad \tilde{\delta}_r = \frac{\delta_n}{\kappa_s^2+\kappa_t^2},$$

respectively. On the other hand, the invariants $\tilde{\delta}_o$ and $\tilde{\delta}_r$ of $f(u,0) = ND_f(u,0)$ as a curve on ND_f are

$$\tilde{\delta}_o = -\frac{\delta_n}{\kappa_s^2+\kappa_t^2}, \quad \tilde{\delta}_r = \frac{\delta_o}{\kappa_\nu^2+\kappa_t^2}.$$

For the invariants $\kappa_g, \kappa_n, \tau_g$ of a curve $\boldsymbol{\gamma}$ on a regular surface, $\boldsymbol{\gamma}$ is an asymptotic curve of f if and only if $\kappa_n \equiv 0$, $\boldsymbol{\gamma}$ is a geodesic of f if and only if $\kappa_g \equiv 0$, and $\boldsymbol{\gamma}$ is a line of curvature of f if and only if $\tau_g \equiv 0$. It is natural to expect this type of explanation about invariants $\kappa_s, \kappa_\nu, \kappa_t$ of cuspidal edge. The singular curvature κ_s (respectively, the limiting normal curvature κ_ν) is defined as a limit of the geodesic curvatures with sign (respectively, the normal curvatures) of curves approaching the singular set of the cuspidal edge, and one can see the same explanation about κ_s and κ_ν [27, 20]. Here, we study κ_t from this point of view. For a regular curve $c : I \longrightarrow U$, it is classically known that $\boldsymbol{\gamma} = \boldsymbol{X}\circ c$ is a line of curvature if and only if the ruled surface with the normal director curve $\boldsymbol{\gamma}(s) + t\boldsymbol{n}(s)$ is a developable surface (i.e., Theorem of

[1]In [14], $\tilde{\delta}_o$ is denoted by δ.

[2]In [7], $\tilde{\delta}_r$ is denoted by δ_r.

Bonnet [29, Page 295]). On the other hand, let $f : I \times \mathbb{R} \longrightarrow \mathbb{R}^3$ be a frontal, and suppose $S(f) = I \times \{0\}$ consists of singular points of the first kind. Assume that $\kappa_t \equiv 0$ on I. Then $\overline{D}_r(u) = \pm\boldsymbol{\nu}(u)$, so that ND_f is a ruled surface with base curve $f|_{S(f)}$ and director curve $\boldsymbol{\nu}$, and it is, by definition, developable. Thus it is natural to expect that $S(f)$ of a frontal with vanishing κ_t can be considered as a line of curvature.

Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a map-germ and 0 a cuspidal edge. Suppose that (u, v) is an adapted coordinate system. Since $f_v(u, 0) = 0$, there exists a vector $h(u, v)$ such that $f_v(u, v) = vh(u, v)$. Set

$$\widetilde{E} = \langle f_u, f_u\rangle\,,\ \widetilde{F} = \langle f_u, h\rangle\,,\ \widetilde{G} = \langle h, h\rangle\,,\ \widetilde{L} = -\langle f_u, \nu_u\rangle\,,\ \widetilde{M} = -\langle h, \nu_u\rangle\,,\ \widetilde{N} = -\langle h, \nu_v\rangle\,.$$

Then

$$E = \widetilde{E}, \quad F = v\widetilde{F}, \quad G = v^2\widetilde{G}, \quad L = \widetilde{L}, \quad M = v\widetilde{M}, \quad N = v\widetilde{N} \tag{7.1}$$

holds, where E, F, G (respectively, L, M, N) are the coefficients of the first fundamental form (respectively, the second fundamental form). Consider the equation

$$(EM - FL)\,du^2 + (EN - GL)\,dudv + (FN - GM)\,dv^2 = 0 \tag{7.2}$$

for a tangent vector $a(u, v)\partial_u + b(u, v)\partial_v \in T_{(u,v)}\mathbb{R}^2$. It is known that if $u'(t)\partial_u + v'(t)\partial_v$ satisfies (7.2), then the curve $(u(t), v(t))$ is a principal curve of f. Substituting (7.1) to (7.2) and factoring v out, we obtain the equation

$$(\widetilde{E}\widetilde{M} - \widetilde{F}\widetilde{L})\,du^2 + (\widetilde{E}\widetilde{N} - v\widetilde{G}\widetilde{L})\,dudv + (v\widetilde{F}\widetilde{N} - v^2\widetilde{G}\widetilde{M})\,dv^2 = 0.$$

Thus if $(\widetilde{E}\widetilde{M} - \widetilde{F}\widetilde{L})(u, 0) \equiv 0$, then we can regard the curve $(u, 0)$ as a line of curvature. By (5.1) of [20], $\kappa_t(u)$ is proportional to $(\widetilde{E}\widetilde{M} - \widetilde{F}\widetilde{L})(u, 0)$. Summarizing the above arguments, $S(f)$ can be regarded as a line of curvature if $\kappa_t \equiv 0$ holds.

Appendix A. Support functions

In this appendix we study invariants of a cuspidal edge using a family of functions on a curve. It is well-known that this method is useful for studying singular curves on singular surfaces. Although the results are the same as we have obtained above, we believe that it is worth mentioning that one can get the same result as Theorems 5.1 and 5.3 by this method.

For a unit speed curve $\boldsymbol{\gamma} : I \longrightarrow M \subset \mathbb{R}^3$ and a vector field $k : I \to TM$ along $\boldsymbol{\gamma}$, we define a function $G_{\boldsymbol{k}} : I \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ by $G_{\boldsymbol{k}}(u, \boldsymbol{x}) = \langle \boldsymbol{x} - \boldsymbol{\gamma}(u), \boldsymbol{k}(u)\rangle$. We call $G_{\boldsymbol{k}}$ a *support function on* $\boldsymbol{\gamma}$ *with respect to* $\boldsymbol{k}$. We denote that $g_{\boldsymbol{k},\boldsymbol{x}_0}(u) = G_{\boldsymbol{k}}(u, \boldsymbol{x}_0)$ for any $\boldsymbol{x}_0 \in \mathbb{R}^3$.

Let $f : I \times (-\varepsilon, \varepsilon) \to \mathbb{R}^3$ be a frontal with a unit normal vector $\boldsymbol{\nu}$, where I is an open interval or a circle, and $\varepsilon > 0$. Assume that $I \times \{0\}$ consists of singular points of the first kind, and we take an adapted coordinate system (u, v) of $I \times (-\varepsilon, \varepsilon)$. Let $\boldsymbol{e}, \boldsymbol{b}, \boldsymbol{\nu}$ be the Darboux frame of $S(f)$. We consider

$$G_{\boldsymbol{\nu}}(u, \boldsymbol{x}), \quad g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u), \quad G_{\boldsymbol{b}}(u, \boldsymbol{x}), \quad g_{\boldsymbol{b},\boldsymbol{x}_0}(u)$$

We have the following propositions.

Proposition A.1. *Under the above setting, we have the following*:
(A) *Suppose that* $\kappa_\nu(u)^2 + \kappa_t(u)^2 \neq 0$. *Then*

(A1) $g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u) = 0$ *if and only if there exist* $\alpha(u)$ *and* $\beta(u)$ *such that*

$$\boldsymbol{x}_0 - f(u, 0) = \alpha\boldsymbol{e}(u) + \beta\boldsymbol{b}(u).$$

(A2) $g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u) = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)' = 0$ *if and only if there exists* $l(u)$ *such that*

$$\boldsymbol{x}_0 - f(u, 0) = -l(u)\overline{D_o}(u).$$

(AI) *Suppose that* $\delta_o(u) \neq 0$. *Then*

(A3) $g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u) = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)' = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)' = 0$ *if and only if*

$$x_0 - f(u,0) = -\frac{\kappa_\nu}{\delta_o}\overline{D_o}(u). \tag{A.1}$$

(A4) $g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u) = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)' = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)' = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)'' = 0$ *if and only if* (A.1) *and* $\sigma_o = 0$.

(A5) $g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u) = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)' = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)' = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)'' = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)''' = 0$ *if and only if* (A.1) *and* $\sigma_o = \sigma_o' = 0$.

(AII) *Suppose that* $\delta_o(u) = 0$. *Then*

(A3') $g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u) = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)' = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)'' = 0$ *if and only if* $\kappa_\nu = 0$. *We remark that under this condition,* $\delta_o = \kappa_s\kappa_t - \kappa_\nu'$.

(AII-1) *Set* $\delta_{\nu 1} = \kappa_t\kappa_s' + 2\kappa_s\kappa_t' - \kappa_\nu''$ *and suppose that* $\delta_o(u) = 0, \delta_{\nu 1}(u) \neq 0$. *Then*

(A4') $g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u) = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)' = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)'' = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)''' = 0$ *if and only if* $\kappa_s = 0$ *and* $x_0 - f(u,0) = -\kappa_s\kappa_t\boldsymbol{e}(u)/\delta_{\nu 1}$.

(A5') $g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u) = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)' = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)'' = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)''' = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)'''' = 0$ *if and only if* $\kappa_s = 0$, $x_0 - f(u,0) = -\kappa_\nu\kappa_t\boldsymbol{e}(u)/\delta_{\nu 1}$ *and*

$$-2\kappa_s^4\kappa_t^2 - (2\kappa_t\kappa_s' - 3\kappa_\nu'')(\kappa_t\kappa_s' - \kappa_\nu'') - 3\kappa_s^2(2(\kappa_t')^2 + \kappa_t\kappa_t'') - \kappa_s(\kappa_t^2\kappa_s'' - 9\kappa_t'\kappa_\nu'' - \kappa_t(-10\kappa_s'\kappa_t' + \kappa_\nu''')) = 0.$$

(AII-2) *Suppose that* $\delta_o(u) = 0, \delta_{\nu 1}(u) = 0$. *Then*

(A4") $g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u) = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)' = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)'' = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)''' = 0$ *if and only if* $\kappa_s = \kappa_\nu = 0$ *and there exists* $l(u)$ *such that* $x_0 - f(u,0) = l(u)\boldsymbol{e}(u)$. *We remark that under this condition,* $\delta_{\nu 1} = -\kappa_t\kappa_s' + \kappa_\nu''$.

(AII-2-1) *Set* $\delta_{\nu 2} = 3\kappa_s'\kappa_t' + \kappa_t\kappa_s'' - \kappa_\nu'''$, *and suppose that* $\delta_o(u) = 0, \delta_{\nu 1}(u) = 0, \delta_{\nu 2}(u) \neq 0$. *Then*

(A5") $g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u) = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)' = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)'' = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)''' = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)'''' = 0$ *if and only if* $\kappa_s = \kappa_\nu = 0$ *and* $x_0 - f(u,0) = -\kappa_t\kappa_s'\boldsymbol{e}(u)/\delta_{\nu 2}$.

(AII-2-2) *Suppose that* $\delta_o(u) = 0, \delta_{\nu 1}(u) = 0, \delta_{\nu 2}(u) = 0$. *Then*

(A5"') $g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u) = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)' = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)'' = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)''' = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)'''' = 0$ *if and only if* $\kappa_s = \kappa_\nu = \kappa_s' = 0$ *and there exists* $l(u)$ *such that* $x_0 - f(u,0) = l(u)\boldsymbol{e}(u)$. *We remark that under this condition,* $\delta_{\nu 2} = \kappa_t\kappa_s'' - \kappa_\nu'''$.

(B) *Suppose that* $\kappa_s(u)^2 + \kappa_t(u)^2 \neq 0$. *Then*

(B1) $g_{\boldsymbol{b},\boldsymbol{x}_0}(u) = 0$ *if and only if there exist* $\alpha(u)$ *and* $\beta(u)$ *such that*

$$\boldsymbol{x}_0 - f(u,0) = \alpha\boldsymbol{e}(u) + \beta\boldsymbol{\nu}(u).$$

(B2) $g_{\boldsymbol{b},\boldsymbol{x}_0}(u) = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)' = 0$ *if and only if there exists* $l(u)$ *such that*

$$\boldsymbol{x}_0 - f(u,0) = l(u)\overline{D}_r(u).$$

(BI) *Suppose that* $\delta_n(u) \neq 0$. *Then*

(B3) $g_{\boldsymbol{b},\boldsymbol{x}_0}(u) = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)' = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)'' = 0$ *if and only if*

$$x_0 - f(u,0) = \frac{-\kappa_s}{\delta_n}D_r(u). \tag{A.2}$$

(B4) $g_{\boldsymbol{b},\boldsymbol{x}_0}(u) = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)' = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)'' = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)''' = 0$ *if and only if* (A.2) *and* $\sigma_n(u) = 0$.

(B5) $g_{\boldsymbol{b},\boldsymbol{x}_0}(u) = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)' = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)'' = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)''' = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)'''' = 0$ *if and only if* (A.2) *and* $\sigma_n(u) = \sigma_n'(u) = 0$.

(BII) *Suppose that* $\delta_n(u) = 0$. *Then*

(B3') $g_{\boldsymbol{b},\boldsymbol{x}_0}(u) = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)' = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)'' = 0$ *if and only if* $\kappa_s = 0$. *We remark that under this condition,* $\delta_n = \kappa_\nu\kappa_t^2 + \kappa_t\kappa_s'$.

(BII-1) *Set* $\delta_{b1} = \kappa_t\kappa_\nu' + 2\kappa_\nu\kappa_t' + \kappa_s''$ *and suppose that* $\delta_n(u) = 0, \delta_{b1}(u) \neq 0$. *Then*

(B4') $g_{\boldsymbol{b},\boldsymbol{x}_0}(u) = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)' = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)'' = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)''' = 0$ *if and only if* $\kappa_s = 0$ *and* $x_0 - f(u,0) = -\kappa_\nu\kappa_t\boldsymbol{e}/\delta_{b1}$.

(B5') $g_{\boldsymbol{b},\boldsymbol{x}_0}(u) = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)' = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)'' = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)''' = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)'''' = 0$ *if and only if* $\kappa_s = 0$, $x_0 - f(u,0) = -\kappa_\nu \kappa_t \boldsymbol{e}/\delta_{b1}$ *and*

$$2\kappa_\nu^4\kappa_t^2+(\kappa_t\kappa_\nu'+\kappa_s'')(2\kappa_t\kappa_\nu+3\kappa_s'')+3\kappa_\nu^2(2(\kappa_t')^2+\kappa_t\kappa_t'')+\kappa_\nu\big(9\kappa_t'\kappa_s''+\kappa_t^2\kappa_\nu''-\kappa_t(-10\kappa_\nu'\kappa_t'-\kappa_s''')\big) = 0.$$

(BII-2) *Suppose that* $\delta_n(u) = 0, \delta_{b1}(u) = 0$. *Then*

(B4") $g_{\boldsymbol{b},\boldsymbol{x}_0}(u) = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)' = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)'' = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)''' = 0$ *if and only if* $\kappa_s(u) = \kappa_\nu(u) = 0$. *We remark that under this condition,* $\delta_{b1} = \kappa_t\kappa_\nu' + \kappa_s''$.

(BII-2-1) *Set* $\delta_{b2} = 3\kappa_\nu'\kappa_t' + \kappa_t\kappa_\nu'' + \kappa_s'''$, *and suppose that* $\delta_n(u) = 0, \delta_{b1}(u) = 0$, $\delta_{b2}(u) \neq 0$. *Then*

(B5") $g_{\boldsymbol{b},\boldsymbol{x}_0}(u) = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)' = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)'' = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)''' = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)'''' = 0$ *if and only if* $\kappa_s = \kappa_\nu = 0$ *and* $x_0 - f(u,0) = -\kappa_\nu'\kappa_t\boldsymbol{e}(u)/\delta_{b2}$.

(BII-2-2) *Suppose that* $\delta_n(u) = 0, \delta_{b1}(u) = 0, \delta_{b2}(u) = 0$. *Then*

(B5"') $g_{\boldsymbol{b},\boldsymbol{x}_0}(u) = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)' = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)'' = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)''' = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)'''' = 0$ *if and only if* $\kappa_s = \kappa_\nu = \kappa_\nu' = 0$ *and, there exists* $l(u)$ *such that*

$$x_0 - f(u,0) = l(u)\boldsymbol{e}(u).$$

We remark that under this condition, $\delta_{b2} = \kappa_t\kappa_\nu'' + \kappa_s'''$.

If $g_{\boldsymbol{b},\boldsymbol{x}_0}(u) = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)' = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)'' = 0$, $g_{\boldsymbol{b},\boldsymbol{x}_0}(u)''' \neq 0$ *or*

$$g_{\boldsymbol{b},\boldsymbol{x}_0}(u) = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)' = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)'' = g_{\boldsymbol{b},\boldsymbol{x}_0}(u)''' = 0,$$

$g_{\boldsymbol{b},\boldsymbol{x}_0}(u)'''' = 0$ *hold, then* $G_{\boldsymbol{b}}$ *is a* $\mathcal{K}$*-versal unfolding of* $g_{\boldsymbol{b},\boldsymbol{x}_0}$ *at* $(u, \boldsymbol{x}_0)$.

If $g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u) = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)' = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)'' = 0$, $g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)''' \neq 0$ *or*

$$g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u) = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)' = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)'' = g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)''' = 0,$$

$g_{\boldsymbol{\nu},\boldsymbol{x}_0}(u)'''' \neq 0$ *hold, then* $G_{\boldsymbol{\nu}}$ *is a* $\mathcal{K}$*-versal unfoldings of* $g_{\boldsymbol{\nu},\boldsymbol{x}_0}$ *at* $(u, \boldsymbol{x}_0)$.

See [1] or [10, Appendix] for $\mathcal{K}$-versal unfolding (written as $\mathcal{K}$-versal deformations). Using Proposition A.1, and by some general results for the singularity theory for families of function germs, one can also show Theorems 5.1 and 5.3. Detailed descriptions on general results in the singularity theory are found in the book[2].

On the other hand, the calculations by using support functions are rather complicated comparing with the direct use of the criteria for frontals in the proof of Theorems 5.1 and 5.3. However, one of the advantages of the method using the support functions is that we can clarify the geometric meanings of the singularities from the contact viewpoint. Let $\boldsymbol{\Gamma} : I \longrightarrow \mathbb{R}^3 \times S^2$ be a regular curve and $F : \mathbb{R}^3 \times S^2 \longrightarrow \mathbb{R}$ a submersion. We say that $\boldsymbol{\Gamma}$ and $F^{-1}(0)$ have contact of *at least order* k for $t = t_0$ if the function $g(t) = F \circ \boldsymbol{\Gamma}(t)$ satisfies $g(t_0) = g'(t_0) = \cdots = g^{(k)}(t_0) = 0$. If $\boldsymbol{\gamma}$ and $F^{-1}(0)$ have contact of at least order k for $t = t_0$ and satisfy the condition that $g^{(k+1)}(t_0) \neq 0$, then we say that $\boldsymbol{\Gamma}$ and $F^{-1}(0)$ have *contact of order* k for $t = t_0$. For any $\boldsymbol{x} \in \mathbb{R}^3$, we define a function $\mathfrak{g}_{\boldsymbol{x}} : \mathbb{R}^3 \times S^2 \longrightarrow \mathbb{R}$ by $\mathfrak{g}_{\boldsymbol{x}}(\boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{x} - \boldsymbol{u}, \boldsymbol{v}\rangle$. Then we have

$$\mathfrak{g}_{\boldsymbol{x}}^{-1}(0) = \{(\boldsymbol{u}, \boldsymbol{v}) \in \mathbb{R}^3 \times S^2 \mid \langle \boldsymbol{u}, \boldsymbol{v}\rangle = \langle \boldsymbol{x}, \boldsymbol{v}\rangle\}.$$

If we fix $\boldsymbol{v} \in S^2$, then $\mathfrak{g}_{\boldsymbol{x}}^{-1}(0)|\mathbb{R}^3 \times \{\boldsymbol{v}\}$ is an affine plane defined by $\langle \boldsymbol{u}, \boldsymbol{v}\rangle = c$, where $c = \langle \boldsymbol{x}, \boldsymbol{v}\rangle$. Since this plane is orthogonal to $\boldsymbol{v}$, it is parallel to the tangent plane $T_{\boldsymbol{v}}S^2$ at $\boldsymbol{v}$. Here we have a representation of the tangent bundle of S^2 as follows:

$$TS^2 = \{(\boldsymbol{u}, \boldsymbol{v}) \in \mathbb{R}^3 \times S^2 \mid \langle \boldsymbol{u}, \boldsymbol{v}\rangle = 1\}.$$

We consider the canonical projection $\pi_2|\mathfrak{g}_{\boldsymbol{x}}^{-1}(0) : \mathfrak{g}_{\boldsymbol{x}}^{-1}(0) \longrightarrow S^2$, where $\pi_2 : \mathbb{R}^3 \times S^2 \longrightarrow S^2$. Then $\pi_2|\mathfrak{g}_{\boldsymbol{x}}^{-1}(0) : \mathfrak{g}_{\boldsymbol{x}}^{-1}(0) \longrightarrow S^2$ is a plane bundle over S^2. Moreover, we define a map

$$\Psi : \mathfrak{g}_{\boldsymbol{x}}^{-1}(0) \longrightarrow TS^2$$

by $\Phi(\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{u}/\langle \boldsymbol{x},\boldsymbol{v}\rangle,\boldsymbol{v})$. Then Φ is a bundle isomorphism. Therefore, we write $TS^2(\boldsymbol{x})=\mathfrak{g}_{\boldsymbol{x}}^{-1}(0)$ and call it an *affine tangent bundle over S^2 through $\boldsymbol{x}$*. With the same notations as above, we distinguish two cases.

(A) Suppose that $(\kappa_\nu(u),\kappa_t(u))\neq(0,0)$ and $\delta_o(u)\neq 0$. We consider

$$\boldsymbol{s}_{OD}(u)=f(u,0)-\frac{\kappa_\nu(u)}{\delta_o(u)}\overline{D}_o(u).$$

By (5.3), we have

$$\boldsymbol{s}'_{OD}(u)=\frac{\sigma_o(u)}{\delta_o(u)}(\kappa_t(u)\boldsymbol{e}(u)-\kappa_\nu(u)\boldsymbol{b}(u)).$$

If we assume that $\sigma_o(u)\equiv 0$, then $\boldsymbol{s}_{OD}$ is a constant vector $\boldsymbol{x}_0$. Then

$$f(u,0)-\boldsymbol{x}_0=\frac{\kappa_\nu(u)}{\delta_o(s)}\overline{D}_o(u)$$

Therefore

$$\mathfrak{g}_{\boldsymbol{x}_0}(f(u,0),\boldsymbol{\nu}(u))=g_{\boldsymbol{\nu},\boldsymbol{x}_0}(s)=\langle \boldsymbol{x}_0-f(u,0),\boldsymbol{\nu}(u)\rangle=0.$$

If there exists $\boldsymbol{x}_0\in\mathbb{R}^3$ such that $\mathfrak{g}_{\boldsymbol{x}_0}(f(u,0),\boldsymbol{\nu}(u))=0$, then we have

$$f(u,0)-\boldsymbol{x}_0=\frac{\kappa_\nu(u)}{\delta_o(s)}\overline{D}_o(u).$$

and $\sigma_o(u)\equiv 0$. We consider a regular curve $(f|_{S(f)},\boldsymbol{\nu}):I\longrightarrow\mathbb{R}^3\times S^2$.

(B) Suppose that $(\kappa_s(u),\kappa_t(u))\neq(0,0)$ and $\delta_n(u)\neq 0$. Then we have similar results to case (A), so that we have the following proposition.

Proposition A.2. *With the same notations as above, we have the following*:
(A) *Suppose that $(\kappa_\nu(u),\kappa_t(u)\neq(0,0)$ and $\delta_o(u)\neq 0$. Then there exists $\boldsymbol{x}_0\in\mathbb{R}^3$ such that $(f|_{S(f)},\boldsymbol{\nu})(I)\subset TS^2(\boldsymbol{x}_0)$ if and only if $\sigma_o\equiv 0$.*
(B) *Suppose that $(\kappa_s(u),\kappa_t(u)\neq(0,0)$ and $\delta_n(u)\neq 0$. Then there exists $\boldsymbol{x}_0\in\mathbb{R}^3$ such that $(f|_{S(f)},\boldsymbol{b})(I)\subset TS^2(\boldsymbol{x}_0)$ if and only if $\sigma_n\equiv 0$.*

The results of Proposition A.1 can be interpreted from the contact viewpoint as follows.

Proposition A.3. *With the same notations as above, we have the following*:
(A) *Suppose that $(\kappa_\nu(u),\kappa_t(u)\neq(0,0)$ and $\delta_o(u)\neq 0$. For $\boldsymbol{x}_0=OD_f(u_0,t_0)$, we have the following*:
(1) *The order of contact of $(f|_{S(f)},\boldsymbol{\nu})$ with $TS^2(\boldsymbol{x}_0)$ at $u=u_0$ is two if and only if*

$$t_0=-\frac{\kappa_\nu(u_0)}{\delta_o(u_0)} \tag{A.3}$$

and $\sigma_o(u_0)\neq 0$. In this case OD_f is a cuspidal edge at (u_0,t_0).
(2) *The order of contact of $(f|_{S(f)},\boldsymbol{\nu})$ with $TS^2(\boldsymbol{x}_0)$ at $u=u_0$ is three if and only if (A.3) and $\sigma_o(u_0)=0$ and $\sigma'_o(u_0)\neq 0$. In this case OD_f is a swallowtail at (u_0,t_0).*
(B) *Suppose that $(\kappa_s(u),\kappa_t(u)\neq(0,0)$ and $\delta_n(u)\neq 0$. For $\boldsymbol{x}_0=ND_f(u_0,t_0)$, we have the following*:
(1) *The order of contact of $(f|_{S(f)},\boldsymbol{b})$ with $TS^2(\boldsymbol{x}_0)$ at $u=u_0$ is two if and only if*

$$t_0=-\frac{\kappa_s(u_0)}{\delta_n(u_0)} \tag{A.4}$$

and $\sigma_n(u_0)\neq 0$. In this case ND_f is a cuspidal edge at (u_0,t_0).
(2) *The order of contact of $(f|_{S(f)},\boldsymbol{b})$ with $TS^2(\boldsymbol{x}_0)$ at $u=u_0$ is three if and only if (A.4) and $\sigma_n(u_0)=0$ and $\sigma'_n(u_0)\neq 0$. In this case ND_f is a swallowtail at (u_0,t_0).*

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