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INTERSECTION SPACES, PERVERSE SHEAVES AND STRING THEORY

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Abstract. We survey recent results describing a perverse sheaf realization of Banagl's intersection space homology in the context of projective hypersurfaces with only isolated singularities. Intersection space homology has been recently proved to be relevant in type IIB string theory, as it provides the correct count of massless 3-branes arising during a Calabi-Yau conifold transition.

1. Introduction

In addition to the four dimensions that model our space-time, string theory requires six dimensions for a string to vibrate. By supersymmetry, these six real dimensions must be realized by a Calabi-Yau space. However, given the multitude of known topologically distinct Calabi-Yau 3-folds, the string model remains undetermined. Therefore, it is important to have mechanisms that allow one to move from one Calabi-Yau space to another. In Physics, a solution to this problem was first proposed by Green-Hübsch [GH1, GH2] who, motivated by Reid's "fantasy" [Re87], conjectured that topologically distinct Calabi-Yau 3-folds are connected to each other by means of conifold transitions, which induce a phase transition between the corresponding string models.

A conifold transition starts out with a smooth Calabi-Yau 3-fold, passes through a singular variety — *the conifold* — by a deformation of complex structure, and arrives at a topologically distinct smooth Calabi-Yau 3-fold by a small resolution of singularities. The deformation collapses embedded three-spheres (called *vanishing cycles*) to isolated ordinary double points, while the resolution resolves the singular points by replacing each of them with a $\mathbb{CP}^1$. In Physics, the topological change resulted from passing from one of the Calabi-Yau's to the conifold was interpreted by Strominger [Str95] by the condensation of massive black holes to massless ones. In type IIA string theory, there are charged two-branes that wrap around the $\mathbb{CP}^1$ 2-cycles, and which become massless when these 2-cycles are collapsed to points by the resolution map. Goresky-MacPherson's intersection homology [GM80, GM83] of the conifold accounts for all of these massless two-branes, and since it also satisfies Poincaré duality, it may be viewed as a physically correct homology theory for type IIA string theory. Similarly, in type IIB string theory there are charged three-branes wrapped around the vanishing cycles, and which become massless as these vanishing cycles are collapsed by the deformation of complex structure. Neither ordinary homology nor intersection homology of the conifold account for these massless three-branes; see [Ba10][Section 3.7] for more details. So a natural problem is to find a physically correct homology theory for type IIB string theory. A solution to this question was suggested by Banagl in [Ba10] via his intersection space homology theory.

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In [Ba10], Banagl developed a homotopy-theoretic method which associates to certain types of singular spaces X (e.g., a conifold) a CW complex IX, called the *intersection space* of X, whose reduced rational homology groups satisfy Poincaré Duality. Roughly speaking, the intersection space IX associated to a singular space X is constructed by replacing links of singularities of X by their corresponding Moore approximations, a process called *spatial homology truncation*. The *intersection space homology*

$$HI_*(X;\mathbb{Q}) := H_*(IX;\mathbb{Q}) \tag{1}$$

is not isomorphic to the intersection homology of the space X, and in fact it can be seen that in the middle degree and for isolated singularities, this new theory takes more cycles into account than intersection homology. For a conifold X, Banagl showed that the dimension of $HI_3(X)$ equals the number of physically present massless 3-branes in IIB theory, so intersection space homology can be viewed as a physically correct homology theory for type IIB string theory.

Our approach for studying intersection space homology is motivated by *mirror symmetry*. In mirror symmetry, given a Calabi-Yau 3-fold X, the mirror map associates to it another Calabi-Yau 3-fold Y so that type IIB string theory on $\mathbb{R}^4 \times X$ corresponds to type IIA string theory on $\mathbb{R}^4 \times Y$. If X and Y are smooth, their Betti numbers are related by precise algebraic identities (e.g., see [CK99]), e.g.,

$$\beta_3(Y) = \beta_2(X) + \beta_4(X) + 2, \tag{2}$$

etc. Morrison [Mor99] conjectured that the mirror of a conifold transition is again a conifold transition, but performed in the reverse order (i.e., by exchanging resolutions and deformations). Thus, if X and Y are mirrored conifolds (in mirrored conifold transitions), the intersection space homology of one space and the intersection homology of the mirror space form a *mirror-pair*, in the sense that

$$\beta_3(IY) = I\beta_2(X) + I\beta_4(X) + 2, \tag{3}$$

etc., where $I\beta_i$ denotes the i-th intersection homology Betti number (see [Ba10] for details). This suggests that it should be possible to compute the intersection space homology $HI_*(X;\mathbb{Q})$ of a variety X in terms of the topology of a smoothing deformation, by "mirroring" known results (e.g., [BBD, dCM, GM82]) relating the intersection homology groups $IH_*(X;\mathbb{Q})$ of X to the topology of a resolution of singularities.

This point of view was successfully exploited in [BM11, BBM], where we considered the case of a hypersurface $X \subset \mathbb{CP}^{n+1}$ with only isolated singularities, this being the main source of examples for conifold transitions. In this note, we review some of the main constructions and results from these works.

Convention: By "manifold" we mean a "complex projective manifold", and by "singular space" we mean a "complex projective variety of pure complex dimension n". We are only interested in "middle-perversity" calculations, so any mentioning of other perversity functions will be ignored. Unless otherwise specified, all (intersection) (co)homology groups will be computed with rational coefficients. Spaces considered in this paper will have at most isolated singularities.

Some of the properties of *intersection (co)homology* of a singular space X which are relevant for the above-mentioned "mirror" approach are:

(a) Intersection homology $IH_*(X)$ satisfies Poincaré duality.
(b) If $\widetilde{X}$ is a resolution of singularities of X, then $IH_*(X)$ is a sub-vector space of $H_*(\widetilde{X})$. Moreover, if $\widetilde{X}$ is a *small* resolution, then $IH_*(X) \cong H_*(\widetilde{X})$.

(c) $IH^*(X)$ is realized by a *perverse*, *self-dual*, *constructible* sheaf complex $\mathcal{IC}_X$, i.e.

$$IH^i(X) \cong \mathbb{H}^i(X, \mathcal{IC}_X[-n]). \tag{4}$$

(d) $IH^*(X)$ carries the *Kähler package*, including Hodge structures, as well as weak and hard Lefschetz theorems.

Based on the above considerations, it is therefore natural to try to "mirror" such properties in the context of intersection space (co)homology. As already mentioned, the Poincaré duality property is satisfied by the intersection spaces. We will thus focus on the properties (b), (c) and (d).

2. Hypersurface singularities and smoothing invariance of intersection space homology

Let X be a complex projective hypersurface of dimension $n > 2$, which, for simplicity, is assumed to have only one isolated singular point x. Let L_x, F_x and $T_x : H_n(F_x) \to H_n(F_x)$ denote the link, Milnor fiber and local monodromy operator of the isolated hypersurface singularity germ (X, x), respectively. By [Mi68], the link L_x is an $(n-2)$-connected closed oriented $(2n-1)$-dimensional manifold. Moreover, the Milnor fiber F_x is a parallelizable $(n-1)$-connected $2n$-dimensional manifold, which has the homotopy type of $\bigvee S^n$, a wedge of n-spheres. The number $\mu_x = \mathrm{rk} H_n(F_x)$ of these n-spheres (also called vanishing cycles) is the local Milnor number at x. It is known that all eigenvalues of T_x are roots of unity. We say that the local monodromy operator T_x is trivial if all eigenvalues of T_x are equal to 1.

The assumption on the dimension of X is needed to assure that the link L_x of x is simply-connected, so the intersection space IX can be defined as in [Ba10]. The actual definition of an intersection space is not needed here, only the calculation of Betti numbers, as described in Theorem 2.1 below, will be used in the sequel. Nevertheless, let us indicate briefly how IX is obtained from X. Let M be the complement of an open cone neighborhood of x so that M is a compact manifold with boundary $\partial M = L_x$. The spatial homology n-truncation of L_x is a topological space $L_x^{<n}$ such that $H_i(L_x^{<n}) = 0$ for $i \geq n$, together with a continuous map $f : L_x^{<n} \to L$ which induces a homology isomorphism in degrees $i < n$. The intersection space IX is then defined as the homotopy cofiber of the composition

$$L_x^{<n} \xrightarrow{f} L_x = \partial M \overset{\text{incl}}{\hookrightarrow} M$$

(see [Ba10] for complete details, and [BM12] for a mild introduction).

The following result can be viewed as a generalization of the Betti calculation from [Ba10] in the context of conifold transitions:

Theorem 2.1. ([BM11][Thm.4.1, Thm.5.2]) *Let X_s be a nearby smoothing of X. Then, under the above assumptions and notations, the following holds:*

$$\dim HI^i(X; \mathbb{Q}) = \begin{cases} \dim H^i(X_s; \mathbb{Q}) & \text{if } i \neq n, 2n; \\ \dim H^i(X_s; \mathbb{Q}) - rk(T_x - 1) & \text{if } i = n; \\ 0 & \text{if } i = 2n. \end{cases} \tag{5}$$

Moreover, under some mild technical assumption on the homology of the link (that is, if $H_{n-1}(L_x; \mathbb{Z})$ is torsion-free), the above identities are derived via a continuous map $IX \to X_s$, and we obtain a smoothing invariance of the intersection space (co)homology $\widetilde{H}^*(IX)$ if, and only if, the local monodromy operator T_x is trivial. So this result can be viewed as mirroring property (b) of intersection homology, expressing the intersection cohomology of X as a sub-vector space of the cohomology of any resolution, with an isomorphism in the case of a small resolution.

Moreover, the local trivial monodromy condition (or the existence of "small deformations") should be regarded as mirroring that of the existence of small resolutions.

3. Perverse sheaf approach to intersection space homology

3.1. **Summary of results.** Guided by a similar philosophy derived from mirror symmetry, in [BBM] we constructed a perverse sheaf[1] $\mathcal{IS}_X$, the *intersection-space complex*, whose global hypercohomology calculates (abstractly) the intersection space cohomology groups of a projective hypersurface $X \subset \mathbb{CP}^{n+1}$ with one isolated singular point.

Theorem 3.1. *([BBM]) Let X_s be a nearby smoothing of X. Then there exists a perverse sheaf complex $\mathcal{IS}_X$ on X so that there are (abstract) isomorphisms*

$$\mathbb{H}^i(X; \mathcal{IS}_X[-n]) \simeq \begin{cases} H^i(IX) & \text{if } i \neq 2n \\ H^{2n}(X_s) = \mathbb{Q} & \text{if } i = 2n. \end{cases} \tag{6}$$

Our construction (see Section 3.2.2 for a sketch) can be viewed as mirroring the fact that the intersection cohomology groups can be computed from a perverse sheaf, namely the intersection cohomology complex $\mathcal{IC}_X$. We would like to point out that for general X there cannot exist a perverse sheaf $\mathcal{P}$ on X such that $HI^*(X;\mathbb{Q})$ can be computed from the hypercohomology group $\mathbb{H}^*(X;\mathcal{P}[-n])$. Indeed, the stalk vanishing conditions that such a perverse sheaf $\mathcal{P}$ satisfies would give $\mathbb{H}^i(X;\mathcal{P}[-n]) = H^i(M)$, for $i < n$, while $H^i(IX) = H^i(M, \partial M)$ if $i < n$. (Here M denotes as before the complement of an open cone neighborhood of x in X.) However, due to the high-connectivity of the links, this goal can be achieved in the case when X is a hypersurface with only isolated singularities, this being in fact the main source of examples for conifold transitions. This fact motivates our study of intersection spaces associated to hypersurfaces with only isolated singularities.

Furthermore, by construction, the intersection space complex $\mathcal{IS}_X$ underlies a mixed Hodge module, therefore its hypercohomology groups carry canonical mixed Hodge structures. This result mirrors the corresponding one for the intersection cohomology complex $\mathcal{IC}_X$.

It follows from the above interpretation of intersection space cohomology that the groups $\mathbb{H}^*(X;\mathcal{IS}_X)$ satisfy Poincaré duality globally, which raises the question whether this duality is induced by a more powerful (Verdier-) self-duality isomorphism $\mathcal{D}(\mathcal{IS}_X) \simeq \mathcal{IS}_X$ in the derived category of constructible bounded sheaf complexes on X. In [BBM], we showed the following:

Theorem 3.2. *([BBM]) If the local monodromy T_x at the singular point x is semi-simple in the eigenvalue 1, then the intersection space complex $\mathcal{IS}_X$ is Verdier self-dual. In particular, for any integer i, there is a non-degenerate pairing*

$$\mathbb{H}^{-i}(X;\mathcal{IS}_X) \times \mathbb{H}^i(X;\mathcal{IS}_X) \to \mathbb{Q}.$$

The assumption on the semi-simplicity of local monodromy in the eigenvalue 1 is satisfied by a large class of isolated singularities, e.g., the weighted homogeneous ones.

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[1] Let us recall here the definition of a *perverse sheaf* on a singular space X with only one isolated singular point x. Such a space can be given a Whitney stratification $\mathcal{X}$ with only two strata: $\{x\}$ and $X \setminus \{x\}$. Denote by $i : \{x\} \hookrightarrow X$ and $j : X \setminus \{x\} \hookrightarrow X$ the corresponding closed and open embeddings. Then a complex $K \in D^b_c(X)$, which is constructible with respect to $\mathcal{X}$, is perverse on X if $j^*K[-n]$ is cohomologically a local system on $X \setminus \{x\}$ and, moreover, the following two (stalk and, respectively, co-stalk vanishing) conditions hold:

$$H^j(i^*K) = 0, \text{ for any } j > 0,$$
$$H^j(i^!K) = 0, \text{ for any } j < 0.$$

Let us next recall that the Beilinson-Bernstein-Deligne decomposition [BBD] for the pushforward $Rf_*\mathbb{Q}_{\widetilde{X}}[n]$ of the constant sheaf $\mathbb{Q}_{\widetilde{X}}$ under an algebraic resolution map $f : \widetilde{X} \to X$ splits off the intersection sheaf $\mathcal{IC}_X$ of X plus contributions from the singularities of X. Suppose now that X sits as $X = \pi^{-1}(0)$ in a family $\pi : \widetilde{X} \to S$ of projective hypersurfaces over a small disc around $0 \in \mathbb{C}$ such that $\widetilde{X}$ is smooth, and $X_s = \pi^{-1}(s)$ is smooth over nearby $s \in S$, $s \neq 0$. In this situation, the *nearby cycle functor* ψ_π for π can be defined, and we have the following result:

Theorem 3.3. *([BBM]) If the local monodromy T_x at the singular point x is semi-simple in the eigenvalue 1, then the intersection space complex $\mathcal{IS}_X$ is a direct summand of the nearby cycle complex $\psi_\pi\mathbb{Q}_{\widetilde{X}}[n]$.*

The summand complementary to $\mathcal{IS}_X$ has the interpretation as being contributed by the singularity x, since it is supported only over $\{x\}$. We regard this splitting of nearby cycles as mirroring the above Beilinson-Bernstein-Deligne decomposition theorem in the following sense. For s sufficiently close to 0, there is a map $sp : X_s \to X$, the specialization map, which should be viewed as mirroring a resolution map. Moreover, the nearby cycle complex $\psi_\pi\mathbb{Q}_{\widetilde{X}}[n]$ can be computed by the (derived) pushforward $R(sp)_*\mathbb{Q}_{X_s}[n]$ of the constant sheaf on a nearby smoothing of X. Altogether, we have a decomposition

$$R(sp)_*\mathbb{Q}_{X_s}[n] \simeq \mathcal{IS}_X \oplus \mathcal{C}, \tag{7}$$

with $\mathcal{C}$ a perverse sheaf supported on the singular set $\{x\}$.

Finally, in [BBM] we prove the following result which mirrors the existence of the Kähler package on intersection cohomology groups:

Theorem 3.4. *([BBM]) If the local monodromy T_x at x is semi-simple in the eigenvalue 1, and the global monodromy T acting on $H^*(X_s)$ is semi-simple in the eigenvalue 1, then the hypercohomology groups $\mathbb{H}^*(X;\mathcal{IS}_X)$ carry pure Hodge structures satisfying the Hard Lefschetz theorem.*

3.2. **Intersection space complex.** Let us now sketch the construction of the perverse sheaf $\mathcal{IS}_X$, see [BBM] and references therein for complete details. We will try to keep the technical details at a minimum, in order not to obscure the presentation.

3.2.1. *Nearby and vanishing cycles.* Let us consider, as before, a hypersurface $X \subset \mathbb{CP}^{n+1}$ with $Sing(X) = \{x\}$. Let $\pi : \widetilde{X} \to S \subset \mathbb{C}$ be a family of hypersurfaces over a small disc S centered at the origin, with $X = \pi^{-1}(0)$, and so that $\widetilde{X}$ is smooth and $X_s := \pi^{-1}(s)$ for $s \neq 0$ is a smooth hypersurface in $\mathbb{CP}^{n+1}$. Let

$$\psi_\pi, \varphi_\pi : D^b_c(\widetilde{X}) \to D^b_c(X)$$

be the *nearby* and *vanishing cycle functors* for π, with monodromy T and resp. $\widetilde{T}$. Then

$$H^i(X_s;\mathbb{Q}) \cong \mathbb{H}^i(X;\psi_\pi\mathbb{Q}_{\widetilde{X}}), \tag{8}$$

and, for the point inclusion $i_x : \{x\} \hookrightarrow X$, with F_x denoting as before the Milnor fiber of the hypersurface singularity germ (X, x), we have

$$H^i(F_x;\mathbb{Q}) \cong H^i(i_x^*\psi_\pi\mathbb{Q}_{\widetilde{X}}) \quad \text{and} \quad \widetilde{H}^i(F_x;\mathbb{Q}) \cong H^i(i_x^*\varphi_\pi\mathbb{Q}_{\widetilde{X}}), \tag{9}$$

with compatible monodromy actions. Note that $Supp(\varphi_\pi\mathbb{Q}_{\widetilde{X}}) = Sing(X) = \{x\}$.

There are canonical morphisms:

$$can : \psi_\pi \to \varphi_\pi \quad \text{and} \quad var : \varphi_\pi \to \psi_\pi$$

so that $can \circ var = \widetilde{T} - 1$, $var \circ can = T - 1$.

The monodromy automorphisms T and $\widetilde{T}$ have Jordan decompositions

$$T = T_u \circ T_s = T_s \circ T_u,$$

where T_s is semisimple (and locally of finite order) and T_u is unipotent, and similarly for $\widetilde{T}$. For any $\lambda \in \mathbb{Q}$ and $K \in D^b_c(\widetilde{X})$, denote by $\psi_{\pi,\lambda}K$ the generalized λ-eigenspace for T, and similarly for $\phi_{\pi,\lambda}K$. There are decompositions

$$\psi_\pi = \psi_{\pi,1} \oplus \psi_{\pi,\neq 1} \quad \text{and} \quad \varphi_\pi = \varphi_{\pi,1} \oplus \varphi_{\pi,\neq 1}$$

so that $T_s = 1$ on $\psi_{\pi,1}$, $\widetilde{T}_s = 1$ on $\varphi_{\pi,1}$, and T_s and $\widetilde{T}_s$ have no 1-eigenspace on $\psi_{\pi,\neq 1}$ and $\varphi_{\pi,\neq 1}$, respectively. Moreover, $can : \psi_{\pi,\neq 1} \to \varphi_{\pi,\neq 1}$ and $var : \varphi_{\pi,\neq 1} \to \psi_{\pi,\neq 1}$ are isomorphisms.

Let $N := \log(T_u)$, and similarly for $\widetilde{N}$. The morphism $\varphi_\pi K \xrightarrow{Var} \psi_\pi K$ is defined by the cone of the pair $(0, N)$. Then $can \circ Var = \widetilde{N}$ and $Var \circ can = N$.

The functors ${}^p\psi_\pi := \psi_\pi[-1]$ and ${}^p\varphi_\pi := \varphi_\pi[-1]$ from $D^b_c(\widetilde{X})$ to $D^b_c(X)$ commute with the Verdier duality functor $\mathcal{D}$ (up to natural isomorphisms), and send perverse sheaves to perverse sheaves. These functors and their decompositions into unipotent and non-unipotent parts lift to the category of mixed Hodge modules, as do the functors can, N, $\widetilde{N}$ and Var. For an introduction to Saito's theory of mixed Hodge modules, the interested reader is advised to consult [Sa89].

3.2.2. *Intersection space complex: construction.* First note that $\psi_\pi \mathbb{Q}_{\widetilde{X}}[n]$, $\varphi_\pi \mathbb{Q}_{\widetilde{X}}[n]$ are perverse sheaves on X. Consider the perverse sheaf

$$\mathcal{C} := \text{Image}(\widetilde{T} - 1) \subseteq \varphi_\pi \mathbb{Q}_{\widetilde{X}}[n], \tag{10}$$

and denote by

$$\iota_\varphi : \mathcal{C} \hookrightarrow \varphi_\pi \mathbb{Q}_{\widetilde{X}}[n]$$

the corresponding inclusion in the abelian category $Perv(X)$. Then $Supp(\mathcal{C}) = \{x\}$, and we have

$$\mathbb{H}^i(X;\mathcal{C}) = \begin{cases} 0 & , \text{ if } i \neq 0, \\ \text{Image}(T_x - 1) & , \text{ if } i = 0. \end{cases} \tag{11}$$

Let

$$\iota := var \circ \iota_\varphi : \mathcal{C} \longrightarrow \psi_\pi \mathbb{Q}_{\widetilde{X}}[n].$$

In view of (8), (11) and the Betti calculation of Theorem 2.1, it is natural to define the intersection space complex by:

$$\mathcal{IS}_X := \text{Coker}\left(\iota : \mathcal{C} \longrightarrow \psi_\pi \mathbb{Q}_{\widetilde{X}}[n]\right) \in Perv(X). \tag{12}$$

Remark 3.5. *If π is a small deformation of X, i.e., if the local monodromy operator T_x is trivial, then $\mathcal{C} \simeq 0$, so we get an isomorphism of perverse sheaves $\mathcal{IS}_X \simeq \psi_\pi \mathbb{Q}_{\widetilde{X}}[n]$. In view of the Betti identity of Theorem 2.1, this isomorphism can be interpreted as a sheaf-theoretical enhancement of the stability result from* [BM11] *mentioned in the Section 2.*

Remark 3.6. *The above construction can be easily adapted to the situation of hypersurfaces with multiple isolated singular points. It then follows from Theorem 3.1 and* [Ba11a]*[Prop.3.6] that the hypercohomology of $\mathcal{IS}_X$ for conifolds X provides the correct count of massless* 3*-branes in type IIB string theory.*

Some of the results described in Section 3.1 can be obtained as direct consequences of the definition of the intersection complex $\mathcal{IS}_X$. We describe some of these instances below. Others require more intricate proofs based on Saito's theory of (mixed) Hodge modules, or, alternatively, on the theory of zig-zags; see [BBM] for complete details on these proofs.

By using the facts stated in Section 3.2.1, it is not hard to see that the intersection space complex $\mathcal{IS}_X$ underlies a mixed Hodge module. More precisely, we have that:

$$\mathcal{IS}_X = \text{coker}\left(\text{Image}(\widetilde{N}) \xrightarrow{Var} {}^p\psi_{\pi,1}\mathbb{Q}_{\widetilde{X}}[n+1]\right) \tag{13}$$

and, as already mentioned, the functors $\widetilde{N}$, *Var* and ${}^p\psi_{\pi,1}$ admit lifts to the category of mixed Hodge modules.

It can also be seen that if T_x is semi-simple in the eigenvalue 1, then:

$$\mathcal{IS}_X \cong \psi_{\pi,1}\mathbb{Q}_{\widetilde{X}}[n].$$

So in this case $\mathcal{IS}_X$ is self-dual, since $\psi_\pi\mathbb{Q}_{\widetilde{X}}[n]$ is self-dual and $\mathcal{D}$ respects the decomposition $\psi_\pi = \psi_{\pi,1} \oplus \psi_{\pi,\neq 1}$. Moreover, in this case, the weight filtration on $\mathbb{H}^i(X, \mathcal{IS}_X)$ coincides (up to a shift) with the monodromy filtration defined by the nilpotent endomorphism N acting on $\mathbb{H}^i({}^p\psi_{\pi,1}) := \mathbb{H}^i(X; {}^p\psi_{\pi,1}\mathbb{Q}_{\widetilde{X}}[n+1])$. So the mixed Hodge structure on $\mathbb{H}^i(X, \mathcal{IS}_X) \cong \mathbb{H}^i({}^p\psi_{\pi,1})$ is pure if and only if $N = 0$, or equivalently if $T = T_s$ on $\mathbb{H}^i({}^p\psi_{\pi,1})$. In other words, one has purity if the action of T on $H^*(X_s)$ is semi-simple in the eigenvalue 1. Moreover, if this is the case, one can show as in [DMSS][Section 3] that the Hard Lefschetz theorem also holds for the hypercohomology groups $\mathbb{H}^i(X; \mathcal{IS}_X)$.

4. Concluding remarks

A natural problem is to extend the construction and study of intersection spaces of complex hypersurfaces beyond the case of isolated singularities. This problem is motivated by string theoretic considerations since, given the success of the use of intersection homology on the one hand and homology of the intersection space on the other hand in the context of the conifold transition, it is natural to investigate the use of such Poincaré duality homology theories in more singular situations encountered in string theory, e.g., for the fibre singularities in F-theory. This is particularly important as the non-uniqueness of the (small) resolutions of singular elliptic fibrations calls for a more model-independent procedure to determine the homology relevant for the physical theory. Recent progress in this direction has been recently made by Banagl and his students, e.g, see [Ba11b]. On the other hand, the sheaf-theoretic approach presented in this note is valid in more general settings, so it can be used to define an intersection space (co)homology theory directly, without having to construct an intersection space at all.

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References

[Ba10] Banagl, M., *Intersection Spaces, Spatial Homology Truncation, and String Theory*, Lecture Notes in Mathematics **1997** (2010), Springer Verlag Berlin-Heidelberg. DOI: 10.1007/978-3-642-12589-8

[Ba11a] Banagl, M., *Foliated Stratified Spaces and a De Rham Complex Describing Intersection Space Cohomology.* arXiv: 1102.4781

[Ba11b] Banagl, M., *Intersection Spaces and Stratification Depth 2*, Journal of Singularities, **5** (2012), 57–84.

[BBM] Banagl, M., Budur, N., Maxim, L., *Intersection spaces, perverse sheaves and type IIB string theory*, Adv. Theor. Math. Phys. 18 (2014), no. 2, 363–399.

[BM11] Banagl, M., Maxim, L., *Deformation of Singularities and the Homology of Intersection Spaces*, Journal of Topology and Analysis **4** (2012), no. 4, 413–448.

[BM12] Banagl, M., Maxim, L., *Intersection Spaces and Hypersurface Singularities* Journal of Singularities **5** (2012), 48–56.

[BBD] Beilinson, A. A., Bernstein, J., Deligne, P., *Faisceaux pervers*, Analysis and topology on singular spaces, I (Luminy, 1981), 5–171, Astérisque **100**, Soc. Math. France, Paris, 1982.

[dCM] de Cataldo, M. A., Migliorini, L., *The decomposition theorem, perverse sheaves and the topology of algebraic maps*, Bull. Amer. Math. Soc. (N.S.) **46** (2009), no. 4, 535–633.

[CK99] Cox, D. A., Katz, S., *Mirror symmetry and algebraic geometry*, Math. Surveys and Monographs, vol. **68**, Amer. Math. Soc., 1999. DOI: 10.1090/surv/068

[DMSS] Davison, B., Maulik, D., Schürmann, J., Szedröi, B., *Purity for graded potentials and quantum cluster positivity*. arXiv: 1307.3379

[GM80] Goresky, M.; MacPherson, R., *Intersection homology theory*, Topology **19** (1980), no. 2, 135–162.

[GM82] Goresky, M.; MacPherson, R., *On the topology of complex algebraic maps*. Algebraic geometry (La Rábida, 1981), 119129, Lecture Notes in Math., **961**, Springer, Berlin, 1982.

[GM83] Goresky, M.; MacPherson, R., *Intersection homology. II.*, Invent. Math. **72** (1983), no. 1, 77–129.

[GH1] Green, P.S., Hübsch, T., *Possible phase transitions among Calabi-Yau compactifications*, Phys. Rev. Lett. **61** (1988), 1163–1166. DOI: 10.1103/PhysRevLett.61.1163

[GH2] Green, P.S., Hübsch, T., *Connecting moduli spaces of Calabi-Yau threefolds*, Comm. Math. Phys. **119** (1988), 431–441. DOI: 10.1007/BF01218081

[Mi68] Milnor, J., *Singular points of complex hypersurfaces*, Annals of Mathematics Studies, No. **61**, Princeton University Press, Princeton, N.J., 1968.

[Mor99] Morrison, D., *Through the looking glass*, Mirror symmetry, III (Montreal, PQ, 1995), 263–277, AMS/IP Stud. Adv. Math., 10, Amer. Math. Soc., Providence, RI, 1999.

[Re87] Reid, M., *The moduli space of* 3*-folds with* $K = 0$ *may nevertheless be irreducible*, Math. Ann. **287** (1987), 329–334. DOI: 10.1007/BF01458074

[Sa89] Saito, M., *Introduction to mixed Hodge modules*, Actes du Colloque de Théorie de Hodge (Luminy, 1987). Astérisque **179-180** (1989), no. 10, 145–162.

[Str95] Strominger, A., *Massless black holes and conifolds in string theory*, Nucl. Phys. B **451** (1995), 96–108. DOI: 10.1016/0550-3213(95)00287-3

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