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VANISHING RESULTS FOR THE AOMOTO COMPLEX OF REAL HYPERPLANE ARRANGEMENTS VIA MINIMALITY

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Abstract. We prove vanishing results for the cohomology groups of the Aomoto complex over an arbitrary coefficient ring for real hyperplane arrangements. The proof uses the minimality of arrangements and descriptions of the Aomoto complex in terms of chambers.

Our methods are used to present a new proof for the vanishing theorem of local system cohomology groups, a result first proved by Cohen, Dimca, and Orlik.

1. Introduction

The theory of hypergeometric integrals originated with Gauss, and has been generalized to higher dimensions for applications in various areas of mathematics and physics ([1, 9, 17]). In this generalization, the notion of local system cohomology groups of the complement of a hyperplane arrangement plays a crucial role.

Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an arrangement of affine hyperplanes in $\mathbb{C}^\ell$ and let

$$M(\mathcal{A}) = \mathbb{C}^\ell \smallsetminus \bigcup_{H\in\mathcal{A}} H$$

be its complement. Let us fix a defining equation α_i of H_i. An arrangement $\mathcal{A}$ is called essential if the normal vectors of hyperplanes generate $\mathbb{C}^\ell$. The first homology group $H_1(M(\mathcal{A}), \mathbb{Z})$ is a free abelian group generated by the meridians $\gamma_1, \dots, \gamma_n$ of hyperplanes. We denote their dual basis by $e_1, \dots, e_n \in H^1(M(\mathcal{A}), \mathbb{Z})$. The element e_i can be identified with $\frac{1}{2\pi\sqrt{-1}} d\log\alpha_i$ via the de Rham isomorphism.

The isomorphism class of a rank one complex local system $\mathcal{L}$ is determined by a homomorphism $\rho : H_1(M(\mathcal{A}), \mathbb{Z}) \longrightarrow \mathbb{C}^\times$, which is also determined by an n-tuple $q = (q_1, \dots, q_n) \in (\mathbb{C}^\times)^n$, where $q_i = \rho(\gamma_i)$.

For a generic parameter $(q_1, \dots, q_n)$, it is known that the following vanishing result holds.

$$\dim H^k(M(\mathcal{A}), \mathcal{L}) = \begin{cases} 0, & \text{if } k \neq \ell, \\ \\ |\chi(M(\mathcal{A}))|, & \text{if } k = \ell. \end{cases} \tag{1}$$

Several sufficient conditions for the vanishing of (1) are known ([1, 8]). Cohen, Dimca and Orlik ([4]) proved the following.

Theorem 1.1. *(CDO-type vanishing theorem) Suppose that $q_X \neq 1$ for each dense edge X contained in the hyperplane at infinity. Then, the vanishing (1) holds. (See §2.1 for description of the notation.)*

The above result is stronger than many other vanishing results. Indeed, for the case $\ell = 2$, it has been proved ([19]) that the vanishing (1) with an additional property holds if and only if the assumption of Theorem 1.1 holds.

The local system cohomology group $H^k(M(\mathcal{A}), \mathcal{L})$ is computed using the twisted de Rham complex $(\Omega^\bullet_{M(\mathcal{A})}, d + \omega\wedge)$ with $\omega = \sum \lambda_i d\log\alpha_i$, where λ is a complex number such that

$\exp(-2\pi\sqrt{-1}\lambda_i) = q_i$ (we denote $\mathcal{L} = \exp(\omega)$). The algebra of rational differential forms $\Omega^\bullet_{M(\mathcal{A})}$ has a natural $\mathbb{C}$-subalgebra $A^\bullet_{\mathbb{C}}(\mathcal{A})$ generated by $e_i = \frac{1}{2\pi\sqrt{-1}} d\log\alpha_i$. This subalgebra is known to be isomorphic to the cohomology ring $H^\bullet(M(\mathcal{A}), \mathbb{C})$ of $M(\mathcal{A})$ ([3]) and to have a combinatorial description, the so-called Orlik-Solomon algebra [11] (see §2.1 for details). The Orlik-Solomon algebra provides a subcomplex $(A^\bullet_{\mathbb{C}}(\mathcal{A}), \omega\wedge)$ of the twisted de Rham complex, which is called the Aomoto complex. There exists a natural morphism

$$(A^\bullet_{\mathbb{C}}(\mathcal{A}), \omega\wedge) \hookrightarrow (\Omega^\bullet_{M(\mathcal{A})}, d + \omega\wedge) \tag{2}$$

of complexes. The Aomoto complex $(A^\bullet_{\mathbb{C}}(\mathcal{A}), \omega\wedge)$ has a purely combinatorial description. Furthermore, it can be considered as a linearization of the twisted de Rham complex $(\Omega^\bullet_{M(\mathcal{A})}, d+\omega\wedge)$. Indeed, there exists a Zariski open subset $U \subset (\mathbb{C}^\times)^n$ that contains $(1, 1, \ldots, 1) \in (\mathbb{C}^\times)^n$ such that (2) is a quasi-isomorphism for $q \in U$ ([7, 16, 10]). However, this is not an isomorphism in general.

The following vanishing result for the cohomology of the Aomoto complex has been obtained by Yuzvinsky.

Theorem 1.2. *([21, 22]) Let $\omega = \sum_{i=1}^n 2\pi\sqrt{-1}\lambda_i e_i \in A^1_{\mathbb{C}}(\mathcal{A})$. Suppose $\lambda_X \neq 0$ for all dense edges X in $L(\mathcal{A})$. Then, we have*

$$\dim H^k(A^\bullet_{\mathbb{C}}(\mathcal{A}), \omega\wedge) = \begin{cases} 0, & \text{if } k \neq \ell, \\ |\chi(M(\mathcal{A}))|, & \text{if } k = \ell. \end{cases} \tag{3}$$

Note that the assumptions in Theorem 1.1 and Theorem 1.2 are somewhat complementary: the first one requires nonresonant conditions along the hyperplane at infinity, whereas the second imposes nonresonant conditions on all dense edges in the affine space.

Recently, Papadima and Suciu proved that the dimension of the local system cohomology group for a torsion local system is bounded by that of the Aomoto complex with finite field coefficients.

Theorem 1.3. *([14]) Let $p \in \mathbb{Z}$ be a prime. Suppose $\omega = \sum_{i=1}^n \lambda_i e_i \in A^1_{\mathbb{F}_p}(\mathcal{A})$ and $\mathcal{L}$ is the local system determined by $q_i = \exp(\frac{2\pi\sqrt{-1}}{p}\lambda_i)$. Then,*

$$\dim_{\mathbb{C}} H^k(M(\mathcal{A}), \mathcal{L}) \leq \dim_{\mathbb{F}_p} H^k(A^\bullet_{\mathbb{F}_p}(\mathcal{A}), \omega\wedge), \tag{4}$$

for all $k \geq 0$.

In view of the Papadima-Suciu inequality (4), it is natural to expect that a CDO-type vanishing theorem for a p-torsion local system may be deduced from that of the Aomoto complex with finite field coefficients. Furthermore, Papadima and Suciu ([15]) clarified the relationship between multinet structures and $H^1(A^\bullet_{\mathbb{F}_p}(\mathcal{A}), \omega\wedge)$. These results motivate the study of the Aomoto complex with coefficients in an arbitrary commutative ring R. The main result of this paper is the following CDO-type vanishing theorem.

Theorem 1.4. *(Theorem 3.1) Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be an essential affine hyperplane arrangement in $\mathbb{R}^\ell$. Let R be a commutative ring with multiplicative unit 1. Let $\omega = \sum_{i=1}^n \lambda_i e_i \in A^1_R(\mathcal{A})$. Suppose that $\lambda_X \in R^\times$ for any dense edge X contained in the hyperplane at infinity. Then, the following holds.*

$$H^k(A^\bullet_R(\mathcal{A}), \omega\wedge) \simeq \begin{cases} 0, & \text{if } k \neq \ell, \\ R^{|\chi(M(\mathcal{A}))|}, & \text{if } k = \ell. \end{cases} \tag{5}$$

Our proof relies on several results ([18, 19, 20]) concerning the minimality of arrangements. We also provide an alternative proof of Theorem 1.1 for real arrangements.

Remark 1.5. If $R = \mathbb{C}$, one can deduce Theorem 1.4 from Theorem 1.1. We present a sketch of the argument. If $\omega = \sum_{i=1}^{n} \lambda_i e_i \in A_R^1(\mathcal{A})$ satisfies the assumption of Theorem 1.4, then so does $t\omega$ for $t \in \mathbb{C}^\times$. Clearly, we have $H^k(A_{\mathbb{C}}^\bullet(\mathcal{A}), \omega\wedge) \simeq H^k(A_{\mathbb{C}}^\bullet(\mathcal{A}), t\omega\wedge)$. However, the tangent-cone theorem ([6, 7]) gives, for $0 < |t| \ll 1$, an isomorphism $H^k(A_{\mathbb{C}}^\bullet(\mathcal{A}), t\omega\wedge) \simeq H^k(M(\mathcal{A}), \mathcal{L}_t)$, where $\mathcal{L}_t = \exp(t\omega)$. Then, Theorem 1.1 gives $H^k(M(\mathcal{A}), \mathcal{L}_t) = 0$.

The remainder of this paper is organized as follows.

In §2, we give some basic terminology and a description of the Aomoto complex in terms of chambers developed in [18, 19, 20]. We also recall the description of a twisted minimal complex in terms of chambers. Two cochain complexes $(R[\mathrm{ch}^\bullet(\mathcal{A})], \nabla_{\omega_\lambda})$ and $(\mathbb{C}[\mathrm{ch}^\bullet(\mathcal{A})], \nabla_{\mathcal{L}})$ are constructed using the real structures of $\mathcal{A}$ (adjacency relations of chambers). These cochain complexes provide a parallel description of the cohomology of the Aomoto complex and the local system cohomology group. Indeed, using these complexes, we can simultaneously prove CDO-type vanishing results for both cases.

In §3, we state the main result and describe the strategy for the proof. The proof consists of an easy part and a hard part. The easy part of the proof mainly uses elementary arguments relating to cochain complexes, which are also stated in this section. The hard part is tackled in §4.

§4 is devoted to an analysis of the polyhedral structures of chambers that are required for matrix presentations of the coboundary map of $(R[\mathrm{ch}^\bullet(\mathcal{A})], \nabla_{\omega_\lambda})$.

2. Notation and Preliminaries

2.1. **Orlik-Solomon algebra and Aomoto complex.** Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be an affine hyperplane arrangement in $V = \mathbb{R}^\ell$. Denote the complement of the complexified hyperplanes by $M(\mathcal{A}) = \mathbb{C}^\ell \smallsetminus \cup_{i=1}^n H_i \otimes \mathbb{C}$. By identifying $\mathbb{R}^\ell$ with $\mathbb{P}_{\mathbb{R}}^\ell \smallsetminus \overline{H}_\infty$, define the projective closure by $\overline{\mathcal{A}} = \{\overline{H}_1, \ldots, \overline{H}_n, \overline{H}_\infty\}$, where $\overline{H}_i \subset \mathbb{P}_{\mathbb{R}}^\ell$ is the closure of H_i in the projective space. We denote the intersection posets of $\mathcal{A}$ and $\overline{\mathcal{A}}$ as $L(\mathcal{A})$ and $L(\overline{\mathcal{A}})$, respectively; these are the posets of subspaces obtained as intersections of some hyperplanes with reverse inclusion order. An element of $L(\mathcal{A})$ (and $L(\overline{\mathcal{A}})$) is also called an edge. We denote the set of all k-dimensional edges by $L_k(\mathcal{A})$. For example, $L_\ell(\mathcal{A}) = \{V\}$ and $L_{\ell-1}(\mathcal{A}) = \mathcal{A}$. Then, $\mathcal{A}$ is essential if and only if $L_0(\mathcal{A}) \neq \emptyset$.

Let R be a commutative ring. Orlik and Solomon gave a simple combinatorial description of the algebra $H^*(M(\mathcal{A}), R)$, which is the quotient of the exterior algebra on classes dual to the meridians, modulo a certain ideal determined by $L(\mathcal{A})$ (see [11]). More precisely, by associating a generator $e_i \simeq \frac{1}{2\pi\sqrt{-1}} d\log\alpha_i$ to any hyperplane H_i, the Orlik-Solomon algebra $A_R^\bullet(\mathcal{A})$ of $\mathcal{A}$ is the quotient of the exterior algebra generated by the elements e_i, $1 \leq i \leq n$, modulo the ideal $I(\mathcal{A})$ generated by:

- elements of the form $\{e_{i_1} \wedge \cdots \wedge e_{i_s} \mid H_{i_1} \cap \cdots \cap H_{i_s} = \emptyset\}$,
- elements of the form $\{\partial(e_{i_1} \wedge \cdots \wedge e_{i_s}) \mid H_{i_1} \cap \cdots \cap H_{i_s} \neq \emptyset \text{ and } \operatorname{codim}(H_{i_1} \cap \cdots \cap H_{i_s}) < s\}$, where $\partial(e_{i_1} \wedge \cdots \wedge e_{i_s}) = \sum_{\alpha=1}^{s} (-1)^{\alpha-1} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_\alpha}} \wedge \cdots \wedge e_{i_s}$.

Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in R^n$ and $\omega_\lambda = \sum_{i=1}^n \lambda_i e_i \in A_R^1(\mathcal{A})$. The cochain complex

$$(A_R^\bullet(\mathcal{A}), \omega_\lambda\wedge) = \{A_R^\bullet(\mathcal{A}) \xrightarrow{\omega_\lambda\wedge} A_R^{\bullet+1}(\mathcal{A})\}$$

is called *the Aomoto complex*.

We say that an edge $X \in L(\overline{\mathcal{A}})$ is *dense* if the localization $\overline{\mathcal{A}}_X = \{\overline{H} \in \overline{\mathcal{A}} \mid X \subseteq \overline{H}\}$ is indecomposable (see [13] for more details). Each hyperplane $\overline{H} \in \overline{\mathcal{A}}$ is considered to be a dense edge. In this paper, the set of dense edges of $\overline{\mathcal{A}}$ contained in $\overline{H}_\infty$ plays an important role. We denote this set by $\mathsf{D}_\infty(\overline{\mathcal{A}})$. We will characterize $X \in \mathsf{D}_\infty(\overline{\mathcal{A}})$ in terms of chambers in Proposition 2.6.

Set $\lambda_\infty := -\sum_{i=1}^n \lambda_i$. For any $X \in L(\overline{\mathcal{A}})$, $\lambda_X := \sum_{\overline{H}_i \supset X} \lambda_i$, where the index i-runs through the set $\{1, 2, \dots, n, \infty\}$.

The isomorphism class of a rank one local system $\mathcal{L}$ on the complexified complement $M(\mathcal{A})$ is determined by the monodromy $q_i \in \mathbb{C}^\times$ around each hyperplane H_i. As in the case of the Aomoto complex, we denote $q_\infty = (q_1 q_2 \cdots q_n)^{-1}$ and $q_X = \prod_{\overline{H}_i \supset X} q_i$ for an edge $X \in L(\overline{\mathcal{A}})$.

2.2. **Chambers and minimal complexes.** In this section, we recall the description of the minimal complex in terms of real structures from [18, 19, 20]. Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an essential hyperplane arrangement in $\mathbb{R}^\ell$. A connected component of $\mathbb{R}^\ell \smallsetminus \bigcup_{i=1}^n H_i$ is called a chamber. The set of all chambers of $\mathcal{A}$ is denoted by $\mathrm{ch}(\mathcal{A})$. A chamber $C \in \mathrm{ch}(\mathcal{A})$ is called a bounded chamber if C is bounded. The set of all bounded chambers of $\mathcal{A}$ is denoted by $\mathrm{bch}(\mathcal{A})$. For a chamber $C \in \mathrm{ch}(\mathcal{A})$, denote the closure of C in $\mathbb{P}^\ell_{\mathbb{R}}$ by $\overline{C}$. It is easily seen that a chamber C is bounded if and only if $\overline{C} \cap \overline{H}_\infty = \emptyset$.

Given two chambers $C, C' \in \mathrm{ch}(\mathcal{A})$, we denote the set of separating hyperplanes of C and C' by

$$\mathrm{Sep}(C, C') := \{H_i \in \mathcal{A} \mid \ H_i \text{ separates } C \text{ and } C'\}.$$

To describe the minimal complex, we must fix a generic flag. Let

$$\mathcal{F} : \emptyset = F^{-1} \subset F^0 \subset F^1 \subset \cdots \subset F^\ell = \mathbb{R}^\ell$$

be a generic flag (i.e., F^k is a generic k-dimensional affine subspace, that is,

$$\dim(\overline{X} \cap \overline{F}^k) = \dim \overline{X} + k - \ell$$

for any $\overline{X} \in L(\overline{\mathcal{A}})$). The genericity of $\mathcal{F}$ is equivalent to

$$F^k \cap L_i(\mathcal{A}) = L_{k+i-\ell}(\mathcal{A} \cap F^k)$$

for $k + i \geq \ell$.

Definition 2.1. We say that the hyperplane $F^{\ell-1}$ is near to $\overline{H}_\infty$ when $F^{\ell-1}$ does not separate 0-dimensional edges $L_0(\mathcal{A}) \subset \mathbb{R}^\ell$. Similarly, we say the flag $\mathcal{F}$ is near to $\overline{H}_\infty$ when F^{k-1} does not separate $L_0(\mathcal{A} \cap F^k)$ for all $k = 1, \dots, \ell$.

From this point, we assume that the flag $\mathcal{F}$ is near to $\overline{H}_\infty$. For a generic flag $\mathcal{F}$ near to $\overline{H}_\infty$, we define

$$\begin{aligned}
\mathrm{ch}^k(\mathcal{A}) &= \{C \in \mathrm{ch}(\mathcal{A}) \mid C \cap F^k \neq \emptyset, C \cap F^{k-1} = \emptyset\} \\
\mathrm{bch}^k(\mathcal{A}) &= \{C \in \mathrm{ch}^k(\mathcal{A}) \mid C \cap F^k \text{ is bounded}\} \\
\mathrm{uch}^k(\mathcal{A}) &= \{C \in \mathrm{ch}^k(\mathcal{A}) \mid C \cap F^k \text{ is unbounded}\}.
\end{aligned}$$

It is then clear that

$$\mathrm{ch}^k(\mathcal{A}) = \mathrm{bch}^k(\mathcal{A}) \sqcup \mathrm{uch}^k(\mathcal{A})$$

$$\mathrm{ch}(\mathcal{A}) = \bigsqcup_{k=0}^{\ell} \mathrm{ch}^k(\mathcal{A}).$$

Note that $\mathrm{bch}^\ell(\mathcal{A}) = \mathrm{bch}(\mathcal{A})$. However, for $k < \ell$, $C \in \mathrm{bch}^k(\mathcal{A})$ is an unbounded chamber.

Definition 2.2. ([19, Definition 2.1]) Let $C \in \mathrm{bch}(\mathcal{A})$. There exists a unique chamber, denoted by $C^\vee \in \mathrm{uch}(\mathcal{A})$, which is the opposite with respect to $\overline{C} \cap \overline{H}_\infty$, where $\overline{C}$ is the closure of C in the projective space $\mathbb{P}^\ell_{\mathbb{R}}$.

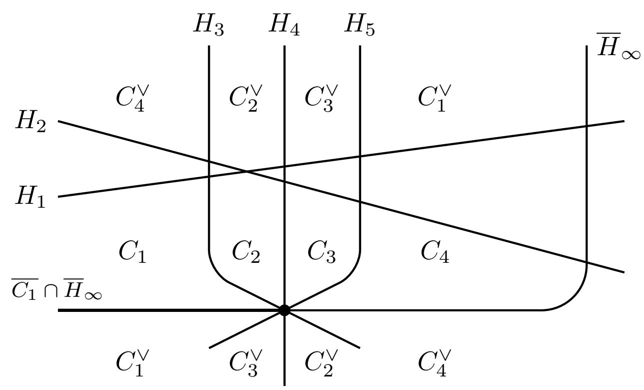

FIGURE 1. Opposite chambers

Let us denote the projective subspace generated by $\overline{C} \cap \overline{H}_\infty$ as $X(C) = \langle \overline{C} \cap \overline{H}_\infty \rangle$.

Proposition 2.3. *Let* $C \in \mathrm{bch}(\mathcal{A})$*. Then*

$$\mathrm{Sep}(C, C^\vee) = \{H \in \mathcal{A} \mid \overline{H} \not\supset X(C)\} = \overline{\mathcal{A}} \smallsetminus \overline{\mathcal{A}}_{X(C)}. \tag{6}$$

Proof. Let $p \in C$ and p' be a point in the relative interior of $\overline{C} \cap \overline{H}_\infty$. Take the line

$$L = \langle p, p' \rangle \subset \mathbb{P}^\ell_\mathbb{R},$$

and choose a point $p'' \in C^\vee \cap L$. Then, consider the segment $[p, p''] \subset \mathbb{R}^\ell = \mathbb{P}^\ell_\mathbb{R} \smallsetminus \overline{H}_\infty$ (see Figure 2). On the projective space $\mathbb{P}^\ell_\mathbb{R}$, the line $L = \langle p, p' \rangle$ must intersect every hyperplane $\overline{H} \in \overline{\mathcal{A}}$ exactly once. Furthermore, L intersects $\overline{H} \in \overline{\mathcal{A}}_{X(C)}$ at p'. Additionally, the segment $[p, p'']$ intersects $H \in \mathrm{Sep}(C, C^\vee)$. Hence, we have (6). $\square$

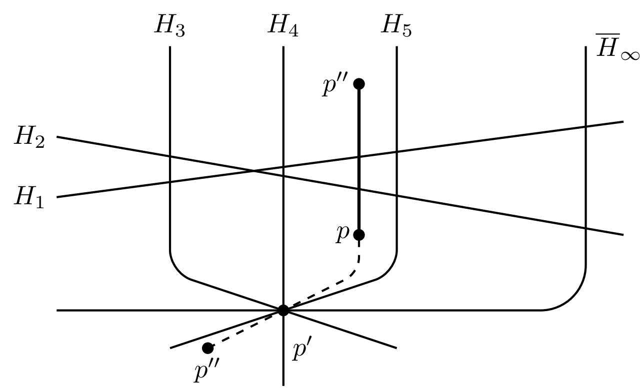

FIGURE 2. The segment $[p, p'']$ (thick segment).

Corollary 2.4. *If* $\dim X(C) = \ell - 1$*, then* $\mathrm{Sep}(C, C^\vee) = \mathcal{A}$.

Proof. In this case, $\overline{\mathcal{A}}_{X(C)} = \{\overline{H}_\infty\}$. Proposition 2.3 implies that $\mathrm{Sep}(C, C^\vee) = \mathcal{A}$. $\square$

Proposition 2.5. *(*[18, 19]*)*

(*i*) $\#\,\mathrm{ch}^k(\mathcal{A}) = b_k$*, where* $b_k = b_k(M(\mathcal{A}))$.

(*ii*) $\# \operatorname{bch}^k(\mathcal{A}) = \# \operatorname{uch}^{k+1}(\mathcal{A})$.
(*iii*) $\# \operatorname{bch}^k(\mathcal{A}) = b_k - b_{k-1} + \cdots + (-1)^k b_0$.

Concerning Proposition 2.5 (*ii*), an explicit bijection is given by the map to the opposite chamber,

$$\iota : \operatorname{bch}^k(\mathcal{A}) \xrightarrow{\simeq} \operatorname{uch}^{k+1}(\mathcal{A}), C \longmapsto C^\vee.$$

The next result characterizes the dense edges contained in $\overline{H}_\infty$.

Proposition 2.6. *(*[19, Proposition 2.4]*) Let $\mathcal{A}$ be an affine arrangement in $\mathbb{R}^\ell$. An edge $X \in L(\overline{\mathcal{A}})$ with $X \subseteq \overline{H}_\infty$ is dense if and only if $X = X(C)$ for some chamber $C \in \operatorname{uch}(\mathcal{A})$. In particular, we have*

$$\mathsf{D}_\infty(\mathcal{A}) = \{X(C) \mid C \in \operatorname{uch}(\mathcal{A})\}. \tag{7}$$

Next, we define the degree map

$$\deg : \operatorname{ch}^k(\mathcal{A}) \times \operatorname{ch}^{k+1}(\mathcal{A}) \longrightarrow \mathbb{Z}.$$

Let $B = B^k \subset F^k$ be a k-dimensional ball of sufficiently large radius so that every 0-dimensional edge $X \in L_0(\mathcal{A} \cap F^k) \simeq L_{\ell-k}(\mathcal{A})$ is contained in the interior of B^k. Let $C \in \operatorname{ch}^k(\mathcal{A})$ and $C' \in \operatorname{ch}^{k+1}(\mathcal{A})$. Then, there exists a vector field $U^{C'}$ on F^k ([18]) that satisfies the following conditions.

- $U^{C'}(x) \neq 0$ for $x \in \partial\overline{C} \cap B^k$.
- Let $x \in \partial(B^k) \cap \overline{C}$. Then, $T_x(\partial B^k)$ can be considered as a hyperplane of T_xF^k. We impose the condition that $U^{C'}(x) \in T_xF^k$ is contained in the half-space corresponding to the inside of B^k.
- If $x \in H \cap F^k$ for a hyperplane $H \in \mathcal{A}$, then $U^{C'}(x) \notin T_x(H \cap F^k)$ and is directed to the side in which C' is lying with respect to H.

When the vector field $U^{C'}$ satisfies the above conditions, we say that *the vector field $U^{C'}$ is directed to the chamber C'*. The above conditions imply that if either $x \in H \cap F^k$ or $x \in \partial B^k$, then $U^{C'}(x) \neq 0$. Thus, for $C \in \operatorname{ch}^k(\mathcal{A})$, U is not vanishing on $\partial(\overline{C} \cap B^k)$. Hence, we can consider the following Gauss map.

$$\frac{U^{C'}}{|U^{C'}|} : \partial(\overline{C} \cap B^k) \longrightarrow S^{k-1}.$$

Fixing an orientation on F^k induces an orientation on $\partial(\overline{C} \cap B^k)$.

Definition 2.7. Define the degree $\deg(C, C')$ between $C \in \operatorname{ch}^k(\mathcal{A})$ and $C' \in \operatorname{ch}^{k+1}(\mathcal{A})$ by

$$\deg(C, C') := \deg\left(\left.\frac{U^{C'}}{|U^{C'}|}\right|_{\partial(\overline{C} \cap B^k)} : \partial(\overline{C} \cap B^k) \longrightarrow S^{k-1} \right) \in \mathbb{Z}.$$

This is independent of the choice of $U^{C'}$ ([18]).

If the vector field $U^{C'}$ does not have zeros on $\overline{C} \cap B^k$, then the Gauss map can be extended to the map $\overline{C} \cap B^k \longrightarrow S^{k-1}$. Hence, $\frac{U^{C'}}{|U^{C'}|} : \partial(\overline{C} \cap B^k) \longrightarrow S^{k-1}$ is homotopic to a constant map, and we have the following result.

Proposition 2.8. *If the vector field $U^{C'}$ is nowhere zero on $\overline{C} \cap B^k$, then $\deg(C, C') = 0$.*

Example 2.9. Let $p_0 \in F^k$ be such that $p_0 \notin \bigcup_{H \in \mathcal{A}} H \cup \partial B^k$. Define the pointing vector field U^{p_0} by

$$U^{p_0}(x) = \overrightarrow{x; p_0} \in T_x F^k, \tag{8}$$

where $\overrightarrow{x; p_0}$ is a tangent vector at x pointing toward p_0 (see Figure 3). The vector field U^{p_0} is directed to the chamber containing p_0. Note that $U^{p_0}(x) = 0$ if and only if $x = p_0$. Hence, if $p_0 \notin C \cap B^k$, the Gauss map $\frac{U^{p_0}}{|U^{p_0}|} : \partial(\overline{C} \cap B^k) \longrightarrow S^{k-1}$ satisfies $\deg\left(\frac{U^{p_0}}{|U^{p_0}|}\right) = 0$. Otherwise, if $p_0 \in C \cap B^k$, then $\deg\left(\frac{U^{p_0}}{|U^{p_0}|}\right) = (-1)^k$.

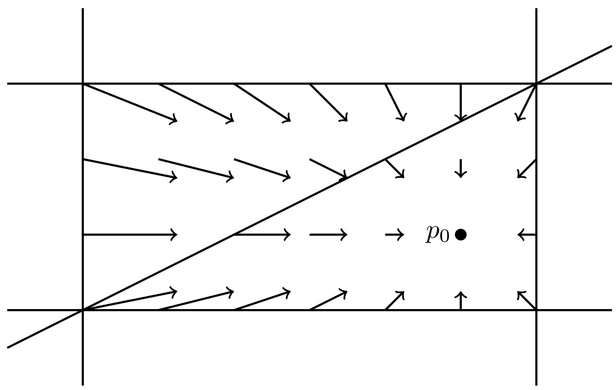

FIGURE 3. Pointing vector field $\frac{1}{4}U^{p_0}$

Consider the Orlik-Solomon algebra $A_R^\bullet(\mathcal{A})$ over the commutative ring R. Let

$$\omega_\lambda = \sum_{i=1}^{n} \lambda_i e_i \in A_R^1(\mathcal{A}) \qquad (\lambda_i \in R).$$

We will describe the Aomoto complex $(A_R^\bullet(\mathcal{A}), \omega_\lambda \wedge)$ in terms of chambers. For two chambers $C, C' \in \mathrm{ch}(\mathcal{A})$, define $\lambda_{\mathrm{Sep}(C,C')}$ by

$$\lambda_{\mathrm{Sep}(C,C')} := \sum_{H_i \in \mathrm{Sep}(C,C')} \lambda_i.$$

Proposition 2.10. *Let C be an unbounded chamber. Then,*

$$\lambda_{\mathrm{Sep}(C,C^\vee)} = -\lambda_{X(C)}.$$

Proof. By Proposition 2.3, we have $\overline{\mathcal{A}} = \overline{\mathcal{A}}_{X(C)} \sqcup \mathrm{Sep}(C, C^\vee)$. Hence, from the definition of $\lambda_\infty = -\sum_{i=1}^n \lambda_i$, we obtain $\lambda_{\mathrm{Sep}(C,C^\vee)} + \lambda_{X(C)} = 0$. $\square$

Let $R[\mathrm{ch}^k(\mathcal{A})] = \bigoplus_{C \in \mathrm{ch}^k(\mathcal{A})} R \cdot [C]$ be the free R-module generated by $\mathrm{ch}^k(\mathcal{A})$. Let

$$\nabla_{\omega_\lambda} : R[\mathrm{ch}^k(\mathcal{A})] \longrightarrow R[\mathrm{ch}^{k+1}(\mathcal{A})]$$

be the R-homomorphism defined by

$$\nabla_{\omega_\lambda}([C]) = \sum_{C' \in \mathrm{ch}^{k+1}} \deg(C, C') \cdot \lambda_{\mathrm{Sep}(C,C')} \cdot [C']. \tag{9}$$

Proposition 2.11. *([20]) $(R[\mathrm{ch}^\bullet(\mathcal{A})], \nabla_{\omega_\lambda})$ is a cochain complex. Furthermore, there is a natural isomorphism of cochain complexes,*

$$(R[\mathrm{ch}^\bullet(\mathcal{A})], \nabla_{\omega_\lambda}) \simeq (A_R^\bullet(\mathcal{A}), \omega_\lambda\wedge).$$

In particular,

$$H^k(R[\mathrm{ch}^\bullet(\mathcal{A})], \nabla_{\omega_\lambda}) \simeq H^k(A_R^\bullet(\mathcal{A}), \omega_\lambda\wedge).$$

Let $\mathcal{L}$ be a rank one local system on $M(\mathcal{A})$ with monodromy $q_i \in \mathbb{C}^\times$ around H_i $(i = 1, \dots, n)$. Fix $q_i^{1/2} = \sqrt{q_i}$ and define $q_\infty^{1/2}$ and $\Delta(C, C')$ by $q_\infty^{1/2} := \left(q_1^{1/2} \cdots q_n^{1/2}\right)^{-1}$ and

$$\Delta(C, C') := \prod_{H_i \in \mathrm{Sep}(C,C')} q_i^{1/2} - \prod_{H_i \in \mathrm{Sep}(C,C')} q_i^{-1/2}.$$

The local system cohomology groups can then be computed in a similar way to the cohomology groups of the Aomoto complex. Indeed, let us define the linear map

$$\nabla_{\mathcal{L}} : \mathbb{C}[\mathrm{ch}^k(\mathcal{A})] \longrightarrow \mathbb{C}[\mathrm{ch}^{k+1}(\mathcal{A})]$$

by

$$\nabla_{\mathcal{L}}([C]) = \sum_{C' \in \mathrm{ch}^{k+1}} \deg(C, C') \cdot \Delta(C, C') \cdot [C'].$$

Then, we have the following.

Proposition 2.12. *([18]) $(\mathbb{C}[\mathrm{ch}^\bullet(\mathcal{A})], \nabla_{\mathcal{L}})$ is a cochain complex. Furthermore, there is a natural isomorphism of cohomology groups:*

$$H^k(\mathbb{C}[\mathrm{ch}^\bullet(\mathcal{A})], \nabla_{\mathcal{L}}) \simeq H^k(M(\mathcal{A}), \mathcal{L}).$$

3. Main results and strategy

3.1. **Main theorems.** In this section, let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a hyperplane arrangement in $\mathbb{R}^\ell$ and R be a commutative ring with 1.

Theorem 3.1. *If $\lambda_X \in R^\times$ for all $X \in \mathsf{D}_\infty(\overline{\mathcal{A}})$, then*

$$H^k(\mathbb{C}[\mathrm{ch}^\bullet(\mathcal{A})], \nabla_{\omega_\lambda}) \simeq \begin{cases} 0, & \text{if } k < \ell, \\ R[\mathrm{bch}(\mathcal{A})], & \text{if } k = \ell. \end{cases}$$

More generally, we can prove the following.

Corollary 3.2. *Let $0 \le p < \ell$. If $\lambda_X \in R^\times$ for all $X \in \mathsf{D}_\infty(\overline{\mathcal{A}})$ with $\dim(X) \ge p$, then*

$$H^k(\mathbb{C}[\mathrm{ch}^\bullet(\mathcal{A})], \nabla_{\omega_\lambda}) = 0, \text{ for all } 0 \le k < \ell - p.$$

Proof. We prove Corollary 3.2 based on Theorem 3.1 with $\mathcal{A} \cap F^{\ell-p}$. The Orlik-Solomon algebra $A_R^\bullet(\mathcal{A} \cap F^{\ell-p})$ is isomorphic to $A_R^{\le \ell-p}(\mathcal{A})$. Hence, we have an isomorphism

$$H^k(A_R^\bullet(\mathcal{A} \cap F^{\ell-p}), \omega_\lambda\wedge) \simeq H^k(A_R^\bullet(\mathcal{A}), \omega_\lambda\wedge), \tag{10}$$

for $k < \ell - p$. Note that $L(\mathcal{A} \cap F^{\ell-p}) \simeq L^{\ge p}(\mathcal{A})$. By assumption, we have that $\lambda_X \in R^\times$ for any $X \in \mathsf{D}_\infty(\mathcal{A} \cap F^{\ell-q})$. Hence, by Theorem 3.1, the left-hand side of (10) vanishes. □

The following vanishing result for the Aomoto complex follows from Proposition 2.11.

Corollary 3.3. *Let $0 \le p < \ell$. If $\lambda_X \in R^\times$ for all $X \in \mathsf{D}_\infty(\overline{\mathcal{A}})$ with $\dim(X) \ge p$, then*

$$H^k(A_R^\bullet(\mathcal{A}), \omega_\lambda\wedge) = 0, \text{ for all } 0 \le k < \ell - p.$$

Remark 3.4. A very similar proof can be used for the case of local systems. Namely, if the local system $\mathcal{L}$ satisfies $q_X \neq 1$ for all $X \in \mathsf{D}_\infty(\overline{\mathcal{A}})$ with $\dim(X) \geq p$, then

$$H^k(\mathbb{C}[\mathrm{ch}^\bullet(\mathcal{A})], \nabla_{\mathcal{L}}) = 0, \text{ for all } k < \ell - p.$$

Using Proposition 2.12, this implies

$$H^k(M(\mathcal{A}), \mathcal{L}) = 0, \text{ for all } k < \ell - p,$$

which gives an alternative proof of Theorem 1.1 given by Cohen, Dimca, and Orlik.

3.2. **Strategy for the proof of Theorem 3.1.** To analyze the cohomology group

$$H^k(R[\mathrm{ch}^\bullet(\mathcal{A})], \nabla_\omega) = \frac{\ker\Big(\nabla_\omega : R[\mathrm{ch}^k(\mathcal{A})] \longrightarrow R[\mathrm{ch}^{k+1}(\mathcal{A})]\Big)}{\mathrm{im}\Big(\nabla_\omega : R[\mathrm{ch}^{k-1}(\mathcal{A})] \longrightarrow R[\mathrm{ch}^k(\mathcal{A})]\Big)},$$

we will use the direct decomposition $R[\mathrm{ch}^k(\mathcal{A})] = R[\mathrm{bch}^k(\mathcal{A})] \oplus R[\mathrm{uch}^k(\mathcal{A})]$, and then consider the map

$$\overline{\nabla}_{\omega_\lambda} : R[\mathrm{bch}^k(\mathcal{A})] \hookrightarrow R[\mathrm{ch}^k(\mathcal{A})] \xrightarrow{\nabla_\omega} R[\mathrm{ch}^{k+1}(\mathcal{A})] \twoheadrightarrow R[\mathrm{uch}^{k+1}(\mathcal{A})]. \tag{11}$$

We will study the map $\overline{\nabla}_{\omega_\lambda} : R[\mathrm{bch}^k(\mathcal{A})] \longrightarrow R[\mathrm{uch}^{k+1}(\mathcal{A})]$ in detail below. Recall that there is a natural bijection $\iota : \mathrm{bch}^k(\mathcal{A}) \xrightarrow{\simeq} \mathrm{uch}^{k+1}(\mathcal{A})$ (see Proposition 2.5 and subsequent remarks). Once we fix an ordering $C_1, \dots, C_b$ of $\mathrm{bch}^k(\mathcal{A})$, we obtain a matrix expression of the map $\overline{\nabla}_{\omega_\lambda}$. We will prove the following.

- (i) Let $C \in \mathrm{bch}^k(\mathcal{A})$. Then, $\deg(C, C^\vee) = (-1)^{\ell-1-\dim X(C)}$.
- (ii) For an appropriate ordering of $\mathrm{bch}^k(\mathcal{A}) = \{C_1, \dots, C_b\}$, the matrix expression of
$$\overline{\nabla}_{\omega_\lambda} : R[\mathrm{bch}^k(\mathcal{A})] \longrightarrow R[\mathrm{uch}^{k+1}(\mathcal{A})]$$
is upper-triangular.
- (iii) $\det \overline{\nabla}_\omega \in R^\times$
- (iv) These imply Theorem 3.1.

(i) and (ii) will be proved in §4.

Here, we prove (iii) and (iv) based on (i) and (ii). First, note that Proposition 2.10, along with the definition (9) of the coboundary map of the complex $(R[\mathrm{ch}^\bullet(\mathcal{A})], \nabla_\omega)$ and the upper-triangularity in (ii) above implies that

$$\det \overline{\nabla}_\omega = \pm \prod_{C \in \mathrm{bch}^k(\mathcal{A})} \deg(C, C^\vee) \lambda_{X(C)}.$$

Then, from the assumption that $\lambda_X \in R^\times$ for $X \in \mathsf{D}_\infty(\mathcal{A})$ (see also Proposition 2.6), we have (iii) directly. Because $\overline{\nabla}_\omega : R[\mathrm{bch}^k(\mathcal{A})] \xrightarrow{\simeq} R[\mathrm{uch}^{k+1}(\mathcal{A})]$, which are diagonals of the following diagram, are isomorphisms of free R-modules, we have $H^k(R[\mathrm{ch}^\bullet(\mathcal{A})], \nabla_\omega) = 0$ for $k < \ell$ and $H^\ell(R[\mathrm{ch}^\bullet(\mathcal{A})], \nabla_\omega) \simeq R[\mathrm{bch}^\ell(\mathcal{A})]$.

$$\begin{array}{ccccccccccc}
R[\mathrm{ch}^0] & \xrightarrow{\nabla_\omega} & R[\mathrm{ch}^1] & \xrightarrow{\nabla_\omega} & \cdots & \xrightarrow{\nabla_\omega} & R[\mathrm{ch}^k] & \xrightarrow{\nabla_\omega} & R[\mathrm{ch}^{k+1}] & \xrightarrow{\nabla_\omega} & \cdots \\
\| & & \| & & & & \| & & \| & & \\
R[\mathrm{bch}^0] & & R[\mathrm{bch}^1] & & \cdots & & R[\mathrm{bch}^k] & & R[\mathrm{bch}^{k+1}] & & \\
& \searrow & \oplus & \searrow & & \searrow & \oplus & \searrow & \oplus & & \\
& & R[\mathrm{uch}^1] & & \cdots & & R[\mathrm{uch}^k] & & R[\mathrm{uch}^{k+1}] & &
\end{array}$$

4. Proofs

In this section, we prove (i) and (ii) stated in §3.2 for $k = \ell - 1$. Namely:

(i') For a chamber $C \in \mathrm{bch}^{\ell-1}(\mathcal{A})$, $\deg(C, C^\vee) = (-1)^{\ell-1-\dim X(C)}$.

(ii') For an appropriate ordering of $\{C_1, \dots, C_b\} = \mathrm{bch}^{\ell-1}(\mathcal{A})$, the matrix expression of $\overline{\nabla}_{\omega_\lambda} : R[\mathrm{bch}^{\ell-1}(\mathcal{A})] \longrightarrow R[\mathrm{uch}^{\ell}(\mathcal{A})]$ is upper-triangular.

For other $k < \ell$, note that the assertions can be proved in a similar way to that for $k = l - 1$ by using the generic section F^{k+1} (see the arguments in the proof of Corollary 3.2).

4.1. **Structure of Walls.** For simplicity, we will set $F = F^{\ell-1}$. Recall that

$$\mathrm{bch}^{\ell-1}(\mathcal{A}) = \{C \in \mathrm{ch}(\mathcal{A}) \mid C \cap F \text{ is a bounded chamber of } F \cap \mathcal{A}\}.$$

Let $C \in \mathrm{bch}^{\ell-1}(\mathcal{A})$. A hyperplane $H \in \mathcal{A}$ is said to be a wall of C if $H \cap F$ is a supporting hyperplane of a facet of $\overline{C} \cap F$. For any $C \in \mathrm{bch}^{\ell-1}(\mathcal{A})$, we denote the set of all walls of C by $\mathrm{Wall}(C)$.

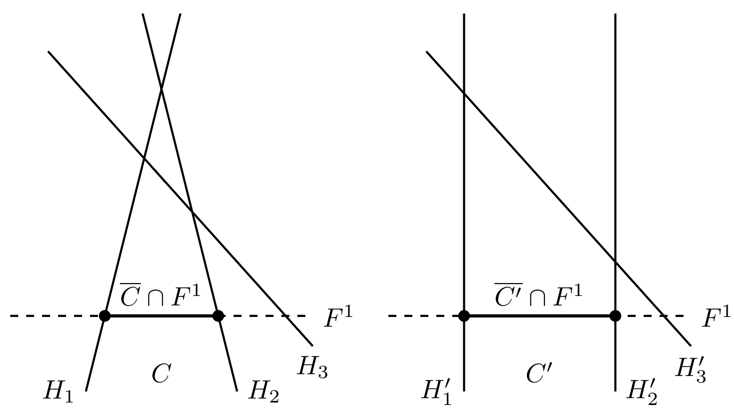

Figure 4. $\mathrm{Wall}(C) = \mathrm{Wall}_2(C) = \{H_1, H_2\}$, $\mathrm{Wall}(C') = \mathrm{Wall}_1(C') = \{H_1', H_2'\}$

We divide the set of walls into two types.

Definition 4.1. A wall $H \in \mathrm{Wall}(C)$ is said to be of the first kind if $\overline{H} \supset X(C)$. Otherwise, we say that H is a wall of the second kind. The sets of the first and the second kind of walls are denoted by $\mathrm{Wall}_1(C)$ and $\mathrm{Wall}_2(C)$, respectively. We have $\mathrm{Wall}(C) = \mathrm{Wall}_1(C) \sqcup \mathrm{Wall}_2(C)$ (see Figure 4 and 5).

Let $C \in \mathrm{bch}^{\ell-1}(\mathcal{A})$ and $\mathrm{Wall}_1(C) = \{H_{i_1}, \dots, H_{i_k}\}$ be walls of the first kind. We choose defining equations $\alpha_{i_1}, \dots, \alpha_{i_k}$ for the walls in $\mathrm{Wall}_1(C)$ so that

$$C \subset \{\alpha_{i_1} > 0\} \cap \cdots \cap \{\alpha_{i_k} > 0\}.$$

Note that $\widetilde{C} := \{\alpha_{i_1} > 0\} \cap \cdots \cap \{\alpha_{i_k} > 0\}$ is a chamber of $\mathrm{Wall}_1(\mathcal{A})$. Let $D \in \mathrm{uch}(\mathcal{A})$ be another unbounded chamber of $\mathcal{A}$. Then, D is said to be inside $\mathrm{Wall}_1(C)$ if

$$D \subset \widetilde{C} = \{\alpha_{i_1} > 0\} \cap \cdots \cap \{\alpha_{i_k} > 0\}.$$

This condition is equivalent to $\mathrm{Sep}(C, D) \cap \mathrm{Wall}_1(C) = \emptyset$.

Recall that the opposite chamber of $C \in \mathrm{bch}^{\ell-1}(\mathcal{A})$ is defined as the opposite chamber with respect to $X(C) \subset \overline{H}_\infty$. Using (6), we have the following.

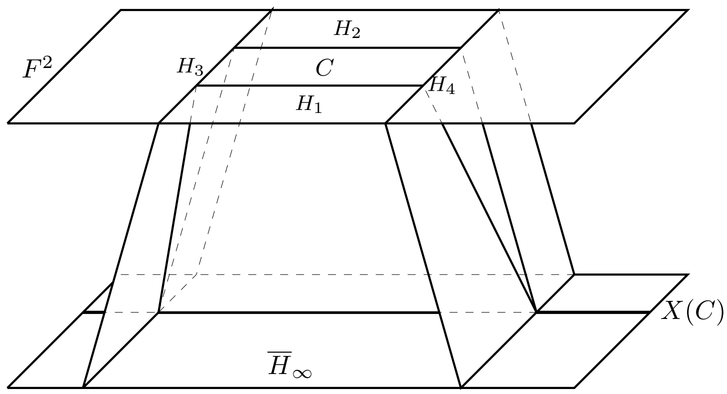

FIGURE 5. $\mathrm{Wall}_1(C) = \{H_1, H_2\}, \mathrm{Wall}_2(C) = \{H_3, H_4\}$.

Proposition 4.2. *Let $C \in \mathrm{bch}^{\ell-1}(\mathcal{A})$. Then,* $\mathrm{Sep}(C, C^\vee) \cap \mathrm{Wall}(C) = \mathrm{Wall}_2(C)$.

Remark 4.3. Let $C \in \mathrm{bch}^{\ell-1}(\mathcal{A})$. If D is inside $\mathrm{Wall}_1(C)$, then

$$X(D) \subset X(C) \quad \text{and} \quad \dim X(D) \leq \dim X(C).$$

4.2. **Fibered structure of chambers.** Let $C \in \mathrm{bch}^{\ell-1}(\mathcal{A})$ and $d = \dim X(C)$. As above, we let $\widetilde{C} \in \mathrm{ch}(\mathrm{Wall}_1(C))$ be the unique chamber such that $C \subset \widetilde{C}$.

For each point $p \in \overline{\widetilde{C}}$, denote $G_1(p) := \langle X(C), p\rangle \cap F$ (see Figure 6). Then, $G_1(p)$ is a d-dimensional affine subspace that is parallel to each $H \in \mathrm{Wall}_1(C)$. Fix a base point $p_0 \in \widetilde{C}$. We also fix an $(\ell-1-d)$-dimensional subspace $G_2(p_0) \subset F$ that passes through p_0 and is transversal to $G_1(p_0)$ (see Figure 6). Let us call $Q_0 := G_2(p_0) \cap \overline{\widetilde{C}}$ the base polytope.

Consider the map $\pi_C : \overline{C} \cap F \longrightarrow Q_0, p \longmapsto G_1(p) \cap Q_0$. For each $q \in Q_0$, the fiber $\pi_C^{-1}(q) = G_1(q) \cap \overline{C}$ is a d-dimensional polytope. This is a conclusion of the assumption that F is generic and near to $\overline{H}_\infty$ and the following elementary proposition.

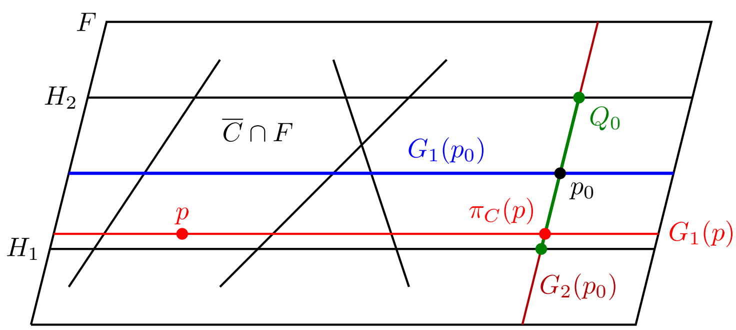

FIGURE 6. Base polytope Q_0 ($\mathrm{Wall}_1(C) = \{H_1, H_2\}$)

Proposition 4.4. *Let $P \subset \mathbb{R}^\ell$ be an ℓ-dimensional polytope. Let $X \subset P$ be a d-dimensional face ($0 \leq d \leq \ell$). We denote by $\langle X\rangle$ the d-dimensional affine subspace spanned by X. Then, for $\varepsilon \in \mathbb{R}^\ell$ with sufficiently small $0 \leq |\varepsilon| \ll 1$, $(\langle X\rangle + \varepsilon) \cap P$ is either an empty set or a d-dimensional polytope.*

Remark 4.5. As $\pi_C : \overline{C} \cap F \longrightarrow Q_0$ is a fibration with contractible fibers, there exists a continuous section $\sigma_C : Q_0 \longrightarrow \overline{C} \cap F$ such that $\pi_C \circ \sigma_C = \mathrm{id}_{Q_0}$.

4.3. **Upper-triangularity.** Let us fix an ordering of the chambers of $\mathrm{bch}^{\ell-1}(\mathcal{A}) = \{C_1, \ldots, C_b\}$ in such a way that

$$\dim X(C_1) \geq \dim X(C_2) \geq \cdots \geq \dim X(C_b).$$

The main result of this section is the following.

Theorem 4.6. *The matrix* $(\deg(C_i, C_j^{\vee}))_{i,j=1,\ldots,b}$ *is upper-triangular. In other words, if* $i > j$, $\deg(C_i, C_j^{\vee}) = 0$.

Proof. Let $C, D \in \mathrm{bch}^{\ell-1}(\mathcal{A})$. Suppose $\dim X(D) \geq \dim X(C)$ and $C \neq D$. Then, we will show that $\deg(C, D^{\vee}) = 0$. The idea of the proof is to construct a vector field $U^{D^{\vee}}$ directed to $D^{\vee}$ on F that is nowhere vanishing on a neighbourhood of $\overline{C} \cap F \subset F$. Then, by Proposition 2.8, we have $\deg(C, D^{\vee}) = 0$.

Let us study the following three cases separately.

(a) $\dim X(C) = \ell - 1$.
(b) $\dim X(C) < \ell - 1$ and D is not inside $\mathrm{Wall}_1(C)$.
(c) $\dim X(C) < \ell - 1$ and D is inside $\mathrm{Wall}_1(C)$.

First, we consider case (a). In this situation, because $\dim X(D) \geq \dim X(C)$, we have $\dim X(D) = \ell - 1$. Choose a point $p \in D \cap F$, and define the vector field U on F by

$$U(x) = \overrightarrow{x;p} \in T_x F.$$

Then, the vector field is directed to p and nowhere vanishing on $\overline{C} \cap F$ (because $p \notin \overline{C}$). By Corollary 2.4, $-U$ is a vector field directed to $D^{\vee}$ that is also nowhere vanishing on $\overline{C} \cap F$. Hence, $\deg(C, D^{\vee}) = 0$.

From this point, we assume $\dim X(C) < \ell - 1$. If D is inside $\mathrm{Wall}_1(C)$, then $X(D) \subset X(C)$ by Remark 4.3, and we have $\overline{\mathcal{A}}_{X(D)} \supset \overline{\mathcal{A}}_{X(C)}$. Proposition 4.2 indicates $\mathrm{Sep}(D, D^{\vee}) \cap \overline{\mathcal{A}}_{X(C)} = \emptyset$. We can conclude that $D^{\vee}$ is also inside $\mathrm{Wall}_1(C)$. Conversely, if D is not inside $\mathrm{Wall}_1(C)$, then $D^{\vee}$ is also not inside $\mathrm{Wall}_1(C)$.

Second, we consider case (b). In this situation, $\mathrm{Sep}(C, D^{\vee}) \cap \mathrm{Wall}_1(C) \neq \emptyset$. Choose a hyperplane $H_{i_0} \in \mathrm{Sep}(C, D^{\vee}) \cap \mathrm{Wall}_1(C)$ and let α_{i_0} be the defining equation of H_{i_0}. Without loss of generality, we may assume that

$$H_{i_0}^{+} = \{\alpha_{i_0} > 0\} \supset D^{\vee}$$
$$H_{i_0}^{-} = \{\alpha_{i_0} < 0\} \supset C.$$

We will construct a vector field $U^{D^{\vee}}$ on F that is directed to $D^{\vee}$ and satisfies

$$U^{D^{\vee}}(x)\alpha_{i_0} > 0 \tag{12}$$

for $x \in \overline{C} \cap F$, where the left hand side of (12) is the derivative of α_{i_0} with respect to the vector field. In particular, we obtain a vector field directed to $D^{\vee}$ that is nowhere vanishing on $\overline{C} \cap F$. It is sufficient to show that, at any point $x_0 \in \overline{C}$, there exists a local vector field around x_0 that satisfies (12). Then, we will obtain a global vector field that satisfies (12) using a partition of unity.

It is sufficient to show the existence of such a vector field around each vertex x_0 of $\overline{C} \cap F$. By the genericity of F, $Z := \bigcap \mathcal{A}_{x_0} = \bigcap_{x_0 \in H \in \mathcal{A}} H$ is a 1-dimensional flat of $\mathcal{A}$, which is transversal to F. By the assumption that F does not separate 0-dimensional flats of $\mathcal{A}$, we have

$$\overline{Z} \cap \overline{H}_{\infty} \subset \overline{C} \cap \overline{H}_{\infty}. \tag{13}$$

(See Figure 7.)

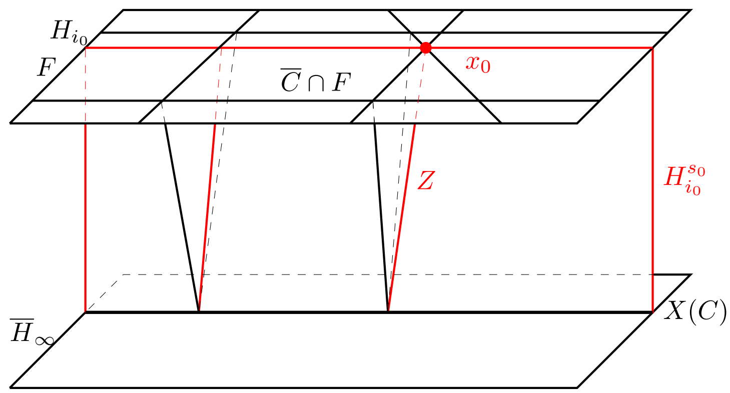

FIGURE 7. Z and $H_{i_0}^{s_0}$.

Set $s_0 := \alpha_{i_0}(x_0)$ and let $H_{i_0}^{s_0} = \{\alpha_{i_0} = s_0\}$ be the hyperplane passing through x_0 that is parallel to H_{i_0}. Then, we have $Z \subset H_{i_0}^{s_0}$, as otherwise we have a contradiction with (13). The hyperplanes $\mathcal{A}_{x_0} = \mathcal{A}_Z$ determine chambers (cones), one of which (denoted by Γ) contains $D^\vee$ (see Figure 8). Hence, the tangent vector $U^{D^\vee}(x_0)$ must be contained in Γ. Furthermore,

$$D \subset \Gamma \cap H_{i_0}^{+} \subset \Gamma \cap H_{i_0}^{>s_0}. \tag{14}$$

In particular, we have $\Gamma \cap H_{i_0}^{>s_0} \neq \emptyset$. Thus, we can construct a vector field $U^{D^\vee}$ around x_0 so that $U^{D^\vee}(x_0) \in \Gamma \cap H_{i_0}^{>s_0}$ and (12) is satisfied around x_0. Hence, we have $\deg(C, D^\vee) = 0$ in case (b).

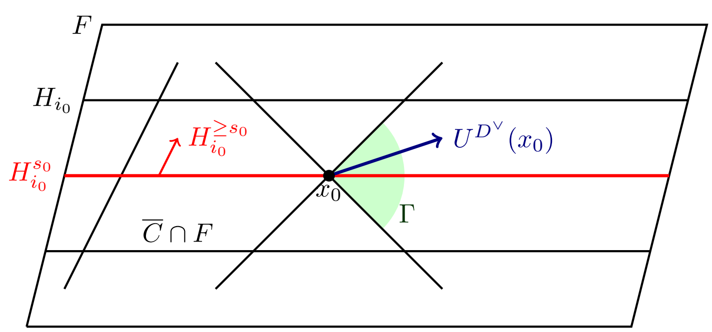

FIGURE 8. Construction of the vector field $U^{D^\vee}$

Third, suppose D is inside $\mathrm{Wall}_1(C)$ (equivalently, $D \subset \widetilde{C}$), and let us handle case (c). As $X(D) \subset X(C)$ and $\dim X(D) \geq \dim X(C)$, we have $X(D) = X(C)$. In this case,

$$\mathrm{Wall}_1(C) = \mathrm{Wall}_1(D) \quad \text{and} \quad \widetilde{C} = \widetilde{D}.$$

We consider the fibration $\pi_D : \overline{D} \cap F \longmapsto Q_0$ that also has d-dimensional polytopes as fibers. Because the fibers are contractible, there exists a continuous section $\sigma_D : Q_0 \longmapsto \overline{D} \cap F$ such that $\pi_D \circ \sigma_D = \mathrm{id}_{Q_0}$.

We now move to the construction of a vector field. For each $p \in \overline{C} \cap F$, we denote the $(\ell - 1 - d)$-dimensional subspace that passes through p and is parallel to $G_2(p_0)$ (see Figure 9).

Let $\{p'\} = G_2(p) \cap G_1(p_0)$. The tangent space decomposes into the direct sum

$$T_pF = T_pG_1(p) \oplus T_pG_2(p).$$

Let us first construct a vector field on the second component. For this, define the tangent vector $V_2(p) \in T_pG_2(p) \subset T_pF$ by

$$V_2(p) = \overrightarrow{p; p'}. \tag{15}$$

The vector field V_2 is obviously inward with respect to $\mathrm{Wall}_1(C)$, and vanishes on the reference fiber $G_1(p_0) \cap \overline{C}$.

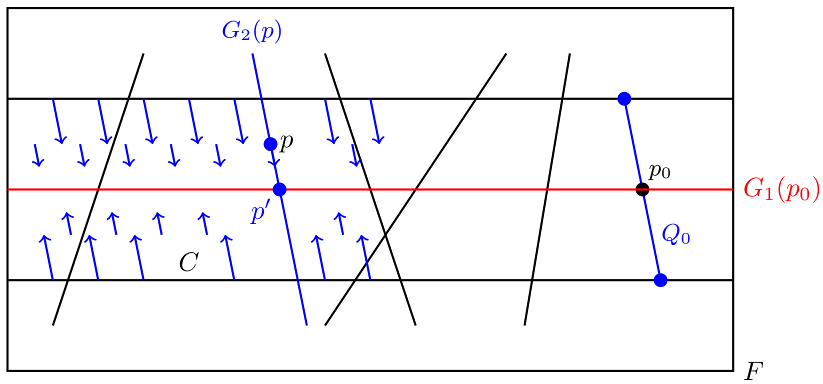

FIGURE 9. V_2.

Let us now construct a vector field V_1 along the fibers $G_1(p)$. Using the section

$$\sigma_D : Q_0 \longrightarrow \overline{C} \cap F \qquad \text{(Remark 4.5)},$$

define V_1 by

$$V_1(p) = \overrightarrow{p; \sigma_D(\pi_C(p))}. \tag{16}$$

(See Figure 10.)

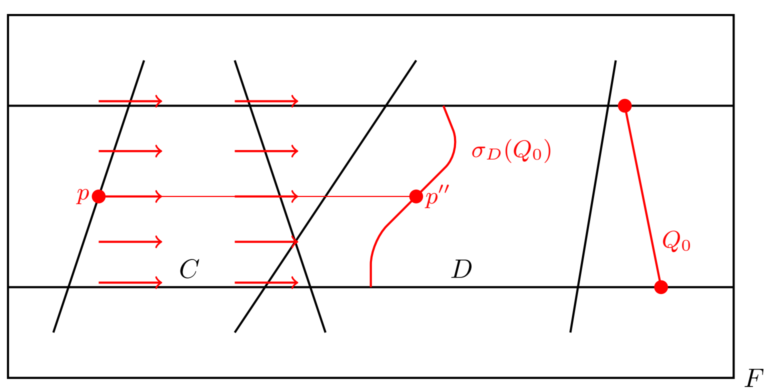

FIGURE 10. V_1, $p'' = \sigma_D(\pi_C(p))$.

Proposition 4.7. *For sufficiently large $t \gg 0$, the vector field tV_1+V_2 is directed to D. Similarly, $-tV_1+V_2$ is a vector field directed to $D^\vee$.*

Proof. Let $p \in H \in \mathrm{Wall}_1(C)$. Recall that D is inside $\mathrm{Wall}_1(C)$. As V_2 is inward and V_1 is tangent to H, the vector field $\pm tV_1+V_2$ is also inward. Let $H \in \mathrm{Wall}_2(C)$ and $p \in H \cap F$. Then, V_1 (resp. $-V_1$) is directed to D (resp. $D^\vee$) with respect to H. Hence, for sufficiently large t, tV_1+V_2 (resp. $-tV_1+V_2$) is directed to D (resp. $D^\vee$). □

Because V_1 is a nowhere vanishing vector field on $\overline{C} \cap F$, $-tV_1+V_2$ is a nowhere vanishing vector field around $\overline{C} \cap F$ that is directed to $D^\vee$. Hence, $\deg(C, D^\vee) = 0$. This completes the proof of Theorem 4.6. □

4.4. **The degree formula.** This section is devoted to a prove the following theorem.

Theorem 4.8. *Let $C \in \mathrm{bch}^{\ell-1}(\mathcal{A})$ and $d = \dim X(C)$. Then,*

$$\deg(C, C^\vee) = (-1)^{\ell-1-d}. \tag{17}$$

We will construct a vector field around $\overline{C} \cap F$ that is directed to $C^\vee$. The vector field V_2 is the same as in the previous section (§4.3). Define the vector field V_1 along the fibers of π_C by

$$V_1(p) = \overrightarrow{p; \sigma_C(\pi_C(p))}. \tag{18}$$

(See Figure 11.)

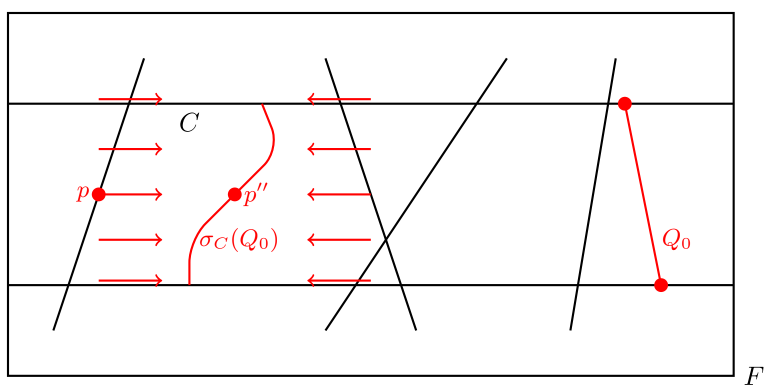

FIGURE 11. V_1, $p'' = \sigma_C(\pi_C(p))$.

Then, tV_1+V_2 is a vector field directed to C (for $t \gg 0$). Since C and $C^\vee$ are separated by $H \in \mathcal{A} \smallsetminus \mathrm{Wall}_1(C)$, the vector field $-tV_1+V_2$ is directed to $C^\vee$. We can compute the degree $\deg(C, C^\vee)$ using the vector field $-tV_1+V_2$. Note that $-tV_1(p)$ is an outward vector field along a d-dimensional space $G_1(p)$, and $V_2(p)$ is an inward vector field that is tangent to an $(\ell-1-d)$-dimensional space $G_2(p)$. Hence, $\deg(C, C^\vee)$ is equal to the index of the following vector field in $\mathbb{R}^{\ell-1}$ at the origin:

$$V = \sum_{i=1}^{d} x_i \frac{\partial}{\partial x_i} - \sum_{i=d+1}^{\ell-1} x_i \frac{\partial}{\partial x_i}. \tag{19}$$

Finally, recall that the de Rham cohomology group $H^{\ell-1}(S^{\ell-2})$ is generated by the differential form ([2])

$$\sum_{i=1}^{\ell-1}(-1)^{i-1}x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{\ell-1}.$$

It is easily seen that the self-map of $H^{\ell-1}(S^{\ell-2})$ induced by the Gauss map of the vector field (19) is equal to multiplication by $(-1)^{\ell-1-d}$. This completes the proof of Theorem 4.8.

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