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ON THE TOPOLOGY OF A RESOLUTION OF ISOLATED SINGULARITIES

VINCENZO DI GENNARO AND DAVIDE FRANCO

Abstract. Let Y be a complex projective variety of dimension n with isolated singularities, $\pi : X \to Y$ a resolution of singularities, $G := \pi^{-1}\mathrm{Sing}(Y)$ the exceptional locus. From the Decomposition Theorem one knows that the map $H^{k-1}(G) \to H^k(Y, Y\backslash\mathrm{Sing}(Y))$ vanishes for $k > n$. Assuming this vanishing, we give a short proof of the Decomposition Theorem for π. A consequence is a short proof of the Decomposition Theorem for π in all cases where one can prove the vanishing directly. This happens when either Y is a normal surface, or when π is the blowing-up of Y along $\mathrm{Sing}(Y)$ with smooth and connected fibres, or when π admits a natural Gysin morphism. We prove that this last condition is equivalent to saying that the map $H^{k-1}(G) \to H^k(Y, Y\backslash\mathrm{Sing}(Y))$ vanishes for all k, and that the pull-back $\pi_k^* : H^k(Y) \to H^k(X)$ is injective. This provides a relationship between the Decomposition Theorem and Bivariant Theory.

1. Introduction

Consider an n-dimensional complex projective variety Y with *isolated singularities*. Fix a *desingularization* $\pi : X \to Y$ of Y. This paper is addressed at the study of some topological properties of the map π. In a previous paper [14], we already observed that, even though π is never a *local complete intersection* map, in some very special cases it may nonetheless admit a *natural Gysin morphism*. By a natural Gysin morphism, we mean a *topological bivariant class* [20, §7], [7]

$$\theta \in T^0(X \xrightarrow{\pi} Y) := Hom_{D^b(Y)}(R\pi_*\mathbb{Q}_X, \mathbb{Q}_Y),$$

commuting with restrictions to the smooth locus of Y (here and in the following $D^b(Y)$ denotes the *bounded derived category* of sheaves of $\mathbb{Q}$-vector spaces on Y).

In this paper, we give a complete characterization of morphisms like π admitting a natural Gysin morphism by means of the *Decomposition Theorem* [2], [6], [8], [9]. In some sense, what we are going to prove is that π admits a natural Gysin morphism if and only if Y is a $\mathbb{Q}$-*intersection cohomology manifold*, i.e., $IC_Y^\bullet \simeq \mathbb{Q}_Y[n]$ in $D^b(Y)$ ($IC_Y^\bullet$ denotes the *intersection cohomology complex* of Y [17, p. 156], [27]). Furthermore, in this case, there is a unique natural Gysin morphism θ, and it arises from the Decomposition Theorem (compare with Theorem 1.2 below).

The Decomposition Theorem is a beautiful and very deep result about algebraic maps. In the words of MacPherson, "it contains as special cases the deepest homological properties of algebraic maps that we know"[26], [34]. As observed in [34, Remark 2.14], since the proof of the Decomposition Theorem proceeds by induction on the dimension of the strata of the singular locus, a key point is the case of varieties with isolated singularities:

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Theorem 1.1 (The Decomposition Theorem for varieties with isolated singularities)**.** *In $D^b(Y)$, we have a decomposition*

$$R\pi_*\mathbb{Q}_X \cong IC^\bullet_Y[-n] \oplus \mathcal{H}^\bullet,$$

where $\mathcal{H}^\bullet$ is quasi-isomorphic to a skyscraper complex on $\mathrm{Sing}(Y)$*. Furthermore, we have*

(1) $\mathcal{H}^k(\mathcal{H}^\bullet) \cong H^k(G)$*, for all* $k \geq n$*,*
(2) $\mathcal{H}^k(\mathcal{H}^\bullet) \cong H_{2n-k}(G)$*, for all* $k < n$*,*

where $G := \pi^{-1}(\mathrm{Sing}(Y))$*, and* $H^k(G)$ *and* $H_{2n-k}(G)$ *have* $\mathbb{Q}$*-coefficients.*

The relationship between the Gysin morphism and the Decomposition Theorem is closely related to an important topological property of the morphism π. Specifically, in [22] and [32] one proves that Theorem 1.1 implies the following vanishing

$$H^{k-1}(G) \to H^k(Y,U) \text{ vanishes for } k > n, \tag{1}$$

where $U = Y\backslash\mathrm{Sing}(Y)$.

One of the main points we would like to stress in this paper (compare with Theorem 3.1) is that

the vanishing (1) is equivalent to the Decomposition Theorem.

More precisely, what we are going to do in this paper is to prove that assuming (1), one can prove Theorem 1.1 in few pages. Actually this equivalence is already implicit in the argument developed by Navarro Aznar in order to prove [30, (6.3) Corollaire, p. 293]. In fact, after proving (1) using Hodge Theory, Navarro Aznar proves the relative Hard Lefschetz Theorem and observes that the Decomposition Theorem follows from Deligne's results on degeneration of spectral sequences. Instead, here we give a simpler and more direct proof, avoiding the use of the relative Hard Lefschetz Theorem. In fact, we deduce the splitting in derived category from a simple result concerning short exact sequences of complexes (compare with Lemma 4.7).

A byproduct of our result is a short proof of the Decomposition Theorem in all cases where one can prove the vanishing (1) directly. This happens when either $2\dim G < n$ (for trivial reasons), or when Y is a *normal surface* in view of *Mumford's theorem* [23], [29], or when $\pi : X \to Y$ is the *blowing-up* of Y along $\mathrm{Sing}(Y)$ with smooth and connected fibres (see Remark 5.1). It is worth remarking that if Y is a locally complete intersection variety, then *Milnor's theorem* on the connectivity of the *link* [16] implies (via Lemma 4.1 below) that the map $H^{k-1}(G) \to H^k(Y,U)$ vanishes for all $k \geq n+2$. Therefore, in this case the question reduces to check that the map $H^n(G) \to H^{n+1}(Y,U)$ vanishes. This in turn is equivalent to require that $H_n(G)$, which is contained in $H_n(X)$ via push-forward, is a nondegenerate subspace of $H_n(X)$ with respect to the natural intersection form $H_n(X) \times H_n(X) \to H_0(X)$ (see Remark 5.1, (i)). Another case is when π admits a natural Gysin morphism. Indeed, in this case it is very easy to prove the stronger property

$$H^{k-1}(G) \to H^k(Y,U) \text{ vanishes for } k > 0.$$

This is the real reason why in our approach the same argument leads to both Theorem 1.1 and the following:

Theorem 1.2. *There exists a natural Gysin morphism for π if and only if Y is a $\mathbb{Q}$-intersection cohomology manifold. In this case, in $D^b(Y)$ we have a decomposition*

$$R\pi_*\mathbb{Q}_X \cong IC^\bullet_Y[-n] \oplus \mathcal{H}^\bullet \cong \mathbb{Q}_Y \oplus \bigoplus_{k\geq 1} R^k\pi_*\mathbb{Q}_X[-k].$$

Moreover, a natural Gysin morphism is unique, and, up to multiplication by a nonzero rational number, it comes from the decomposition above via projection onto $\mathbb{Q}_Y$.

For a more precise and complete statement see Theorem 3.2 and Remark 3.3 below. For instance, from Theorem 3.2, (ix), we deduce that a natural Gysin morphism exists when Y is nodal of even dimension n, or when Y is a cone over a smooth basis M with $H^\bullet(M) \cong H^\bullet(\mathbb{P}^{n-1})$. We stress that the existence of a natural Gysin morphism forces the exceptional locus G to have dimension 0 or $n-1$ (see Remark 6.1).

Last but not least, we have been led to consider the issues addressed in this paper by our previous work on Noether-Lefschetz Theory. We refer to the papers [10], [11], [12], [13] anyone interested in the overlaps between the topological properties investigated here and the Noether-Lefschetz Theorem (specifically, we made an heavy use of the Decomposition Theorem in [12, Remark 3 and Theorem 6, (6.3), p. 169], and in [13, Theorem 2.1, proof of (a), p. 262]).

2. Notations

(i) Let Y be a complex irreducible projective variety of dimension $n \geq 1$, with isolated singularities. Let $\pi : X \to Y$ be a resolution of the singularities of Y. For all $y \in \mathrm{Sing}(Y)$, set $G_y := \pi^{-1}(y)$. Set $G := \bigcup_{y \in \mathrm{Sing}(Y)} G_y = \pi^{-1}(\mathrm{Sing}(Y))$. Let $i : G \hookrightarrow X$ be the inclusion.

(ii) All cohomology and homology groups are with $\mathbb{Q}$-coefficients. For a function $f : A \to B$ we denote by $\Im(f)$ the image of f, i.e., $\Im(f) = f(A)$.

(iii) Set $U := Y \backslash \mathrm{Sing}(Y) \cong X \backslash G$. Denote by $\alpha : U \hookrightarrow Y$ and $\beta : U \hookrightarrow X$ the inclusions. For all k we have the following natural commutative diagram:

$$
\begin{array}{ccc}
H^k(Y) & \xrightarrow{\pi_k^*} & H^k(X) \\
 & {\scriptstyle\alpha_k^*}\searrow \quad \swarrow{\scriptstyle\beta_k^*} & \\
 & H^k(U) &
\end{array}
\tag{2}
$$

where all the maps denote pull-back.

Remark 2.1. From the commutativity of (2) we deduce $\Im(\alpha_k^*) \subseteq \Im(\beta_k^*)$. Since $H^k(Y) \cong H^k(X)$ for $k \leq 0$ or $k \geq 2n$, we have $\Im(\alpha_k^*) = \Im(\beta_k^*)$ for $k \leq 0$ or $k \geq 2n$. It may happen that $\Im(\alpha_k^*) \neq \Im(\beta_k^*)$. We may interpret the condition $\Im(\alpha_k^*) = \Im(\beta_k^*)$ as follows. Combining the Universal Coefficient Theorem with the Lefschetz Duality Theorem [31, p. 248 and p. 297] we have $H^k(U) \cong H_{2n-k}(Y, \mathrm{Sing}(Y))$ for all k. Since $\mathrm{Sing}(Y)$ is finite, we also have

$$H_{2n-k}(Y) \cong H_{2n-k}(Y, \mathrm{Sing}(Y))$$

for $k \leq 2n-2$, and $H_1(Y) \subseteq H_1(Y, \mathrm{Sing}(Y))$. Therefore, for $k \leq 2n-2$, (2) identifies with the diagram:

$$
\begin{array}{ccc}
H^k(Y) & \longrightarrow & H_{2n-k}(X) \\
 & \searrow \quad \swarrow & \\
 & H_{2n-k}(Y) &
\end{array}
$$

where the map $H^k(Y) \to H_{2n-k}(X)$ is the composite of Poincaré Duality $H^k(X) \cong H_{2n-k}(X)$ with the pull-back π_k^*, the map $H_{2n-k}(X) \to H_{2n-k}(Y)$ is the push-forward, and the map $H^k(Y) \xrightarrow{\cdot \cap [Y]} H_{2n-k}(Y)$ is the *duality morphism*, i.e., the cap-product with the fundamental class $[Y] \in H_{2n}(Y)$ [28]. It follows that $\Im(\alpha_k^*) = \Im(\beta_k^*)$ *if and only if every cycle in* $H_{2n-k}(Y)$ *coming from* $H_{2n-k}(X)$ *via push-forward is the cap-product of a cocycle in* $H^k(Y)$ *with the fundamental class* $[Y]$. This holds true also for $k = 2n-1$ because $H_1(Y) \subseteq H_1(Y, \mathrm{Sing}(Y)) \cong H^{2n-1}(U)$.

(iv) Embed Y in some projective space $\mathbb{P}^N$. For all $y \in \mathrm{Sing}(Y)$ choose a small closed ball $S_y \subset \mathbb{P}^N$ around y, and set $B_y := S_y \cap Y$, $D_y := \pi^{-1}(B_y)$, $B := \bigcup_{y\in \mathrm{Sing}(Y)} B_y$, and $D := \pi^{-1}(B)$. B_y is homeomorphic to the cone over the link ∂B_y of the singularity $y \in Y$, with vertex at y [16, p. 23]. B_y is contractible, by excision we have

$$H^k(Y,U) \cong H^k(B, B\backslash \mathrm{Sing}(Y)) \cong H^k(B, \partial B)$$

for all k, and from the cohomology long exact sequence of the pair $(B, \partial B)$ we get

$$H^k(Y,U) \cong H^{k-1}(\partial B)$$

for all $k \geq 2$. We have $\partial D \cong \partial B$ via π, and by excision we have

$$H^k(X,U) \cong H^k(D, D\backslash G) \cong H^k(D, \partial D)$$

for all k [17, p. 38]. Since G is homotopy equivalent to D, we have $H^k(G) \cong H^k(D)$. Putting everything together, from the cohomology long exact sequence of the pair $(D, \partial D)$ we get the following exact sequence

$$H^k(X,U) \to H^k(G) \to H^{k+1}(Y,U) \stackrel{\gamma^*_{k+1}}{\to} H^{k+1}(X,U) \tag{3}$$

for all $k \geq 1$, where γ^*_{k+1} denotes the pull-back. Observe that since $\mathrm{Sing}(Y)$ is finite, we have $H^k(G) = \oplus_{y\in \mathrm{Sing}(Y)} H^k(G_y)$, $H^k(B) = \oplus_{y\in \mathrm{Sing}(Y)} H^k(B_y)$, $H^k(\partial B) = \oplus_{y\in \mathrm{Sing}(Y)} H^k(\partial B_y)$.

Remark 2.2. Assume that Y is a locally complete intersection variety. From the connectivity of the link [16, Milnor's theorem p. 76, and Hamm's theorem p. 80], it follows that *the duality morphism $H^k(Y) \to H_{2n-k}(Y)$ is an isomorphism for all $k \notin \{n-1, n, n+1\}$, is injective for $k = n-1$, and is surjective for $k = n+1$. In particular $\Im(\alpha^*_k) = \Im(\beta^*_k)$ for all $k \notin \{n-1, n\}$.* In order to prove this property, we argue as follows. We may assume $0 < k < 2n$ and $n \geq 2$. From the cohomology long exact sequence of the pair (Y, U) we have:

$$\ldots \to H^k(Y,U) \to H^k(Y) \to H^k(U) \to H^{k+1}(Y,U) \to \ldots, \tag{4}$$

and by excision $H^k(Y,U) \cong H^k(B, \partial B)$. Taking into account that each B_y is contractible and that ∂B_y is path connected [16, loc. cit.], from the cohomology long exact sequence of the pair $(B, \partial B)$ we get $H^1(B, \partial B) = 0$ and $H^k(B, \partial B) \cong H^{k-1}(\partial B)$ for $k \geq 2$. Since

$$H^k(U) \cong H_{2n-k}(Y, \mathrm{Sing}(Y)),$$

and $H_{2n-k}(Y) \cong H_{2n-k}(Y, \mathrm{Sing}(Y))$ for $k \leq 2n-2$, from (4) we get the exact sequence for $k \notin \{1, 2n-1\}$ (compare with [15, p. 5]):

$$H^{k-1}(\partial B) \to H^k(Y) \to H_{2n-k}(Y) \to H^k(\partial B).$$

Each ∂B_y is $(n-2)$-connected by Milnor's theorem [16, loc. cit.], and it is a compact oriented real manifold of dimension $2n-1$, in particular $h^k(\partial B_y) = h^{2n-1-k}(\partial B_y)$ by Poincaré Duality [16, p. 91]. It follows that the map $H^k(Y) \to H_{2n-k}(Y)$ is an isomorphism for

$$k \notin \{1, n-1, n, n+1, 2n-1\}.$$

As for the case $k = 1 \neq n-1$, this follows from (4) because

$$H^1(Y,U) \cong H^1(B, \partial B) = 0,$$

$H^1(U) \cong H_{2n-1}(Y, \mathrm{Sing}(Y)) \cong H_{2n-1}(Y)$, and $H^2(Y,U) \cong H^2(B, \partial B) \cong H^1(\partial B) = 0$ by connectivity of the link. When $k = 2n-1 \neq n+1$, we have

$$H^{2n-1}(Y,U) \cong H^{2n-1}(B, \partial B) = H^{2n-2}(\partial B) = 0.$$

Thus, $H^{2n-1}(Y) \hookrightarrow H^{2n-1}(U)$. On the other hand $H_1(Y) \hookrightarrow H_1(Y, \mathrm{Sing}(Y)) \cong H^{2n-1}(U)$. It follows that the duality morphism $H^{2n-1}(Y) \to H_1(Y)$ is injective. Then it is an isomorphism because we have just seen, in the case $k=1$, that $h^1(Y)=h_{2n-1}(Y)$. Finally notice that, when $n\geq 3$, from previous analysis and (4) we get the exact sequence:

$$0\to H^{n-1}(Y)\to H_{n+1}(Y)\to H^{n-1}(\partial B)\to H^n(Y)\to H_n(Y)$$
$$\to H^n(\partial B)\to H^{n+1}(Y)\to H_{n-1}(Y)\to 0.$$

Therefore, the duality morphism

$$H^{n-1}(Y)\to H_{n+1}(Y)$$

is injective, and the map $H^{n+1}(Y)\to H_{n-1}(Y)$ is onto. This holds true also when $n=2$. In fact, also in this case we have $H^1(B,\partial B)=0$, which implies that the duality morphism $H^1(Y)\to H_3(Y)$ is injective. Moreover, a similar analysis as before shows that the image of $H^3(Y)$ and $H_1(Y)$ have the same codimension in $H^3(U)$. Thus, they are equal. This concludes the proof of the claim.

(v) By [31, Lemma 14, p. 351] we have $H^k(X,U)\cong H_{2n-k}(G)$. Therefore, from the cohomology long exact sequence of the pair (X,U) we get a long exact sequence:

$$\dots\to H^{k-1}(U)\to H_{2n-k}(G)\to H^k(X)\stackrel{\beta_k^*}{\to} H^k(U)\to\dots. \tag{5}$$

(vi) For all $y\in \operatorname{Sing}(Y)$ set:

$$H_y^k:=\begin{cases} H^k(G_y) & \text{if } k\geq n\\ H_{2n-k}(G_y) & \text{if } k<n.\end{cases}$$

Let $\mathcal{H}_y^k$ be the skyscraper sheaf on Y with stalk at y given by H_y^k. Set $H^k:=\oplus_{y\in\operatorname{Sing}(Y)}H_y^k$ and $\mathcal{H}^k:=\oplus_{y\in\operatorname{Sing}(Y)}\mathcal{H}_y^k$. We consider $\mathcal{H}^\bullet$ as a complex of sheaves on Y with vanishing differentials $d^k_{\mathcal{H}^\bullet}=0$.

Remark 2.3. From the Universal Coefficient Theorem [31, p. 248] it follows that the $\mathbb{Q}$-vector spaces H^{n-k} and H^{n+k} are isomorphic for all k. This implies that $\mathcal{H}^\bullet[n]$ *is self-dual, i.e., in the bounded derived category $D^b(Y)$ of Y we have* $\mathcal{H}^\bullet[n]\cong D(\mathcal{H}^\bullet[n])$. Taking into account that in $\mathcal{H}^\bullet[n]$ all the differentials vanish, to prove that $\mathcal{H}^\bullet[n]$ is self-dual it suffices to prove that the complexes $\mathcal{H}^\bullet[n]$ and $D(\mathcal{H}^\bullet[n])$ have isomorphic sheaf cohomology. Since $\mathcal{H}^\bullet[n]$ is supported on a finite set, this amounts to prove that $\mathcal{H}^\bullet[n]$ and $D(\mathcal{H}^\bullet[n])$ have isomorphic hypercohomology, i.e., that

$$\mathbb{H}^k(\mathcal{H}^\bullet[n])\cong\mathbb{H}^k(D(\mathcal{H}^\bullet[n]))$$

for all k. But by Poincaré-Verdier Duality [17, p. 69, Theorem 3.3.10] we have:

$$\mathbb{H}^k(D(\mathcal{H}^\bullet[n]))\cong\mathbb{H}^{-k}(\mathcal{H}^\bullet[n])^\vee\cong\mathbb{H}^{n-k}(\mathcal{H}^\bullet)^\vee\cong(H^{n-k})^\vee\cong H^{n+k}\cong\mathbb{H}^k(\mathcal{H}^\bullet[n]).$$

(vii) We say that a graded morphism $\theta_\bullet: H^\bullet(X)\to H^\bullet(Y)$ is *natural* if for all k one has $\theta_k\circ\pi_k^*=\mathrm{id}_{H^k(Y)}$, and the following diagram commutes [14]:

$$\begin{array}{ccc} H^k(Y) & \stackrel{\theta_k}{\longleftarrow} & H^k(X)\\ {\scriptstyle\alpha_k^*}\searrow & & \swarrow{\scriptstyle\beta_k^*}\\ & H^k(U), & \end{array}$$

i.e., $\alpha_k^*\circ\theta_k=\beta_k^*$.

Remark 2.4. The existence of a natural graded morphism $\theta_\bullet: H^\bullet(X)\to H^\bullet(Y)$ is equivalent to saying that, for all k, the pull-back $\pi_k^*: H^k(Y)\to H^k(X)$ is injective and $\Im(\alpha_k^*)=\Im(\beta_k^*)$ (compare with the proof of (i) $\implies$ (ii) in Theorem 3.2 below).

(*viii*) We say that a (topological) bivariant class $\theta \in Hom_{D^b(Y)}(R\pi_*\mathbb{Q}_X, \mathbb{Q}_Y)$ is *natural* if the induced graded morphism $\theta_\bullet : H^\bullet(X) \to H^\bullet(Y)$ is natural [14], [20].

Remark 2.5. Fix a bivariant class

$$\theta \in H^0(X \stackrel{\pi}{\to} Y) \cong Hom_{D^b(Y)}(R\pi_*\mathbb{Q}_X, \mathbb{Q}_Y).$$

Let $\theta_0 : H^0(X) \to H^0(Y)$ be the induced map. Let $q \in \mathbb{Q}$ be such that

$$\theta_0(1_X) = q \cdot 1_Y \in H^0(Y) \cong \mathbb{Q}$$

[31, p. 238]. Put

$$\deg \theta := q.$$

For all k and all $c \in H^k(Y)$, by the projection formula [20, (G_4), (i), p. 26], and [31, 9, p. 251], we have :

$$\theta_k(\pi_k^*(c)) = \theta_k(1_X \cup \pi_k^*(c)) = \theta_0(1_X) \cup c = \deg \theta \cdot (1_Y \cup c) = \deg \theta \cdot c. \tag{6}$$

It follows that for all k one has:

$$\theta_k \circ \pi_k^* = \deg \theta \cdot \mathrm{id}_{H^k(Y)}. \tag{7}$$

Next consider the independent square:

$$\begin{array}{ccc} U & \stackrel{\beta}{\hookrightarrow} & X \\ \| & & {\scriptstyle\pi}\downarrow \\ U & \stackrel{\alpha}{\hookrightarrow} & Y \end{array}$$

and set $\theta' := \alpha^*(\theta) \in Hom_{D^b(U)}(\mathbb{Q}_U, \mathbb{Q}_U)$ [20, (G_2), p. 26]. Applying [20, (G_2), (ii), p. 26] to the square:

$$\begin{array}{ccc} H^0(U) & \stackrel{\beta_0^*}{\leftarrow} & H^0(X) \\ {\scriptstyle\theta_0'}\downarrow & & {\scriptstyle\theta_0}\downarrow \\ H^0(U) & \stackrel{\alpha_0^*}{\leftarrow} & H^0(Y) \end{array}$$

we get

$$\theta_0'(1_U) = \theta_0'(\beta_0^*(1_X)) = \beta_0^*(\theta_0(1_X)) = \beta_0^*(\deg \theta \cdot 1_Y) = \deg \theta \cdot \beta_0^*(1_Y) = \deg \theta \cdot 1_U.$$

Since $\pi_{|_U} = \mathrm{id}_U$, as in (6) we deduce for all k and all $c \in H^k(U)$:

$$\theta_k'(c) = \theta_k'((\pi|_U)_k^*(c)) = \theta_k'(1_U \cup c) = \theta_0'(1_U) \cup c = \deg \theta \cdot (1_U \cup c) = \deg \theta \cdot c,$$

i.e.,

$$\theta_k' = \deg \theta \cdot \mathrm{id}_{H^k(U)}. \tag{8}$$

From [20, (G_2), (ii), p. 26] it follows that

$$\deg \theta \cdot \beta_k^* = \theta_k' \circ \beta_k^* = \alpha_k^* \circ \theta_k \tag{9}$$

for all k. By (7) and (9) we see that *a bivariant class* θ *is natural if and only if* $\deg \theta = 1$*, and this is equivalent to saying that* $\beta_k^* = \alpha_k^* \circ \theta_k$ *for all* k. Observe that if θ is a bivariant class with $\deg \theta \neq 0$, then $\frac{1}{\deg \theta}\theta$ is natural.

(*ix*) We say that Y is a $\mathbb{Q}$-*cohomology (or homology) manifold* if for all $y \in Y$ and all $k \neq 2n$ one has $H^k(Y, Y\backslash\{y\}) = 0$, and $H^{2n}(Y, Y\backslash\{y\}) \cong \mathbb{Q}$ [27], [28]. Recall that Y is a $\mathbb{Q}$-intersection cohomology manifold if $IC_Y^\bullet \cong \mathbb{Q}_Y[n]$ in $D^b(Y)$, where $IC_Y^\bullet$ denotes the intersection cohomology complex of Y [17, p. 156], [27].

Remark 2.6. By [20, 3.1.4, p. 34] we know that there is a mapping $\phi : X \to \mathbb{R}^m$ such that $(\pi, \phi) : X \to Y \times \mathbb{R}^m$ is a closed imbedding. In this case one has

$$H^0(X \stackrel{\pi}{\to} Y) \cong H^m(Y \times \mathbb{R}^m, Y \times \mathbb{R}^m \backslash X_\phi),$$

where X_ϕ is the image of X in $Y \times \mathbb{R}^m$. If Y is a $\mathbb{Q}$-cohomology manifold, then by Poincaré-Alexander-Lefschetz Duality [1, Theorem 1.1] we have:

$$H^m(Y \times \mathbb{R}^m, Y \times \mathbb{R}^m \backslash X_\phi) \cong H_{2n}(X).$$

It follows that

$$\dim_{\mathbb{Q}} H^0(X \stackrel{\pi}{\to} Y) = 1. \tag{10}$$

On the other hand, since U is smooth, we also have [19, Lemma 2 and (26), p. 217]:

$$H^0(U \stackrel{\mathrm{id}_U}{\to} U) \cong H^m(U \times \mathbb{R}^m, U \times \mathbb{R}^m \backslash U_\phi) \cong H^{BM}_{2n}(U) \cong H^0(U) \cong \mathbb{Q},$$

where $H^{BM}_{2n}(U)$ denotes the Borel-Moore homology. Therefore, the pull-back

$$\alpha^* : H^0(X \stackrel{\pi}{\to} Y) \to H^0(U \stackrel{\mathrm{id}_U}{\to} U)$$

for bivariant classes identifies with the restriction in Borel-Moore homology:

$$H_{2n}(X) \cong H^{BM}_{2n}(U).$$

Comparing with (8) and (10), this proves that *if Y is a $\mathbb{Q}$-cohomology manifold, then there is a unique natural bivariant class.*

(x) Let $\mathcal{I}^\bullet$ be an injective resolution of $\mathbb{Q}_X$. Let $\mathcal{J}^\bullet := \pi_*(\mathcal{I}^\bullet)$ be the derived direct image $R\pi_*\mathbb{Q}_X$ of $\mathbb{Q}_X$ in $D^b(Y)$. When $k \geq 1$ the cohomology sheaves $R^k\pi_*\mathbb{Q}_X = H^k(\mathcal{J}^\bullet)$ are supported on $\mathrm{Sing}(Y)$, and for all $y \in \mathrm{Sing}(Y)$ we have $H^k(\mathcal{J}^\bullet)_y = H^k(G_y)$.

Remark 2.7. The complex $\mathcal{J}^\bullet[n]$ is self-dual. In fact, by [17, p. 69, Proposition 3.3.7, (ii)], we have:

$$D(\mathcal{J}^\bullet[n]) = D(R\pi_*\mathbb{Q}_X[n]) = R\pi_*(D(\mathbb{Q}_X[n])) = R\pi_*(\mathbb{Q}_X[n]) = \mathcal{J}^\bullet[n].$$

(xi) Since Y has only isolated singularities, we have [17, Proposition 5.4.4, p. 157]:

$$IH^k(Y) \cong \begin{cases} H^k(Y) & \text{if } k > n \\ \Im(\alpha^*_n) & \text{if } k = n \\ H^k(U) & \text{if } k < n. \end{cases} \tag{11}$$

3. The main results

Theorem 3.1 below is essentially already known. Property (i) implies (ii) by [32, Theorem 1.11, p. 518]. That property (ii) implies (i) is implicit in the argument developed by Navarro in order to prove [30, (6.3) Corollaire, p. 293] using a relative version of the Hard Lefschetz Theorem. Here we give a simpler and more direct proof that (ii) implies (i), avoiding the use of the relative Hard Lefschetz Theorem.

Theorem 3.1. *The following properties are equivalent.*

(i) In the derived category of Y there is an isomorphism:

$$R\pi_*\mathbb{Q}_X \cong IC^\bullet_Y[-n] \oplus \mathcal{H}^\bullet.$$

(ii) The map $H^{k-1}(G) \to H^k(Y, U)$ vanishes for all $k > n$.

The equivalences of properties (v), (vi) and (vii) in the next Theorem 3.2 are already known [4], [28], [27]. We insert them in the claim for Reader's convenience. We refer to [27] for other equivalences concerning a $\mathbb{Q}$-cohomology manifold.

Theorem 3.2. *The following properties are equivalent.*

(i) The map $H^{k-1}(G) \to H^k(Y,U)$ vanishes for all $k>0$ and the pull-back π_k^* is injective.

(ii) There exists a natural graded morphism $\theta_\bullet : H^\bullet(X) \to H^\bullet(Y)$.

(iii) There exists a natural bivariant class $\theta \in Hom_{D^b(Y)}(R\pi_*\mathbb{Q}_X, \mathbb{Q}_Y)$.

(iv) The natural map $H^\bullet(Y) \to IH^\bullet(Y)$is an isomorphism;

(v) Y is a $\mathbb{Q}$-intersection cohomology manifold.

(vi) Y is a $\mathbb{Q}$-cohomology manifold.

(vii) The duality morphism $H^\bullet(Y) \stackrel{\cdot\cap[Y]}{\longrightarrow} H_{2n-\bullet}(Y)$ is an isomorphism (i.e., Y satisfies Poincaré Duality).

(viii) In $D^b(Y)$ there exists a decomposition

$$R\,\pi_*\mathbb{Q}_X \cong \mathbb{Q}_Y \oplus \bigoplus_{k\geq 1} R^k\pi_*\mathbb{Q}_X[-k]. \tag{12}$$

Moreover, if $\pi : X \to Y$ is the blowing-up of Y along $\operatorname{Sing}(Y)$ with smooth and connected fibres, then previous properties are equivalent to the following property:

(ix) For all $y \in Sing(Y)$ one has $H^\bullet(G_y) \cong H^\bullet(\mathbb{P}^{n-1})$.

Remark 3.3. (i) Projecting onto $\mathbb{Q}_Y$, from the decomposition (12), we obtain a bivariant class

$$\eta \in Hom_{D^b(Y)}(R\pi_*\mathbb{Q}_X, \mathbb{Q}_Y),$$

whose induced Gysin morphisms $\eta_k : H^k(X) \to H^k(Y)$ are surjective. In particular $\deg\eta \neq 0$. By Remark 2.6 it follows that $\frac{1}{\deg\eta}\eta$ is the unique natural bivariant class.

(ii) The natural morphism $\theta_\bullet : H^\bullet(X) \to H^\bullet(Y)$ is unique and identifies with the push-forward via Poincaré Duality:

$$H^\bullet(X) \cong H_{2n-\bullet}(X) \to H_{2n-\bullet}(Y) \cong H^\bullet(Y).$$

In fact, by Remark 2.1 we know that, for $k < 2n-1$, the restriction map $\alpha_k^* : H^k(Y) \to H^k(U)$ is nothing but the duality (iso)morphism because $H^k(U) \cong H_{2n-k}(Y)$. Therefore, we have $\theta_k = (\alpha_k^*)^{-1} \circ \beta_k^*$. The case $k = 2n-1$ is similar because $H_1(Y) \subseteq H^{2n-1}(U)$ (again compare with Remark 2.1).

4. Preliminaries

Lemma 4.1. *The following sequences are exact:*

$$0 \to H^k(Y) \stackrel{\pi_k^*}{\to} H^k(X) \stackrel{i_k^*}{\to} H^k(G) \to 0 \quad \textit{for all } k>n,$$

$$H^n(Y) \stackrel{\pi_n^*}{\to} H^n(X) \stackrel{i_n^*}{\to} H^n(G) \to 0,$$

$$0 \to H_{2n-k}(G) \to H^k(X) \stackrel{\beta_k^*}{\to} H^k(U) \to 0 \quad \textit{for all } k<n.$$

Proof. By [18, p. 84, 6*] we know that $H^k(Y, \mathrm{Sing}(Y)) \cong H^k(X, G)$ for all k. Since $\mathrm{Sing}(Y)$ is finite, we also have $H^k(Y, \mathrm{Sing}(Y)) \cong H^k(Y)$ for $k \geq 1$. Therefore, the long exact sequence of the pair:

$$\dots \to H^k(X, G) \to H^k(X) \xrightarrow{i_k^*} H^k(G) \to H^{k+1}(X, G) \to \dots$$

identifies, when $k \geq 1$, with the long exact sequence:

$$\dots \to H^k(Y) \xrightarrow{\pi_k^*} H^k(X) \xrightarrow{i_k^*} H^k(G) \to H^{k+1}(Y) \to \dots. \tag{13}$$

In order to prove that the first two sequences are exact, it suffices to prove that i_k^* is surjective for all $k \geq n$. To this purpose, let L be a general hyperplane section of Y, and put $Y_0 := Y \backslash L$, and $X_0 := \pi^{-1}(Y_0)$. As before, we have a long exact sequence:

$$\dots \to H^k(Y_0) \to H^k(X_0) \to H^k(G) \to H^{k+1}(Y_0) \to \dots$$

and by Deligne's theorem [33, Proposition 4.23], we know that the pull-back maps

$$H^k(X) \xrightarrow{i_k^*} H^k(G) \quad \text{and} \quad H^k(X_0) \to H^k(G)$$

have the same image. Then we are done. In fact, since Y_0 is affine, we have $H^{k+1}(Y_0) = 0$ for all $k \geq n$ by stratified Morse Theory [21, p. 23-24].

In order to examine the last sequence, assume $k < n$. Then $2n - k > n$, and we just proved that the pull-back $H^{2n-k}(X, G) \cong H^{2n-k}(Y) \to H^{2n-k}(X)$ is injective. Combining the Poincaré Duality Theorem with the Lefschetz Duality Theorem [31, p. 297] we have $H^{2n-k}(X) \cong H_k(X)$ and $H^{2n-k}(X, G) \cong H_k(U)$. Therefore, the push-forward $H_k(U) \to H_k(X)$ is injective. Hence, the restriction $H^k(X) \to H^k(U)$ is onto for all $k < n$. Now our assertion follows from (5). □

Lemma 4.2. *Fix an integer k, and let $\gamma_k^* : H^k(Y, U) \to H^k(X, U)$ be the pull-back. Assume that $\pi_k^* : H^k(Y) \to H^k(X)$ is injective. Then the following properties are equivalent.*

(i) γ_k^* is injective;

(ii) $\Im(\alpha_{k-1}^*) = \Im(\beta_{k-1}^*)$;

(iii) $H^{k-1}(G) \to H^k(Y, U)$ is the zero map.

Proof. Consider the natural commutative diagram with exact rows:

$$\begin{array}{ccccccc} H^{k-1}(X) & \xrightarrow{\beta_{k-1}^*} & H^{k-1}(U) & \longrightarrow & H^k(X, U) & \longrightarrow & H^k(X) \\ {\scriptstyle \pi_{k-1}^*}\uparrow & & \| & & {\scriptstyle \gamma_k^*}\uparrow & & {\scriptstyle \pi_k^*}\uparrow \\ H^{k-1}(Y) & \xrightarrow{\alpha_{k-1}^*} & H^{k-1}(U) & \longrightarrow & H^k(Y, U) & \longrightarrow & H^k(Y). \end{array}$$

If γ_k^* is injective, then

$$\ker(H^{k-1}(U) \to H^k(X, U)) = \ker(H^{k-1}(U) \to H^k(Y, U)).$$

It follows that $\Im(\alpha_{k-1}^*) = \Im(\beta_{k-1}^*)$ because $\Im(\alpha_{k-1}^*) = \ker(H^{k-1}(U) \to H^k(Y, U))$ and

$$\Im(\beta_{k-1}^*) = \ker(H^{k-1}(U) \to H^k(X, U)).$$

Conversely, assume that $\Im(\alpha_{k-1}^*) = \Im(\beta_{k-1}^*)$, and fix an element $c \in \ker \gamma_k^*$. Since π_k^* is injective, there exists some $c' \in H^{k-1}(U)$ which maps to c via $H^{k-1}(U) \to H^k(Y, U)$. Since $c \in \ker \gamma_k^*$, a fortiori c' belongs to $\Im(\beta_{k-1}^*)$. Hence, $c' \in \Im(\alpha_{k-1}^*)$ and $c = 0$. The equivalence of (i) with (iii) follows from (3). □

Corollary 4.3. *Let $H_k(G) \to H^{2n-k}(G)$ be the map obtained by composing the map $H_k(G) \to H^{2n-k}(X)$ with the pull-back $H^{2n-k}(X) \to H^{2n-k}(G)$. Assume $k \geq n$ and that $\Im(\alpha_k^*) = \Im(\beta_k^*)$. Then the map $H_k(G) \to H^{2n-k}(G)$ is injective.*

Proof. By Lemma 4.1, Lemma 4.2, and (3), we deduce that the map $H^k(X,U) \to H^k(G)$ is onto. Dualizing we get an injective map $H_k(G) \to H_k(X,U)$. We are done because, by excision and the Lefschetz Duality Theorem [31, p. 298], we have

$$H_k(X,U) \cong H_k(D, \partial D) \cong H^{2n-k}(D) \cong H^{2n-k}(G).$$

□

Corollary 4.4. *We have:*

$$H^k(X) \cong \begin{cases} IH^k(Y) \oplus H^k(G) & \text{if } k > n, \\ IH^k(Y) \oplus H_{2n-k}(G) & \text{if } k < n. \end{cases}$$

Moreover, if $\Im(\alpha_n^*) = \Im(\beta_n^*)$, *then*

$$H^n(X) \cong IH^n(Y) \oplus H^n(G).$$

Proof. In view of Lemma 4.1 we only have to examine the case $k = n$. Since $\beta_n^* \circ \pi_n^* = \alpha_n^*$, there exists a subspace $P \subseteq \Im(\pi_n^*) \subseteq H^n(X)$, which is mapped isomorphically to

$$\Im(\beta_n^*) = \Im(\alpha_n^*) = IH^n(Y)$$

via β_n^*. In particular $P \cap \ker \beta_n^* = \{0\}$, and so $H^n(X) = IH^n(Y) \oplus \ker \beta_n^*$. On the other hand $\ker \beta_n^* = \Im(H^n(X,U) \to H^n(X))$. By Corollary 4.3 we know that the map $H^n(X,U) \to H^n(X)$ is injective because so is the composite $H^n(X,U) \cong H_n(G) \to H^n(X) \to H^n(G)$. Therefore, $\ker \beta_n^* = \Im(H^n(X,U) \to H^n(X)) \cong H^n(X,U) \cong H_n(G) \cong H^n(G)$. □

Lemma 4.5. *Assume that* $\Im(\alpha_k^*) = \Im(\beta_k^*)$ *for all* $k \geq n$. *Then there is an injective map of complexes*

$$0 \to \mathcal{H}^\bullet \to \mathcal{J}^\bullet.$$

Proof. It is enough to prove that for all k there is a monomorphism of sheaves

$$\mathcal{H}^k \hookrightarrow \ker(\mathcal{J}^k \to \mathcal{J}^{k+1}).$$

First, we examine the case $k \geq n$.

To this aim, set $\Gamma^\bullet := \Gamma(\mathcal{J}^\bullet)$ and denote by $d^k : \Gamma^k \to \Gamma^{k+1}$ the differential. Then we have $H^k(X) = H^k(\Gamma^\bullet)$. By Lemma 4.1 every element a of $H^k = H^k(G)$ can be lifted to an element $c \in \ker\, d^k$. We claim that every $a \in H^k(G)$ can be lifted to an element $b \in \ker\, d^k \subseteq \Gamma(\mathcal{J}^k)$ which is supported on $\mathrm{Sing}(Y)$. Proving this claim amounts to show that every $a \in H^k(G)$ can be lifted to an element $b \in \ker\, d^k \subset \Gamma(\mathcal{J}^k) = \Gamma(\mathcal{I}^k)$ such that $b\mid_U = 0 \in \Gamma(\mathcal{J}^k\mid_U)$. But $c\mid_U$ projects to a cohomology class living in $\Im(H^k(X) \to H^k(U))$. By our assumption we have

$$\Im(H^k(X) \stackrel{\beta_k^*}{\to} H^k(U)) = \Im(H^k(Y) \stackrel{\alpha_k^*}{\to} H^k(U)).$$

Since

$$H^k(Y) \cong H^k(Y, \mathrm{Sing}(Y)) \cong H^k(X,G)$$

[18, p. 84, 6*], we find

$$\Im(H^k(Y) \stackrel{\alpha_k^*}{\to} H^k(U)) = \Im(H^k(X,G) \to H^k(U)).$$

On the other hand we have

$$H^k(X,G) \cong H^k(X, \beta_! \mathbb{Q}_U)$$

[5, Theorem 12.1], [17, Remark 2.4.5, (ii)]. By definition of direct image with proper support [24, §2.6], [17, Definition 2.3.21], the sheaf $\beta_! \mathbb{Q}_U$ identifies with the subsheaf of $\mathbb{Q}_X$ consisting of sections with support contained in U. It follows that there exists $e_U \in \Gamma(\mathcal{J}^{k-1} \mid_U)$ and $g \in \Gamma(\mathcal{J}^k)$ supported in U such that

$$c \mid_U - d^{k-1}(e_U) = g \mid_U .$$

Moreover, there exists $e \in \Gamma(\mathcal{J}^{k-1})$ with $e \mid_U = e_U$, because $\mathcal{J}^{k-1}$ is injective (hence flabby). We conclude that the section

$$c - g - d^{k-1}(e) \in \Gamma(\mathcal{J}^k)$$

is supported on $\mathrm{Sing}(Y)$. Our claim is proved because $g + d^{k-1}(e) \in \Gamma(\mathcal{J}^k)$ vanishes in $H^k(G)$. To conclude the proof in the case $k \geq n$, fix a basis $a_r \in H^k = H^k(G)$ and lift every a_r to a $b_r \in \ker\, d^k \subseteq \Gamma(\mathcal{J}^k)$ as in the claim. We get an isomorphism between $H^k(G)$ and a subspace of $\Gamma(\mathcal{J}^k)$ consisting of sections supported on $\mathrm{Sing}(Y)$. We are done because such an isomorphism projects to a monomorphism of sheaves $\mathcal{H}^k \hookrightarrow \ker\,(J^k \to J^{k+1})$.

Now we assume $k < n$.

By Lemma 4.1 every element a of $H^k = H_{2n-k}(G) \subseteq H^k(X)$ can be lifted to an element $c \in \ker\, d^k$. Since a restricts to 0 in $H^k(U)$, there exists $e \in \Gamma(\mathcal{J}^{k-1} \mid_U)$ such that $c \mid_U = d_U^{k-1}(e)$. Since $\mathcal{J}^{k-1}$ is flabby, we may assume $e \in \Gamma(\mathcal{J}^{k-1})$. Therefore, $b := c - d^{k-1}(e) \in \Gamma(\mathcal{J}^k)$ represents a and is supported on $\mathrm{Sing}(Y)$. As in the case $k \geq n$, applying this argument to a basis of $H^k = H_{2n-k}(G)$, we define a monomorphism of sheaves $\mathcal{H}^k \hookrightarrow \ker\,(\mathcal{J}^k \to \mathcal{J}^{k+1})$. □

With the same assumption as in Lemma 4.5, let $\mathcal{K}^\bullet$ be the cokernel of the inclusion $0 \to \mathcal{H}^\bullet \to \mathcal{J}^\bullet$:

$$0 \to \mathcal{H}^\bullet \to \mathcal{J}^\bullet \to \mathcal{K}^\bullet \to 0.$$

All the sheaves of these complexes are injective. Previous sequence gives rise to a long exact sequence of sheaf cohomology:

$$\ldots \to \mathcal{H}^k \to \mathcal{H}^k(\mathcal{J}^\bullet) \to \mathcal{H}^k(\mathcal{K}^\bullet) \to \ldots,$$

and for all $k \geq 1$ these sheaves are supported on $\mathrm{Sing}(Y)$.

Proposition 4.6. *For all k the sequence*

$$0 \to \mathcal{H}^k \to \mathcal{H}^k(\mathcal{J}^\bullet) \to \mathcal{H}^k(\mathcal{K}^\bullet) \to 0$$

is exact.

Proof. It suffices to prove that the map $H_y^k \to \mathcal{H}^k(\mathcal{J}^\bullet)_y$ is injective for all $y \in \mathrm{Sing}(Y)$ and all $k > 0$. If $k \geq n$ this is obvious because $H^k(\mathcal{J}^\bullet)_y = H^k(G_y) = H_y^k$. When $1 \leq k < n$ we have $H_y^k = H_{2n-k}(G_y)$. And the map $H_{2n-k}(G_y) \to H^k(\mathcal{J}^\bullet)_y = H^k(G_y)$ is injective by Corollary 4.3. □

Lemma 4.7. *Let $0 \to \mathcal{H}^\bullet \xrightarrow{f^\bullet} \mathcal{J}^\bullet \xrightarrow{g^\bullet} \mathcal{K}^\bullet \to 0$ be an exact sequence of complexes of sheaves. Assume that $\mathcal{H}^\bullet$ is a complex of injective sheaves with vanishing differential $d_{\mathcal{H}^\bullet}^k = 0$ for all k. The following properties are equivalent.*

(i) The sequence coming from the cohomology long exact sequence:

$$0 \to \mathcal{H}^k(\mathcal{H}^\bullet) \to \mathcal{H}^k(\mathcal{J}^\bullet) \to \mathcal{H}^k(\mathcal{K}^\bullet) \to 0 \tag{14}$$

is exact for all k.

(ii) There is a complex map $s^\bullet : \mathcal{K}^\bullet \to \mathcal{J}^\bullet$ such that $g^\bullet \circ s^\bullet = \mathrm{id}_{\mathcal{K}^\bullet}$.

Proof. We only have to prove that (i) implies (ii).

Since $\mathcal{H}^0$ is injective, the exact sequence sequence $0 \to \mathcal{H}^0 \to \mathcal{J}^0 \to \mathcal{K}^0 \to 0$ admits a section $s^0 : \mathcal{K}^0 \to \mathcal{J}^0$, with $g^0 \circ s^0 = \mathrm{id}_{\mathcal{K}^0}$. Therefore, we may construct $s^\bullet = \{s^i\}_{i \geq 0}$ using induction on i. Assume $i \geq 0$ and that there are sections $s^0, \dots, s^i$, with $s^h : \mathcal{K}^h \to \mathcal{J}^h$, $g^h \circ s^h = \mathrm{id}_{\mathcal{K}^h}$, and $s^h \circ d^{h-1}_{\mathcal{K}^\bullet} = d^{h-1}_{\mathcal{J}^\bullet} \circ s^{h-1}$ for all $0 \leq h \leq i$. As before, since $\mathcal{H}^{i+1}$ is injective and the sequence $0 \to \mathcal{H}^{i+1} \to \mathcal{J}^{i+1} \to \mathcal{K}^{i+1} \to 0$ is exact, there exists a section $\sigma^{i+1} : \mathcal{K}^{i+1} \to \mathcal{J}^{i+1}$, with $g^{i+1} \circ \sigma^{i+1} = \mathrm{id}_{\mathcal{K}^{i+1}}$. A priori it may happen that $\sigma^{i+1} \circ d^i_{\mathcal{K}^\bullet}$ is different from $d^i_{\mathcal{J}^\bullet} \circ s^i$, so we have to modify σ^{i+1}. To this purpose set:

$$\delta := \sigma^{i+1} \circ d^i_{\mathcal{K}^\bullet} - d^i_{\mathcal{J}^\bullet} \circ s^i \in Hom(\mathcal{K}^i, \mathcal{J}^{i+1}).$$

Since

$$g^{i+1} \circ \delta = g^{i+1} \circ \sigma^{i+1} \circ d^i_{\mathcal{K}^\bullet} - g^{i+1} \circ d^i_{\mathcal{J}^\bullet} \circ s^i = d^i_{\mathcal{K}^\bullet} - d^i_{\mathcal{K}^\bullet} = 0,$$

it follows that

$$\Im(\delta) \subseteq \mathcal{H}^{i+1}. \tag{15}$$

Since (14) is exact, the map g^i sends $\ker d^i_{\mathcal{J}^\bullet}$ onto $\ker d^i_{\mathcal{K}^\bullet}$, i.e.,

$$g^i(\ker d^i_{\mathcal{J}^\bullet}) = \ker d^i_{\mathcal{K}^\bullet}. \tag{16}$$

In view of the exactness of the sequence $0 \to \mathcal{H}^\bullet \xrightarrow{f^\bullet} \mathcal{J}^\bullet \xrightarrow{g^\bullet} \mathcal{K}^\bullet \to 0$, and of the assumption $d^i_{\mathcal{H}^\bullet} = 0$, we also have

$$\ker g^i = \Im(f^i) \subseteq \ker d^i_{\mathcal{J}^\bullet}. \tag{17}$$

Combining (16) and (17) we deduce that:

$$\ker d^i_{\mathcal{J}^\bullet} = (g^i)^{-1}(\ker d^i_{\mathcal{K}^\bullet}). \tag{18}$$

In fact, by (16) we have $\ker d^i_{\mathcal{J}^\bullet} \subseteq (g^i)^{-1}(\ker d^i_{\mathcal{K}^\bullet})$. On the other hand, if $x \in (g^i)^{-1}(\ker d^i_{\mathcal{K}^\bullet})$, then $g^i(x) \in \ker d^i_{\mathcal{K}^\bullet}$, and by (16) we may write $g^i(x) = g^i(y)$ for some $y \in \ker d^i_{\mathcal{J}^\bullet}$. Hence, $x - y \in \ker g^i$, and from (17) it follows that $x \in \ker d^i_{\mathcal{J}^\bullet}$. From (18) we get:

$$s^i(\ker d^i_{\mathcal{K}^\bullet}) \subseteq \ker d^i_{\mathcal{J}^\bullet}. \tag{19}$$

To prove this, recall that $g^i \circ s^i = \mathrm{id}_{\mathcal{K}^i}$. Therefore, $g^i(s^i(\ker d^i_{\mathcal{K}^\bullet})) = \ker d^i_{\mathcal{K}^\bullet}$, and so, taking into account (18), we have:

$$s^i(\ker d^i_{\mathcal{K}^\bullet}) \subseteq (g^i)^{-1}(\ker d^i_{\mathcal{K}^\bullet}) = \ker d^i_{\mathcal{J}^\bullet}.$$

By (19) we deduce that:

$$\ker d^i_{\mathcal{K}^\bullet} \subseteq \ker \delta, \tag{20}$$

and from (15) and (20) we get

$$\delta \in Hom(\mathcal{K}^i / \ker d^i_{\mathcal{K}^\bullet}, \mathcal{H}^{i+1}).$$

Since $\mathcal{H}^{i+1}$ is injective, we may extend δ to a map $\tilde{\delta} \in Hom(\mathcal{K}^{i+1}, \mathcal{H}^{i+1})$ such that

$$\tilde{\delta} \circ d^i_{\mathcal{K}^\bullet} = \delta. \tag{21}$$

We have

$$\tilde{\delta} \in Hom(\mathcal{K}^{i+1}, \mathcal{J}^{i+1})$$

because $\mathcal{H}^{i+1}$ maps to $\mathcal{J}^{i+1}$ via f^{i+1}. Now we define:

$$s^{i+1} := \sigma^{i+1} - \tilde{\delta}.$$

From (21) it follows that

$$s^{i+1} \circ d^i_{\mathcal{K}^\bullet} = d^i_{\mathcal{J}^\bullet} \circ s^i,$$

and since $\Im(\tilde{\delta}) \subseteq \mathcal{H}^{i+1}$, we also have

$$g^{i+1} \circ s^{i+1} = \mathrm{id}_{\mathcal{K}^{i+1}}.$$

□

5. Proof of Theorem 3.1

As we have seen in Section 3, by [32, Theorem 1.11, p. 518] one knows that the Decomposition Theorem implies (ii). Therefore, we only have to prove that (ii) implies (i).

In view of Lemma 4.1 and Lemma 4.2 we have $\Im(\alpha_k^*) = \Im(\beta_k^*)$ for all $k \geq n$. From Lemma 4.5, Proposition 4.6, and Lemma 4.7, we get:

$$R\pi_*\mathbb{Q}_X = \mathcal{J}^\bullet = \mathcal{K}^\bullet \oplus \mathcal{H}^\bullet. \tag{22}$$

Hence, we only have to prove that

$$\mathcal{K}^\bullet \cong IC_Y[-n],$$

where $IC_Y^\bullet = IC_Y^{top}[-n]$ denotes the intersection cohomology complex of Y [17, p. 156]. Observe that the restriction $\alpha^{-1}\mathcal{K}^\bullet$ of $\mathcal{K}^\bullet$ to U is $\mathbb{Q}_U$, and that, by (22), we have $\mathcal{K}^\bullet \in D_c^b(Y)$ [17, p. 81-82]. Therefore, $\mathcal{K}^\bullet[n]$ is an extension of $\mathbb{Q}_U[n]$ [17, p. 134]. So to prove that $\mathcal{K}^\bullet \cong IC_Y[-n]$ it suffices to prove that $\mathcal{K}^\bullet[n] \cong \alpha_{!*}\mathbb{Q}_U[n]$, i.e., that $\mathcal{K}^\bullet[n]$ is the intermediary extension of $\mathbb{Q}_U[n]$ [17, p.156 and p.135]. By [17, Proposition 5.2.8, p. 135], this in turn reduces to prove that for all $y \in \mathrm{Sing}(Y)$ the following two conditions hold true ($i_y : \{y\} \to Y$ denotes the inclusion):

(a) $\mathcal{H}^k i_y^{-1}\mathcal{K}^\bullet[n] = 0$ for all $k \geq 0$;

(b) $\mathcal{H}^k i_y^{!}\mathcal{K}^\bullet[n] = 0$ for all $k \leq 0$.

As for condition (a) we notice that [17, p.130]:

$$\mathcal{H}^k i_y^{-1}\mathcal{K}^\bullet[n] = \mathcal{H}^k(\mathcal{K}^\bullet[n])_y = \mathcal{H}^{k+n}(\mathcal{K}^\bullet)_y,$$

and $\mathcal{H}^{k+n}(\mathcal{K}^\bullet)_y = 0$ because $\mathcal{J}^\bullet = \mathcal{K}^\bullet \oplus \mathcal{H}^\bullet$, and $\mathcal{H}^{k+n}(\mathcal{J}^\bullet)_y = H^{k+n}(G_y) = \mathcal{H}^{k+n}(\mathcal{H}^\bullet)_y$ for $k \geq 0$.

For the condition (b), first notice that combining (22) with Remarks 2.3 and 2.7, we deduce that $\mathcal{K}^\bullet[n]$ is self-dual. Therefore, condition (b) reduces to (a). In fact, we have [17, p. 130, proof of Lemma 5.1.15]:

$$\mathcal{H}^k i_y^{!}\mathcal{K}^\bullet[n] = \mathcal{H}^{-k}(i_y^{-1}D(\mathcal{K}^\bullet[n]))^\vee = \mathcal{H}^{-k}(i_y^{-1}(\mathcal{K}^\bullet[n]))^\vee = \mathcal{H}^{-k+n}(\mathcal{K}^\bullet)_y^\vee = 0$$

because $k \leq 0$.

Remark 5.1. (i) If $n = 2$, then the map $H^{k-1}(G) \to H^k(Y, U)$ vanishes for all $k \geq n+2$ for trivial reasons. In view of the connectivity of the link, combining Remark 2.2 with Lemma 4.1 and Lemma 4.2, we see that this holds true also when Y is locally complete intersection. Therefore, either when $n = 2$ or when Y is locally complete intersection, in order to deduce the decomposition (i) in Theorem 3.1, we need only check that the map $H^n(G) \to H^{n+1}(Y, U)$ is the zero map. On the other hand, the vanishing of the map $H^n(G) \to H^{n+1}(Y, U)$ is equivalent to require that the natural map $H_n(G) \to H^n(G) \cong H_n(G)^\vee$ is onto (compare with (3), (5), and Corollary 4.3). Since $H_n(G)$ is contained in $H_n(X)$ via push-forward (Lemma 4.1), it follows that the map $H_n(G) \to H^n(G) \cong H_n(G)^\vee$ is onto if and only if $H_n(G)$ is a nondegenerate subspace of $H_n(X)$ with respect to the natural intersection form $H_n(X) \times H_n(X) \to H_0(X) \cong \mathbb{Q}$. By Mumford's theorem [23], [29] we know this holds true when Y is a normal surface. Therefore, *in the case Y is a normal surface (or when $2\dim G < n$), our Theorem 3.1 gives a new and simplified proof of the Decomposition Theorem for $\pi : X \to Y$.*

(ii) Assume that $\pi : X \to Y$ is the blowing-up of Y along $\mathrm{Sing}(Y)$, with smooth and connected fibres. By Poincaré Duality we have $H_{2n-k}(G_y) \cong H^{k-2}(G_y)$ for all $y \in \mathrm{Sing}(Y)$. It follows that $H^k(X,U) \cong H_{2n-k}(G) \cong \oplus_{y\in \mathrm{Sing}(Y)} H_{2n-k}(G_y) \cong \oplus_{y\in\mathrm{Sing}(Y)} H^{k-2}(G_y)$. Hence, the map $H^k(X,U) \to H^k(G)$ identifies with the map $\oplus_{y\in\mathrm{Sing}(Y)} H^{k-2}(G_y) \to \oplus_{y\in\mathrm{Sing}(Y)} H^k(G_y)$ given, on each summand $H^{k-2}(G_y) \to H^k(G_y)$, by the self-intersection formula, i.e., by the cup-product with the first Chern class $c_1(N_y) \in H^2(G_y)$ of the normal bundle N_y of G_y in X. Since π is the blowing-up along the finite set $\mathrm{Sing}(Y)$, the dual normal bundle $N_y^\vee \cong \mathcal{O}_{G_y}(1)$ is ample for all $y \in \mathrm{Sing}(Y)$. From the Hard Lefschetz Theorem it follows that the map $H^{k-2}(G_y) \to H^k(G_y)$ is onto for all $k \geq n$, and so also the map $H^k(X,U) \to H^k(G)$ is. By (3), this implies the vanishing of the map $H^k(G) \to H^{k+1}(Y,U)$. Therefore, also in this case our Theorem 3.1 gives a new and simplified proof of the Decomposition Theorem for π.

(iii) More generally, assume only that the fibres of $\pi : X \to Y$ are smooth and connected, so that π is not necessarily the blowing-up along $\mathrm{Sing}(Y)$. Using the extension of the Hard Lefschetz Theorem to bundles of higher rank due to Bloch and Gieseker [3], [25], with a similar argument as before one proves that *if the dual normal bundle $N_y^\vee$ of G_y in X is ample for all $y \in Sing(Y)$, then the map $H^k(G) \to H^{k+1}(Y,U)$ vanishes for all $k \geq n$.* In fact, set

$$h_y := \dim X - \dim G_y$$

for all $y \in \mathrm{Sing}(Y)$. Now the map $H^k(X,U) \to H^k(G)$ identifies with the map

$$\oplus_{y\in\mathrm{Sing}(Y)} H^{k-2h_y}(G_y) \to \oplus_{y\in\mathrm{Sing}(Y)} H^k(G_y)$$

given, on each summand $H^{k-2h_y}(G_y) \to H^k(G_y)$, by the cup-product with the top Chern class $c_{h_y}(N_y) = (-1)^{h_y} c_{h_y}(N_y^\vee) \in H^{2h_y}(G_y)$ of the normal bundle N_y of G_y in X. And such a map is onto for $k \geq n$ by the quoted extension of the Hard Lefschetz Theorem, because $N_y^\vee$ is ample. We refer to [15, Proposition 2.12 and proof] for examples of resolution of singularities verifying previous assumptions.

6. Proof of Theorem 3.2

(i) $\implies$ (ii) By Lemma 4.1 and Lemma 4.2 we have $\Im(\alpha_k^*) = \Im(\beta_k^*)$ for all k. Let $y_1, \dots, y_a, y_{a+1}, \dots, y_b$ be a basis of $H^k(Y)$ such that $\alpha_k^* y_1, \dots, \alpha_k^* y_a$ is a basis for $\Im(\alpha_k^*) = \Im(\beta_k^*)$, and $y_{a+1}, \dots, y_b$ a basis for $\ker \alpha_k^*$. Since $\pi_k^*(\ker \alpha_k^*) \subseteq \ker \beta_k^*$, we may extend $\pi_k^* y_{a+1}, \dots, \pi_k^* y_b$ to a basis $\pi_k^* y_{a+1}, \dots, \pi_k^* y_b, x_{b+1}, \dots, x_c$ of $\ker \beta_k^*$. Then

$$\pi_k^* y_1, \dots, \pi_k^* y_a, \pi_k^* y_{a+1}, \dots, \pi_k^* y_b, x_{b+1}, \dots, x_c$$

is a basis for $H^k(X)$. Define $\theta_k : H^k(X) \to H^k(Y)$ setting $\theta_k(\pi_k^*(y_i)) := y_i$, and $\theta_k(x_i) := 0$. Then $\theta_\bullet$ is a natural morphism.

(ii) $\implies$ (i) The existence of a natural morphism implies that π_k^* is injective and $\Im(\beta_k^*) \subseteq \Im(\alpha_k^*)$ for all k. Since in general we have $\Im(\alpha_k^*) \subseteq \Im(\beta_k^*)$, it follows that $\Im(\alpha_k^*) = \Im(\beta_k^*)$ for all k. By Lemma 4.1 and Lemma 4.2 we get (i).

(ii) $\implies$ (iv) Since π_k^* is injective for all k, using (13) we get a short exact sequence:

$$0 \to H^k(Y) \stackrel{\pi_k^*}{\to} H^k(X) \stackrel{i_k^*}{\to} H^k(G) \to 0$$

for all $k \geq 1$. In particular, for $k \geq 1$, we have

$$H^k(X) \cong H^k(Y) \oplus H^k(G). \tag{23}$$

On the other hand, since $\theta_k \circ \pi_k^* = \mathrm{id}_{H^k(Y)}$, the short exact sequence

$$0 \to \ker \theta_k \to H^k(X) \stackrel{\theta_k}{\to} H^k(Y) \to 0$$

admits π_k^* as a section. It follows another decomposition:

$$H^k(X) = \pi_k^* H^k(Y) \oplus \ker \theta_k. \tag{24}$$

Comparing (23) with (24) we see that

$$\ker \theta_k \cong H^k(G)$$

for all $k \geq 1$. On the other hand, since $\alpha_k^* \circ \theta_k = \beta_k^*$, we have

$$\ker \theta_k \subseteq \ker(H^k(X) \stackrel{\beta_k^*}{\to} H^k(U)) = \Im(H^k(X,U) \to H^k(X)). \tag{25}$$

Since $H^k(X,U) \cong H_{2n-k}(G)$, it follows that

$$\dim H^k(G) \leq \dim H_{2n-k}(G) \tag{26}$$

for all $k \geq 1$. By the Universal-coefficient formula [31, p. 248] we deduce that, for $1 \leq k \leq 2n-1$,

$$\ker \theta_k \cong H^k(G) \cong H_{2n-k}(G). \tag{27}$$

Taking into account that $\Im(\alpha_n^*) = \Im(\beta_n^*)$, combining (23), (27) and Corollary 4.4, it follows that $\dim H^k(Y) = \dim IH^k(Y)$ for all k. Therefore, by (11), it suffices to prove that

$$\alpha_k^* : H^k(Y) \to H^k(U)$$

is surjective for all $k < n$. To this purpose notice that, for $k < n$, β_k^* is surjective by Lemma 4.1. This implies that also α_k^* is by (24) and (25) (compare with diagram (2)).

(iv) $\implies$ (vii) Since intersection cohomology verifies Poincaré Duality [17, p. 158], we have:

$$H^h(Y) = IH^h(Y) = (IH^{2(m+1)-h}(Y))^\vee = (H^{2(m+1)-h}(Y))^\vee = H_{2(m+1)-h}(Y).$$

(vii) $\implies$ (iv) This follows from (11) and Remark 2.1.

(v) $\iff$ (vi) $\iff$ (vii) By [28, Theorem 2, Lemma 2, Lemma 3] we know that the duality morphism is an isomorphism if and only if Y is a $\mathbb{Q}$-cohomology manifold, which is equivalent to saying that Y is a $\mathbb{Q}$-intersection cohomology manifold by [27, Theorem 1.1] (compare also with [4]).

(vii) $\implies$ (ii) Denote by $d_k^Y : H^k(Y) \to H_{2n-k}(Y)$ the duality isomorphism, by

$$d_k^X : H^k(X) \cong H_{2n-k}(X)$$

the Poincaré Duality isomorphism, by $\pi_{*,k} : H_{2n-k}(X) \to H_{2n-k}(Y)$ the push-forward. Set $\theta_k : H^k(X) \to H^k(Y)$ with

$$\theta_k := (d_k^Y)^{-1} \circ \pi_{*,k} \circ d_k^X.$$

Then $\theta_\bullet$ is a natural morphism.

(iii) $\iff$ (ii) We only have to prove that (ii) implies (iii). This follows from Remark 2.6 because Y is a $\mathbb{Q}$-cohomology manifold.

(ii) $\implies$ (viii) Since Y is a $\mathbb{Q}$-intersection cohomology manifold, combining (27) with Theorem 3.1, we get:

$$R\pi_*\mathbb{Q}_X \cong \mathbb{Q}_Y \oplus \mathcal{H}^\bullet \cong \mathbb{Q}_Y \oplus \bigoplus_{k\geq 1} R^k\pi_*\mathbb{Q}_X[-k].$$

(viii) $\implies$ (ii) See Remark 3.3, (i).

(ii) $\iff$ (ix) By [27, Theorem 1.1] we deduce that Y is a $\mathbb{Q}$-intersection cohomology manifold if and only if for all $y \in \mathrm{Sing}(Y)$ the link ∂B_y has the same $\mathbb{Q}$-homology type as a sphere S^{2n-1}. On the other hand, via deformation to the normal cone, we may identify ∂B_y with the link of the vertex of the projective cone over $G_y \subseteq \mathbb{P}^{N-1}$. Restricting the Hopf bundle $S^{2N-1} \to \mathbb{P}^{N-1}$ to G_y, we obtain an S^1-bundle $\partial B_y \to G_y$ inducing the Thom-Gysin sequence [31, p. 260]

$$\dots \to H^k(G_y) \to H^k(\partial B_y) \to H^{k-1}(G_y) \to H^{k+1}(G_y) \to H^{k+1}(\partial B_y) \to \dots$$

And this sequence implies that ∂B_y has the same $\mathbb{Q}$-homology type as a sphere S^{2n-1} if and only if $H^\bullet(G_y) \cong H^\bullet(\mathbb{P}^{n-1})$.

Remark 6.1. By (26) it follows that $h_2(G) \leq h_{2n-2}(G)$. Therefore, if Y is a $\mathbb{Q}$-cohomology manifold, then $\dim G = 0$ or $\dim G = n-1$.

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Vincenzo Di Gennaro, Università di Roma "Tor Vergata", Dipartimento di Matematica, Via della Ricerca Scientifica, 00133 Roma, Italy.
E-mail address: `digennar@axp.mat.uniroma2.it`

Davide Franco , Università di Napoli "Federico II", Dipartimento di Matematica e Applicazioni "R. Caccioppoli", P.le Tecchio 80, 80125 Napoli, Italy.
E-mail address: `davide.franco@unina.it`