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STRATIFIED SUBMERSIONS AND CONDITION (D)

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To my friend David Trotman for his 60th birthday

Abstract. In this paper we investigate Goresky's Condition *(D)* for a stratified submersion between two Whitney stratifications. After revisiting the main results on Condition *(D)* of 1976 and 1981 due to Goresky, we give new equivalent properties[1] and two sufficient analytic conditions and their geometric meaning.

1. Introduction.

Let $f : M \to M'$ be a C^1 map between C^1 manifolds and $\mathcal{W} \subseteq M$ and $\mathcal{W}' \subseteq M'$ Whitney stratified sets such that the restriction $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ is a stratified surjective submersion. Condition (D) for $f : M \to M'$ with respect to $\mathcal{W}$ and $\mathcal{W}'$ was originally introduced by M. Goresky in his Ph.D. Thesis (1976) as a convenient technical condition to define the singular substratified objects $\mathcal{W}$ allowed to represent the geometric chains and cochains of a Thom-Mather abstract stratified space $\mathcal{X}$ (**[5]** 2.3 and 4.1) in the aim of introducing nice geometric homology and cohomology theories.

Condition (D) for $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ at $x \in X \subseteq \overline{Y}$ (where $X < Y$ are strata of $\mathcal{W}$, see §2.2 for the definition) roughly speaking means that for every stratum Y of $\mathcal{W}$, the surjective differential map $f_{Y*} : TY \to TY'$ extends to a surjective map (see Remark 3.7) $f_{*x|C_xY} : C_xY \to C_{x'}Y'$ between the *Nash* tangent cones C_xY and $C_{x'}Y'$ (where $C_xY = \sqcup_{\{y_i\}_i \to x} \lim_i T_{y_i}Y$ is analogous in the real case to the Whitney tangent cone $C_4(Y,x)$ **[21]**).

1.1. ***Historical motivations.*** Using an appropriate definition of stratified cycles (Definition 2.4) Goresky proves that every abstract stratified cycle in a manifold is cobordant to one which is radial on M and that, thanks to the condition (D), this last admits a Whitney cellularisation (**[5]** 3.7).

This result is the main step in proving his important theorems on the bijective representability of the homology of a C^1 *manifold* M by its geometric stratified cycles and of the cohomology of an arbitrary Thom-Mather abstract stratified set (**[5]** 2.4 and 4.5).

For a Whitney stratification $\mathcal{X} = (A, \Sigma)$, in 1981 **[6]** Goresky redefines his geometric homology and cohomology theories using only Whitney (that is (b)-regular) substratified cycles and cocycles of $\mathcal{X}$, denoting them in this case $WH_k(\mathcal{X})$ and $WH^k(\mathcal{X})$, without assuming this time the condition (D) in their definition. With these new definitions and replacing the terminology (but essentially not the meaning) *"radial"* by *"with conical singularities"* (**[6]**, Appendices 1, 2, 3) Goresky again proves the bijectivity of his homology and cohomology representation maps:

Theorem 1.1. *If $\mathcal{X} = (M, \{M\})$ is the trivial stratification of a compact C^1 manifold, the homology representation map $R_k : WH_k(\mathcal{X}) \to H_k(M)$ is a bijection.*

Proof. **[6]** Theorem 3.4. □

Key words and phrases. Stratified sets and maps, Whitney Conditions, regular cellularisations.

[1] Used in **[16]** to give a new proof of the (b)-regularity of stratified mapping cylinders needed to Goresky in 1978 to prove a theorem of Whitney cellularisation of Whitney stratifications with conical singularities.

Theorem 1.2. *If $\mathcal{X} = (A, \Sigma)$ is a compact Whitney stratified space, the cohomology representation map $R^k : WH^k(\mathcal{X}) \to H^k(A)$ is a bijection.*

Proof. **[6]** Theorem 4.7. □

Later such geometric theories were improved by the author of the present paper by introducing a sum operation in $WH_k(M)$ and $WH^k(\mathcal{X})$ geometrically meaning transverse union of stratified cycles **[14, 15]**.

1.2. ***Problems related to condition*** (D)**.** Although in the revised theory of 1981 **[6]**, condition (D) was not assumed in the definitions of the Whitney cycles and cocycles, it was once again the main tool to obtain the two important representation theorems, through a strategy of *using Condition (D) in order to construct Whitney cellularisations of Whitney stratifications with conical singularities* using stratified mapping cylinders whose (b)-regularity is obtained through the condition (D) (**[6]**, App. 1,2,3). We give a short survey of this in §2.2.

We underline here that in the homology case the main result, that $R_k : WH_k(\mathcal{X}) \to H_k(M)$ is a bijection, was established *only* when $\mathcal{X} = (M, \{M\})$ is a trivial stratification of a compact manifold M and that the complete homology statement for $\mathcal{X}$ an arbitrary compact (b)-regular stratification remains a famous problem of Goresky which is still unsolved (**[5]** p. 52, **[6]** p. 178):

Conjecture 1.1. *If $\mathcal{X} = (A, \Sigma)$ is a compact Whitney stratified space the homology, representation map $R_k : WH_k(\mathcal{X}) \to H_k(A)$ is a bijection.*

The proof of this conjecture would follow as a corollary if one could prove the following:

Conjecture 1.2. *Every compact Whitney stratified space $\mathcal{X}$ admits a Whitney cellularisation.*

This would be also a first important step of a possible proof of the celebrated conjecture:

Conjecture 1.3. *Every compact Whitney stratified space $\mathcal{X}$ admits a Whitney triangulation.*

Let us recall that in 2005 M. Shiota proved that semi-algebraic sets admit a Whitney triangulation **[18]** and in 2012 M. Czapla gave new proof of this result **[2]** as a corollary of a more general triangulation theorem for definable sets. On the other hand, our motivation being the applications to Goresky's geometric homology theory, we are interested in the stronger Conjectures 1.2 and 1.3 for stratifications having C^1 strata.

In 1978 Goresky also proved an important triangulation theorem for compact Thom-Mather stratified sets **[7]** whose proof (based on a double inductive step) can be used to obtain a Whitney cellularisation of a Whitney stratification provided that one knows how to obtain Whitney stratified mapping cylinders. Goresky used this idea *based on Condition* (D) for Whitney stratifications having *only conical singularities* (see Proposition 2.4) for which he gave a solution of Conjecture 1.2 and deduced as applications the proof of Theorems 1.1 and 1.2.

The strategy of Goresky could be used for an approach to a more general solution of Conjecture 1.2. In this context it is clear that Goresky's condition (D) might play an important role in answering affirmatively Conjecture 1.4 and in solving the famous conjectures 1.1 and 1.3.

1.3. ***Content of the paper.*** In §2.1 we review quickly some basic notions about the most important regular stratifications concerned by this paper: the Whitney (b)-regular stratifications **[21]** and the abstract stratified sets of Thom-Mather **[9, 10, 19]**. Then in §2.2 we introduce the definition of condition (D) for a stratified submersions $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ as a technical tool to obtain (b)-regularity of stratified mapping cylinders and we recall all results of Goresky of 1976-81 **[5, 7]** necessary to prove that: *"Every Whitney stratification with conical singularities* *and conical control data admits a Whitney cellularisation"* (Proposition 2.4) which is a partial solution of Conjecture 1.2.

In §3.1, we analyze what condition (D) means for a C^1 submersion $f : M \to M'$ between C^1 manifolds at a regular point $y_0 \in M$. First we remark that submersivity can be interpreted as the $C^{0,1}$-regularity of the foliation defined by the fibres of f (from Proposition 3.5 to Corollary 3.2).

When $Y \subseteq M$ are riemannian manifolds, we show that the submersivity at $y_0 \in Y$ of the restriction $f_Y : Y \to Y'$ is equivalent to the continuity at y_0 of the *canonical distribution* $\mathcal{D}(y) = \perp(\ker f_{Y*y}, T_yY)$ (Proposition 3.6).

Then we introduce two test functions h_Y and H_Y (Definition 3.5) given by the minimum and the maximum norm of the isomorphism $f_{Y*y|\mathcal{D}(y)} : \mathcal{D}(y) \to T_{y'}Y'$ and its inverse isomorphism $f^{-1}_{Y*y|\mathcal{D}(y)} : T_{y'}Y' \to \mathcal{D}(y)$, such that $\lim_{y\to y_0} h_Y(y)$ and $\lim_{y\to y_0} H_Y(y)$ characterize the submersivity of f_Y at y_0 (Proposition 3.7).

Finally in §2.2, thanks to this, we prove that submersivity at y_0 is also equivalent to the property "$f_{*y_0}(\lim_{y_i\to y_0} \mathcal{D}(y_i)) \supseteq \lim_i f_{*y_i}(\mathcal{D}(y_i))$" and to Condition (D) for f_Y at y_0, interpreted as stratified map defined on the stratification $Y - \{y_0\} \sqcup \{y_0\}$ (Proposition 3.8).

This preliminary analysis of §3 is necessary in introducing the results of §4.

In §4 we give the main results of this paper.

First in §4.1 we investigate the technical, geometric and analytic content of condition (D) at a point $x \in X < Y$ (X, Y being two strata of $\mathcal{W}$) for a general stratified submersion $f : \mathcal{W} \to \mathcal{W}'$ between two Whitney stratifications.

In Theorem 4.3 we prove that, in the context of stratified spaces, condition (D) at $x \in X < Y$ is equivalent to the key property (which is the most important technical content of Condition (D)):

"For every $\{y_i\}_i \subseteq Y$ *such that* $\lim_i y_i = x \in X$*, every* $v' \in \lim_i T_{y_i}Y$ *can be written as a limit* $\lim_i v'_i = v'$ *of a sequence* $\{v'_i \in T_{f(y_i)}f(Y)\}_i$ *having a bounded sequence of preimages* $\{w_i \in f^{-1}_{*y_i}(v'_i) \subseteq T_{y_i}Y\}_i$*"*

and it is again equivalent to the property of transforming "continuously" the limits of the canonical distributions: $f_{*x}(\lim_{y_i\to x} \mathcal{D}(y_i)) \supseteq \lim_{y_i\to x} f_{*y_i}(\mathcal{D}(y_i))$.

The author of the present paper used this properties in **[16]**, when $f_\mathcal{W} = \pi_{XY|\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ is the restriction of a projection $\pi_{XY} : S^\epsilon_{XY} \to X$, to give a different proof of the essential result of Goresky (Proposition 2.2) that *"Stratified mapping cillynders with conical singularities admit a* (b)*-regular natural stratification"*; the property which allow to prove the important Whitney Cellularisation Theorem (Proposition 2.4) recalled above.

In Theorem 4.4 and Corollary 4.3 we prove that the analytic conditions $\liminf_{y\to x} h_Y(y) > 0$ and $\liminf_{y\to x} H_Y(y) < +\infty$ are sufficient for condition (D) at $x \in X < Y$.

In §4.2 for U, V two vector subspaces of an Euclidian vector space E, we use the usual "distance" functions $\delta(u, V)$ and $\delta(U, V)$ ($u \in E$) to define the *essential minimal distance* $\delta'(U, V)$ between U and V, as the sinus of the *minimum essential angle* $\alpha(U, V)$ between two *essential mutual subspaces* U', V' of U and V and we prove some useful properties of $\delta(u, V)$, $\delta(U, V)$ and $\delta'(U, V)$.

In §4.3 using this new "distance" function $\delta'(U, V)$ we introduce two new *geometric* test functions δ_Y (intrinsic by x) and $\delta_{Y,x}$ (depending on x) for Condition (D) at $x \in X < Y$.

In Theorem 4.5 and Corollary 4.4 we prove, when $f : M \to M'$ is a submersion at x, equivalence between the more geometric condition $\liminf_{y\to x} \delta_Y(y) > 0$ and the analytic condition

$\liminf_{y\to x} h_Y(y) > 0$ (or $\limsup_{y\to x} H_Y(y) < +\infty$) and thanks to this that $\liminf_{y\to x} \delta_Y(y) > 0$ becomes a sufficient condition for Condition (D) at $x \in X < Y$ (Corollary 4.5).

After making precise relations between δ_Y and $\delta_{Y,x}$ (Propositions 4.9 and 4.10) we find that the analogous results of Theorem 4.5 and Corollary 4.4 hold by considering the function $\delta_{Y,x}$ instead of δ_Y (Theorem 4.6 and Corollary 4.6).

We conclude the section by explaining (by two examples) the geometric meaning of the sufficient conditions $\liminf_{y\to x} \delta_Y(y) > 0$ and $\liminf_{y\to x} \delta_{Y,x}(y) > 0$.

2. Stratified Spaces and Maps and Condition (D).

A *stratification* of a topological space A is a locally finite partition Σ of A into C^1 connected manifolds (called the *strata* of Σ) satisfying the *frontier condition*: if X and Y are disjoint strata such that X intersects the closure of Y, then X is contained in the closure of Y. We write then $X < Y$ and $\partial Y = \sqcup_{X<Y} X$ so that $\overline{Y} = Y \sqcup \big(\sqcup_{X<Y} X\big) = Y \sqcup \partial Y$ and $\partial Y = \overline{Y} - Y$ ($\sqcup$ = disjoint union). The pair $\mathcal{X} = (A, \Sigma)$ is called a *stratified space* with *support* A and *stratification* Σ.

A *stratified map* $f : \mathcal{X} \to \mathcal{X}'$ between stratified spaces $\mathcal{X} = (A, \Sigma)$ and $\mathcal{X}' = (B, \Sigma')$ is a continuous map $f : A \to B$ which sends each stratum X of $\mathcal{X}$ into a unique stratum X' of $\mathcal{X}'$, such that the restriction $f_X : X \to X'$ is C^1.

A *stratified submersion* is a stratified map f such that each $f_X : X \to X'$ is a C^1 submersion.

2.1. ***Regular Stratified Spaces and Maps.*** Extra regularity conditions may be imposed on the stratification Σ, such as to be an *abstract stratified set* in the sense of Thom-Mather **[9, 10, 19]** or, when A is a subset of a C^1 manifold, to satisfy conditions (a) or (b) of Whitney **[21]**, or (c) of K. Bekka **[1]** or, when A is a subset of a C^2 manifold, to satisfy conditions (w) of Kuo-Verdier **[22]**, or (L) of Mostowski **[17]**.

In this paper we will consider essentially Whitney ((b)-regular) stratifications so called because they satisfy Condition (b) of Whitney (1965, **[21]**).

Definition 2.1. Let Σ be a stratification of a subset $A \subseteq \mathbb{R}^N$, $X < Y$ strata of Σ and $x \in X$.

One says that $X < Y$ *is* (b)*-regular* (or that it satisfies *Condition* (b) *of Whitney*) at x if for every pair of sequences $\{y_i\}_i \subseteq Y$ and $\{x_i\}_i \subseteq X$ such that $\lim_i y_i = x \in X$ and $\lim_i x_i = x$ and moreover $\lim_i T_{y_i}Y = \tau$ and $\lim_i [y_i - x_i] = L$ in the appropriate Grassmann manifolds (here $[v]$ denotes the vector space spanned by v) then $L \subseteq \tau$.

The pair $X < Y$ *is called* (b)*-regular* if it is (b)-regular at every $x \in X$.

Σ is called a (b)*-regular* (or a *Whitney)* stratification if all $X < Y$ in Σ are (b)-regular.

Most important properties of Whitney stratifications follow because they are in particular abstract stratified sets **[9, 10]**.

Definition 2.2. (Thom-Mather 1970) Let $\mathcal{X} = (A, \Sigma)$ be a stratified space.

A family $\mathcal{F} = \{(\pi_X, \rho_X) : T_X \to X \times [0, \infty[)\}_{X\in\Sigma}$ is called *a system of control data* of $\mathcal{X}$ if for each stratum $X \in \Sigma$ we have that:

(1) T_X is a neighbourhood of X in A (called *tubular neighbourhood of* X);
(2) $\pi_X : T_X \to X$ is a continuous retraction of T_X onto X (called *projection on* X);
(3) $\rho_X : T_X \to [0, \infty[$ is a continuous function such that $X = \rho_X^{-1}(0)$ (called the *distance from* X);

and, furthermore, for every pair of adjacent strata $X < Y$, by considering the restriction maps $\pi_{XY} := \pi_{X|T_{XY}}$ and $\rho_{XY} := \rho_{X|T_{XY}}$, on the subset $T_{XY} := T_X \cap Y$, we have that:

5) the map $(\pi_{XY}, \rho_{XY}) : T_{XY} \to X \times]0, \infty[$ is a C^1 submersion (then $\dim X < \dim Y$);
6) for every stratum Z of $\mathcal{X}$ such that $Z > Y > X$ and for every $z \in T_{YZ} \cap T_{XZ}$ the following *control conditions* are satisfied:
i) $\pi_{XY}\pi_{YZ}(z) = \pi_{XZ}(z)$ (called the *π-control condition*)
ii) $\rho_{XY}\pi_{YZ}(z) = \rho_{XZ}(z)$ (called the *ρ-control condition*).

In what follows for every $\epsilon > 0$ we will pose $T_X^\epsilon := T_X(\epsilon) = \rho_X^{-1}([0, \epsilon[)$, $S_X^\epsilon := S_X(\epsilon) = \rho_X^{-1}(\epsilon)$, and $T_{XY}^\epsilon := T_X^\epsilon \cap Y$, $S_{XY}^\epsilon := S_X^\epsilon \cap Y$ and without loss of generality will assume $T_X = T_X(1)$ **[9,10]**.

The pair $(\mathcal{X}, \mathcal{F})$ is called an *abstract stratified set (ASS)* if A is Hausdorff, locally compact and admits a countable basis for its topology. Since one usually works with a unique system of control data $\mathcal{F}$ of $\mathcal{X}$, in what follows we will omit $\mathcal{F}$.

If $\mathcal{X}$ is an abstract stratified set, then A is metrizable and the tubular neighbourhoods $\{T_X\}_{X\in\Sigma}$ may (and will always) be chosen such that: "$T_{XY} \neq \emptyset \Leftrightarrow X \leq Y$" and

$$\text{"}T_X \cap T_Y \neq \emptyset \Leftrightarrow X \leq Y \text{ or } X \geq Y\text{"}$$

(where both implications $\Leftarrow$ automatically hold for each $\{T_X\}_X$) as in **[9, 10]**, pp. 41-46.

The notion of system of control data of $\mathcal{X}$, introduced by Mather, is very important because it allows one to obtain good extensions of (stratified) vector fields **[9, 10]** which are the fundamental tool in showing that a stratified (controlled) submersion $f : \mathcal{X} \to M$ into a manifold, satisfies Thom's First Isotopy Theorem: the stratified version of Ehresmann's fibration theorem **[3,9,10,19]**.

Moreover by applying it to the projections $\pi_X : T_X \to X$ it follows in particular that $\mathcal{X}$ has a *locally trivial structure* and also a locally trivial topologically conical structure.

This fundamental property allows moreover to prove that ASS are triangulable spaces **[7]**.

Since Whitney (b)-regular) stratifications are ASS, they are locally trivial and triangulable.

2.2. ***Condition* (D) *and Goresky's results.*** The following definition was introduced by Goresky first in his Ph.D. Thesis **[5]** (1976) and later in **[6]** (1981).

Definition 2.3. Let $f : M \to M'$ be a C^1 map between C^1 manifolds and $\mathcal{W} \subseteq M$ and $\mathcal{W}' \subseteq M'$ Whitney stratifications such that the restriction $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ is a <u>surjective</u> stratified submersion (so f takes each stratum Y of $\mathcal{W}$ to only one stratum $Y' = f(Y)$ of $\mathcal{W}' = f(\mathcal{W})$).

One says that $f : M \to M'$ *satisfies condition* (D) *with respect to* $\mathcal{W}$ *and* $\mathcal{W}'$ and we will say for short that the restriction $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ *satisfies the condition* (D) if the following holds:

for every pair of adjacent strata $X < Y$ of $\mathcal{W}$ and every point $x \in X$ and every sequence $\{y_i\}_i \subseteq Y$ such that $\lim_i y_i = x \in X$, $\lim_i T_{y_i}Y = \tau$ and $\lim_i T_{f(y_i)}Y' = \tau'$ in the appropriate Grassmann manifolds, then $f_{*x}(\tau) \supseteq \tau'$. Starting from now we will write this for short by:

$$f_{*x}(\lim_i T_{y_i}Y) \ \supseteq \ \lim_i T_{f(y_i)}Y' \, .$$

and we will extend this notation also to some other limits of subspaces of the $\{T_{y_i}Y\}_i$.

Later on we will also consider given, with the obvious restricted meaning of the definition 2.3, what one intends by: *"$f : M \to M'$ satisfies condition (D) with respect to $X < Y$"* and *"$f : M \to M'$ satisfies condition (D) with respect to $X < Y$ at $x \in X$"* (*"at $x \in X < Y$"*).

In the whole of the paper we will denote $Y' = f(Y)$ and $y' = f(y)$, for every $y \in Y$.

Example 2.1. Let M be the horizontal plane $M = \{z = 1\} \subseteq \mathbb{R}^3$, $M' = L(0, 1, 0) = y$-axis in $\mathbb{R}^3$ and $f : M \to M'$ the standard projection $f(x, y, z) = y$.

Let $\mathcal{W} = (W, \Sigma_{\mathcal{W}})$ be the stratified space with support $W = \{y = tan(x) \ : \ x \geq 0\} \cap M$ the half graph of the tangent map in M and stratification $\Sigma_{\mathcal{W}} = \{R, S\}$ where $R = \{(0,0,1)\}$ and $S = W \cap \{x > 0\}$. Then $R < S$.

Let $\mathcal{W}'$ be the stratified space with support the half y-axis, $W' = M' \cap \{y \geq 0\}$ in M' and stratification $\Sigma_{\mathcal{W}'} = \{R', S'\}$ where $R' = \{(0,0,0)\}$ and $S' = M' \cap \{y > 0\}$. Then $R' < S'$.

Then $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ satisfies condition (D) at $(0,0,1) \in R < S$.

If $\mathcal{W} = (W, \Sigma_{\mathcal{W}})$ is as above but taking now for W the half parabola $W = \{y = x^2, x \geq 0\} \cap M$ in M, then $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ does not satisfy condition (D) at $(0,0,1) \in R < S$. $\square$

Figures 1 and 2 below represents both cases of Example 2.1. In figure 1, $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ satisfies condition (D) at $(0,0,1) \in R < S$ while in figure 2 it does not.

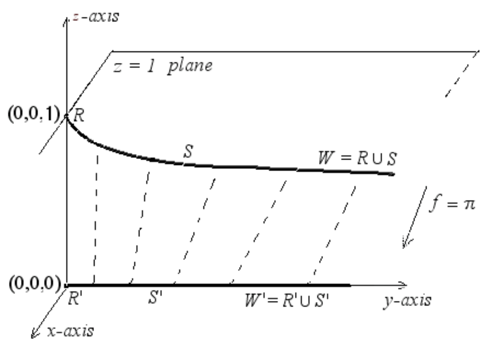

Figure 1

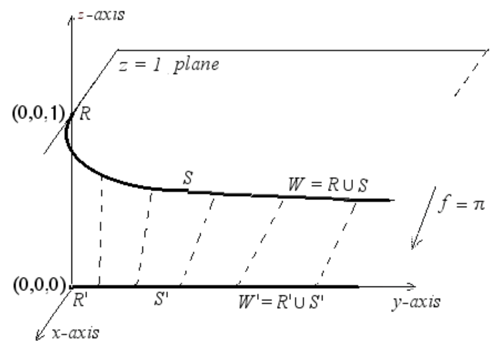

Figure 2

An important example in which condition (D) holds is the case of cellular maps **[5]**, **[16]**:

Proposition 2.1. *Let $f : M \to M'$ be a surjective C^1 submersion and h and h' two smooth cellularisations of two subsets $\mathcal{K} \subseteq M$ and $\mathcal{K}' \subseteq M'$ making the following diagram*

$$\begin{array}{ccc} \mathcal{H} & \stackrel{h}{\to} & \mathcal{K} \subseteq M \\ g \downarrow & & \downarrow f \\ \mathcal{H}' & \stackrel{h'}{\to} & \mathcal{K}' \subseteq M' . \end{array}$$

commutative where $g : \mathcal{H} \to \mathcal{H}'$ is a cellular map of cellular complexes.

Then $f_{\mathcal{K}} : \mathcal{K} \to \mathcal{K}'$ satisfies condition (D). $\square$

In 1976 Goresky used condition (D) to define a convenient class of stratified subspaces $\mathcal{W} \subseteq \mathcal{X}$ of a Thom-Mather ASS $\mathcal{X} = (A, \Sigma)$ equipped with a system of control data

$$\mathcal{F} = \{(\pi_X, \rho_X) : T^1_X \to X \times [0, \infty[\}_{X \in \Sigma}$$

[9, 10] and a *family of lines* of $\mathcal{X}$, $\mathcal{R} = \{r^\epsilon_X : T^1_X - X \to S^\epsilon_X\}_{X \in \Sigma, \epsilon \in]0,\delta[}$ **[7]** retracting every tubular neighbourhood $T^1_X - X$ on its ϵ-sphere S^ϵ_X.

Definition 2.4. (**[5]** 2.3.2). Let $\mathcal{X}$ be a Thom-Mather ASS, equipped with a fixed system of control data $\mathcal{F}$ and a family of lines $\mathcal{R}$ and denote, for every stratum X of $\mathcal{X}$, by C^o_X the open cone operator associated to $\mathcal{R}$, that is: $C^o_X(Q) = {r^\epsilon_X}^{-1}(Q)$ for every $Q \subseteq S^\epsilon_X$.

A Thom-Mather ASS $\mathcal{W} \subseteq \mathcal{X}$ is called a *substratified object of* $\mathcal{X}$ and one says that $\mathcal{W}$ *follows the lines of* $\mathcal{X}$ if the following hold:

(1) Each stratum R of $\mathcal{W}$ is a submanifold of a stratum X of $\mathcal{X}$.
(2) For each stratum X of $\mathcal{X}$, $\mathcal{W} \cap X$ satisfies Whitney's condition (b).
(3) For each stratum X of $\mathcal{X}$, there exists $\epsilon > 0$ such that $\mathcal{W} \cap (T^{\epsilon}_X - X) = C^o_X(\mathcal{W} \cap S^{\epsilon}_X)$.
(4) If X is a stratum of $\mathcal{X}$, there exists $\epsilon > 0$ such that $\pi_{\mathcal{W}\cap S^{\epsilon}_X} : \mathcal{W} \cap S^{\epsilon}_X \to \mathcal{W} \cap X$ is a stratified submersion which satisfies condition (D).

Goresky commented on property 4) above as follows: *"Condition* (D) *is used in section 6.4 to guarantee that certain intersections of substratified objects will be substratified objects. It can be weakened considerably and perhaps omitted completely although this would necessitate considerably more technical analysis when intersections of substratified objects are considered".*

Later in 1981 Goresky redefined his geometric homology $WH_k(\mathcal{X})$ and cohomology $WH^k(\mathcal{X})$ (this time only) for a Whitney stratification $\mathcal{X}$ without asking that the substratified objects representing cycles and cocycles of $\mathcal{X}$ satisfy condition (D) above (**[6]** §3 and §4).

The main reason for which Goresky introduced Condition (D) in 1981 was that it allows one to obtain Condition (b) for the natural stratifications on the mapping cylinder of the stratified submersion:

Proposition 2.2. *Let* $\pi : E \to M'$ *be a* C^1 *riemannian vector bundle and* $M = S^{\epsilon}_{M'}$ *the* ϵ*-sphere bundle of* E*. If* $\mathcal{W} \subseteq M$*,* $\mathcal{W}' = \pi(\mathcal{W}) \subseteq M'$ *are two Whitney stratifications such that* $\pi_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ *is a stratified submersion which satisfies condition* (D)*, then the closed stratified mapping cylinder*

$$C_{\mathcal{W}'}(\mathcal{W}) = \bigsqcup_{Y \subseteq \mathcal{W}} \left[(C_{\pi_{\mathcal{W}}(Y)}(Y) - \pi_{\mathcal{W}}(Y)) \sqcup \pi_{\mathcal{W}}(Y) \sqcup Y \right]$$

is a Whitney (i.e. (b)*-regular) stratified space.*

Proof. **[6]** Appendix A.1 or **[16]** for a different proof. □

Then, in order to use it together with Proposition 2.3 below:

Proposition 2.3. *Every Whitney stratification* $\mathcal{W}$ *in a manifold* M *can be deformed to a Whitney stratification* $\mathcal{W}'$ *having conical singularities.*

Proof. **[6]** Appendix A.3. Proposition. □

Goresky proved that:

Proposition 2.4. *Every Whitney stratified space* $\mathcal{X}$ *with conical singularities and conical control data admits a Whitney cellularisation.*

Proof. **[7]** Appendix A.2. Proposition. □

Proposition 2.4 gives hence a partial solution of Conjecture 1.2 in the introduction and suggests moreover new ideas for an approach to his general solution.

Proposition 2.4 was thus also the main tool which allowed Goresky to prove his two homology representation theorems, Theorem 1.1 and Theorem 1.2, recalled in the introduction.

A detailed account of condition (D), containing a finer analysis, new proofs and equivalent properties of Goresky's results is given in **[16]**.

3. $C^{0,1}$-Regular foliations and condition (D) for C^1 maps.

3.1. ***Regular foliations from C^1 maps.*** In this section we clarify some simple properties of C^1 maps that will be useful in §4.

Remark 3.1. *Let $f : M \to M'$ be a C^1 map between C^1 manifolds, $y_0 \in M$ and $\{y_i\}_i \subseteq M$ a sequence such that $\lim_i y_i = y_0$.*

1) For every sequence of vectors $\{v_i \in \ker f_{*y_i}\}_i$ such that $\lim_i v_i = v_0$ one has $v_0 \in \ker f_{*y_0}$.

2) If, in an appropriate Grassmann manifold, there exists

$$\lim_i \ker f_{*y_i} = \tau,$$

then $\tau \subseteq \ker f_{*y_0}$ (starting from now we will write this for short by: "$\lim_i \ker f_{*y_i} \subseteq \ker f_{*y_0}$").

Proof. Since f is C^1 one obviously has: $f_{*y_0}(v_0) = f_{*y_0}(\lim_i v_i) = \lim_i f_{*y_i}(v_i) = 0$. □

The opposite inclusion $\lim_i \ker f_{*y_i} \supseteq \ker f_{*y_0}$ would follow immediately when two such vector spaces have the same dimension. This happens when f is a submersion:

Proposition 3.5. *Let $f : M \to M'$ be a C^1 submersion on $M - \{y_0\}$ for a point $y_0 \in M$.*

Then the following conditions are equivalent:

1) $f : M \to M'$ is a submersion at y_0;

2) For all $\{y_i\}_i \subseteq M - \{y_0\}$ converging to y_0 there exists $\lim_i \ker f_{*y_i}$ and

$$\lim_i \ker f_{*y_i} = \ker f_{*y_0}.$$

This means that the map $\mathcal{K} : M \longrightarrow \mathbb{G}_k(TM)$, $\mathcal{K}(y) := \ker f_{*y}$ is continuous.

3) For all $\{y_i\}_i \subseteq M - \{y_0\}$ converging to y_0 there exists $\lim_i \ker f_{*y_i}$ and

$$\lim_i \ker f_{*y_i} \supseteq \ker f_{*y_0}.$$

Proof. Since f is a C^1 submersion at $M - \{y_0\}$, for every $y_i \in M - \{y_0\}$, if $y'_i = f(y_i)$, the fibre $f^{-1}(y'_i)$ is a C^1 manifold of dimension $k = \dim M - \dim M'$ such that $T_{y_i} f^{-1}(y'_i) = \ker f_{*y_i}$.

In particular, for every $i \in \mathbb{N}$, $\dim \ker f_{*y_i} = k$.

$(1 \Rightarrow 2)$. Let $\{\ker f_{*y_{i_h}}\}_h$ an arbitrary converging subsequence of the sequence $\{\ker f_{*y_i}\}_i$.

If f is a submersion at y_0, then $f^{-1}(y'_0)$ is a C^1 k-manifold too with tangent spaces

$$T_{y_0} f^{-1}(y'_0) = \ker f_{y_0 *}$$

and $\dim \ker f_{*y_0} = k = \dim \lim_h \ker f_{*y_{i_h}}$.

Since f is a C^1 map, $\lim_h \ker f_{*y_{i_h}} \subseteq \ker f_{*y_0}$ (Remark 3.1) and having both the same dimension k they coincide: $\lim_h \ker f_{*y_{i_h}} = \ker f_{*y_0}$.

All converging subsequences of the sequence $\{\ker f_{*y_i}\}_i$ have then the same limit $\ker f_{*y_0}$ in the Grassmann compact manifold and hence there exists $\lim_i \ker f_{*y_i}$ and

$$\lim_i \ker f_{*y_i} = \ker f_{*y_0}.$$

$(2 \Rightarrow 3)$. Obvious.

$(3 \Rightarrow 1)$. If $\lim_i \ker f_{*y_i} \supseteq \ker f_{*y_0}$, then, for every i, $\dim \ker f_{*y_0} \leq \dim \ker f_{*y_i}$ and by codimension $\dim Im\, f_{*y_0} \geq \dim Im\, f_{*y_i}$. Thus again f being a submersion at y_i one has:

$$\dim Im\, f_{*y_0} \geq \dim Im\, f_{*y_i} = \dim T_{y'_i} M' = \dim T_{y'_0} M'$$

and, since $Im\, f_{*y_0} \subseteq T_{y'_0} M'$, then necessarily $Im\, f_{*y_0} = T_{y'_0} M'$ and f is a submersion at y_0. □

With the same hypotheses and proof of the proposition 3.5 one has:

Remark 3.2. *The following conditions are equivalent:*
1) $f : M \to M'$ *is a submersion at* y_0*;*
2) For all $\{y_i\}_i \subseteq M - \{y_0\}$ *converging to* y_0 *there exists* $\lim_i \ker f_{*y_i}$ *and*

$$\dim \lim_i \ker\, f_{*y_i} = \dim \ker\, f_{*y_0};$$

3) For all $\{y_i\}_i \subseteq M - \{y_0\}$ *converging to* y_0 *there exists* $\lim_i \ker f_{*y_i}$ *and*

$$\dim \lim_i \ker\, f_{*y_i} \geq \dim \ker\, f_{*y_0}. \;\square$$

Corollary 3.1. *If* $f : M \to M'$ *is a* C^1*-submersion, the foliation of* M *defined by* $\mathcal{F} = \{M_y = f^{-1}(y')\}_{y \in M}$*, where* $y' = f(y)$*, is* $C^{0,1}$*-regular. I.e. for every sequence* $\{y_i\}_i \subseteq M$

$$\lim_i y_i = y_0 \Longrightarrow \lim_i T_{y_i} M_{y_i} = T_{y_0} M_{y_0}.$$

Proof. Since f is a C^1 submersion on M, for every $y_i \in M$, $f^{-1}(y_i')$ is a C^1 manifold of dimension $k = \dim M - \dim M'$ and $T_{y_i} f^{-1}(y') = \ker\, f_{*y_i}$. Then, by Proposition 3.5:

$$\lim_i T_{y_i} M_{y_i} \;=\; \lim_i \ker\, f_{*y_i} \;=\; \ker\, f_{y_0 *} \;=\; T_{y_0} M_{y_0}. \quad \square$$

Corollary 3.2. *Let* $f : M \to M'$ *be a* C^1 *map and* $\mathcal{F}' = \{M_i'\}_i$ *an* $C^{0,1}$*-regular foliation of* M' *whose leaves are transverse to* f *and such that there exists a submanifold* V *of* M' *of dimension* $h = \dim M' - \dim \mathcal{F}'$ *transverse to each leaf of* $\mathcal{F}'$ *and intersecting it in a singleton* $V \cap M_i' = \{y_i'\}$*.*
Then the foliation of M *defined by* $\mathcal{F} = \{M_i = f^{-1}(M_{i'}')\}_i$ *is* $C^{0,1}$*-regular.*

Proof. Let us consider the submersion $g : M' \to V$ defined for every $y' \in M'$, by

$$g_{|M_i'} = \text{ constant } = y_i'.$$

Thus g defines the foliation $\mathcal{F}' = \{M_{y'}'\}_{y' \in M'}$ via preimage.
Then the foliation $\mathcal{F} = \{M_i\}_i$ of M is defined by the C^1 submersion $g \circ f : M \to V$. $\square$

Starting from now we will suppose $M = M^n$ to be a riemannian manifold of dimension n.

For a C^1 map $f : M \to M'$ let us consider the distribution of vector subspaces $\mathcal{D}(y) := \mathcal{D}_f(y)$ obtained by splitting every $T_y M$ as the direct orthogonal sum:

$$T_y M \;=\; \mathcal{D}(y) \;\oplus\; \ker\, f_{*y} \qquad \text{where} \qquad \mathcal{D}(y) := \perp (\ker\, f_{*y}, T_y M)\,.$$

We call $\mathcal{D} : M \to \mathbb{G}_{n-k}(TM)$, $\{\mathcal{D}(y) = \perp (\ker\, f_{*y}, T_y M)\}_y$ the *canonical distributions* of f.

We will see that the study of the condition (D) for a submersive restriction $f_Y : Y \to Y'$ ($Y \subseteq M$ and $Y' \subseteq M'$) at a point x in the adherence $\overline{Y}$ of Y is strongly related to good properties of limits of the distribution

$$\mathcal{D}(y) \;=\; \mathcal{D}_{f_Y}(y) \;:= \perp (\ker\, f_{Y*y}, T_y Y).$$

When $f_Y = \pi_{XY|} : S_{XY}^{\epsilon} \to X$ is the restriction of a projection $\pi_{XY} : T_{XY} \to X$ on a stratum $X < Y$, of a system of control data $\{(T_X, \pi_X, \rho_X)\}_X$ of a regular stratification, then $\mathcal{D}_f(y)$ is defined in the same way as the *canonical distribution* $\mathcal{D}_X(y)$ *relative to the stratum* X introduced in [**11, 12, 13**]. In this case, if $\mathcal{W}$ and $\mathcal{W}'$ are Whitney refinements of S_{XY}^{ϵ} and X, Condition (D) implies the (a)*-regularity* (see [**13**]) of a "horizontal" foliation related to $\mathcal{D}_X$ in a particular stratified mapping cylinder $C_{\mathcal{W}'}(\mathcal{W})$ [**16**] (from Lemma 3.1 to Theorem 3.4).

Lemma 3.1. *Let $V \subseteq U$ be two vector subspaces of $\mathbb{R}^n$.*

If $\{V_i\}_i$ and $\{U_i\}_i$ are two sequences of vector subspaces of $\mathbb{R}^n$ with $V_i \subseteq U_i$, $l = \dim V_i$, $k = \dim U_i$ for every i and such that $\lim_i U_i = U$ in $\mathbb{G}_k^n$, then

$$\lim_i V_i \ = \ V \ \text{in} \ \mathbb{G}_l^n \quad \Longleftrightarrow \quad \lim_i \perp (V_i, U_i) \ = \perp (V, U) \ \text{ in } \ \mathbb{G}_{k-l}^n \ .$$

Proof. ($\Rightarrow$). Let us denote $\mathcal{D}_i \ = \perp (V_i, U_i)$ and $\mathcal{D} \ = \perp (V, U)$ and show that $\lim_i \mathcal{D}_i \ = \ \mathcal{D}$.
Since $\dim V_i = l$ and $\dim U_i = k$ then $\dim \mathcal{D}_i = k - l$ for every i.
Since $U \ = \ \lim_i U_i \in \mathbb{G}_k^n$ and $V \ = \ \lim_i V_i$, then $\dim U = k$, $\dim V = l$ and $\dim \mathcal{D} \ = \ k - l$.

Let $\{\mathcal{D}_{i_h}\}_h$ be an arbitrary convergent subsequence of $\{\mathcal{D}_i\}_i$ and $\mathcal{D}' \ = \ \lim_h \mathcal{D}_{i_h}$.
Every vector $w \in \mathcal{D}' = \lim_h \mathcal{D}_{i_h}$ is a limit $w = \lim_h w_{i_h}$ of a sequence of vectors $\{w_{i_h} \in \mathcal{D}_{i_h}\}_h$ so that $< w_{i_h}, v_{i_h} > = \ 0$ for every vector $v_{i_h} \in V_{i_h}$.

On the other hand $V = \lim_i V_i = \lim_h V_{i_h}$, so every vector $v \in V$ is also a limit $v = \lim_h v_{i_h}$ of a sequence of vectors $\{v_{i_h} \in V_{i_h}\}_h$ and we have $< w, v > = \ \lim_h < w_{i_h}, v_{i_h} > = \ 0$ so that $w \in \perp (V, U) = \mathcal{D}'$. Hence $\mathcal{D}' \subseteq \mathcal{D}$ and, since they have the same dimension, $\mathcal{D}' = \mathcal{D}$.

Therefore every convergent subsequence $\{\mathcal{D}_{i_h}\}_h$ of $\{\mathcal{D}_i\}_i$ has limit $\mathcal{D}$ and so $\lim_i \mathcal{D}_i = \mathcal{D}$.

The proof of ($\Leftarrow$) follows from ($\Rightarrow$) because $V_i = \perp (\mathcal{D}_i, U_i)$ and $V = \perp (\mathcal{D}, U)$. $\square$

Proposition 3.6 below anticipates some arguments that will appear in §4.

Proposition 3.6. *Let M^n be a riemannian manifold and $f : M \to M'$ a C^1 submersion on $M - \{y_0\}$ with $y_0 \in M$. Then the following conditions are equivalent:*

1) $f : M \to M'$ is a submersion at y_0;

2) For all $\{y_i\}_i \subseteq M - \{y_0\}$ converging to y_0 there exists $\lim_i \mathcal{D}(y_i)$ and

$$\lim_i \mathcal{D}(y_i) = \perp (\ker f_{*y_0}, T_{y_0} M).$$

I. e.: the map $\mathcal{D} : M \to \mathbb{G}_{n-k}(TM)$, $\mathcal{D}(y) := \perp (\ker f_{*y}, T_y M)$ is continuous;

3) For all $\{y_i\}_i \subseteq M - \{y_0\}$ converging to y_0 there exists $\lim_i \mathcal{D}(y_i)$ and

$$\lim_i \mathcal{D}(y_i) \subseteq \perp (\ker f_{*y_0}, T_{y_0} M).$$

Proof. It follows immediately from Proposition 3.5 and the previous Lemma 3.1. $\square$

Definition 3.5. below will play an important role in the next section.

Definition 3.5. Let $f : M \to M'$ be a C^1 map of riemannian manifolds, $Y \subseteq M$, $Y' \subseteq M'$ two C^1-submanifolds whose restriction $f_Y : Y \to Y'$ is a C^1 surjective submersion; so $Y' = f(Y)$, $T_{y'} Y' = T_{f(y)} f(Y)$, $y' = f(y)$ for all y, and we will assume such notations in the whole of the paper.

Let $x \in \overline{Y} \subseteq M$ (*a priori* x could lie or not in Y) and $x' = f(x)$.

For every point $y \in Y$, let $\mathcal{D}(y) = \perp (\ker f_{Y*y}, T_y Y)$ be the canonical distribution of f_Y.
The restricted differential map:

$$f_{Y*y|\mathcal{D}(y)} \ : \ \mathcal{D}(y) \ \longrightarrow \ T_{y'} Y'$$

is then an isomorphism and for every unit vector $u \in \mathcal{D}(y)$, one has $f_{Y*y}(u) \neq 0$, so that by compactness of each unit sphere of $\mathcal{D}(y)$ one can define the continuous map h_Y:

$$h_Y : Y - \{x\} \to \ ]0, +\infty[\quad , \quad h_Y(y) = \min \big\{ || \ f_{Y*y|\mathcal{D}(y)}(u) \ || \ \ : \ \ || \ u \ || \ = 1 \big\}.$$

Similarly, by considering the inverse map $f^{-1}_{Y*y|\mathcal{D}(y)} \ : \ T_{y'} Y' \to \mathcal{D}(y)$, every vector $v' \in T_y Y'$ has a unique (pre)image $v = f^{-1}_{Y*y|\mathcal{D}(y)}(v')$ such that $v \in \mathcal{D}(y)$ and $f_{Y*y}(v) = v'$.

We call such a vector $v = f^{-1}_{Y*y|\mathcal{D}(y)}(v')$ *the canonical lifting of* v'*:*
it is the unique vector $v \in T_yY$ such that $f_{Y*y}(v) = v'$ and having no component along ker f_{Y*y}.

Of course $v' \neq 0$ if and only if its lift $v \neq 0$.

So, starting from now, every vector that we will lift, will always be supposed $\neq 0$.
We will understand this also in many statements of §4 without say it explicitely every time.

We can then define the *dual* continuous map H_Y:

$$H_Y : Y - \{x\} \to \,]0, +\infty[\quad , \quad H_Y(y) = \max\left\{ || \, f^{-1}_{Y*y|\mathcal{D}(y)}(v') \, || \; : \; || \, v' \, || \; = 1 \right\}.$$

I.e. $H_Y(y)$ is the classical norm of the linear isomorphism $f^{-1}_{Y*y|\mathcal{D}(y)} \; : \; T_{y'}Y' \to \mathcal{D}(y)$.

Remark 3.3. *For every* $y \in Y$ *and every vector* $v' \in T_{y'}Y' - \{0\}$ *we have:*

1) The unit vector $u = \frac{v}{|| \, v \, ||}$ *of the canonical lifting* $v := f^{-1}_{Y*y|\mathcal{D}(y)}(v') \in \mathcal{D}(y)$ *of* $v' \in T_{y'}Y'$ *satisfies:*

$$|| \, v \, || \; = \; \frac{|| \, v' \, ||}{|| \, f_{Y*y|\mathcal{D}(y)}(u)||}.$$

2) If $|| \, v' \, || = 1$ *then:* $|| \, v \, || \; = \; \frac{1}{|| \, f_{Y*y|\mathcal{D}(y)}(u)||}$.

3) $H_Y(y) = \dfrac{1}{h_Y(y)}$.

Proof. For 1) one easily finds:

$$|| \, v' \, || \; = || \, f_{Y*y}(v) \, || \; = || \, f_{Y*y}(\tfrac{v}{|| \, v \, ||}) \, || \; \cdot || \, v \, || \; = || \, f_{Y*y|\mathcal{D}(y)}(u) \, || \; \cdot || \, v \, ||$$

which also obviously implies 2), while 3) follows by 2) thanks to:

$$H_Y(y) = \sup_{|| \, v' \, || = 1} \left\{ \, || \, v \, || \; : \; v' \in T_{y'}Y' \; \right\} = \sup_{|| \, u \, || = 1} \left\{ \; \frac{1}{|| \, f_{Y*y|\mathcal{D}(y)}(u)||} \; : \, u \in \mathcal{D}(y) \, \right\} =$$

$$\frac{1}{\inf_{|| \, u \, || = 1} \left\{ || \, f_{Y*y|\mathcal{D}(y)}(u)|| \; : \; u \in \mathcal{D}(y) \right\}} = \frac{1}{h_Y(y)}. \qquad \square$$

Being interested in the properties of the maps h_Y and H_Y at a regular point we will suppose in Proposition 3.7 below that $Y \cup \{x\} = M$, and we will denote $y_0 = x$, $h = h_Y$ and $H = H_Y$.

Proposition 3.7. *Let* $f : M \to M'$ *be a* C^1 *map, submersion on* $M - \{y_0\}$ *with* $y_0 \in M$*.*

The following conditions are equivalent:

1) $f : M \to M'$ *is a submersion at* y_0*;*

2) There exists $\lim_{y \to y_0} h(y) > 0$*;*

3) There exists $\lim_{y \to y_0} H(y) < +\infty$*.*

Proof. 1) $\Rightarrow$ 2). If y_0 is a regular point of M, and f is a submersion at y_0 then Definition 3.5 of the continuous map h extends naturally to y_0 giving $\lim_{y \to y_0} h(y) = h(y_0) \in \,]0, +\infty[$.

2) $\Rightarrow$ 3). It follows obviously by Remark 3.3.

3) $\Rightarrow$ 1). Let us fix a unit vector $v' \in T_{y'_0}M'$.

By hypothesis for every sequence $\{y_i\}_i \subseteq M$ such that $\lim_i y_i = y_0$ one has $\lim_i H(y_i) < +\infty$.

Given then a sequence of *unit* vectors $\{v'_i \in T_{y'_i}M'\}_i$ such that $\lim_i v'_i = v'$, the sequence of canonical lifts $\{v_i := f^{-1}_{*y_i|\mathcal{D}(y_i)}(v'_i) \in \mathcal{D}(y_i)\}_i$, is bounded: $\sup_i \; ||v_i|| \; \leq \; \sup_i \; H(y_i) < +\infty$.

There exists thus a subsequence $\{v_{ih}\}_h$ converging to a vector $v = \lim_h v_{ih} \in T_{y_0}M$ and $f : M \to M'$ being C^1 at y_0 one finds:

$$f_{*y_0}(v) = f_{*y_0}(\lim_h v_{ih}) = \lim_h f_{*y_{i_h}}(v_{i_h}) = \lim_h v'_{ih} = v' .$$

Therefore $f_{*y_0} : T_{y_0}M \to T_{y'_0}M'$ is surjective and f is a submersion at y_0. □

3.2. ***Condition*** (D) ***at a regular point.*** Let us recall now the definition of the condition (D) for $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ at $x \in X < Y$.

Let $f : M \to M'$ be a C^1 map between C^1 manifolds, $\mathcal{W} \subseteq M$ and $\mathcal{W}' \subseteq M'$ Whitney stratifications and suppose that the restriction $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ is a stratified (stratum for stratum) surjective submersion satisfying condition (D) at $x \in X < Y$.

This means that for every sequence $\{y_i\}_i \subseteq Y$ such that $\lim_i y_i = x \in X$ one has:

$$\exists \lim_i T_{y_i}Y = \tau \quad \text{and} \quad \exists \lim_i T_{y'_i}Y' = \tau' \quad \Longrightarrow \quad f_{*x}(\tau) \supseteq \tau'$$

where $Y' = f(Y)$ and $y' = f(y)$ for every $y \in Y$.

Remark 3.4. *The C^1 smoothness of f on M does not suffice to imply the inclusion $f_{*x}(\tau) \supseteq \tau'$ which as one sees with easy examples is false in general (see Example 2.1).* □

We will show in the next section (Theorem 4.3) that it depends on the possibility of extracting a bounded sequence of vector preimages v_i, one in each fibre $f^{-1}_{*y_i}(v'_i)$ with $\lim_i v'_i \in \tau'$.

We will see moreover that the whole complexity of the condition (D) at x is contained in the behaviour near x of the maps h_Y and/or H_Y.

Remark 3.5. *Condition (D) for $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ at $x \in X < Y$ does not depend on the stratum X containing x: to formulate it, one must consider a map f defined on a C^1 manifold M containing Y and $x \in \overline{Y}$ and which is C^1 on M.* □

Remark 3.6. *With the same hypotheses and notations as above we have:*

i) Since $f : M \to M'$ is C^1 the opposite inclusion $f_{*x}(\tau) \subseteq \tau'$ is always satisfied.

ii) $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ being a stratified submersion, $T_{y'_i}Y' = f_{*y_i}(T_{y_i}Y)$ for every i.

Proof i). If $v \in \tau$ we can write $v = \lim_i v_i$ for a sequence $\{v_i \in T_{y_i}Y\}_i$, hence:

$$f_{*x}(v) = f_{*x}(\lim_i v_i) = \lim_i f_{*y_i}(v_i) \in \lim_i f_{*y_i}(T_{y_i}Y) = \tau' \quad \text{and so:} \quad f_{*x}(\tau) \subseteq \tau' . \quad \square$$

Since $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ is the restriction of a C^1 map $f : M \to M'$ between two manifolds, there exists a differential map $f_{*x} : T_xM \to T_{x'}M'$ and a unique possible way to define the restriction $f_{*x|C_xY}$ to the tangent cone (the Nash fiber) $C_xY := \bigsqcup_{\tau = \lim_i T_{y_i}Y} \tau$ of Y at x.

Condition (D) implies moreover that the "restriction" $f_{*x|C_xY} : C_xY \to C_{x'}Y'$ *must be surjective.* This is the most natural generalisation at a singular point of the submersivity:

Remark 3.7. *If $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ satisfies condition (D) at $x \in X < Y$, then*

i) $f_{*x}(\tau) = \tau'$;

ii) The surjective differential map $f_{Y*} : TY \to TY'$ of the restriction $f_Y : Y \to Y'$ extends surjectively to the union of linear maps:

$$f_{Y*x|C_xY} = \bigsqcup_{\tau = \lim_i T_{y_i}Y} f_{*x|\tau} \ : \ C_xY = \bigsqcup_{\tau = \lim_i T_{y_i}Y} \tau \longrightarrow C_{x'}Y' = \bigsqcup_{\tau' = \lim_i T_{y'_i}Y'} \tau'$$

between the tangent cones C_xY and $C_{x'}Y'$. □

Condition (D) for $f_{\mathcal{W}}$ also morally means that the differential maps $f_{Y*y} : T_yY \to T_{y'}Y'$ have to be surjective including all possible limit maps $\lim_{y_i \to x} f_{Y*y_i} : T_{y_i}Y \to T_{y'_i}Y'$: a kind of *"super-submersivity"* defined in the same spirit as Goresky's *super-transversality* **[5]**.

Look now at what condition (D) "means" at a regular point $y_0 \in Y$.

Let $f : M \to M'$ a C^1 map on a riemannian C^1 manifold M and $Y \subseteq M$ a submanifold.

If the restriction $f_Y : Y \to Y'$ is a surjective submersion out of a point $y_0 \in Y$, then condition (D) for f_Y at y_0 can be naturally defined as condition (D) for $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ by considering for $\mathcal{W}$ and $\mathcal{W}'$ the Whitney stratifications $\mathcal{W} = (Y - \{y_0\}) \sqcup \{y_0\}$ and $\mathcal{W}' = (Y' - \{y'_0\}) \sqcup \{y'_0\}$ with $y'_0 = f(y_0)$ (we also include the possibility $Y = M$).

With such an extended meaning we have :

Proposition 3.8. *Let $f_Y : Y \to Y' = f(Y)$ be a surjective C^1 map and $y_0 \in Y$ such that f_Y is a submersion at every point of $Y - \{y_0\}$. Then the following conditions are equivalent:*

1) $f_Y : Y \to Y'$ is a submersion at y_0;

2) $\lim_i y_i = y_0$ and $\exists \lim_i \mathcal{D}(y_i) \Longrightarrow f_{Y*y_0}\big(\lim_i \mathcal{D}(y_i)\big) \supseteq \lim_i f_{Y*y_i}\big(\mathcal{D}(y_i)\big)$;

3) f_Y satisfies the condition (D) at y_0.

Proof. Since Y and Y' are C^1 manifolds, for every sequence $\{y_i\}_i \subseteq Y - \{y_0\}$ such that $\lim_i y_i = y_0$, we automatically have that both limits exist:

$$\tau = \lim_i T_{y_i}Y_0 = \lim_i T_{y_i}Y = T_{y_0}Y \quad \text{and} \quad \tau' = \lim_i T_{y'_i}Y'_0 = \lim_i T_{y'_i}Y' = T_{y'_0}Y' .$$

Moreover, f_Y being a submersion at every $y_i \in Y - \{y_0\}$, by decomposing $T_{y_i}Y$ in the orthogonal direct sum: $T_{y_i}Y = \mathcal{D}(y_i) \oplus \ker f_{Y*y_i}$, with $\mathcal{D}(y_i) = \perp(\ker f_{Y*y_i}, T_{y_i}Y)$, then $f_{Y*y_i|\mathcal{D}(y_i)} : \mathcal{D}(y_i) \to T_{y'_i}Y'$ is an isomorphism of vector spaces, and hence $\tau' = \lim_i f_{Y*y_i}\big(\mathcal{D}(y_i)\big)$.

$(1 \Rightarrow 2)$. Let us suppose that $f_Y : Y \to Y'$ is a submersion at y_0.
We fix a unit vector $v' \in \lim_i f_{Y*y_i}\big(\mathcal{D}(y_i)\big)$ and we will show that $v' \in f_{Y*y_0}\big(\lim_i \mathcal{D}(y_i)\big)$.

There exists then a sequence of unit vectors $\{v'_i \in f_{*y_i}\big(\mathcal{D}(y_i)\big)\}_i$ such that $v' = \lim_i v'_i$.
For every $v'_i \in f_{Y*y_i}(\mathcal{D}(y_i))$ the canonical lifting v_i satisfies $v_i \in \mathcal{D}(y_i)$ and $f_{Y*y}(v_i) = v'_i$.
Now f_Y being a submersion at y_0, by Proposition 3.7 $(1 \Rightarrow 3)$, we find that $\limsup_{y \to y_0} H_Y(y) < +\infty$ and that the sequence $\{v_i = f^{-1}_{*y_i|\mathcal{D}(y_i)}(v'_i)\}_i$ is bounded and admits a subsequence $\{v_{ih}\}_h$ converging to a vector $v = \lim_h v_{i_h} \in \lim_h \mathcal{D}(y_{ih}) = \lim_i \mathcal{D}(y_i)$ for which

$$f_{Y*y_0}(v) = f_{Y*y_0}(\lim_h v_{ih}) = \lim_h f_{Y*y_0}(v_{ih}) = \lim_h v'_{ih} = v' .$$

Therefore $v' \in f_{Y*y_0}\big(\lim_i \mathcal{D}(y_i)\big)$.

$(2 \Rightarrow 3)$. Chosen a subsequences such that there exists $\lim_h \mathcal{D}(y_{i_h})$ we immediately have :

$$f_{Y*y_0}(\tau) = f_{Y*y_0}\big(\lim_h T_{i_h}Y\big) \supseteq f_{Y*y_0}\big(\lim_h \mathcal{D}(y_{i_h})\big) \supseteq \lim_h f_{Y*y_{i_h}}\big(\mathcal{D}(y_{i_h})\big) = \lim_h T_{y'_{i_h}}Y' = \tau' .$$

Hence Condition (D) holds at y_0 for f_Y.

$(3 \Rightarrow 1)$. If f_Y satisfies condition (D) at y_0, we have $f_{Y*y_0}(\tau) \supseteq \tau'$ and since y_0 is a regular point of the manifold Y, $\tau = \lim_i T_{y_i}Y = T_{y_0}Y$ and $\tau' = \lim_i T_{y'_i}Y' = T_{y'_0}Y'$. Thus $f_{Y*y_0}(T_{y_0}Y) \supseteq T_{y'_0}Y'$.

Hence $f_{Y*y_0} : T_{y_0}Y \to T_{y'_0}Y'$ is surjective, and $f_Y : Y \to Y'$ is a submersion at y_0. $\square$

4. Sufficient conditions, analytic and geometric meanings for condition (D).

In this section we prove the main results of the paper given in Theorems 4.3, 4.4, 4.5, 4.6 and their Corollaries 4.3, 4.4, 4.5, 4.6.

Starting from the analysis of the technical content of condition (D), (Theorem 4.3) we find various equivalent analytic and geometric properties (Theorems 4.4, 4.5, 4.6), which are all sufficient conditions for Condition (D) (Corollaries 4.3, 4.5 and 4.6).

4.1. ***Technical content and sufficient analytic conditions for Condition*** (D).

Theorem 4.3 below explains the essential technical content of the condition (D).

The equivalence $(1 \Leftrightarrow 4)$ has been used by the author of the present paper in **[16]** (Theorem 3.3) when $f_{\mathcal{W}} = \pi_{XY|\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ is the restriction of a projection $\pi_{XY} : S^{\epsilon}_{XY} \to X$, to prove that certain stratified mapping cones $C_{\mathcal{W}'}(\mathcal{W})$ are (b)-regular, to obtain an equivalent version of Goresky's essential Proposition 2.2 and 2.4 (Theorem 3.4 and Corollary 3.2, **[16]**) .

Proposition 2.2 is really the key property in proving Proposition 2.4 which gives a partial solution of Conjecture 1.2, suggests new ideas for a general approach to it and is fundamental for the proof of Theorems 1.1 and 1.2 in the theories WH_*, WH^* of Goresky (see §2).

Theorem 4.3. *Let $f : M \to M'$ be a C^1 map between C^1 manifolds, $\mathcal{W} \subseteq M$ and $\mathcal{W}' \subseteq M'$ Whitney stratifications such that the restriction $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ is a stratified surjective submersion.*

Let $X < Y$ be strata of $\mathcal{W}$, $x \in X$. By denoting $f_Y : Y \to Y' = f(Y)$ the restriction of f, and for all $y \in Y$, $y' = f(y)$ and $\mathcal{D}(y) = \perp(\ker f_{Y*y}, T_yY)$, the following conditions are equivalent:

1. *The map $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ satisfies the condition (D) at $x \in X < Y$.*
2. *For all $\{y_i\}_i \subseteq Y$ such that $\lim_i y_i = x$ and both limits $\tau = \lim_i T_{y_i}Y$ and $\tau' = \lim_i T_{y'_i}Y'$ exist, for every $v' \in \tau' - \{0\}$ <u>there exists a sequence</u> $\{v'_i \in T_{y'_i}Y' - \{0\}\}_i$ such that $\lim_i v'_i = v'$ and having a bounded sequence of preimages $\{w_i \in f^{-1}_{Y*y_i}(v'_i) \in T_{y_i}Y\}_i$.*
3. *For all $\{y_i\}_i \subseteq Y$ such that $\lim_i y_i = x$ and both limits $\tau = \lim_i T_{y_i}Y$ and $\tau' = \lim_i T_{y'_i}Y'$ exist, for every $v' \in \tau' - \{0\}$ there exists a sequence $\{v'_i \in T_{y'_i}Y' - \{0\}\}_i$ such that $\lim_i v'_i = v'$ and having the sequence by canonical lifting $\{v_i \in f^{-1}_{Y*y_i|\mathcal{D}(y_i)}(v'_i) \in \mathcal{D}(y_i)\}_i$ bounded.*
4. *For all $\{y_i\}_i \subseteq Y$ such that $\lim_i y_i = x$ and both limits $\sigma = \lim_i \mathcal{D}(y_i)$ and $\tau' = \lim_i T_{y'_i}Y'$ exist, one has: $f_{*x}(\lim_i \mathcal{D}(y_i)) \supseteq \lim_i f_{Y*y_i}(\mathcal{D}(y_i))$.*

Proof. Let us consider a sequence $\{y_i\}_i \subseteq Y$ such that $\lim_i y_i = x$ and both limits $\tau = \lim_i T_{y_i}Y$ and $\tau' = \lim_i T_{y'_i}Y'$ exist in the appropriate Grassmann manifold.

Remark also that, $f_Y : Y \to Y'$ being submersive, $T_{y'_i}Y' = f_{Y*y_i}(T_{y_i}Y) = f_{*y_i}(T_{y_i}Y)$ for each i.

$(1 \Rightarrow 2)$. If $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ satisfies the condition (D) at $x \in X < Y$, $f_{*x}(\tau) \supseteq \tau'$ then for every vector $v' \in \tau'$ there exists a vector $v \in \tau$ such that $v' = f_{*x}(v)$.

Since $v \in \tau = \lim_i T_{y_i}Y$, there exists a sequence $\{w_i \in T_{y_i}Y\}_i$ such that $v = \lim_i w_i$ and $\{w_i\}_i$ is in particular obviously bounded. The sequence of the images $\{v'_i := f_{*y_i}(w_i)\}_i$ satisfies then:

i) $\lim_i v'_i = \lim_i f_{*y_i}(w_i) = f_{*x}(\lim_i w_i) = f_{*x}(v) = v'$;

ii) $\{v'_i = f_{*y_i}(w_i)\}_i$ admits the bounded sequence of lifting $\{w_i \in f^{-1}_{*y_i}(v'_i)\}_i$.

$(2 \Rightarrow 3)$. Under the hypothesis 2), by decomposing every vector w_i in the orthogonal sum $w_i = v_i + u_i \in \mathcal{D}(y_i) \oplus \ker f_{Y*y_i}$ one immediately has $||v_i|| \leq ||w_i||$ so that if $\{w_i\}_i$ is bounded then $\{v_i\}_i$ is bounded too and moreover: $v_i \in \mathcal{D}(y_i)$ and $f_{*y_i}(v_i) = v_i'$.

$(3 \Rightarrow 4)$. Let $v' \in \lim_i f_{*y_i}(\mathcal{D}(y_i)) \subseteq \tau'$ and let us suppose that $\lim_i \mathcal{D}(y_i) = \sigma$ exists.

By hypothesis 3) for every $v' \in \tau'$ there exists a sequence $\{v_i' \in T_{y_i'}Y'\}_i$ such that $\lim_i v_i' = v'$ whose sequence of canonical lifting $\{v_i \in f^{-1}_{Y*y_i}(v_i') \cap \mathcal{D}(y_i) \subseteq T_{y_i}Y\}_i$ is bounded.

Thus for a convenient subsequence of indexes $\{i_h\}_h$ there exist $v = \lim_h v_{i_h}$, $\tau = \lim_h T_{y_{i_h}}Y$ and (obviously) $\lim_h \mathcal{D}(y_{i_h})$ so that

$$v = \lim_h v_{i_h} \in \lim_h \mathcal{D}(y_{i_h}) = \lim_i \mathcal{D}(y_i)$$

and

$$v' = \lim_h v'_{i_h} = \lim_h f_{Y*y_{i_h}}(v_{i_h}) = f_{*x}(v) \in f_{*x}(\lim_i \mathcal{D}(y_i))$$

and in conclusion:

$$f_{*x}(\lim_i \mathcal{D}(y_i)) \supseteq \lim_i f_{Y*y_i}(\mathcal{D}(y_i)) .$$

$(4 \Rightarrow 1)$. Let $\{y_i\}_i \subseteq Y$ be a sequence such that $\lim_i y_i = x$ and both limits $\tau = \lim_i T_{y_i}Y$ and $\tau' = \lim_i T_{y_i'}Y'$ exist in the appropriate Grassmann manifold.

The Grassmann manifold being compact, there exists a subsequence of indices $(i_h)_h$, such that there exists also $\lim_h \mathcal{D}(y_{i_h}) =: \sigma$.

Thus $f_Y : Y \to Y'$ being a submersion, $T_{y'_{i_h}}Y' = f_{Y*y_{i_h}}(T_{y_{i_h}}Y) = f_{*y_{i_h}}(T_{y_{i_h}}Y)$ and hence:

$$\tau' = \lim_i T_{y_i'}Y' = \lim_h T_{y'_{i_h}}Y' = \lim_h f_{Y*y_{i_h}}(\mathcal{D}(y_{i_h})) = \lim_h f_{*y_{i_h}}(\mathcal{D}(y_{i_h})) \subseteq$$

by the hypothesis 4)

$$\subseteq \; f_{*x}(\lim_h \mathcal{D}(y_{i_h})) \; \subseteq \; f_{*x}(\lim_h T_{y_{i_h}}Y) \; = \; f_{*x}(\lim_i T_{y_i}Y) \; = \; f_{*x}(\tau) \; .$$

Then in conclusion $f : \mathcal{W} \to \mathcal{W}'$ satisfies the condition (D) at $x \in X < Y$. □

Theorem below extends to the stratiffied case the previous Propostion 3.7 and allows to give in Corollary 4.3 a sufficient analytic condition for Condition (D).

Theorem 4.4. *Let $f : M \to M'$ be a C^1 map between C^1 manifolds, $\mathcal{W} \subseteq M$ and $\mathcal{W}' \subseteq M'$ Whitney stratifications such that the restriction $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ is a stratified surjective submersion.*

Let $X < Y$ be adjacent strata of $\mathcal{W}$, $x \in X$, $Y' = f(Y)$ and $y' = f(y)$ for all $y \in Y$.

Let us consider for $f_Y : Y \to Y'$ the distribution $\mathcal{D}(y) = \perp(\ker f_{Y*y}, T_yY)$ and the maps

$$h_Y : Y \to]0, \infty[\quad , \quad h_Y(y) = \min\{||f_{Y*y|\mathcal{D}(y)}(u)|| \; : \; ||u|| = 1\} ,$$

$$H_Y : Y \to \;]0, +\infty[\quad , \quad H_Y(y) = \max\{|| \; f^{-1}_{Y*y|\mathcal{D}(y)}(v') \; || \; : \; ||v'|| \; = 1\} .$$

The following conditions are equivalent:

1) For all $\{y_i\}_i \subseteq Y$ such that $\lim_i y_i = x$ and both limits $\tau = \lim_i T_{y_i}Y$ and $\tau' = \lim_i T_{y_i'}Y'$ exist, for every vector $v' \in \tau' - \{0\}$, <u>every sequence</u> of vectors $\{v_i' \in T_{y_i'}Y' - \{0\}\}_i$ such that $\lim_i v_i' = v'$ has a bounded subsequence of canonical liftings $\{v_{i_h} = f^{-1}_{Y*y_{i_h}|\mathcal{D}(y_{i_h})}(v'_{i_h})\}_h$.

2) For all $\{y_i\}_i \subseteq Y$ such that $\lim_i y_i = x$ and both limits $\tau = \lim_i T_{y_i}Y$ and $\tau' = \lim_i T_{y_i'}Y'$ exist, for every unit vector $u' \in \tau'$, every sequence of unit vectors $\{u_i' \in T_{y_i'}Y'\}_i$ such that $\lim_i u_i' = u'$ has a bounded subsequence of canonical liftings $\{u_{i_h} = f^{-1}_{Y*y_{i_h}|\mathcal{D}(y_{i_h})}(u'_{i_h})\}_h$.

3) $\liminf_{y\to x} h_Y(y) > 0$.

4) $\limsup_{y\to x} H_Y(y) < +\infty$.

Proof 1) $\Rightarrow$ 2). Obvious.

Proof 2) $\Rightarrow$ 1). If $v' \in \tau' - \{0\}$ and $\{v'_i \in T_{y'_i}Y' - \{0\}\}_i$ is a sequence such that $\lim_i v'_i = v'$, then $u' := \frac{v'}{||v'||} \in \tau'$ and $u'_i := \frac{v'_i}{||v'_i||} \in T_{y'_i}Y'$ are unit vectors such that $\lim_i u'_i = u'$.

By the hypothesis 2) the sequence of canonical liftings $\{u_i := f^{-1}_{Y*y_i|\mathcal{D}(y_i)}(u'_i)\}$ admits a bounded subsequence $\{u_{i_h}\}_h$. So there exists $K > 0$ such that

$$||f^{-1}_{Y*y_{i_h}|\mathcal{D}(y_{i_h})}(\frac{v'_{i_h}}{||v'_{i_h}||})|| \le K \quad \text{and hence:} \quad ||f^{-1}_{Y*y_{i_h}|\mathcal{D}(y_{i_h})}(v'_{i_h})|| \le K \cdot ||v'_{i_h}|| \,.$$

The canonical liftings $\{v_{i_h} := f^{-1}_{Y*y_{i_h}|\mathcal{D}(y_{i_h})}(v'_{i_h})\}_h$ of the $\{v'_{i_h}\}_h$ are then bounded by:

$$||v_{i_h}|| = ||f^{-1}_{Y*y_{i_h}|\mathcal{D}(y_{i_h})}(v'_{i_h})|| \le K \cdot ||v'_{i_h}|| \le K \cdot \sup_h ||v'_{i_h}|| = K' < +\infty \,.$$

Proof 2) $\Rightarrow$ 3). Let $l = \liminf_{y\to x} h_Y(y)$ the minimum value of adherence of h_Y.

There exists then a sequence $\{y_i\}_i \subseteq Y$ such that $\lim_i y_i = x \in X$ and $\lim_i h_Y(y_i) = l \in \mathbb{R}$.

By definition of each $h_Y(y_i)$, there exists a sequence of unit vectors $\{u_i \in \mathcal{D}(y_i) \subseteq T_{y_i}Y\}_i$ such that each $h_Y(y_i) = ||f_{Y*y|\mathcal{D}(y_i)}(u_i)||$ realizes the minimum norm defining $h_Y(y_i)$ (Definition 3.5).

There exists a subsequence $\{y_{i_h}\}_h$, such that both limits exist:

$$\lim_i T_{y_{i_h}}Y =: \tau \quad \text{and} \quad \lim_i T_{y'_{i_h}}Y' =: \tau'.$$

Every u_{i_h} being a unit vector $\in \mathcal{D}(y_{i_h}) - \{0\}$, its image $u'_{i_h} := f_{Y*y_{i_h}|\mathcal{D}(y_{i_h})}(u_{i_h}) \in T_{y'_{i_h}}Y' - \{0\}$ is not zero (as well as for all images of vectors in $\mathcal{D}(y_{i_h}) - \{0\}$) and we can write:

$$u_{i_h} = f^{-1}_{Y*y_{i_h}|\mathcal{D}(y_{i_h})}({u'}_{i_h}) \in \mathcal{D}(y_{i_h}) \quad \text{and} \quad \frac{u_{i_h}}{||u'_{i_h}||} = f^{-1}_{Y*y_{i_h}|\mathcal{D}(y_{i_h})}(\frac{{u'}_{i_h}}{||{u'}_{i_h}||}) \in \mathcal{D}(y_{i_h}).$$

For a suitable further subsequence (note it again $\{i_h\}_h$), there exists then the limit :

$$u' \; := \; \lim_h \; \frac{{u'}_{i_h}}{||{u'}_{i_h}||} \; \in \; \lim_h T_{y'_{i_h}}Y' - \{0\} \,.$$

It follows that:

i) The unit vector $u' \; = \; \lim_h \; \frac{{u'}_{i_h}}{||{u'}_{i_h}||} \; \in \; \tau' - \{0\}$.

ii) Every vector $\frac{u_{i_h}}{||u'_{i_h}||} = f^{-1}_{Y*y_{i_h}|\mathcal{D}(y_{i_h})}(\frac{u'_{i_h}}{||u'_{i_h}||})$ is the canonical lifting of the unit vectors $\frac{u'_{i_h}}{||u'_{i_h}||}$.

Hence, by the hypothesis 2), there exists a bounded subsequence (let us denote it again) $\{\frac{u_{i_h}}{||u'_{i_h}||}\}_h$. That is there exists $K > 0$ such that $||f^{-1}_{Y*y_{i_h}|\mathcal{D}(y_{i_h})}(\frac{u'_{i_h}}{||u'_{i_h}||})|| \le K$.

Therefore,

$$1 = ||u_{i_h}|| = ||f^{-1}_{Y*y_{i_h}|\mathcal{D}(y_{i_h})}(u'_{i_h})|| \le K \cdot ||u'_{i_h}|| = K \cdot h_Y(y_{i_h})$$

and in conclusion:

$$l \; = \; \liminf_{y\to x} h_Y(y) \; = \; \lim_i h_Y(y_i) \; = \; \lim_h h_Y(y_{i_h}) \; \ge \; \frac{1}{K} \; > \; 0 \,.$$

Proof 3) $\Rightarrow$ 4). It follows immediately because by Remark 3.3.3 one has: $H_Y(y) = \frac{1}{h_Y(y)}$.

Proof 4) ⇒ 2). Let $\{y_i\}_i \subseteq Y$ be a sequence of points such that $\lim_i y_i = x$, $\lim_i T_{y_i}Y = \tau$, $\lim_i T_{y'_i}Y' = \tau'$ and let us fix $u' \in \tau'$ a unit vector and a sequence of unit vectors $\{u'_i \in T_{y'_i}Y'\}_i$ such that $\lim_i u'_i = u'$.

Since $L := \limsup_{y\to x} H_Y(y) < +\infty$, then $\limsup_i H_Y(y_i) \leq L$ is finite and so, by Definition 3.5 of each $H_Y(y_i)$, the sequence

$$||f^{-1}_{Y*y_i|\mathcal{D}(y_i)}(u'_i)|| \leq H_Y(y_i) \leq L \quad \text{is bounded.} \qquad \square$$

We deduce then, as corollary, a sufficient condition for Goresky's Condition (D):

Corollary 4.3. *Let $f : M \to M'$ be a C^1 map between C^1 manifolds, $\mathcal{W} \subseteq M$ and $\mathcal{W}' \subseteq M'$ Whitney stratifications such that the restriction $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ is a stratified surjective submersion.*

Let $X < Y$ be adjacent strata of $\mathcal{W}$ and x a point of X.

If $\liminf_{y\to x} h_Y(y) > 0$ or equivalently $\limsup_{y\to x} H_Y(y) < +\infty$ then:

$$f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}' \quad \text{satisfies the condition } (D) \text{ at } x \in X < Y\,.$$

Proof. It follows immediately by 3) ⇒ 1) of Theorem 4.4 and 3) ⇒ 1) of Theorem 4.3. □

4.2. ***Distance functions between vector subspaces of an Euclidian space.*** We will give a sufficient condition for Condition (D) in terms of all possible limits of the sequences of *essential angles* $\{\alpha'(T_{y_i}Y, \ker f_{*y_i})\}_i$ between the vector subspaces $T_{y_i}Y$ and $\ker f_{*y_i}$ of $T_{y_i}M$. We introduce then the essential minimal distance between two vector subspaces.

Definition 4.6. Let V be a vector subspace of a Euclidian space E.

For every vector $u \in E$ let us define *the distance of u from V* as usual **[22]** by:

$$\delta(u, V) \;=\; \inf_{v\in V} ||\, u - v\,||\,.$$

Such a minimum value $\inf_{v\in V} ||\,u - v\,||$ is realized when $u - v$ is orthogonal to V, so precisely when $v = p_V(u)$ is the orthogonal projection of u on V. In particular:

$$\delta(u, V) \;=\; \inf_{v\in V} ||\,u - v\,|| \;= ||\; u - p_V(u)||$$

and if $u \neq 0$ we let $\alpha(u, V) := \alpha(u, p_V(u))$ denote the *unoriented* angle $\in [0, \frac{\pi}{2}]$ between u and $p_V(u)$.

Let us recall now some simple properties of the fonction δ:

Remark 4.8. *Under the above hypotheses we have:*

1) $||\, u - p_V(u)\,|| \;= ||\,u\,||\, sin\,\alpha(u, V)$ *and* $||\,u\,|| \;= 1 \;\Rightarrow\; ||\,u - p_V(u)\,|| \;= sin\,\alpha(u, V)$;

2) $||\,p_V(u)\,|| \;=\; ||\,u\,||\, cos\,\alpha(u, V)$ *and* $||\,u\,|| = 1 \;\Rightarrow\; ||\,p_V(u)\,|| \;= cos\,\alpha(u, V)$;

3) $u \in V \iff \delta(u, V) \;=\; 0$;

4) $\delta(a, V) \;=\; ||a\,|| \cdot \delta(\frac{a}{||a\,||}, V)$, *for all* $a \in E - \{0\}$;

5) $\lim_i u_i \;=\; u \implies \lim_i \delta(u_i, V) \;=\; \delta(u, V)$;

6) $\lim_i V_i \;=\; V \implies \lim_i \delta(u, V_i) \;=\; \delta(u, V)$;

7) $\lim_i u_i \;=\; u$ *and* $\lim_i V_i \;=\; V \implies \lim_i \delta(u_i, V_i) \;=\; \delta(u, V)$.

Proof. 1), . . . , 4) are immediate, while 5) follows thanks to: $\lim_i p_V(u_i) = p_V(u)$ and 6) by: $\lim_i p_{V_i}(u) = p_V(u)$. The proof of 7) holds since the inequalities:

$$\delta(u, V) \;=\; ||u - p_V(u)|| \;\;\leq\;\; ||u - u_i|| + \delta(u_i, V_i) + ||p_{V_i}(u_i) - p_V(u)||$$

$$\delta(u_i,V_i) \;=\; ||u_i - p_{V_i}(u_i)|| \;\;\leq\;\; ||u-u_i|| \;+\; \delta(u,V) \;+\; ||p_V(u) - p_{V_i}(u_i)||$$

imply

$$|\delta(u,V) - \delta(u_i,V_i)| \;\leq\; ||u-u_i|| \;+\; ||p_{V_i}(u_i) - p_V(u)||$$

and since the hypotheses $\lim_i u_i = u$ and $\lim_i V_i = V$ imply $\lim_i p_{V_i}(u_i) = p_V(u)$. □

One usually considers as "distance" function between two vector subspaces $U, V \subseteq E$, not necessarily of the same dimension, the following :

$$\delta(U,V) \;:= \sup_{u\in U\;,\;||u||=1} \delta(u,V) = \sup_{u\in U\;,\;||u||=1} \inf_{v\in V} \;||\,u-v\,||\;.$$

Thanks to the equality (true since every $||\;u\;|| \;=1$):

$$\delta(U,V) = \sup_{u\in U\;,\;||u||=1} ||\,u - p_V(u)\,|| \;= \sup_{u\in U\;,\;||u||=1} sin\,\alpha(u,V) \;\in\; [0,1],$$

by denoting $\alpha(U,V)$ the maximum angle $\in [0, \frac{\pi}{2}]$ between a vector of U and its projection on V, one can write:

$$\delta(U,V) \;=\; \sup_{u\in U} \; sin\,\alpha(u,V) \;\;=\;\; sin\;\alpha(U,V)\,.$$

One finds then:

Remark 4.9. *The function $\delta(U,V)$ satisfies the following properties:*

1) $\delta(U,V) = 0 \quad\Longleftrightarrow\quad U \subseteq V$;

2) $\delta(V,U) = 1 \quad\Longleftrightarrow\quad \exists\, v \in V - U \;:\; v \perp U$ (this holds if $U \subset V$ is strictly contained);

3) $\delta(U,V) \;\neq\; \delta(V,U)$ is not symmetric in general;

4) $||u|| = 1 \quad\Longrightarrow\quad \delta(L(u),V) = \delta(u,V)$ where $L(u)$ is the vector subspace spanned by u;

5) $\delta(a,V) \;\leq\; 2||a-b|| + \delta(b,V)$ for every unit vectors $a, b \in E$;

6) $\delta(a,U) \;\leq\; 2\delta(a,V) + \delta(V,U)$ for every unit vector $a \in E$;

7) $\lim_i U_i = U$, and $\lim_i V_i = V \quad\Longrightarrow\quad \lim_i \delta(U_i,V_i) \;=\; \delta(U,V)$.

Proof. $1), \ldots, 4)$ are immediate.

The proof of 5) follows easily by $\delta(a,V) = ||a - p_V(a)||$ and

$$||a - p_V(a)|| \leq ||a-b|| + ||b - p_V(b)|| + ||p_V(b) - p_V(a)|| \;\leq\; ||a-b|| + \delta(b,V) + ||b-a||\,.$$

The proof of 6) follows similarly, since:

$$\delta(a,U) = ||a - p_U(a)|| \leq ||a - p_V(a)|| + ||p_V(a) - p_U(p_V(a))|| + ||p_U(p_V(a)) - p_U(a)|| =$$
$$\delta(a,V) + \delta(p_V(a),U) + ||p_U(a - p_V(a))|| \;\leq\; \delta(a,V) + \delta(V,U) + ||a - p_V(a)|| =$$
$$2\delta(a,V) + \delta(V,U)\,.$$

To prove 7), let u be the unit vectors $\in U$ such that $\delta(U,V) = ||u - p_V(u)|| = \delta(u,V)$

Since $\lim_i U_i = U$ then $\lim_i p_{U_i}(u) = u$, so by Remark 4.8.7 and since every $p_{U_i}(u) \in U_i$ one has:

$$\delta(U,V) \;=\; \delta(u,V) \;=\; \lim_i \delta(p_{U_i}(u),V_i) \;\leq\; \lim_i \delta(U_i,V_i)\,.$$

Simalrly if u_i is the unit vector $\in U_i$ such that $\delta(U_i,V_i) = ||u_i - p_{V_i}(u_i)|| = \delta(u_i,V_i)$ (taking a subsequence if necessary), there exists $\lim_i u_i = a \in U$ and by 5) one finds:

$$\delta(U_i,V_i) \;=\; \delta(u_i,V_i) \;\leq\; 2||u_i - a|| \;+\; \delta(a,V_i) \;\leq\; 2||u_i - a|| \;+\; \delta(U,V_i)$$

hence also that :

$$\lim_i \delta(U_i,V_i) \;\leq\; 2\;\lim_i ||u_i - a|| \;+\; \lim_i \delta(U,V_i) \;=\; \delta(U,V)\,. \qquad \square$$

In order to define a finer "distance" $\delta'(U,V)$ between U and V, we will be interested in the "minimum essential angle", $\alpha'(U,V)$, between U and V, a notions which needs the following more detailed definition.

Definition 4.7. Let $U,V\subseteq E$ two vector subspaces not necessarily of the same dimension.

If $U=\{0\}$ or $V=\{0\}$ let us define $\delta'(U,V)=0$. Suppose then $U\neq\{0\}$ and $V\neq\{0\}$.

If $U\cap V=\{0\}$, every unit vector $u\in U$ does not lie in V so $||\,u-p_V(u)\,||>0$ and using the previous Remark 4.8.1) one can simply define:

$$\delta'(U,V)\;=\min_{u\in U\;,\;||u||=1}\;||\,u-p_V(u)\,||\;=\min_{u\in U\;,\;||u||=1}\;sin\,\alpha(u,p_V(u))\;\in\;]0,1],$$

and denoting $\alpha'(U,V)$ the minimum positive angle between a vector of U and its projection on V, one can write

$$\delta'(U,V)=sin\;\alpha'(U,V)\,.$$

Thus using that $\alpha'(U,V)=\alpha'(V,U)$, one has:

Remark 4.10. *If $U,V\neq\{0\}$, then:*

$$U\cap V=\{0\}\quad\Longrightarrow\quad U\not\subseteq V\;\;\textit{and}\;\;V\not\subseteq U\quad\Longrightarrow\quad\delta'(U,V)\;=\;\delta'(V,U)>0.\qquad\square$$

Our definition 4.7 of $\delta'(U,V)$, in the case $U\neq\{0\}$ and $V\neq\{0\}$ and $U\cap V=\{0\}$, coincides with the definition given in **[8]** (p. 534, where it is denoted by $\delta(U,V)$).

On the other hand the definition in **[8]** in the case $U\cap V\neq\{0\}$ satisfies $\delta(U,V)=0$.

This is not convenient enough for our aims, so we have to extend it in a finer way:

Definition 4.8. If $U\cap V\neq\{0\}$, we consider their *essential mutual subspaces*:

$$U'\;:=\perp(U\cap V;U)\quad\text{and}\quad V'\;:=\perp(U\cap V;V)\,,$$

that easily satisfy $U'\cap V'=\{0\}$ and define

$$\delta'(U,V)\;:=\;\delta'(U',V')\;=\min_{u'\in U'\;,\;||u'||=1}\;||\,u'-p_{V'}(u')\,||\;=\;sin\;\alpha'(U',V')$$

and call $\alpha'(U,V):=\alpha'(U',V')$ the *minimum essential angle between U and V* and similarly we call $\delta'(U,V):=\delta'(U',V')$ the *minimum essential distance between U and V.*

Definition 4.8 and Remark 4.9, obviously imply:

Remark 4.11. *For every two arbitrary vector subspaces U,V of E :*
1) $U\cap V=\{0\}\iff U'=U$ and $V'=V\iff U'=U$ or $V'=V$.
2) $\delta'(U,V):=\delta'(U',V')=\delta'(V',U')=\delta'(V,U)$. $\square$

Thus Definition 4.8 extends Definition 4.7 and allows us to obtain that the fonction:

$$\delta'\;:\;\mathbb{G}(E)\times\mathbb{G}(E)\longrightarrow[0,1]\quad,\quad\delta'(U,V):=\delta'(U',V')$$

is a symmetric function, where $\mathbb{G}(E)$ denotes the Grassmann manifold of all vector subspaces of E. Moreover we have:

Remark 4.12. *For every pair of vector subspaces U,V of E:*

1) $\delta'(U,V)=0\iff U\subseteq V$ *or* $U\supseteq V$.

2) *If* $\dim U=\dim V$; $\delta'(U,V)=0\iff U=V$.

3) $\delta'(U,V):=\delta'(U',V')=\delta'(U',V)=\delta'(U,V')$.

Proof 1), 2). It follows easily since: $U \subseteq V$ if and only if $U' = \{0\}$ and then $\delta'(U,V) = 0$.

Proof 3). Since $V = (U \cap V) \oplus V'$ is an orthogonal sum, for every $u' \in U'$ its projection $p_V(u')$ on V decomposes into the orthogonal sum $p_V(u') = p_{U\cap V}(u') + p_{V'}(u')$.

Moreover, since u', lying in U', is orthogonal to $U \cap V$, one has $p_{U\cap V}(u') = 0$ and $p_V(u) = p_{V'}(u')$.

By definition 4.8,

$$\delta'(U,V) \quad = \quad \delta'(U',V') \quad = \quad \min_{u'\in U'\,,\;||u'\;||=1} \quad ||\,u' - p_{V'}(u')\,||\,.$$

Since $U' \cap V \subseteq U \cap U' \cap V = U' \cap (U \cap V) = \{0\}$, then $U' \cap V = \{0\}$ and

$$\delta'(U',V) \quad = \quad \min_{u'\in U'\,,\;||u'||=1} \quad ||\,u' - p_V(u')\,||\;.$$

Since $p_V(u') = p_{V'}(u')$ for every $u' \in U'$ one finds: $\delta'(U,V) := \delta'(U',V') = \delta'(U',V)$.

Finally, δ' being a symmetric function (Remark 4.11.2), this last equality also implies:

$$\delta'(U,V) \;:=\; \delta'(U',V') \;=\; \delta'(V',U') \;=\; \delta'(V',U) \;=\; \delta'(U,V')\,. \qquad \square$$

One sees moreover easily that δ' is a *decreasing* function with respect to both variables U, V.

As one can see with simple examples, δ' *is not a metric* also when restricted to a family of subspaces of the same dimension, except for the 1-dimensional case.

4.3. ***Sufficient conditions and geometric meaning.*** With the same hypotheses and notations as in §4.1 and §4.2, if U, V are the two vector subspaces $U := T_yY$ and $V := \ker f_{*y}$ of $E := T_yM$, the essential mutual subspace U' is:

$$U' \;:=\; [T_yY]' \;=\perp (T_yY \cap \ker f_{*y}; T_yY) \;=\perp\; (\ker f_{Y*y}; T_yY) = \mathcal{D}(y)\,.$$

We can then define (using also Remark 4.12.3) the function

$$\delta_Y : Y \to [0,\infty[\quad , \quad \delta_Y(y) := \delta'(T_yY, \ker f_{*y}) = \delta'(\mathcal{D}(y), \ker f_{*y})$$

and we have:

Theorem 4.5. *Let $f : M \to M'$ be a C^1 map between C^1 manifolds, $\mathcal{W} \subseteq M$ and $\mathcal{W}' \subseteq M'$ Whitney stratifications such that the restriction $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ is a stratified surjective submersion.*

Let $X < Y$ be strata of $\mathcal{W}$ and $x \in X$ and consider the function δ_Y defined by

$$\delta_Y : Y \to [0,\infty[\quad , \quad \delta_Y(y) := \delta'(T_yY, \ker f_{*y}) = \delta'(\mathcal{D}(y), \ker f_{*y})\,.$$

If $f : M \to M'$ is a submersion at x, the following conditions are equivalent:

1) $\liminf_{y\to x} \delta_Y(y) > 0$.

2) For every sequence $\{y_i\}_i \subseteq Y$ such that $\lim_i y_i = x \in X$ and $\lim_i \mathcal{D}(y_i) = \sigma$ exists, for every unit vector $u \in \lim_i \mathcal{D}(y_i)$ and every sequence $\{u_i \in \mathcal{D}(y_i)\}_i$, of unit vectors converging to $u = \lim_i u_i$, there exists a subsequence of images $\{u'_{i_h} = f_{Y*y_{i_h}}(u_{i_h})\}_h$ such that $\inf_h ||u'_{i_h}|| > 0$.

3) For every sequence $\{y_i\}_i \subseteq Y$ such that $\lim_i y_i = x \in X$ and $\lim_i f_{Y*y_i}(T_{y_i}Y) = \tau'$ exists, for every $v' \in \lim_i f_{Y*y_i}(T_{y_i}Y) - \{0\}$, every sequence $\{v'_i \in f_{Y*y_i}(T_{y_i}Y) - \{0\}\}_i$ converging to $v' = \lim_i v'_i$, has an upper bounded subsequence of canonical liftings $\{v_{i_h} = f^{-1}_{Y*y_{i_h}|\mathcal{D}(y_{i_h})}(v'_{i_h})\}_h$.

Proof $(1 \Rightarrow 2)$. Let suppose that 2) does not hold.

Then, for a sequence $\{y_i\}_i \subseteq Y$, $\lim_i y_i = x \in X$, $\lim_i \mathcal{D}(y_i) = \sigma$ and there exists a unit vector $u \in \lim_i \mathcal{D}(y_i)$ which is a limit of a sequence of unit vectors $\{u_i \in \mathcal{D}(y_i)\}_i$ such that $\lim_i \; ||f_{Y*y_i}(u_i)|| = 0$ and hence necessarily $\lim_i \; f_{Y*y_i}(u_i) \;= 0$.

As f is C^1 at x, one has:

$$f_{*x}(u) \;=\; f_{*x}(\lim_i u_i) \;=\; \lim_i f_{*y_i}(u_i) \;=\; 0 \qquad \text{that is:} \qquad u \in \ker f_{*x}\,.$$

Since, for every i, $\mathcal{D}(y_i) \cap \ker f_{*y_i} = \{0\}$ and $\delta_Y(y_i)$ is the essential *minimal* distance

$$\delta_Y(y_i) \;=\; \delta'(\mathcal{D}(y_i), \ker f_{*y_i}) \;=\; \min_{u'_i \in \mathcal{D}(y_i)\;,\;||u'_i\;||=1} \delta(u'_i,\; \ker\, f_{*y_i})\,,$$

and as $u_i \in \mathcal{D}(y_i)$ by Remark 4.9.6, we can write:

$$0 \le \delta_Y(y_i) = \delta'(\mathcal{D}(y_i),\; \ker\, f_{*y_i}) \le \delta(u_i, \ker\, f_{*y_i}) \le 2\delta(u_i, \ker\, f_{*x}) + \delta(\ker\, f_{*x}, \ker\, f_{*y_i})\,.$$

Since $\lim_i u_i = u$, and $u \in \ker\, f_{*x}$ (by Remark 4.8.5) we have: $\lim_i \delta(u_i, \ker\, f_{*x}) = 0$.

By hypothesis $f : M \to M'$ is a submersion at x[1] so by Proposition 3.5 and Remark 4.9.7:

$$\lim_i \ker\, f_{*y_i} = \ker\, f_{*x} \qquad \text{and} \qquad \lim_i \delta(\ker\, f_{*x}, \ker\, f_{*y_i}) = 0\,.$$

These two limits being 0, one concludes that $\lim_i \delta_Y(y_i) = 0$ which implies

$$\liminf_{y \to x} \delta_Y(y) = 0$$

in opposition to the hypothesis 1).

Proof $(2 \Rightarrow 1)$. Let us suppose in opposite that $\liminf_{y\to x} \delta_Y(y) = 0$.
There exists then a sequence $\{y_i\} \subseteq Y$ such that

$$\lim_i y_i = x \qquad \text{and} \qquad \lim_i \delta'(\mathcal{D}(y_i), \ker f_{*y_i}) \;=\; \lim_i \delta_Y(y_i) \;=\; 0\,.$$

Being δ' the essential *minimal* distance and $\mathcal{D}(y_i) \cap \ker f_{*y_i} = \{0\}$ for everi i, there exists then a sequence of unit vectors $\{u_i \in \mathcal{D}(y_i)\}_i$ realizing such a minimal essential distances, i.e. such that:

$$\lim_i \delta(u_i, \ker\, f_{*y_i}) = 0\,.$$

By Remark 4.9.6) one has:

$$(*): \qquad \delta(u_i, \ker\, f_{*x}) \;\le\; 2\delta(u_i, \ker\, f_{*y_i}) \;+\; \delta(\ker\, f_{*y_i}, \ker\, f_{*x})\,.$$

Now since f is C^1 at x, $\lim_i \ker\, f_{*y_i} \subseteq \ker\, f_{*x}$ (Remark 3.1) so by Remarks 4.9.7 and 4.9.1 one has[2]:

$$\lim_i \delta(\ker\, f_{*y_i}, \ker\, f_{*x}) \;=\; \delta(\lim_i \ker\, f_{*y_i}, \ker\, f_{*x}) \;=\; 0\,.$$

Then since one also has $\lim_i \delta(u_i, \ker\, f_{*y_i}) = 0$ by the $(*)$ above using Remark 4.8.5.($\Leftarrow$) one finds:

$$\lim_i \delta(u_i, \ker\, f_{*x}) = 0\,.$$

Every $u_i \in \mathcal{D}(y_i)$ being a unit vector, there exists a subsequence of indexes $\{i_k\}_k$ such that both limits $\lim_k \mathcal{D}(y_{i_k}) = \sigma$ and $u = \lim_k u_{i_k} \in \lim_k \mathcal{D}(y_{i_k})$ exist.

Then by Remark 4.8.3 one has:

$$\delta(u, \ker\, f_{*x}) = \lim_k \delta(u_{i_k}, \ker\, f_{*x}) = 0 \qquad \text{and hence} \qquad u \in \ker\, f_{*x}\,.$$

In conclusion, the sequence of images $u'_{i_k} := f_{*y_{i_k}}(u_{i_k})$ of the unit vectors $\{u_{i_k} \in \mathcal{D}(y_{i_k})\}_k$ satisfies:

$$\lim_k f_{*y_{i_k}}(u_{i_k}) \;=\; f_{*x}(\lim_k u_{i_k}) \;=\; f_{*x}(u) \;=\; 0$$

[1]If f is not a submersion at x, $\ker f_{*x} \supset \lim_i \ker f_{*y_i}$ strictly and by Remark 4.9.2: $\delta(\ker\, f_{*x}, \lim_i \ker f_{*y_i}) = 1$.

[2]Here we did not need the hypothesis: $f : M \to M'$ is a submersion at x.

and cannot have a subsequence such that $\inf_h ||u'_{i_{k_h}}|| > 0$.

Proof. $(3 \Leftrightarrow 2)$. If $v' \in \lim_i f_{Y*y_i}(T_{y_i}Y) - \{0\}$ and $\{v'_i \in f_{Y*y_i}(T_{y_i}Y) - \{0\}\}_i$ is a sequence such that $\lim_i v'_i = v'$, by Remark 3.3.1) the unit vectors $u_i := \frac{v_i}{|| \, v_i \, ||}$ of the canonical liftings $v_i := f^{-1}_{Y*y|\mathcal{D}(y_i)}(v'_i) \in \mathcal{D}(y_i) - \{0\}$ of the v'_i satisfy:

$$|| \, v_i \, || \;=\; \frac{|| \, v'_i \, ||}{|| \, f_{Y*y_i|D(y_i)}(u_i)||} = \frac{|| \, v'_i \, ||}{|| \, f_{Y*y_i}(u_i)||} \,.$$

Hence, being $\{v'_i\}_i$ converging to v', the sequence of canonical liftings $\{v_i\}_i$ has an upper bounded subsequence $\{v_{i_h}\}_h$ if and only if the sequence of images $\{u'_i := f_{Y*y_i}(u_i)\}_i$ admits a subsequence $\{u'_{i_h} := f_{Y*y_{i_h}}(u_{i_h})\}_h$ such that $\inf_h ||u'_{i_h}|| > 0$. □

By recalling the definition 3.5 of the fonctions h_Y and H_Y with the same proof as above, Theorem 4.5 can be simply and analytically stated as follows:

Corollary 4.4. *Let $f : M \to M'$ be a C^1 map between C^1 manifolds, $\mathcal{W} \subseteq M$ and $\mathcal{W}' \subseteq M'$ Whitney stratifications such that the restriction $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ is a stratified surjective submersion.*

Let $X < Y$ be strata of $\mathcal{W}$, $x \in X$ and δ_Y the function:

$$\delta_Y : Y \to [0, \infty[\qquad , \qquad \delta_Y(y) = \delta'(T_yY, \ker f_{*y}) = \delta'(\mathcal{D}(y), \ker f_{*y}) \,.$$

If $f : M \to M'$ is a submersion at x, the following conditions are equivalent:

1) $\liminf_{y \to x} \delta_Y(y) > 0$ *;*

2) $\liminf_{y \to x} h_Y(y) > 0$ *;*

3) $\limsup_{y \to x} H_Y(y) < +\infty$ *.* □

We deduce then the following analytic sufficient condition for $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ to satisfy condition (D) at $x \in X < Y$:

Corollary 4.5. *Let $f : M \to M'$ be a C^1 map between C^1 manifolds, $\mathcal{W} \subseteq M$ and $\mathcal{W}' \subseteq M'$ Whitney stratifications such that the restriction $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ is a stratified surjective submersion.*

Let $X < Y$ be strata of $\mathcal{W}$ and $x \in X$. If $f : M \to M'$ is a submersion at x, we have:

$$\liminf_{y \to x} \; \delta_Y(y) > 0 \qquad \Longrightarrow \qquad f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}' \textit{ satisfies condition } (D) \textit{ at } x \in X < Y.$$

Proof. The proof follows easily by Theorem 4.5 (or Corollary 4.4) and Corollary 4.3. □

In Theorem 4.5 and its Corollaries 4.4 and 4.5, we gave sufficient conditions to obtain condition (D) at a point $x \in X < Y$ using a function $\delta_Y(y) = \delta'(T_yY, \ker f_{*y}) = \delta'(\mathcal{D}(y), \ker f_{*y})$ depending on the stratum Y and *intrinsically* defined with respect to the point $x \in X \subseteq \overline{Y}$.

We can also obtain a similar result using a function depending on Y and x, by setting this time $U := T_yY$ and $V := \ker f_{*x}$. In this case the essential mutual subspace U' is:

$$U' \;:=\; [T_yY]' \;= \perp (T_yY \cap \ker f_{*x} \; ; \; T_yY)$$

and we can define the function:

$$\delta_{Y,x} : Y \to [0, \infty[\qquad , \qquad \delta_{Y,x}(y) := \delta'(T_yY, \ker f_{*x}) \,.$$

A priori, $[T_yY]'$ is not equal to $\mathcal{D}(y)$ and $\delta_{Y,x}(y)$ is not equal to $\delta'(\mathcal{D}(y), \ker f_{*x})$.

Later on we will denote $\mathcal{D}'(y)$ for $[T_yY]'$.

Proposition 4.9. *Let $f : M \to M'$ be a C^1 map, $\mathcal{W} \subseteq M$ and $\mathcal{W}' \subseteq M'$ Whitney stratifications such that the restriction $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ is a stratified surjective submersion.*

Let $X < Y$ be strata of $\mathcal{W}$, $x \in X$ and $\{y_i\}_i \subseteq Y$ a sequence such that $\lim_i y_i = x$ and both limit below exist. If $f : M \to M'$ is a submersion at x, then:

$$\liminf_i \delta_{Y,x}(y_i) = 0 \qquad \Longleftrightarrow \qquad \liminf_i \delta_Y(y_i) = 0 \,.$$

Proof. For every $i \in \mathbb{N}$, let $\mathcal{D}'(y_i) := [T_{y_i}Y]'$ and $\mathcal{D}(y_i)$ be the vectors subspaces of $T_{y_i}Y$:

$$\mathcal{D}'(y_i) := \perp (T_{y_i}Y \cap \ker f_{*x} \,;\, T_{y_i}Y) \qquad \text{then} \quad \mathcal{D}'(y_i) \cap \ker f_{*x} = \{0\}$$

$$\mathcal{D}(y_i) := \perp (T_{y_i}Y \cap \ker f_{*y_i} \,;\, T_{y_i}Y) \qquad \text{then} \quad \mathcal{D}(y_i) \cap \ker f_{*y_i} = \{0\}\,.$$

By considering possibly subsequences we can suppose that both the limits exist:

$$\sigma' := \lim_i \mathcal{D}'(y_i) \qquad \text{and} \qquad \sigma := \lim_i \mathcal{D}(y_i)\,.$$

and since $f : M \to M'$ is a submersion at x, $\lim_i \ker f_{*y_i} = \ker f_{*x}$ (Proposition 3.5) and $\sigma' = \sigma$.

By Remark 4.12.3 and being every $\delta_{Y,x}(y_i) = \delta'(\mathcal{D}'(y_i), \ker f_{*x})$ a minimal essential distance, there exists, for every i, a unit vector $v_i \in \mathcal{D}'(y_i) \subseteq T_{y_i}Y$ such that:

$$\delta_{Y,x}(y_i) = \delta'(\mathcal{D}'(y_i), \ker f_{*x}) = \min_{u'_i \in \mathcal{D}'(y_i)\,,\, \|u'_i\|=1} \delta(u'_i,\ \ker\ f_{*x}) = \delta(v_i,\ \ker\ f_{*x})$$

and (by taking possibly a subsequence) we can also suppose that there exists $\lim_i v_i = v \in \sigma'$.

Similarly there exists a unit vector $w_i \in \mathcal{D}(y_i) \subseteq T_{y_i}Y$ such that:

$$\delta_Y(y_i) = \delta'(\mathcal{D}(y_i), \ker f_{*y_i}) = \min_{u_i \in \mathcal{D}(y_i)\,,\, \|u_i\|=1} \delta(u_i,\ \ker\ f_{*y_i}) = \delta(w_i,\ \ker\ f_{*y_i})$$

and such that there exists $\lim_i w_i = w \in \sigma$.

Proof ($\Rightarrow$). If $\liminf_i \delta_{Y,x}(y_i) = 0$, by extracting possibly a subsequence, one can write:

$$0 \;=\; \lim_i \delta_{Y,x}(y_i) \;=\; \lim_i \delta(v_i,\ \ker\ f_{*x}) \;=\; \delta(v,\ \ker\ f_{*x}) \qquad \text{and so:} \quad v \;\in\; \ker\ f_{*x}.$$

Let $p_i : T_{y_i}Y \to \mathcal{D}(y_i)$ be the orthogonal projection on $\mathcal{D}(y_i)$ and $\omega_i :=\ p_i(v_i) \in \mathcal{D}(y_i)$. Then:

$$\lim_i \omega_i \;=\; \lim_i p_i(v_i) \;=\; p_\sigma(v) \;=\; v \qquad \text{as} \qquad v \in \sigma' = \sigma.$$

Since $\omega_i \in \mathcal{D}(y_i)$ and by Remark 4.9.6) we find:

$$\delta_Y(y_i) \;=\; \delta(w_i, \ker f_{*y_i}) \;\le\; \delta(\omega_i, \ker f_{*y_i}) \;\le 2\delta(\omega_i, \ker f_{*x}) \;+\; \delta(\ker f_{*x}, \ker f_{*y_i})$$

and being $\lim_i \omega_i = v \in \ker f_{*x}$ and $\lim_i \ker f_{*y_i} = \ker f_{*x}$ we conclude:

$$0 \;\le\; \lim_i \delta_Y(y_i) \;\le\; 2\delta(v, \ker f_{*x}) \;+\; \delta(\ker f_{*x}, \lim_i \ker f_{*y_i}) \;= 0 + 0 = 0.$$

Proof ($\Leftarrow$). It is completely dual to the proof ($\Rightarrow$) and it could be omitted.

If $\liminf_i \delta_Y(y_i) = 0$, by extracting possibly a subsequence, one can write:

$$0 = \lim_i \delta_Y(y_i) = \lim_i \delta(w_i,\ \ker\ f_{*y_i}) = \delta(w,\ \lim_i \ker\ f_{*y_i}) \qquad \text{and so:} \quad w \in \lim_i \ker\ f_{*y_i} \subseteq \ker\ f_{*x}.$$

Let $p'_i : T_{y_i}Y \to \mathcal{D}'(y_i)$ be the orthogonal projection on $\mathcal{D}'(y_i)$ and $\theta_i := p'_i(w_i) \in \mathcal{D}'(y_i)$. Then:

$$\lim_i \theta_i = \lim_i p'_i(w_i) = p_{\sigma'}(w) = w \qquad \text{as} \qquad w \in \sigma = \sigma'.$$

Since $\theta_i \in \mathcal{D}'(y_i)$ and by Remark 4.9.6) we find:

$$\delta_{Y,x}(y_i) \;=\; \delta(w_i, \ker f_{*y_i}) \;\le\; \delta(\theta_i, \ker f_{*y_i}) \;\le 2\delta(\theta_i, \ker f_{*y_i}) \;+\; \delta(\ker f_{*y_i}, \ker f_{*x})$$

and being $\lim_i \theta_i = w \in \lim_i \ker f_{*y_i} = \ker f_{*x}$ we conclude:

$$0 \leq \lim_i \delta_{Y,x}(y_i) \leq 2\delta(w, \lim_i \ker f_{*y_i}) + \delta(\lim_i \ker f_{*y_i}, \ker f_{*x}) = 0+0=0. \qquad \square$$

Proposition 4.10. *With the same notations as in Theorem 4.5 and Proposition 4.9:*

$$\liminf_{y \to x} \delta_{Y,x}(y) > 0 \qquad \Longleftrightarrow \qquad \liminf_{y \to x} \delta_Y(y) > 0\,.$$

Proof. Both implications follow by Proposition 4.9 using that $\liminf_{y \to x} \delta(y)$ is the minimum value of adherence of any function δ. $\square$

Using the specific (to x) function $\delta_{Y,x}$, instead of the intrinsic (by x) δ_Y, Corollary 4.4 gives:

Theorem 4.6. *Let $f : M \to M'$ be a C^1 map between C^1 manifolds, $\mathcal{W} \subseteq M$ and $\mathcal{W}' \subseteq M'$ Whitney stratifications such that the restriction $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ is a stratified surjective submersion.*

Let $X < Y$ be strata of $\mathcal{W}$, $x \in X$ and $\delta_{Y,x}$ the function defined by

$$\delta_{Y,x} : Y \to [0,\infty[\quad , \quad \delta_{Y,x}(y) = \delta'(T_yY, \ker f_{*x}) = \delta'(\mathcal{D}'(y_i), \ker f_{*x})\,.$$

If $f : M \to M'$ is a submersion at x, the following conditions are equivalent:

1) $\liminf_{y \to x} \delta_{Y,x}(y) > 0$ *;*

2) $\liminf_{y \to x} h_Y(y) > 0$ *;*

3) $\limsup_{y \to x} H_Y(y) < +\infty$ *.*

Proof. ($1 \Leftrightarrow 2$). It follow by Proposition 4.10 and Corollary 4.4.

Proof. ($2 \Leftrightarrow 3$). It is formally the same of the proof of Theorem, 4.5. $\square$

By Theorem 4.6 and Theorem 4.4 (or Corollary 4.3) one has:

Corollary 4.6. *Let $f : M \to M'$ be a C^1 map between C^1 manifolds, $\mathcal{W} \subseteq M$ and $\mathcal{W}' \subseteq M'$ Whitney stratifications such that the restriction $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ is a stratified surjective submersion.*

For every strata $X < Y$ of $\mathcal{W}$ and $x \in X$ we have:

$$\liminf_{y \to x} \delta_{Y,x}(y) > 0 \qquad \Longrightarrow \qquad f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}' \textit{ satisfies condition } (D) \textit{ at } x \in X < Y\,. \qquad \square$$

Geometric meanings. The analytic conditions $\liminf_{y \to x} \delta_Y(y) > 0$ (in Theorem 4.5 and Corollary 4.4), and $\liminf_{y \to x} \delta_{Y,x}(y) > 0$ (in Theorem 4.6 and Corollary 4.6) for $f_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}'$ at $x \in X < Y$, have respectively the following geometric meanings:

"No limit of essential subspaces $\lim_{y_i \to x} \mathcal{D}(y_i)$ *has a common direction with* $\lim_i \ker f_{*y_i}$*".*

"No limit of essential subspaces $\lim_{y_i \to x} \mathcal{D}'(y_i)$ *has a common direction with* $\ker f_{*x}$*".*

So, in Exemple 2.1 for $f : \mathbb{R}^2 \times \{1\} \to \{0\} \times \mathbb{R} \times \{0\}$, $f(a,b,1) = (0,b,0)$ and $x = (0,0,1)$ one has:

$$\lim_{y \to x} \ker f_{*y} = \ker f_{*x} = L(1,0,0) \qquad \text{and for both choices of } Y \qquad \mathcal{D}(y) = \mathcal{D}'(y) = T_yY.$$

Hence the limits of the essential subspaces $\mathcal{D}(y)$ and the limits of the test function $\delta_Y(y)$ are:

1) For $\mathcal{W} = Y \cup \{x\} = \{y = (a, \tan(a), 1) : a > 0\} \cup \{x\}$, when Condition (D) holds (Fig. 1):

$$\begin{cases} \lim_{y\to x} \mathcal{D}(y) \;=\; \lim_{a\to 0} L\Big(1, \dfrac{1}{\cos^2(a)}, 0\Big) \;=\; L(1,1,0) \;\not\subseteq\; L(1,0,0) \\ \\ \text{and} \\ \\ \lim_{y\to x} \delta_Y(y) \;=\; \lim_{a\to 0}\; \sin\arctan\, \dfrac{1}{\cos^2(a)} \;=\; \dfrac{\sqrt{2}}{2} \;>\; 0\,. \end{cases}$$

2) For $\mathcal{W} = Y \cup \{x\} = \{y = (a, a^2, 1) : a > 0\} \cup \{x\}$ when Condition (D) does not hold (Fig. 2):

$$\begin{cases} \lim_{y\to x} \mathcal{D}(y) \;=\; \lim_{a\to 0} L(1, 2a, 0) \;=\; L(1,0,0) \subseteq\; L(1,0,0) \\ \\ \text{and} \\ \\ \lim_{y\to x} \delta_Y(y) \;=\; \lim_{a\to 0}\; \sin\arctan\,(2a) \;=\; 0\,. \qquad \square \end{cases}$$

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