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ON CHARACTERISTIC CLASSES OF SINGULAR HYPERSURFACES AND INVOLUTIVE SYMMETRIES OF THE CHOW GROUP

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Abstract. For every choice of an integer and a line bundle on an algebraic scheme we construct an associated involution on its Chow group, and show that various notions of characteristic class for singular hypersurfaces are interchanged via such involutions. As an application, we apply our formulas to effectively compute some non-trivial characteristic classes associated with a graph hypersurface. In the case of projective space we show that such involutions are induced by involutive correspondences.

1. Introduction

Fix an algebraically closed field $\mathfrak{K}$ of characteristic zero, let M be a smooth $\mathfrak{K}$-variety and let $X \subset M$ be a hypersurface. For singular X there exists a generalization of the notion of 'Milnor number' to arbitrary singularities which is a characteristic class supported on the singular locus of X referred to in the literature as the *Milnor class* of X, which we denote by $\mathcal{M}(X)$. Milnor classes have received significant interest in the recent literature [17][8][21][18][11][10], and –for a general closed subscheme $Y \hookrightarrow M$– are defined (up to sign) as the difference between the *Fulton class* $c_{\mathrm{F}}(Y)$ and its *Chern-Schwartz-MacPherson* class $c_{\mathrm{SM}}(Y)$. Both the Fulton class and CSM class are elements of the Chow group which are generalizations of Chern classes to the realm of singular varieties in the sense that the classes both agree with the total homology Chern class in the smooth case[1]. Another characteristic class supported on the singular locus of a hypersurface X is the *Lê-class* of X, denoted $\Lambda(X) \in A_*X$, which was first defined in [10] and named as such as the *Lê-class* is closely related to the so-called *Lê-cycles* of X, which were initially defined and studied independently of Milnor classes [16]. The main result announced in [10] was that if $\mathscr{O}(X)$ is very ample then both $\mathcal{M}(X)$ and $\Lambda(X)$ determine each other in a completely symmetric way, i.e.,

$$\mathcal{M}_k(X) = \sum_{j=0}^{d-k} (-1)^{j+k} \binom{j+k}{k} c_1(\mathscr{O}(X))^j \cap \Lambda_{j+k}, \tag{1.1}$$

and

$$\Lambda_k(X) = \sum_{j=0}^{d-k} (-1)^{j+k} \binom{j+k}{k} c_1(\mathscr{O}(X))^j \cap \mathcal{M}(X)_{j+k}, \tag{1.2}$$

where d is the dimension of the singular locus of X and an ith subscript on a class denotes its component of dimension i.

However, it was soon discovered that formulas (1.1) and (1.2) did *not* in fact hold, as an erratum appeared stating that there had been a subtle error which lead to a misidentification of the global *Lê-class* $\Lambda(X)$ with the Segre class $s(X_s, M)$ of the singular scheme X_s of X in M [9]. In any case, a direct corollary of Theorem 4.3 which we prove in §4 is that formulas (1.1)

[1] We give a more in-depth discussion of all classes mentioned here in §2.

and (1.2) *do* in fact hold once the components of $\Lambda(X)$ in formulas (1.1) and (1.2) are replaced by components of a class $\widetilde{\Lambda}(X)$ closely related to the Segre class $s(X_s, M)$, namely

$$\widetilde{\Lambda}(X) = c(\mathscr{O}(X))c(T^*M \otimes \mathscr{O}(X)) \cap s(X_s, M) \in A_*X_s, \tag{1.3}$$

where again X_s denotes the *singular scheme* of X, i.e., the subscheme of X whose ideal sheaf is locally generated by all partial derivatives of a defining equation for X. Moreover, we require no assumption that $\mathscr{O}(X)$ be very ample.

In [3], the class $c(T^*M \otimes \mathscr{O}(X)) \cap s(X_s, M)$ was taken as the definition of a class referred to as the *μ-class* of the singular scheme X_s of X (as it generalized Parusiǹski's 'μ-number' [19]), denoted $\mu(X_s)$, thus the class $\widetilde{\Lambda}(X)$ properly realizing formulas (1.1) and (1.2) is precisely given by

$$\widetilde{\Lambda}(X) = c(\mathscr{O}(X)) \cap \mu(X_s) \in A_*X_s.$$

Moreover, if we define $\widetilde{\Lambda}^{(k)}(X)$ for $k \in \mathbb{Z}$ as

$$\widetilde{\Lambda}^{(k)}(X) = c(\mathscr{O}(X))^k \cap \mu(X_s) \in A_*X_s,$$

we show that symmetric formulas analogous to (1.1) and (1.2) hold between the Milnor class $\mathcal{M}(X)$ and $\widetilde{\Lambda}^{(k)}(X)$ for all $k \in \mathbb{Z}$. As such, it is essentially the μ-class which is at the heart of this duality with the Milnor class. Applications of μ-classes to the study of dual varieties varieties and contact schemes of hypersurfaces were also considered in [3].

The symmetry of formulas (1.1) and (1.2) seem to suggest the existence of some non-trivial involutive symmetry of A_*X which exchanges $\mathcal{M}(X)$ and $\widetilde{\Lambda}(X)$, which we show in §4 is in fact the case. Furthermore, we show in §3 that for every integer $n \in \mathbb{Z}$ and line bundle $\mathscr{L} \to X$ there exists an associated involution

$$i_{n,\mathscr{L}} : A_*X \to A_*X,$$

and that other notions of characteristic class for singular varieties are interchanged via such involutions as well.

In what follows we give a brief review of the characteristic classes under consideration in §2. In §3 we define the maps $i_{n,\mathscr{L}}$ and show they are in fact involutions. In §4 we prove involutive formulas which relate different characteristic classes, and we give an application of our formulas by computing the Segre class and μ-class of a highly non-reduced scheme which is the singular scheme of a graph hypersurface. Such classes would be extremely difficult to compute solely from their definitions. We then close in §5 with an interpretation of the involutions $i_{n,\mathscr{L}}$ for X projective in terms of involutive correspondences on projective spaces.

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2. Characteristic classes of singular hypersurfaces

The total Chern class $c(X)$ of a smooth $\mathfrak{K}$-variety X is the most basic characteristic class for $\mathfrak{K}$-varieties in the sense that all other reasonable notions of characteristic class are linear combinations of Chern classes over a suitable ring. For those interested in singularities, it is then only natural that one would want to generalize the notion of Chern class to the realm of singular varieties (and schemes) in such a way that that they agree with the usual Chern class for smooth varieties. The CSM class $c_{\mathrm{SM}}(X)$ of a possibly singular variety X is in some sense the most direct generalization, since for $\mathfrak{K} = \mathbb{C}$ it generalizes the Poincaré-Hopf (or Gauß-Bonnet) theorem to the realm of singular varieties, i.e.,

$$\int_X c_{\mathrm{SM}}(X) = \chi(X),$$

where $\chi(X)$ denotes the topological Euler characteristic with compact support, and the integral sign is notation for taking the dimension zero component of a class[2]. For arbitrary $\mathfrak{K}$ (algebraically closed of characteristic zero) we simply *define* the Euler characteristic of a $\mathfrak{K}$-variety as the 'integral' of its CSM class. Moreover, CSM classes are a generalization of counting in the sense that they obey inclusion-exclusion (which of course is very useful for computations). In [4], Aluffi obtained a very nice formula for the CSM class of a hypersurface in terms of the Segre class (see Definition 2.1) of its singular scheme, and since we are only concerned with hypersurfaces in this note we may use his formula as a working definition (we recall Aluffi's formula in §4, after introducing some useful notations).

Another class generalizing the Chern class to the realm of singular varieties and schemes is the Fulton class, which is defined for any subscheme of a smooth $\mathfrak{K}$-variety M. From here on we will refer to such schemes as *embeddable schemes*. For X a (possibly singular) local complete intersection, its Fulton class $c_{\mathrm{F}}(X)$ agrees (after pushforward to M) with the total Chern class of a smooth variety in the same rational equivalence class as X, and so $c_{\mathrm{SM}}(X)$ differs from $c_{\mathrm{F}}(X)$ only in terms of dimension less than or equal to the dimension of its singular locus. The difference $c_{\mathrm{SM}}(X) - c_{\mathrm{F}}(X)$ then measures the discrepancy of $c_{\mathrm{SM}}(X)$ from the Chern class of a smooth deformation of X (parametrized by $\mathbb{P}^1$), and is an invariant precisely of the *singularities* of X. For X with only isolated singularities (over $\mathbb{C}$) the integral of $c_{\mathrm{SM}}(X) - c_{\mathrm{F}}(X)$ agrees (up to sign) precisely with the sum of the Milnor numbers of each singular point of X, thus it seemed natural to refer to this class generalization of global Milnor number as the 'Milnor class' of X, which we denote by $\mathcal{M}(X) := c_{\mathrm{SM}}(X) - c_{\mathrm{F}}(X)$[3].

To define the Fulton class of an arbitrary embeddable scheme, we first need the following

Definition 2.1. Let M be a smooth $\mathfrak{K}$-variety and $Y \hookrightarrow M$ a subscheme. For Y regularly embedded (so that its normal cone is in fact a vector bundle, which we denote by $N_Y M$), the *Segre class* of Y relative to M is denoted $s(Y, M)$, and is defined as

$$s(Y, M) := c(N_Y M)^{-1} \cap [Y] \in A_* Y.$$

For Y 'irregularly' embedded, let $f : \widetilde{M} \to M$ be the blowup of M along Y and denote the exceptional divisor of f by E. The Segre class of Y relative to M is then defined as

$$s(Y, M) := f|_{E*} s(E, \widetilde{M}) \in A_* Y,$$

where $f|_{E*}$ denotes the proper pushforward of f restricted to E. As E is always regularly embedded, this is enough to define the Segre class of Y (relative to M) in any case.

The Fulton class is then given by the following

Definition 2.2. Let Y be a subscheme of some smooth variety M. It's *Fulton class* is denoted $c_{\mathrm{F}}(Y)$, and is defined as

$$c_{\mathrm{F}}(Y) := c(TM) \cap s(Y, M) \in A_* Y.$$

Remark 2.1. As shown in [12] (Example 4.2.6), $c_{\mathrm{F}}(Y)$ is intrinsic to Y, i.e., it is independent of an embedding into some smooth variety (thus justifying the absence of an ambient M anywhere in its notation).

[2]We note that while CSM classes were first defined over $\mathbb{C}$ [15], their definition was later generalized to an arbitrary algebraically closed field of characteristic zero in [13].

[3]We blindly ignore any sign conventions some may associate with this class in the literature.

Remark 2.2. While the Fulton class is sensitive to scheme structure, the CSM class of a scheme by definition coincides with that of its support with natural reduced structure, and thus is *not* sensitive to any non-trivial scheme structure. As for Milnor classes, since they are defined as the difference between the CSM and Fulton classes, they are scheme-theoretic invariants as well. More precisely, in the case of a possibly singular/non-reduced hypersurface X, $\mathcal{M}(X)$ is an invariant of the *singular scheme* of X, i.e., the subscheme of X whose ideal sheaf is locally generated by the partial derivatives of a local defining equation for X. We note that at present it is not clear what scheme structure on the singular locus of an arbitrary local complete intersection determines its Milnor class, though for a large class of global complete intersections it was shown in [11] that the Milnor class is determined by a direct generalization of the notion of singular scheme of a hypersurface to complete intersections.

As noted in Remark 2.2, while Fulton classes are sensitive to scheme structure, in some sense they are not sensitive to the singularities of a hypersurface (or more generally a local complete intersection), since (as mentioned earlier) the Fulton class of a local complete intersection coincides with that of a smooth representative of its rational equivalence class (e.g, the Fulton class of two distinct lines in the plane is the same as the Fulton class of a smooth conic). A scheme-theoretic characteristic class which is also sensitive to the singularities of an embeddable scheme Y is the *Aluffi class* of Y, denoted by $c_{\mathrm{A}}(Y)$, which may be integrated to yield the Donaldson-Thomas type invariant of Y. Aluffi classes were first defined by Aluffi in [5], where he referred to them as *weighted Chern-Mather classes*. Behrend then later coined the term 'Aluffi class' in [7], where he makes the first connection between Aluffi's weighted Chern-Mather classes (albeit with a different sign convention) and Donaldson-Thomas invariants of Calabi-Yau threefolds. For Y the singular scheme of a hypersurface X it was shown in [5] that (up to sign) $c_{\mathrm{A}}(Y) = c(\mathscr{O}(X)) \cap \mathcal{M}(X)$, and since this is the only context in which we consider Aluffi classes we refer the reader to both [7][5] for precise definitions and further discussion.

3. The involutions $i_{n,\mathscr{L}}$

Let X be an algebraic $\mathfrak{K}$-scheme. For every $(n, \mathscr{L}) \in \mathbb{Z} \times \mathrm{Pic}(X)$ we now define a map $i_{n,\mathscr{L}} : A_*X \to A_*X$, and show that it is an involutive automorphism of A_*X (these will be precisely the involutions which relate various characteristic classes alluded to above). But before doing so, we first introduce two intersection theoretic operations, which will not only provide an efficient way for defining the involutions $i_{n,\mathscr{L}}$, but will also be of computational utility.

So let $\alpha \in A_*X$ be written as $\alpha = \alpha^0 + \cdots + \alpha^n$, where α^i is the component of α of *codimension* i (in X). We denote by $\alpha^\vee$ the class

$$\alpha^\vee := \sum (-1)^i \alpha^i,$$

and refer to it as the 'dual' of α.

We now define an action of $\mathrm{Pic}(X)$ on A_*X. Given a line bundle $\mathscr{L} \to X$ we denote its action on $\alpha = \sum \alpha^i \in A_*X$ by $\alpha \otimes_X \mathscr{L}$ [4], which we define as

$$\alpha \otimes_X \mathscr{L} := \sum \frac{\alpha^i}{c(\mathscr{L})^i}.$$

It is straightforward to show that this defines an honest action (i.e.,

$$(\alpha \otimes_X \mathscr{L}) \otimes_X \mathscr{M} = \alpha \otimes_X (\mathscr{L} \otimes \mathscr{M})$$

[4] The notation '$\otimes_X$' is not to be confused with a similar notation used in a different context in [14] §8.1

for any line bundles $\mathscr{L}$ and $\mathscr{M}$), and we refer to this action as 'tensoring by a line bundle'. For $\mathscr{E}$ a rank r class in the Grothendieck group of vector bundles on X (note that r may be non-positive), the formulas

$$(c(\mathscr{E}) \cap \alpha)^{\vee} = c(\mathscr{E}^{\vee}) \cap \alpha^{\vee} \tag{3.1}$$

$$(c(\mathscr{E}) \cap \alpha) \otimes_X \mathscr{L} = \frac{c(\mathscr{E} \otimes \mathscr{L})}{c(\mathscr{L})^r} \cap (\alpha \otimes_X \mathscr{L}) \tag{3.2}$$

were proven in [2] (along with the first appearance of the 'tensor' and 'dual' operations), and will be indispensable throughout the remainder of this note[5]. We now arrive at the following

Proposition 3.1. *Let X be an algebraic $\mathfrak{K}$-scheme, $n \in \mathbb{Z}$ and $\mathscr{L} \to X$ be a line bundle. Then the map $i_{n,\mathscr{L}} : A_*X \to A_*X$ given by*

$$\alpha \mapsto c(\mathscr{L})^n \cap (\alpha^{\vee} \otimes_X \mathscr{L})$$

is an involutive automorphism of A_*X (i.e., $i_{n,\mathscr{L}} \circ i_{n,\mathscr{L}} = \mathrm{id}_{A_*X}$).

Proof. Let $\alpha \in A_*X$ and denote $i_{n,\mathscr{L}}(\alpha)$ by β, i.e.,

$$\beta = c(\mathscr{L})^n \cap (\alpha^{\vee} \otimes_X \mathscr{L})\,. \tag{3.3}$$

We will show that $i_{n,\mathscr{L}}(\beta) = \alpha$, which implies the conclusion of the proposition. Capping both sides of the equation 3.3 by $c(\mathscr{L})^{-n}$ we get

$$c(\mathscr{L})^{-n} \cap \beta = \alpha^{\vee} \otimes_X \mathscr{L}. \tag{3.4}$$

By formula 3.2, for any line bundle $\mathscr{M} \to X$ we have

$$\big(c(\mathscr{L})^{-n} \cap \beta\big) \otimes_X \mathscr{M} = \frac{c(\mathscr{M})^n}{c(\mathscr{L} \otimes \mathscr{M})^n} \cap (\beta \otimes_X \mathscr{M}),$$

thus tensoring both sides of equation 3.4 by $\mathscr{L}^{\vee}$ yields

$$c(\mathscr{L}^{\vee})^n \cap (\beta \otimes_X \mathscr{L}^{\vee}) = \alpha^{\vee}. \tag{3.5}$$

Finally, taking the 'dual' (i.e. applying formula 3.1) to both sides of equation 3.5 we have

$$\alpha = c(\mathscr{L})^n \cap (\beta^{\vee} \otimes_X \mathscr{L}) = i_{n,\mathscr{L}}(\beta),$$

as desired.

The fact that $i_{n,\mathscr{L}}$ is a homomorphism (i.e. $\mathbb{Z}$-linear) follows from the fact that dualizing, tensoring by a line bundle and capping with Chern classes are all linear operations. $\square$

Remark 3.1. The map $\alpha \mapsto \alpha^{\vee}$ sending a class to its dual coincides with $i_{n,\mathscr{O}}$ for every $n \in \mathbb{Z}$.

4. Symmetric formulas abound

We now assume M is a smooth proper $\mathfrak{K}$-variety and $X \subset M$ is an arbitrary hypersurface (i.e., the zero-*scheme* associated with a non-trivial section of line bundle on M). We denote the singular scheme of X by X_s, which is the subscheme of X whose ideal sheaf is the restriction to X of the ideal sheaf on M which is locally generated by a defining equation for X and each of its partial derivatives. In what follows, as we prefer to work mostly in M, we will not distinguish between classes in A_*X and their pushforwards (via the natural inclusion) to A_*M. We will call two classes *k -$\mathscr{L}$ dual* if one is the image of the other (and so vice-versa) under the map $i_{k,\mathscr{L}}$. In this section, we show formulas (1.1) and (1.2) both hold when $\Lambda(X)$ is replaced by $\widetilde{\Lambda}(X)$ as defined via 1.3, and that these symmetric relations are consequences of the fact that $\mathcal{M}(X)$

[5] The tensor and dual operations, along with formulas 3.1 and 3.2 are what we refer to as Aluffi's 'intersection-theoretic calculus' in §1.

and $\widetilde{\Lambda}(X)$ are simply $\dim(M)$-$\mathscr{O}(X)$ dual. Similar relations are then derived for other notions of characteristic class for singular varieties.

We now recall Aluffi's formula for the CSM class of X, which as mentioned earlier we will take as a working definition.

Theorem 4.1 (Aluffi, [4])**.**

$$c_{\mathrm{SM}}(X) = \frac{c(TM)}{c(\mathscr{O}(X))} \cap \left([X] + s(X_s, M)^{\vee} \otimes_M \mathscr{O}(X)\right).$$

We then immediately arrive at the following

Corollary 4.2.

$$\mathcal{M}(X) = \frac{c(TM)}{c(\mathscr{O}(X))} \cap \left(s(X_s, M)^{\vee} \otimes_M \mathscr{O}(X)\right).$$

Proof. This follows directly from definitions of Fulton class and Milnor class, as

$$\mathcal{M}(X) = c_{\mathrm{SM}}(X) - c_{\mathrm{F}}(X) \quad \text{and} \quad c_{\mathrm{F}}(X) = c(TM) \cap s(X, M) = \frac{c(TM)}{c(\mathscr{O}(X))} \cap [X].$$

□

The fact that formulas (1.1) and (1.2) hold after replacing Λ by $\widetilde{\Lambda}$ are a special case of the following

Theorem 4.3. *Let n be an integer. Then*

$$\mathcal{M}(X) = i_{n,\mathscr{O}(X)}(\alpha_X(n)) \quad \textit{and} \quad \alpha_X(n) = i_{n,\mathscr{O}(X)}(\mathcal{M}(X)),$$

where

$$\alpha_X(n) := c(T^*M \otimes \mathscr{O}(X)) c(\mathscr{O}(X))^{n+1-\dim(M)} \cap s(X_s, M).$$

Proof. By Corollary 4.2 we have

$$\begin{aligned}
\mathcal{M}(X) &= \frac{c(TM)}{c(\mathscr{O}(X))} \cap \left(s(X_s, M)^{\vee} \otimes_M \mathscr{O}(X)\right) \\
&= c(\mathscr{O}(X))^n \cap \left(\frac{c(TM) c(\mathscr{O})^{n+1-\dim(M)}}{c(\mathscr{O}(X))^{n+1}} \cap \left(s(X_s, M)^{\vee} \otimes_M \mathscr{O}(X)\right) \right) \\
&\overset{3.2}{=} c(\mathscr{O}(X))^n \cap \left(\left(c(TM \otimes \mathscr{O}(-X)) c(\mathscr{O}(-X))^{n+1-\dim(M)} \cap s(X_s, M)^{\vee} \right) \otimes_M \mathscr{O}(X) \right) \\
&\overset{3.1}{=} c(\mathscr{O}(X))^n \cap \left(\left(c(TM^* \otimes \mathscr{O}(X)) c(\mathscr{O}(X))^{n+1-\dim(M)} \cap s(X_s, M) \right)^{\vee} \otimes_M \mathscr{O}(X) \right) \\
&= i_{n,\mathscr{O}(X)}(\alpha_X(n)).
\end{aligned}$$

The formula $\alpha_X(n) = i_{n,\mathscr{O}(X)}(\mathcal{M}(X))$ then follows as $i_{n,\mathscr{O}(X)}$ is an involution by Proposition 3.1. □

Remark 4.1. The most natural case of Theorem 4.3 is when $n = \dim(X)$, in which case we have the formulas

$$\mathcal{M}(X) = i_{\dim(X),\mathscr{O}(X)}(\mu(X_s)) \quad \text{and} \quad \mu(X_s) = i_{\dim(X),\mathscr{O}(X)}(\mathcal{M}(X)),$$

where we recall $\mu(X_s)$ denotes the μ-class of the singular scheme X_s of X, which is defined via the formula

$$\mu(X_s) = c(T^*M \otimes \mathscr{O}(X)) \cap s(X_s, M) \in A_* X_s. \tag{4.1}$$

The μ-class was first defined by Aluffi [3], and is an intrinsic invariant of the singularities of X. Such classes arise often in the study of projective duality [20] (though they are actually referred to as 'Milnor classses' in that text!), have applications to the study of contact schemes of hypersurfaces [3], and are closely related to the Donaldson-Thomas type invariant of X_s [7].

Remark 4.2. As n varies over $\mathbb{Z}$, writing out the formula for the kth dimensional piece $\mathcal{M}_k(X)$ of the Milnor class of X via Theorem 4.3 yields infinitely many symmetric formulas similar to (1.1) and (1.2). In particular, for $n = \dim(M)$ we have $\alpha_X(\dim(M)) = \widetilde{\Lambda}(X)$ as defined in (1.3), a fact which implies formulas (1.1) and (1.2) indeed hold after $\Lambda(X)$ is replaced by $\widetilde{\Lambda}(X)$, which we now state and prove via

Corollary 4.4. *Formulas* (1.1) *and* (1.2) *hold after* Λ *is replaced by* $\widetilde{\Lambda}$.

Proof. Denote the dimension of M by d. By Theorem 4.3,

$$\begin{aligned}
\mathcal{M}(X) &= i_{d,\mathscr{O}(X)}(\alpha_X(d)) \\
&= i_{d,\mathscr{O}(X)}(\widetilde{\Lambda}(X)) \\
&= c(\mathscr{O}(X))^d \cap \left(\widetilde{\Lambda}(X)^\vee \otimes_M \mathscr{O}(X)\right) \\
&= c(\mathscr{O}(X))^d \cap \left(\sum_{i=0}^{d} \frac{(-1)^i \widetilde{\Lambda}_{d-i}(X)}{c(\mathscr{O}(X))^i}\right) \\
&= \sum_{i=0}^{d} (-1)^i c(\mathscr{O}(X))^{d-i} \cap \widetilde{\Lambda}_{d-i}(X) \\
&= \sum_{i=0}^{d} (-1)^i (1 + c_1(\mathscr{O}(X)))^{d-i} \cap \widetilde{\Lambda}_{d-i}(X) \\
&= \sum_{i=0}^{d} \sum_{j\geq 0} (-1)^i \binom{d-i}{j} c_1(\mathscr{O}(X))^j \cap \widetilde{\Lambda}_{d-i}(X).
\end{aligned}$$

In the last equality the term $c_1(\mathscr{O}(X))^j \cap \widetilde{\Lambda}_{d-i}(X)$ is of dimension $d-i-j$, and so $\mathcal{M}_k(X)$ corresponds to setting $i = d-k-j$, which yields

$$\mathcal{M}_k(X) = \sum_{j\geq 0} (-1)^{d-k-j} \binom{j+k}{j} c_1(\mathscr{O}(X))^j \cap \widetilde{\Lambda}_{j+k}(X),$$

which is equivalent (up to sign) to formula (1.1) with Λ replaced by $\widetilde{\Lambda}$ via the identity

$$\binom{a+b}{a} = \binom{a+b}{b}.$$

The (possible) disparity in sign comes from the fact that in [10] their definition of Milnor class differs from ours by a factor of $(-1)^d$. The analogue of formula (1.2) then immediately follows as $\mathcal{M}(X)$ and $\widetilde{\Lambda}(X)$ are d-$\mathscr{O}(X)$ dual. □

Remark 4.3. We note that it was much more work to write out formulas for the individual components $\mathcal{M}_k(X)$ than that of the total Milnor class $\mathcal{M}(X)$ (as in Theorem 4.3). And this is a general principle when computing characteristic classes, i.e., it is often simpler to compute a *total* class rather than its individual components.

Remark 4.4. As mentioned in §2, in [5] Aluffi defined a scheme-theoretic characteristic class for arbitrary embeddable $\mathfrak{K}$-schemes which Behrend refers to as the 'Aluffi class' in his theory of Donaldson-Thomas type invariants [7]. The analogue of the Gauß-Bonnet theorem in this theory is the formula

$$\int_Y c_{\mathrm{A}}(Y) = \chi_{\mathrm{DT}}(Y),$$

where Y is an embeddable scheme with Aluffi class $c_{\mathrm{A}}(Y)$, and $\chi_{\mathrm{DT}}(Y)$ denotes the Donaldson-Thomas type invariant of Y. If Y is the singular scheme of a hypersurface X it was shown in [5] that

$$c_{\mathrm{A}}(Y) = c(\mathscr{O}(X)) \cap \mathcal{M}(X).$$

Thus capping both sides of the formulas constituting Theorem 4.3 with $c(\mathscr{O}(X))$ then yields

Corollary 4.5. *Let n be an integer, Y be the singular scheme of a hypersurface X and let $\alpha_X(n)$ be defined as in Theorem 4.3. Then*

$$c_{\mathrm{A}}(Y) = i_{n+1,\mathscr{O}(X)}(\alpha_X(n)) \quad \textit{and} \quad \alpha_X(n) = i_{n+1,\mathscr{O}(X)}(c_{\mathrm{A}}(Y)).$$

We now give an application of such formulas by computing classes that would be considerably difficult using only their definitions.

Example 4.6. Let X be the hypersurface in $\mathbb{P}^4$ given by

$$X : (t_1t_2t_3t_4 + t_1t_2t_3t_5 + t_1t_2t_4t_5 + t_1t_3t_4t_5 + t_2t_3t_4t_5 = 0) \subset \mathbb{P}^4.$$

Such a hypersurface is the graph hypersurface associated with the 'banana graph' with 5 edges [6]. The homogeneous ideal associated with its singular scheme X_s is then

$$(t_2t_3t_4 + t_2t_3t_5 + t_2t_4t_5 + t_3t_4t_5, \ldots, t_1t_2t_3 + t_1t_2t_4 + t_1t_3t_4 + t_2t_3t_4).$$

In [6], the Milnor class of X was computed as

$$\mathcal{M}(X) = 60H^4 - 10H^3,$$

where H denotes the class of a hyperplane in $\mathbb{P}^4$. By Theorem 4.3 we have

$$\begin{aligned} \mu(X_s) &= c(\mathscr{O}(X))^3 \cap (\mathcal{M}(X)^\vee \otimes_{\mathbb{P}^4} \mathscr{O}(X)) \\ &= (1+4H)^3 \cdot \left(\frac{60H^4}{(1+4H)^4} + \frac{10H^3}{(1+4H)^3}\right) \\ &= \frac{60H^4}{(1+4H)} + 10H^3 \\ &= 60H^4(1-4H) + 10H^3 \\ &= 60H^4 + 10H^3, \end{aligned}$$

so that the μ-class of the singular scheme of X is in fact the dual of the Milnor class. The Aluffi class of X_s is then given by

$$c_{\mathrm{A}}(X_s) = c(\mathscr{O}(X)) \cap \mathcal{M}(X) = (1+4H)(60H^4 - 10H^3) = 20H^4 - 10H^3,$$

so that the Donaldson-Thomas type invariant of X_s is 20. By definition of the μ-class (4.1) we may compute the Segre class of X_s in $\mathbb{P}^4$ via the formula

$$s(X_s, \mathbb{P}^4) = c(T^*\mathbb{P}^4 \otimes \mathscr{O}(X))^{-1} \cap \mu(X_s),$$

thus

$$s(X_s, \mathbb{P}^4) = \frac{(1+4H)}{(1+3H)^5} \cdot (60H^4 + 10H^3) = -50H^4 + 10H^3.$$

We conclude this section by identifying the 'n-$\mathscr{O}(X)$ dual partners' of the CSM class of X, which we state via the following

Theorem 4.7. *Let n be an integer. Then*

$$c_{\mathrm{SM}}(X) = i_{n,\mathscr{O}(X)}(\nu_X(n)+\alpha_X(n)) \quad \textit{and} \quad \nu_X(n)+\alpha_X(n) = i_{n,\mathscr{O}(X)}(c_{\mathrm{SM}}(X)),$$

where

$$\nu_X(n) = c(T^*M\otimes \mathscr{O}(X))c(\mathscr{O}(X))^{n-\dim(M)}\cap -[X]$$

and $\alpha_X(n)$ is as defined in Theorem 4.3.

Proof. By Proposition 3.1 and Theorem 4.3, the proof amounts to showing

$$c_{\mathrm{F}}(X) = i_{n,\mathscr{O}(X)}(\nu_X(n)),$$

as $c_{\mathrm{SM}}(X) = c_{\mathrm{F}}(X) + \mathcal{M}(X)$. Thus

$$\begin{aligned}
c_{\mathrm{F}}(X) &= c(TM)\cap s(X,M) \\
&= c(TM)\cap \left(c(N_XM)^{-1}\cap [X]\right) \\
&= c(TM)\cap ([X]\otimes_M \mathscr{O}(X)) \\
&= c(\mathscr{O}(X))^n \cap \left(\frac{c(TM)c(\mathscr{O})^{n-\dim(M)}}{c(\mathscr{O}(X))^n}\cap ([X]\otimes_M \mathscr{O}(X))\right) \\
&\overset{3.2}{=} c(\mathscr{O}(X))^n \cap \left(\left(c(TM\otimes \mathscr{O}(-X))c(\mathscr{O}(-X))^{n-\dim(M)}\cap [X]\right)\otimes_M \mathscr{O}(X)\right) \\
&\overset{3.1}{=} c(\mathscr{O}(X))^n \cap \left(\left(c(T^*M\otimes \mathscr{O}(X))c(\mathscr{O}(X))^{n-\dim(M)}\cap -[X]\right)^{\vee}\otimes_M \mathscr{O}(X)\right) \\
&= i_{n,\mathscr{O}(X)}(\nu_X(n)),
\end{aligned}$$

as desired. □

5. $i_{n,\mathscr{L}}$ via involutive correspondences

Let M and N be smooth proper $\mathfrak{K}$-varieties. A *correspondence* from M to N is a class $\alpha \in A_*(M\times N)$, and such an α induces homomorphisms

$$\alpha_* \in \mathrm{Hom}(A_*M, A_*N) \quad \text{and} \quad \alpha^* \in \mathrm{Hom}(A_*N, A_*M)$$

given by

$$\beta \overset{\alpha_*}{\longmapsto} q_*(\alpha\cdot p^*\beta), \quad \gamma \overset{\alpha^*}{\longmapsto} p_*(\alpha\cdot q^*\gamma),$$

where p is the projection $M\times N\to M$, q is the projection $M\times N \to N$ and '$\cdot$' denotes the intersection product in $A_*(M\times N)$ (which is well defined via the smoothness assumption on $M$and N). Correspondences are at the heart of Grothendieck's theory of motives, and generalize algebraic morphisms in the sense that we think of an arbitrary class $\alpha \in A_*(M\times N)$ as a generalization of the graph Γ_f of a (proper) morphism $f\in \mathrm{Hom}(M,N)$. Just as a morphism $f\in \mathrm{Hom}(M,N)$ induces morphisms on the corresponding Chow groups via proper pushforward (f_*) and flat pullback (f^*), the morphisms α_* and α^* are direct generalizations of proper pushforward and flat pullback as $f_* = (\Gamma_f)_*$ and $f^* = (\Gamma_f)^*$. Moreover, correspondences may be composed in such a way that the functorial properties of proper pushforward and flat pullback still hold, i.e., $(\alpha\circ\vartheta)_* = \alpha_*\circ\vartheta_*$ and $(\alpha\circ\vartheta)^* = \vartheta^*\circ\alpha^*$ for composable correspondences α and ϑ. From this perspective we were naturally led to the question of whether or not for an algebraic scheme X the involutions $i_{n,\mathscr{L}}$ defined in §3 are induced by involutive correspondences in $A_*(X\times X)$. We answer this question for $X=\mathbb{P}^N$ via the following

Theorem 5.1. *Let N be a positive integer and $(n,m) \in \mathbb{Z} \times \mathbb{Z}$. Then there exists a unique $\alpha = \sum_{i+j\leq N} a_{i,j}x^i y^j \in \mathbb{Z}[x,y]/(x^{N+1}, y^{N+1}) \cong A_*(\mathbb{P}^N \times \mathbb{P}^N)$ such that $i_{n,\mathscr{O}(m)} = \alpha_*$*[6]*, and the coefficients of α are given by*

$$a_{N-j,i} = (-1)^j \binom{n-j}{i-j} m^{i-j}.$$

Proof. Consider $\mathbb{P}^N \times \mathbb{P}^N$ with the natural projections onto its first and second factors, which we denote by p and q respectively. Denote by x the hyperplane class in the first factor and by y the hyperplane class in the second factor (we use the same notations for their pullbacks via the natural projections). Let $\beta = \sum_{i=0}^N \beta_i x^i \in A_*\mathbb{P}^N$. It follows directly from the definition of $i_{n,\mathscr{O}(m)}$ and induction that

$$i_{n,\mathscr{O}(m)}(\beta) = \sum_{i=0}^N \left(\sum_{j=0}^N (-1)^j \binom{n-j}{i-j} m^{i-j}\beta_j \right) y^i.$$

We now let $\alpha = \sum_{i+j\leq N} a_{i,j}x^i y^j \in A_*(\mathbb{P}^N \times \mathbb{P}^N)$ be arbitrary, compute $\alpha_*(\beta) = q_*(\alpha \cdot p^*\beta)$, set its coefficients equal to those of $i_{n,\mathscr{O}(m)}(\beta)$, and then observe that this determines the $a_{i,j}$ uniquely. Since we are not using a notational distinction for x and its pullback p^*x, $p^*\beta$ retains exactly the same form as β in its expansion with respect to x. Now $\alpha \cdot p^*\beta$ is just usual multiplication in the ring $\mathbb{Z}[x,y]/(x^{N+1}, y^{N+1})$, and $q_*(\alpha \cdot p^*\beta)$ is just the coefficient of x^N in the expansion of $\alpha \cdot p^*\beta$ with respect to x, which yields

$$\alpha_*(\beta) = \sum_{i=0}^N \left(\sum_{j=0}^N a_{N-j,i}\beta_j \right) y^i.$$

By setting $\alpha_*(\beta) = i_{n,\mathscr{O}(m)}(\beta)$ the $a_{i,j}$ are then uniquely determined to be as stated in the conclusion of the theorem.

To see that $q_*(\gamma)$ for arbitrary $\gamma \in A_*(\mathbb{P}^N \times \mathbb{P}^N)$ is indeed the coefficient of x^N in the expansion of γ with respect to x, one may first view q as the natural projection of the projective bundle $\mathbb{P}(\mathscr{E})$ with $\mathscr{E}$ the trivial rank $N+1$ bundle over $\mathbb{P}^N$ and $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1) = x$. Then by the projection formula, to compute $q_*(\gamma)$ we need only to compute $q_*(x^i)$ in the expansion of γ with respect to x, which we do using the notion of *Segre class* of a vector bundle[7]. By definition of the *Segre class* of $\mathscr{E}$, denoted $s(\mathscr{E})$, we have

$$s(\mathscr{E}) := q_*(1 + x + x^2 + \cdots).$$

And since $s(\mathscr{E}) = c(\mathscr{E})^{-1} = 1$, matching terms of like dimension we see that all powers of x map to 0 except for x^N which maps to 1. ☐

It would be interesting to determine objects of the bounded derived category of $\mathbb{P}^N \times \mathbb{P}^N$ whose Chern characters coincide with α as given in Theorem 5.1. And certainly there must be a larger class of varieties (other than projective spaces) for which an analogue of Theorem 5.1 holds.

[6]Note that $\alpha_* = \alpha^*$ in this case.

[7]We note that the notion of Segre class of a vector bundle is different than the *relative* Segre class we define in §2 (see [12], Chapter 3 for a precise definition).

References

[1] P. Aluffi. Remarks on Lê cycles and Milnor classes. *Unpublished note.*

[2] P. Aluffi. MacPherson's and Fulton's Chern classes of hypersurfaces. *Internat. Math. Res. Notices*, (11):455ff., approx. 11 pp. (electronic), 1994. doi:10.1155/S1073792894000498.

[3] P. Aluffi. Singular schemes of hypersurfaces. *Duke Math. J.*, 80(2):325–351, 1995. doi:10.1215/S0012-7094-95-08014-4.

[4] P. Aluffi. Chern classes for singular hypersurfaces. *Trans. Amer. Math. Soc.*, 351(10):3989–4026, 1999. doi:10.1090/S0002-9947-99-02256-4.

[5] P. Aluffi. Weighted Chern-Mather classes and Milnor classes of hypersurfaces. In *Singularities—Sapporo 1998*, volume 29 of *Adv. Stud. Pure Math.*, pages 1–20. Kinokuniya, Tokyo, 2000.

[6] Paolo Aluffi and Matilde Marcolli. Feynman motives of banana graphs. *Commun. Number Theory Phys.*, 3(1):1–57, 2009. doi:10.4310/CNTP.2009.v3.n1.a1.

[7] Kai Behrend. Donaldson-Thomas type invariants via microlocal geometry. *Ann. of Math. (2)*, 170(3):1307–1338, 2009. doi:10.4007/annals.2009.170.1307.

[8] J.-P. Brasselet, D. Lehmann, J. Seade, and T. Suwa. Milnor classes of local complete intersections. *Trans. Amer. Math. Soc.*, 354(4):1351–1371 (electronic), 2002. doi:10.1090/S0002-9947-01-02846-X.

[9] R. Callejas-Bedregal, M. F. Z. Morgado, and J. Seade. Erratum to: Lê cycles and Milnor classes. *Invent. Math.*, 197(2):483–489, 2014. doi:10.1007/s00222-014-0529-9.

[10] R. Callejas-Bedregal, M. F. Z. Morgado, and J. Seade. Lê cycles and Milnor classes. *Invent. Math.*, 197(2):453–482, 2014. doi:10.1007/s00222-013-0450-7.

[11] J. Fullwood. On Milnor classes via invariants of singular schemes. *Journal of singularities*, 8:1–10, 2014.

[12] William Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge*. Springer-Verlag, Berlin, second edition, 1998. doi:10.1007/978-1-4612-1700-8.

[13] Gary Kennedy. MacPherson's Chern classes of singular algebraic varieties. *Comm. Algebra*, 18(9):2821–2839, 1990. doi:10.1080/00927879008824054.

[14] Robert Lazarsfeld. *Positivity in algebraic geometry. II*, volume 49 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2004. Positivity for vector bundles, and multiplier ideals. doi:10.1007/978-3-642-18808-4.

[15] R. D. MacPherson. Chern classes for singular algebraic varieties. *Ann. of Math. (2)*, 100:423–432, 1974.

[16] D. B. Massey. The Lê varieties. I. *Invent. Math.*, 99(2):357–376, 1990. doi:10.1007/BF01234423.

[17] Laurentiu Maxim. On Milnor classes of complex hypersurfaces. In *Topology of stratified spaces*, volume 58 of *Math. Sci. Res. Inst. Publ.*, pages 161–175. Cambridge Univ. Press, Cambridge, 2011.

[18] Laurentiu Maxim, Morihiko Saito, and Jörg Schürmann. Hirzebruch-Milnor classes of complete intersections. *Adv. Math.*, 241:220–245, 2013. doi:10.1016/j.aim.2013.04.001.

[19] Adam Parusiński. A generalization of the Milnor number. *Math. Ann.*, 281(2):247–254, 1988. doi:10.1007/BF01458431.

[20] E. A. Tevelev. *Projective duality and homogeneous spaces*, volume 133 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2005. Invariant Theory and Algebraic Transformation Groups, IV.

[21] Shoji Yokura. Verdier-Riemann-Roch for Chern class and Milnor class. *Asian J. Math.*, 6(1):1–22, 2002.

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