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A LONG AND WINDING ROAD TO DEFINABLE SETS

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To David Trotman on the occasion of his 60th birthday.

Abstract. We survey the development of o-minimal structures from a geometric point of view and compare them with subanalytic sets insisting on the differences. The idea is to show the long way from semi-analytic to definable sets, from normal partitions to cell decompositions. Some recent results are discussed in the last section.

Introduction

This paper was conceived as a historical survey. In a sense it is a follow up of the book [DS1]. It does contain some recent results (mostly in the last section, e.g. on the Kuratowski convergence of definable sets from [DD]) and some results that are not new, but are not very well known; albeit, its aim is mostly didactical and historical. The younger author appreciated this historical insight as well as the intertwining of subanalytic geometry, Pfaffian geometry and o-minimal structures, and wishes to share it with others, as it proved useful to himself.

We have the feeling that definable sets and their cell decompositions have replaced nowadays every other kind of special sets and stratifications, especially in applications (for instance in control theory, cf. our later quotes). The cell decompositions have not necessarily the same proprieties as subanalytic stratifications (not only they may not be analytic, but even not $\mathcal{C}^\infty$-smooth cf. [LGR]). Other wrong beliefs are also quite popular (for instance that subanalytic sets form an o-minimal structure, which is not true). We spotted, as well, numerous omissions in various references by different authors. This is due partially to the fact that many important papers (especially those written in French) got forgotten.

This survey has two authors, which are (easily identifiable) mother and son. The older author worked in Łojasiewicz's group ever since 1967, presented Gabrielov's work [G] at Łojasiewicz's seminar (this was a starting point for the theory of subanalytic sets *à la polonaise*), wrote (with J. Stasica) the preprint [DS*] presenting the results obtained by Łojasiewicz's group and was even, by pure chance, present in Dijon when the Pfaffian sets were born there (in 1989, this was an idea of Robert Moussu developed this year by Claude Roche and Jean-Marie Lion and continued later cf. [L], [MR]...). The older author can be therefore considered as a witness to the development we describe here, which began in 1965, when Łojasiewicz published his IHES preprint on semi-analytic sets [Ł1], now accessible on line on the site of Michel Coste [CŁ]. Our survey will present the way that led from semianalytic to subanalytic, Pfaffian and definable sets (the order here is not as linear as most people tend to believe).

The younger author appreciated the historical knowledge that let him understand better definable sets and wishes to share it with others. He also contributed to the much modernized and completed book version [DS1] of the preprint [DS*].

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Let us remark that E. Bierstone and P. Milman, the authors of the very well written IHES text *Semi-analytic and subanalytic sets* [BM] were among the first to quote the preprint [DS*] that served them as a basis for their presentation of subanalytic sets (the *Fiber-Cutting Lemma* is, for instance, *lemme B* from the initial work [DŁS] of Denkowska, Łojasiewicz, Stasica). L. Van den Dries, who can be considered as a father of definable sets (cf. the book [vdD]) also knew the preprint [DS*].

As to our friend, David Trotman, we owe him a lot. We met very early in our careers and David, a world known specialist in singularities and in particular in stratifications, encouraged our work, asked questions that led to the writing of some of our papers, especially those concerning stratifications (like [DSW], [DW]) and, together with Bernard Teissier popularized the preprint [DS*]. Later, Trotman and Teissier played a very important role in the publication of its book version [DS1]. Many thanks to both of them.

The stratifications, a tool largely used by René Thom , were brought to Poland by Łojasiewicz, who was one of Thom's close friends. As we mention in the survey, Łojasiewicz had his own way of constructing different stratifications, to begin with normal partitions (they were a main ingredient used in Łojasiewicz's theory of subanalytic sets, as opposed to that of Hironaka, based on desingularization).

The so called 'Łojasiewicz group' in Kraków consisted of (in order in which they joined the group), the following Łojasiewicz's students: Krystyna Wachta, Zofia Denkowska, Jacek Stasica, Wiesław Pawłucki, Krzysztof Kurdyka and Zbigniew Hajto.

There are many sources of information about semi-analytic sets ([Ł1]), subanalytic sets ([H2], [DS1], [ŁZ]) and definable sets ([vdD], [vdDM], [C2]). In this paper we are only trying to put all this together in some order and in its historical context, with special interest given to stratifications. We also gathered in this survey a lot of information otherwise scattered in the literature (the bibliography is still far from being exhaustive, we included in it what we feel represents the different facets of the subject).

May it serve the younger!

1. A reminder

For a start, recall one of the (equivalent) definitions of an o-minimal structure (see [C2], [vdD]):

Definition 1.1. A *structure* on the field $(\mathbb{R}, +, \cdot)$ is a collection $\mathcal{S} = \{\mathcal{S}_n\}_{n\in\mathbb{N}}$, where each $\mathcal{S}_n$ is a family of subsets of $\mathbb{R}^n$ satisfying the following axioms:

(1) $\mathcal{S}_n$ contains all the algebraic subsets of $\mathbb{R}^n$;
(2) $\mathcal{S}_n$ is a Boolean algebra ([1]) of the powerset of $\mathbb{R}^n$;
(3) If $A \in \mathcal{S}_m$, $B \in \mathcal{S}_n$, then $A \times B \in \mathcal{S}_{m+n}$;
(4) If $\pi\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is the natural projection and $A \in \mathcal{S}_{n+1}$, then $\pi(A) \in \mathcal{S}_n$.

The elements of $\mathcal{S}_n$ are called *definable* (or *tame*) subsets of $\mathbb{R}^n$.
The structure $\mathcal{S}$ is *o-minimal* (*o* stands for *order*) if it satisfies the additional condition

(5) $\mathcal{S}_1$ is nothing else but all the finite unions of points and intervals of any type.

It is natural to introduce the following notion:

Definition 1.2. Given a structure $\mathcal{S}$, we call *definable* (in $\mathcal{S}$) any function $f\colon A \to \mathbb{R}^n$, where $A \subset \mathbb{R}^m$, such that its graph, again denoted f, belongs to $\mathcal{S}_{m+n}$.

[1] Recall that a family $\mathcal{S}$ of sets, subsets of $\mathbb{R}^n$ in our case, is a *Boolean algebra*, if $\varnothing \in \mathcal{S}$ and for every $A, B \in \mathcal{S}$, there is $A \cap B, A \cup B, \mathbb{R}^n \setminus A \in \mathcal{S}$.

Remark 1.3. Clearly, axiom (4) implies that if f is definable, its definition set $A \in \mathcal{S}_m$. The image, $f(A) \in \mathcal{S}_n$ since it coincides with $\pi(f \cap (A \times \mathbb{R}^n))$, where π is the natural projection onto $\mathbb{R}^n$, and $A \times \mathbb{R}^n \in \mathcal{S}_{m+n}$. Finally, the definability of $f = (f_1, \ldots, f_n)$ is equivalent to the definability of its components f_i.

Proposition 1.4. *Every o-minimal structure contains semi-algebraic sets. (cf. subsection 1.1)*

Proof. Indeed, by condition (1) it contains algebraic sets and thus it suffices to show that it contains all the sets of the form $\{x \in \mathbb{R}^n \mid P(x) > 0\}$ with P being a polynomial (axiom (2)). Any such set can be written as $\{x \in \mathbb{R}^n \mid \exists \varepsilon > 0\colon P(x) = \varepsilon\}$ and thus it can be written as the projection $\pi(A)$ by $\pi(x,t) = x$ of the algebraic set $\{(x,t) \in \mathbb{R}^n \times \mathbb{R} \mid t^2 P(x) = 1\}$. Condition (4) yields $\pi(A) \in \mathcal{S}_n$. □

Remark 1.5. It is easy to see that if $A \in \mathcal{S}_{m+n}$ and $B \in \mathcal{S}_n$, then the set

$$\{x \in \mathbb{R}^m \mid \exists y \in B\colon (x,y) \in A\}$$

is in $\mathcal{S}_m$, this set being the projection onto $\mathbb{R}^m$ of $A \cap (\mathbb{R}^m \times B)$. Since taking the complement changes the quantifier $\forall$ to $\exists$, the same is true for $\{x \in \mathbb{R}^m \mid \forall y \in B, (x,y) \in B\}$, i.e., this set belongs to $\mathcal{S}_m$.

1.1. **Semi-algebraic geometry.** (See e.g. [C1] or [BCR]). The definition of semi-algebraic sets is global. In fact, Łojasiewicz [Ł1] used the notion of sets 'described by' the functions of a given subring $\mathcal{A}$ of the ring of continuous real functions defined in $\mathbb{R}^n$. These are the sets of the form

$$A = \bigcup_{i=1}^{p} \bigcap_{j=1}^{q} \{x \in \mathbb{R}^n \mid f_{ij}(x) * 0\}$$

where $*$ stands for any of the signs $>, <, =$. Such sets form a Boolean algebra denoted $S(\mathcal{A})$.

If $\mathcal{A}$ is the ring of polynomials of n variables, $S(\mathcal{A})$ is the Boolean algebra of semi-algebraic sets.

Clearly, semi-algebraic sets verify the conditions (1), (2), (3), (5) of o-minimal structures. It suffices to check the condition (4) (projection property), the others being easy. This condition is verified thanks to the following theorem of Tarski-Seidenberg:

Theorem 1.6 (Tarski-Seidenberg)**.** *Let $\pi\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be the natural projection and let $A \subset \mathbb{R}^n \times \mathbb{R}$ be a semi-algebraic set. Then $\pi(A)$ is semi-algebraic, too.*

The classical geometric approach to this theorem is based on the following lemma.

Lemma 1.7 (Cohen — *Lemme de saucissonage*)**.** ***Classical version:*** *Let $P(x,t)$ be a polynomial in $n+1$ variables. Then there exists a finite partition of $\mathbb{R}^n$: $\mathbb{R}^n = \bigcup_{i=1}^p A_j$ into semi-algebraic sets A_j such that for any $i = 1, \ldots, p$, either $P(x,t)$ has constant sign for $x \in A_i$ and all $t \in \mathbb{R}$, or there is a finite number of continuous semi-algebraic functions $\xi_1 < \ldots < \xi_{p_i}$ on A_i such that for $x \in A_i$, $\{P(x,t) = 0\} = \{\xi_j(x), j = 1, \ldots, p_i\}$ and the sign of $P(x,t)$ depends only on the signs of $t - \xi_j(x)$, $j = 1, \ldots, p_i$.*

Łojasiewicz's version: *Let $\mathcal{A}$ be a ring of real continuous functions defined on a topological space X. Assume that each set from $S(\mathcal{A})$ has only a finite number of connected components, each of them belonging to $S(\mathcal{A})$. Then for any $E \in S(\mathcal{A}[t])$ there exists a finite partition $X = \bigcup_{i=1}^p A_i$ with $A_i \in S(\mathcal{A})$ and real functions $\xi_{A_i,1} < \ldots < \xi_{A_i,p_i}$, continuous on A_i (it may happen that there are none for some i), such that E is the union of sets from $S(\mathcal{A}[t])$ of one of the two forms below:*

$$B_{ik} := \{(x,t) \in A_i \times \mathbb{R} \mid \xi_{A_i,k}(x) < t < \xi_{A_i,k+1}(x)\}, k = 0, \ldots, p_i + 1,$$
$$\text{or } C_{ik} := \{(x, \xi_{A_i,\ell}(x)) \mid x \in A_i\}, \ell = 1, \ldots, p_i,$$

where $\xi_{A_i,0} \equiv -\infty$ *and* $\xi_{A_i,p_i+1} \equiv +\infty$.

Clearly, Łojasiewicz's version implies the classical one, as the assumption on the finiteness of the number of connected components follows by induction. Below we quote the original Łojasiewicz's proof of his version:

Proof. The set E is described by some $f_i(x,t) = \sum_{j=0}^m a_{ij}(x)t^{m-j}$, $i = 1, \dots n$ with $a_{ij} \in \mathcal{A}$. Let φ_{ik} denote the kth derivative of f_i with respect to t, here $k = 1, \dots, m$. Put $f_J := \prod_{(i,k)\in J} \varphi_{ik}$, where $J \subset I := \{1, \dots, n\} \times \{1, \dots, m\}$. Define for $r = 1, \dots, m, \infty$,

$$A_{J,r} := \{x \in X \mid f_J(x,t) = 0 \quad \text{has exactly} \quad r \quad \text{complex roots} \quad t\}.$$

It is easy to check that each $A_{J,r} \in S(\mathcal{A})$. For any fixed J, the sets $A_{J,r}$, $r = 1, \dots, m, \infty$ form a partition of X, whence we recover a partition of X from the connected components of the intersections $\bigcap_J A_{J,r_J}$. We call them $A_1, \dots, A_p$.

It is easy to see by applying Rouché's Theorem (in fact, Hurwitz theorem, which is a corollary for analytic functions) that for any A_j and any $J = \{(i,k) \in I \mid \varphi_{ik} \neq 0 \,\text{on}\, A_j \times \mathbb{R}\}$ one can find continuous functions $\xi_{A_j,1}(x) < \dots < \xi_{A_j,p_j}(x)$ such that

$$\{x \in A_j \mid f_J(x,t) = 0\} = \bigcup_{i=1}^{p_j} \xi_{A_j,i},$$

the latter denoting the graphs of $\xi_{A_j,i}$.

Now, since $f_J \neq 0$ on B_{jk}, then on this set either $\varphi_{ik} \neq 0$, or $\varphi_{ik} \equiv 0$, depending on whether $(i,k) \in J$ or not. On the other hand, for C_{jk} either $\varphi_{ik} \equiv 0$ on $A_j \times \mathbb{R}$ which is the trivial case, or $\varphi_{ik} \not\equiv 0$ on it. If the latter occurs, then the roots of $\varphi_{ik}(x,t) = 0$ over A_j are continuous functions $\xi_1(x) < \ldots < \xi_r(x)$. Since each graph ξ_ρ is contained in $\bigcup_{\iota=1}^{p_j} \xi_{A_j,\iota}$ and the graphs $\xi_{A_j,\iota}$ are open-closed in this union, there is a unique ι_ρ such that $\xi_\rho = \xi_{A_j,\iota_\rho}$. Hence, on C_{jk} one has either $\varphi_{ik} \equiv 0$, or $\varphi_{ik} \neq 0$ depending on whether $k = \iota_\rho$ for some ρ or not.

Finally, we show that $B_{jk}, C_{jk} \in S(\mathcal{A}[t])$. Let D be one of these sets. Then

$$D \subset T := \bigcap_{i=1}^n \bigcap_{k=0}^m \{x \in A_j \mid \varphi_{ik} \in \Theta_{ik}\},$$

where Θ_{ik} is either $\{t < 0\}$, or $\{0\}$, or $\{t > 0\}$. It suffices to prove now that in fact $D = T$. If there were a point $(a,t) \in T \setminus D$, then for some t' there would be $(a,t') \in D$. By Thom's Lemma ([2]), the set $(\{a\} \times \mathbb{R}) \cap T$ is convex, whence $\{a\} \times [t,t'] \subset T$. Whatever the form of D (either B_{jk} or C_{jk}), there exists $t_1, t_2 \in [t,t']$ such that $f_J(t_1,a) = 0$ while $f_J(t_2,a) \neq 0$. That is a contradiction, since there must be either $f_J \equiv 0$, or $f_J \neq 0$ on T depending on whether $\Theta_{ik} = \{0\}$ for some $(i,k) \in J$, or not.

It remains to observe that the sets B_{ik}, C_{ik} form a partition of $X \times \mathbb{R}$ and on each of them one has either $f_i \equiv 0$, or $f_i \neq 0$, which implies that E is the union of some of them. □

Remark 1.8. Under the assumptions of the Lemma above on $\mathcal{A}$ we have:

(1) Each $E \in S(\mathcal{A}[t])$ has only a finite number of connected components, each of them belonging to $S(\mathcal{A}[t])$; therefore, by induction, the same is true in $S(\mathcal{A}[t_1, \dots, t_n])$.
(2) If $\pi\colon X \times \mathbb{R} \to X$ is the natural projection, then $\pi(E) \in S(\mathcal{A})$ for $E \in S(\mathcal{A}[t])$; therefore, by induction, the same is true for $\pi\colon X \times \mathbb{R}^n \to X$ and $S(\mathcal{A}[t_1, \dots, t_n])$.

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[2] Thom's Lemma: *Let $P(t)$ be a polynomial of degree n. Then each set $\Delta_P := \bigcap_{k=0}^n \{t \in \mathbb{R} \mid P^{(k)}(t) \in \Theta_k\}$, where Θ_k is either $\{t < 0\}$, or $\{0\}$, or $\{t > 0\}$, is connected: an open interval, a point, or possibly void.* Indeed, for $n = 0$ there is nothing to do. If the lemma holds for $n-1$ take n and apply the lemma to P'. Then $\Delta_P = \Delta_{P'} \cap \{P(t) \in \Theta_0\}$. If $\Delta_{P'}$ is an open interval, then $P'(t) \neq 0$ in it and thus P is strictly monotone on $\Delta_{P'}$ and the lemma follows.

Taking $X = \{0\}$ and $\mathcal{A} = \mathbb{R}$ the first remark above yields by induction:

Theorem 1.9. *Every semi-algebraic set has a finite number of connected components, each of them semi-algebraic.*

The second remark for $\mathcal{A} = \mathbb{R}[x_1, \ldots, x_m]$ implies the Tarski-Seidenberg Theorem, also by induction.

Remark 1.10. The theorem of Tarski-Seidenberg itself implies that the image of a semi-algebraic set under any semi-algebraic mapping is semi-algebraic as in Remark 1.3. It is clear that semi-algebraic sets form an o-minimal structure.

The theory of semi-algebraic sets is well exposed in [C1], [C2], [BR], [BCR]. We list here some of their basic properties:

Theorem 1.11. *The Euclidean distance to a nonempty semi-algebraic set is semi-algebraic (i.e., has semi-algebraic graph).*

The obvious proof follows from the description of the graph and we easily obtain the following corollary.

Corollary 1.12. *If A is semi-algebraic, then the closure $\overline{A}$, the interior* $\mathrm{int}A$ *and the border ∂A are semi-algebraic as well.*

Remark 1.13. The theorem and corollary above still hold true if one changes the words *semi-algebraic* to *definable* (partly due to Proposition 1.4).

The most striking property of semi-algebraic sets is the existence of *explicit* uniform bounds, for example on the number of connected components. These bounds are nicely gathered in the book [YC] by G. Comte and Y. Yomdin.

1.2. **Definable sets.** By 'definable sets' we always mean 'definable in some given o-minimal structure $\mathcal{S}$'. For this part we refer the reader to the works [vdD], [C3] and the survey [vdDM].

It is worth saying a few words about the point of view of mathematical logic: o-minimal structures can be introduced in the following way. Given a family of functions (the 'vocabulary' of a language) $\mathcal{F} = \{\mathcal{F}_n\}_{n\in\mathbb{N}}$, $\mathcal{F}_n \subset \mathbb{R}^{\mathbb{R}^n}$, one considers the sets described by first-order formulæ, or, in other words, by the 'operations' $=$, $<$, $+$, $\cdot$ and quantifiers applied to functions from $\mathcal{F}$ or real numbers. The collection of all the sets obtained in this way in the spaces $\mathbb{R}^n$ is the structure denoted by $\mathbb{R}_\mathcal{F}$. To be more precise, a subset of $\mathbb{R}^m$ is said to be definable in $\mathbb{R}_\mathcal{F}$, if it belongs to the smallest collection of subsets of $\mathbb{R}^n$, $n \in \mathbb{N}$, which

(1) contains the graphs of addition and multiplication, and all the graphs of functions in $\mathcal{F}$, and of constant maps;
(2) contains the graph of the order relation $<$, and of the equality;
(3) is closed under taking Cartesian products, finite unions or intersections, complements, and images under linear projections.

As earlier, a function $f\colon \mathbb{R}^m \to \mathbb{R}^n$ is said to be definable if its graph is definable. If each definable set has finitely many connected components, then $\mathbb{R}_\mathcal{F}$ is o-minimal.

The model theoretic notion of the structure $\mathbb{R}_\mathcal{F}$ generated by $\mathcal{F}$ provides useful information about the real geometry of the sets and functions obtained this way. The starting point of this approach is the question of how much we have to extend a given language in order to describe the solutions of systems of differential equations written in it, for instance: to what class does the solution of analytic differential equations belong?

Note that functions of one variable are particularly important since they carry most of the information about the structure (in some sense the whole structure is obtained through projections of graphs).

For $\mathcal{F} = \varnothing$, the structure $\mathbb{R}_\varnothing$ is just the class of semi-algebraic sets studied already by Tarski. The o-minimality of such a structure $\mathbb{R}_\mathcal{F}$ means precisely that all its sets have a finite number of connected components. This fact is important e.g. for differential equations as it excludes oscillations. If we take $\mathcal{F}$ to be the convergent power series in a given polidisc, extendable by zero outside it ([3]), then $\mathbb{R}_\mathcal{F}$ is usually denoted $\mathbb{R}_{\mathrm{an}}$ (*restricted analytic functions*). It is model complete (it follows from [G], see below for this notion) and contains all the *globally* subanalytic sets (of which we will speak later on). The structure $\mathbb{R}_{\mathrm{Pfaff}}$ generated by the so-called Pfaffian functions (see later on) is o-minimal as well (cf. [W2]). This implies the o-minimality of $\mathbb{R}_{\mathrm{exp}}$ which is the structure generated by the exponential function.

One more remark: among the first four axioms of a structure on $\mathbb{R}$ the difficulties arise mostly for two of them, namely the projection property (4) (or *elimination of quantifiers*) and the operation of taking the complement in (2). The projection property is what is missing for semi-analytic sets (see Example 4.1) and thus the larger class of subanalytic sets is needed, but when these were introduced, the problem with axiom (2) appeared: how to prove that the complement of a subanalytic set is again subanalytic? This was solved first by A. Gabrielov [G]. That property is called *model completeness* of the structure (notion introduced by A. Robinson). In other words, if in the definition of $\mathbb{R}_\mathcal{F}$ the operation of taking the complement is superfluous, the structure is said to be model complete.

The most important tool from the geometric point of view is the *cell decomposition*:

Definition 1.14. A set $C \subset \mathbb{R}^m$ is called a *definable cell* if
(1) for $m = 1$, C is a point or an open, nonempty interval;
(2) for $m > 1$,

- either $C = f$ is the graph of a continuous, definable function $f\colon C' \to \mathbb{R}$, where $C' \subset \mathbb{R}^{m-1}$ ($\mathbb{R}^{m-1}$ is the subspace of the first $m-1$ variables in $\mathbb{R}^m$) is a definable cell; such a cell we shall call *thin*;
- or $C = (f_1, f_2)$ is a definable *prism*, i.e. $(f_1, f_2) = \{(x,t) \in \mathbb{R}^{m-1} \times \mathbb{R} \mid x \in C',\ f_1(x) < t < f_2(x)\}$, where $C' \subset \mathbb{R}^{m-1}$ is a definable cell and both functions $f_j\colon C' \to \mathbb{R} \cup \{\pm\infty\}$ are continuous, definable and such that $f_1 < f_2$ on C' and each f_j either takes all values in $\mathbb{R}$, or is constant.

Definition 1.15. Let $C \subset \mathbb{R}^{n+1}$ be a definable cell over a cell $C' \subset \mathbb{R}^n$. Then its dimension $\dim C$ is defined to be either $\dim C'$, if C is thin, or $\dim C' + 1$ if C is a prism. Of course, in $\mathbb{R}$, $\dim\{a\} = 0$ and $\dim(a,b) = 1$.

It is easy to check that for a cell $C \subset \mathbb{R}^n$ one has $\dim C = n$ iff C is open and $\dim C < n$ iff C is nowhere-dense. Moreover, there is always a definable homeomorphism sending C, call it h_C, on an open cell in $\mathbb{R}^{\dim C}$.

Definition 1.16. A cell C defined over a cell C' is said to be of class $\mathscr{C}$ ([4]), if for the defining function f, or f_i respectively, the composition $f \circ h_{C'}^{-1}$ ($f_i \circ h_{C'}^{-1}$ respectively) is of that class ([5]).

Definition 1.17. A cylindrical cell decomposition of $\mathbb{R}^{n+1}$ is a finite decomposition of $\mathbb{R}^{n+1}$ into pairwise disjoint cells whose projections onto the first n coordinates yield a cylindrical cell

[3] To be more precise: $\mathcal{F}_n$ consists of functions $f\colon \mathbb{R}^n \to \mathbb{R}$ which are analytic in $[-1,1]^n$ and vanish off this cube.

[4] e.g. class $\mathscr{C}^k$ with $k = 1, 2, \ldots, \infty, \omega$ (where ω means analycity)

[5] In particular, a $\mathscr{C}^k$ cell is a $\mathscr{C}^k$ submanifold of dimension $\dim C$.

decomposition of $\mathbb{R}^n$. The cell decomposition is said to be of class $\mathscr{C}$ or $\mathscr{C}^k$, $k = 1, 2, \ldots, \infty, \omega$, if all the cells are of that class.

A cell decomposition need not be a stratification in the sense of definition 2.23, since the frontier condition of the latter definition may fail to hold. To see this consider the decomposition of $\mathbb{R}^2$ into the following five cells: $C_1 = \{(x, y) \in \mathbb{R}^2 \mid x < 0\}$, $C_2 = \{0\} \times \mathbb{R}$, $C_3 = (0, +\infty) \times \{0\}$, $C_4 = \{x, y > 0\}$ and $C_5 = \{x > 0, y < 0\}$. Then $\overline{C_3} \setminus C_3$ cannot be obtained from the other cells. Turning a cell decomposition into a stratification requires a further refinement.

The following notion is identical with that of definition 2.24.

Definition 1.18. If $A_1, \ldots A_n \in \mathcal{S}_n$, then a cell decomposition $\mathcal{C}$ is said to be *compatible* (or *adapted to*) with these sets if for any $C \in \mathcal{C}$ and any i, there is $C \cap A_i \neq \varnothing \Rightarrow C \subset A_i$. In that case each A_i is the union of some cells from $\mathcal{C}$.

Cohen's Lemma 1.7 provides a semi-algebraic cell decomposition of a given semi-algebraic set. The generalization of this to arbitrary o-minimal structure is the following theorem (compare to Theorem 2.25):

Theorem 1.19 (Cylindrical cell decomposition of class $\mathscr{C}^k$)**.** *Given a finite family of definable sets $A_1, \ldots, A_n$ and a $k \in \mathbb{N}$ there is always a cylindrical cell decomposition of class $\mathscr{C}^k$ of $\mathbb{R}^n$ compatible with this family.*

Remark 1.20*.* Until quite recently it has been an open question whether an arbitrary o-minimal structure admits a $\mathscr{C}^\infty$ cell decomposition. The negative answer was given by O. Le Gal and J.-Ph. Rolin in [LGR], where an explicit example is given. Actually, most of the known o-minimal structures on the field $\mathbb{R}$ admit analytic cell decomposition. An earlier result — that the o-minimal structures generated by convenient quasianalytic Denjoy-Carleman classes admit $\mathscr{C}^\infty$ cell decomposition but no analytic cell decomposition was obtained in [RSW]. See also Remark 2.63.

Corollary 1.21. *A definable cell being connected, the theorem above implies that any definable set A has only finitely many connected components* ([6]) *and they all are definable, too (cf. Theorem 1.9). Moreover, they are open-closed in A.*

For a given set $E \subset \mathbb{R}^n$ let $cc(E)$ denote the family of its connected components. If $A \subset \mathbb{R}^m \times \mathbb{R}^n$, then we put $A_x := \{y \in \mathbb{R}^n \mid (x, y) \in A\}$. The following holds:

Theorem 1.22. *For any definable set $A \subset \mathbb{R}^m \times \mathbb{R}^n$ there is an N such that for all $x \in \mathbb{R}^m$, $\#cc(A_x) \leq N$.*

The possibility of obtaining a $\mathscr{C}^k$ cell decomposition for any k is based on the following:

Theorem 1.23. *Let $f\colon \Omega \to \mathbb{R}$ be a definable function on an open set $\Omega \subset \mathbb{R}^n$. Then for each $k \in \mathbb{N}$ there is a closed definable and nowhere-dense set $Z \subset \Omega$ apart from which f is of class $\mathscr{C}^k$.*

In particular:

Theorem 1.24. *For any definable $f\colon A \to \mathbb{R}$, $A \subset \mathbb{R}^n$, and any $k \in \mathbb{N}$, there is a $\mathscr{C}^k$ cell decomposition of $\mathbb{R}^n$, compatible with A and such that on any of its cells contained in A, f is of class $\mathscr{C}^k$.*

---

[6] Actually, they are even definably arcwise connected.

Remark 1.25. The Cell Decomposition Theorem provides also an interesting and useful observation:

Let $A \subset \mathbb{R}^n$ be definable and let $L \subset \mathbb{R}^n$ be a linear subspace. If for any $a \in \mathbb{R}^n$ the set $A \cap (L + a)$ is nowhere-dense in $L + a$, then A is nowhere-dense.

This clearly follows from the fact that A is nowhere-dense iff it does not contain an open cell and the trace of an open cell on $L + a$ is open.

Definition 1.26. One can define the dimension of a definable set to be

$$\dim A := \max\{\dim C \mid C \text{ is a cell}\colon C \subset A\}.$$

Proposition 1.27. *If $A \subset \mathbb{R}^n$ is definable, then $\dim A = n$ if and only if $\mathrm{int}A \neq \varnothing$ and $\dim A < n$ if and only if A is nowhere-dense. Moreover, for any definable $B \subset \mathbb{R}^m$ one has $\dim A \times B = \dim A + \dim B$; if $m = n$, then $\dim A \cup B = \max\{\dim A, \dim B\}$ and if $B \subset A$, then $\dim B \leq \dim A$. Finally, if $f\colon A \to \mathbb{R}^m$ is definable, then $\dim f(A) \leq \dim A$ ([7]).*

Remark 1.28. One can also prove that there is a definable bijection $f\colon A \to B$ between two given definable sets (in different ambient spaces), then $\dim A = \dim B$.

The next proposition shows how a cell decomposition induces a cell decomposition in subspaces:

Proposition 1.29. *Let $\mathcal{C}$ be a cell decomposition of $\mathbb{R}^m \times \mathbb{R}^n$ and, for $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$, let $\pi(x, y) = x$. Then*

(1) *$\tilde{\mathcal{C}} := \{\pi(C) \mid C \in \mathcal{C}\}$ is a cell decomposition of $\mathbb{R}^m$;*
(2) *Let $D \in \tilde{\mathcal{C}}$ and let $\mathcal{C}_D := \{C \in \mathcal{C} \mid \pi(C) = D\}$. Then for any $x \in D$ the sections $\{C_x \mid C \in \mathcal{C}_D\}$ are a cell decomposition of $\mathbb{R}^n$ and $\dim C_x = \dim C - \dim D$.*

Finally, o-minimal structures offer the possibility of triangulating definable sets:

Theorem 1.30. *Let $A \subset \mathbb{R}^n$ be a compact definable set and let $B_i \subset A$, $i = 1, \dots, k$ be definable. Then there is a simplicial complex $\mathcal{K}$, with vertices in $\mathbb{Q}^n$, and a definable homeomorphism $\phi\colon |\mathcal{K}| \to A$ sich that each B_i is a union of images by ϕ of open simplices from $\mathcal{K}$.*

One important fact that excludes from o-minimal structures such an untame behaviour as that of the graph of $\sin 1/x$ is the following theorem:

Theorem 1.31. *Let $A \subset \mathbb{R}^n$ be definable. Then $\dim \overline{A} \setminus A < \dim A$.*

We end with the following useful lemma:

Lemma 1.32 (Curve Selecting Lemma)**.** *If $A \subset \mathbb{R}^n$ is definable and $a \in \overline{A \setminus \{a\}}$, then there is a definable curve $\gamma\colon [0, 1) \to \mathbb{R}^n$, homeomorphic on its image and such that $\gamma(0) = a$, $\gamma((0, 1)) \subset A$.*

2. Locally semi-algebraic, semi-analytic and subanalytic sets

The properties of locally semi-algebraic, semi-analytic and subanalytic sets are often richer than these of general o-minimal structures. We are now in the local situation. We will still have Boolean algebras with the properties (1), (2), (3) and (5) of the definition of o-minimal structures but the projection property is not satisfied in general without additional hypotheses like the set being bounded in the direction of the projection.

[7]This expresses well the tameness of the topology involved. No pathologies as that of the Peano curve are permitted.

Definition 2.1 (Łojasiewicz)**.** Let $E \subset M$ where M is a real analytic variety ([8]). Then $\dim E = -1$, if $E = \varnothing$, or, if E is nonempty,

$$\dim E = \max\{\dim \Gamma \mid \Gamma \text{ an analytic submanifold} \colon \Gamma \subset E\}.$$

Definition 2.2. A point $a \in E$ is called smooth or *regular*, if $E \cap U$ is an analytic submanifold for some neighbourhood U of a. Then we define $\dim_a E := \dim E \cap U$ (it does not depend on the choice of U).

Remark 2.3. Clearly $\dim E = \max\{\dim_a E \mid a \text{ regular in } E\}$.

In the case of the dimension of a definable set A, we have for any $k \in \mathbb{N}$,

$$\dim A = \max\{\dim \Gamma \mid \Gamma \text{ a definable } \mathscr{C}^k \text{ submanifold} \colon \Gamma \subset A\}.$$

Proposition 2.4. *In any of the classes of sets discussed in this part the assertions of Proposition 1.27 and of Theorem 1.31 remain true.*

2.1. **Semi-algebraic and locally semi-algebraic sets.** An important feature of semi-algebraic functions is that their smoothness implies analycity. Even more, the smoothness of a semi-algebraic function is equivalent to it being an *analytic-algebraic* or *Nash* function (see [Ł1]):

Definition 2.5. An analytic function $f \colon U \to \mathbb{R}$, where $U \subset \mathbb{R}^n$ is open, is called a *Nash function* if for any $x_0 \in U$ there is a neighbourhood $V \ni x_0$ and a non-zero polynomial $P(x,t)$ for which there is $P(x, f(x)) \equiv 0$ in V ([9]).

Example 2.6. The analycity assumption in the definition is better understood in view of the following example of a (semi-algebraic) function $f(t) = \sqrt[3]{t^2}$ for $t \in \mathbb{R}$. The polynomial

$$P(x,y) = y^3 - x^2$$

annihilates the graph, but f is not even differentiable at the origin.

Theorem 2.7 (see [BCR])**.** *Given a semi-algebraic open set $U \subset \mathbb{R}^n$ and a semi-algebraic function $f \colon U \to \mathbb{R}$ the following equivalence holds:*

$$f \text{ is of class } \mathscr{C}^\infty \iff f \text{ is a Nash function.}$$

For what follows we refer the reader to [Ł1] where locally semi-algebraic sets were introduced (later they were known as Nash sets).

Definition 2.8. A locally semi-algebraic set in an open set $\Omega \subset \mathbb{R}^n$ is a set which in a neighbourhood of any point $a \in \Omega$ can be described by a finite number of polynomial equations or inequalities.

Remark 2.9. In particular, any set $E \subset \Omega$ described by Nash functions in an open semi-algebraic set Ω is locally semi-algebraic. This implies that a *semi-Nash set*, i.e., a set described locally by Nash functions, is a locally semi-algebraic set (and vice versa).
Recall that a *Nash submanifold* is a submanifold admitting an atlas of Nash functions. Let us observe that a point of a locally semi-algebraic set is regular if and only if in a small neighbourhood of this point the set is a Nash submanifold.

Proposition 2.10. *For any semi-algebraic set $E \subset \mathbb{R}^n$ there exists an algebraic set $V \subset \mathbb{R}^n$ such that $V \supset E$ and $\dim V = \dim E$.*

---

[8] In this text 'variety' and 'manifold' mean the same.

[9] If U is connected, it is easy to check that the same polynomial is good at each point, i.e., $P(x, f(x)) \equiv 0$ in the whole of U.

Proposition 2.11. *Each connected Nash submanifold $N \subset \mathbb{R}^n$ which is closed in a semi-algebraic set is semi-algebraic. In particular, the frontier $\overline{N} \setminus N$ is semi-algebraic iff N is semi-algebraic.*

The following proposition provides a link between semi-algebraic and locally semi-algebraic sets:

Proposition 2.12. *If U is an affine chart of $\mathbb{P}_n$ and $A \subset U$, then A is semi-algebraic in U if and only if A is locally semi-algebraic in $\mathbb{P}_n$.*

It can be proved that any semialgebraic function $f\colon \Omega \to \mathbb{R}$, where $\Omega \subset \mathbb{R}^n$ open, is Nash ouside a nowhere-dense semialgebraic set $Z \subset \Omega$. This implies that the category of semi-algebraic sets admits Nash-analytic cell decomposition.

Two more facts about semi-algebraic functions, that we provide with Łojasiewicz's proofs:

Lemma 2.13. *Let $f\colon (a, +\infty) \to \mathbb{R}$ be semi-algebraic. Then for some $b, N > 0$, there is $|f(x)| \leq x^N$, when $x > b$.*

Proof. Write $f = \bigcup_i \bigcap_j \{P_i = 0, Q_{ij} > 0\}$ and observe that due to univalence of the graph, for each i there is $P_i \not\equiv 0$. Let $P = \prod_i P_i$. Since $P(x, f(x)) \equiv 0$, then $f(x)$ is the root of the polynomial $P(x, \cdot)$ with polynomial coefficients $a_i(x)$, $i = 1, \ldots, d$. If $a_0(x)$ is the leading coefficient, then for some $b > 0$ there is $a_0(x) \neq 0$, if $x > b$. Now, $f(x)$ being a root, one has

$$|f(x)| \leq 2 \max_{i=1}^{d} \left(\frac{|a_i(x)|}{|a_0(x)|} \right)^{1/j}, \quad x > b,$$

and the lemma follows. □

Theorem 2.14 (Łojasiewicz's inequality)**.** *If $f, g\colon K \to \mathbb{R}$ are continuous semi-algebraic functions on a compact semi-algebraic set K and $f^{-1}(0) \subset g^{-1}(0)$, then for some $C, N > 0$ there is*

$$|f(x)| \geq C|g(x)|^N, \quad x \in K.$$

Proof. For $t > 0$ let $G_t := \{x \in K \mid t|g(x)| = 1\}$. These are compact semi-algebraic sets. If $G_t \neq \varnothing$, then let $m(t) := \max_{G_t} 1/|f|$, otherwise put $m(t) = 0$. The function $m\colon (0, +\infty) \to \mathbb{R}$ is semi-algebraic and thus by the preceding lemma, $m(t) \leq t^N$ for $t > b$. This means that for all $x \in K$, $|g(x)| \in (0, 1/b)$ implies $|g(x)|^N \leq |f(x)|$. Finally let

$$M := \max\{|g(x)|^N / |f(x)| \mid x \in K\colon |g(x)| \geq 1/b\}$$

and $C := \max\{M, 1\}$. The assertion follows. □

Remark 2.15. Taking $g(x) := \mathrm{dist}(x, f^{-1}(0))$ we obtain the semi-algebraic version of the general Łojasiewicz inequality:

$$(\#) \qquad |f(x)| \geq \mathrm{const.dist}(x, f^{-1}(0))^N, \quad x \in K.$$

On the other hand, by applying the theorem to the functions G and F defined as

$$G\colon K \times K \ni (x, y) \mapsto |f(x) - f(y)|$$

and $F(x, y) = ||x - y||$ we obtain the Hölder continuity of f (with exponent $1/N$).

Corollary 2.16 (Regular separation)**.** *If A, B are compact nonempty semi-algebraic sets, then for some constants $C, N > 0$,*

$$\mathrm{dist}(x, A) \geq C \mathrm{dist}(x, A \cap B)^N, \quad x \in B.$$

Proof. Apply the preceding theorem to $f(x) = \mathrm{dist}(x, A)$ and $g(x) = \mathrm{dist}(x, A \cap B)$. □

Remark 2.17. Both inequalities exclude any kind of *flatness*. In particular regular separation means that the possible tangency of two sets at a common point is not of infinite order.

Example 2.18. The above properties may not be satisfied in general o-minimal structures — for instance, $\mathbb{R}_{\exp}$ contains $\exp(t)$ and $\exp(-1/t^2)$ as definable functions: the first one does not satisfy the inequality in the lemma above, the second one does not satisfy the Łojasiewicz inequality where g is the distance to the origin (neither is its graph regularly separated from its domain).

Let us also note the following theorem, whose direct and elegant proof is presented in [S]:

Theorem 2.19. *Let A be semi-algebraic and let $A^{(k)} = \{x \in A \mid A \cap U$ is a k-dimensional analytic (Nash) manifold for some neighbourhood $U \ni x\}$. Then $A^{(k)}$ is semi-algebraic. In particular, the set of singular (i.e., non regular) points is semi-algebraic of dimension $< \dim A$.*

Remark 2.20. Finally, observe that for $\mathbb{R}^n$ the semi-algebraic homeomorphism $h(x) = x/(1+||x||)$ sends any semialgebraic set onto a semi-algebraic bounded set. This remark is important in view of the fact that subanalytic sets form an o-minimal structure only if we restrict ourselves to those of them which are 'bounded at infinity'. In that case we have of course an analogy between that class of sets (considered already in [T]) and semi-algebraic sets. See Definition 2.59.

2.2. **Semi-analytic sets (Łojasiewicz 1965).**

Definition 2.21. A set $A \subset \mathbb{R}^n$ (or, more generally $A \subset M$, where M is an analytic variety) is called *semi-analytic*, if for any $x \in \mathbb{R}^n$, there are a neighbourhood $U \ni x$ and analytic functions f_i, g_{ij} in U such that

$$A \cap U = \bigcup_{i=1}^{p} \bigcap_{j=1}^{q} \{x \in U \mid f_i(x) = 0, g_{ij}(x) > 0\}.$$

A mapping $f\colon E \to \mathbb{R}^n$ with $E \subset \mathbb{R}^m$ is said to be semi-analytic if its graph is a semi-analytic set in $\mathbb{R}^{m+n}$.

Example 2.22. Note that the description is *local* but not in the sense that we are moving along the set in question. The difference is better understood on the following example:
the graph $G := \{(x, \sin(1/x)) \mid x > 0\}$ is semi-analytic in $\mathbb{R}_+ \times \mathbb{R}$ but not in the whole of $\mathbb{R}^2$ because no point $(0, y)$ with $|y| \leq 1$ has a neighbourhood in which G can be described by a finite number of analytic equations and inequalities.

It is easy to check that the sets semi-analytic in a given analytic manifold form a Boolean algebra. Moreover, the union of a locally finite family of semi-analytic sets and the pre-image of a semi-analytic set by a semi-analytic mapping are semi-analytic. Semi-analytic sets have almost all the nice properties of semi-algebraic sets except that they need not be stable under proper projections.

The theory of semi-analytic and subsequently subanalytic sets originates in Łojasiewicz's solution to Laurent Schwartz's famous Division Problem (1957), see [Ł2] for an account. S. Łojasiewicz was the first person who meticulously built the fully systematized theory of semi-analytic sets (as in his preprint [Ł1]), using normal partitions which are a very clever tool, being a particular instance of a stratification:

Definition 2.23. A family of submanifolds of a manifold M is called a *stratification* of M if

- M is the union of the sets of the family
- the family is locally finite,
- the sets of the family are pairwise disjoint,

- for any leaf (or stratum) Γ belonging to this family, its frontier $\overline{\Gamma}\setminus\Gamma$ is the union of some members of the family with dimensions strictly smaller than $\dim\Gamma$.

Definition 2.24. Let f be a function of nonvanishing germ at a, a point of a real analytic manifold M. A stratification of a neighbourhood of a is said to be *compatible with* f if on any leaf of the stratification either $f\equiv 0$, or $f\neq 0$.
Let $E\subset M$. A stratification $\mathcal{N}$ is *compatible with the set* E if, for any stratum $\Gamma\in\mathcal{N}$, either $\Gamma\subset E$, or $\Gamma\cap E=\varnothing$ ([10]).

In 1965, Łojasiewicz presented a construction of the so called *normal partitions* which are special stratifications of *normal neighbourhoods*. The normal neighbourhoods form a topological basis of neighbourhoods. The normal partition of a neighbourhood starts with choosing the direction that is good for the Weierstrass Preparation Theorem and replacing the zeroes of an analytic germ by the zeroes of a distinguished polynomial. Then the construction goes down. At each step a good direction must be chosen (this makes the construction non-explicit), the distinguished polynomials are complexified and their determinants are studied in order to control multiple zeroes. All this ends up as a very detailed stratification called normal partition. For a thorough construction, consult [Ł1] and [DS1].

Theorem 2.25 (Łojasiewicz)**.** *Let $f_1,\dots,f_r$ be analytic functions defined in a neighbourhood of the origin of a finite dimensional real vector space. Then there exists a normal partition $\mathcal{N}$ at 0 compatible with $f_1,\dots,f_r$. (The same is true on any real analytic manifold.)*

Normal partitions play a crucial role in the theory of semi-analytic sets. The striking fact about the normal partitions is that the existence of such a partition compatible with a given set is a necessary and sufficient condition for the set to be semi-analytic:

Theorem 2.26 ([Ł1])**.** *A set $E\subset M$ is semi-analytic if and only if at any point $a\in M$ there is a normal partition compatible with E.*

Remark 2.27. Of course, given a finite family of semi-analytic sets in a real analytic manifold we can always find a normal partition compatible with them, which is just a restatement of Theorem 2.25.

Normal partitions are also used to prove the semi-analytic version of the Bruhat-Cartan-Wallace *Curve Selecting Lemma*:

Lemma 2.28 (Semi-analytic curve selecting lemma)**.** *Let E be a semi-analytic set and suppose that $a\in\overline{E\setminus\{a\}}$. Then there exist an analytic function $\gamma\colon(0,1)\to E$ yielding a semi-analytic curve and such that $\lim_{t\to 0+}\gamma(t)=a$.*

As the construction of normal partitions is somehow tiring, this strong (but elementary) tool was used almost uniquely by Polish mathematicians, with one important exception: Pfaffian varieties, the theory of which started in Dijon (see section 3).

Although the distance to a semi-analytic set need not be semi-analytic (it is subanalytic — see last section Theorem 4.3) we have the following:

Theorem 2.29. *The statement of Corollary 1.12 is true in the semi-analytic category. Moreover, the Łojasiewicz inequalities 2.14 and (#) as well as the regular separation 2.16 and Hölder continuity hold for semi-analytic sets.*

To finish this part let us quote two important theorems:

[10] In other words, $\Gamma\cap E\neq\varnothing\Rightarrow\Gamma\subset E$, just as in Definition 1.18.

Theorem 2.30 (Łojasiewicz)**.** *For any semi-analytic set A, the family $cc(A)$ is locally finite and each component $C \in cc(A)$ is semi-analytic.*

Theorem 2.31 (Łojasiewicz)**.** *An obvious analogon of Theorem 2.19 holds for semi-analytic sets.*

2.3. **Subanalytic sets (1975).** For this part we refer the reader to [DS1] for the most detailed presentation. Otherwise, there are: [BM] (a much more concise presentation but including an elementary approach to uniformization), and still less detailed, Łojasiewicz's book [ŁZ] written in Spanish and Łojasiewicz's short survey [Ł2]. And of course there is the preprint of H. Hironaka presenting his approach via desingularization [H2].

After completing the theory of semi-analytic sets in 1965 S. Łojasiewicz tried to study the projections of relatively compact semi-analytic sets but was stopped by the difficulty of the theorem of the complement.

The theorem of the complement was finally proved, independently, by H. Hironaka and A. Gabrielov. For H. Hironaka the theory of subanalytic sets was a kind of by-product of his famous desingularization theorem (compare [H1]). Gabrielov in [G] proved the theorem in an elementary way, reducing it to the study of complements of the graphs of functions. S. Łojasiewicz decided to build the theory of subanalytic sets from a scratch, using normal partitions and an idea of René Thom, which was later given the name of *Fibre-Cutting Lemma*.

Many mathematicians proved very interesting subanalytic results using Hironaka's approach. Let us quote M. Tamm or R. Hardt and his very interesting stratification theorems [Ht1]. All theorems about subanalytic sets can be obtained by Łojasiewicz's methods, too. They are gathered in [DS1].

Definition 2.32. A set E in a real analytic variety M is called *subanalytic* if for any $x \in M$ there is a neighbourhood $U \ni x$ such that $E \cap U = \pi(A)$, where $\pi \colon M \times N \to M$ is the natural projection, N is a real variety and A is semi-analytic and relatively compact in $M \times N$.

Remark 2.33. Projections of semi-analytic sets need not be subanalytic even if the sets are relatively compact and the projections are *proper*([11]) — see Example 4.1. That is a major difference with the definable case that should be borne in mind.

Remark 2.34. The union of a locally finite family of subanalytic sets and the intersection of a finite family of subanalytic sets are subanalytic.

Let us speak now about a very useful concept of S. Łojasiewicz, namely *N-relatively compact* sets and their projections.

Definition 2.35 ([Ł1])**.** Let M, N be two analytic varieties and let $\pi \colon M \times N \to M$ the natural projection. A subset $E \subset M \times N$ is called *N-relatively compact* if for any $A \subset M$ relatively compact the set $(A \times N) \cap E$ is relatively compact, too.

Remark 2.36. If the set E in the definition above is subanalytic in $M \times N$, then $\pi(E)$ is subanalytic, too.

Definition 2.37. A map $f \colon E \to N$, where $E \subset M$ is a nonempty subanalytic set, is subanalytic iff its graph f is subanalytic in $M \times N$.

Note that the domain of a subanalytic map need not be subanalytic, especially if its graph is not N-relatively compact.

[11] The pre-image of any compact set is compact

Definition 2.38. A map $f \subset M \times N$ is said to be *h-relatively compact* if the pre-image of any relatively compact subset of N is relatively compact. The map f is called *v-relatively compact* if the image of any relatively compact subset of M is relatively compact ([12]).

Remark 2.39. f is h-relatively compact iff its graph est M-relatively compact and f is v-relatively compact iff its graph is N-relatively compact. If f is proper, then it is h-relatively compact and each continuous $f\colon M \to N$ with closed domain is v-relatively compact.

Proposition 2.40. *Let $f \subset M \times N$ be a map and E a subanalytic subset of N. Any of the following conditions guarantees that $f^{-1}(E)$ is subanalytic:*

(a) *f is subanalytic v-relatively compact ([13]),*
(b) *E is relatively compact and f is subanalytic.*

Proof. Observe that $f^{-1}(E) = \pi(f \cap (M \times E))$, where $\pi\colon M \times N \to M$ is the natural projection, and apply 2.36. $\square$

Remark 2.41. Note that in o-minimal structures the assertion holds without any extra assumptions on the definable function f.

Proposition 2.42. *Let $f \subset M \times N$ be a map and H a subanalytic subset of M. Then any of the following conditions implies that $f(H)$ is subanalytic in N:*

(a) *f is subanalytic h-relatively compact ([14]);*
(b) *H is relatively compact and f is subanalytic;*
(c) *H is relatively compact and f is analytic in a neighbourhood of $\overline{H}$;*
(d) *f is analytic in a neighbourhood of $\overline{H}$ and $f|_{\overline{H}}$ is proper.*

Proof. It suffices to apply Remark 2.36 and observe that, if $\pi\colon M \times N \to N$ is the natural projection, then $f(H) = \pi(f \cap (H \times N))$. $\square$

We give below three other definitions of subanalytic sets (they all are equivalent):

Definition 2.43. A subset E of a real analytic variety M is called *subanalytic* if for each $x \in M$ there is a neighbourhood V such that $E \cap V$ is the image of a semi-analytic set by a proper analytic mapping.

Proposition 2.42 implies that this definition is equivalent to the previous one.

Theorem 2.44 (Gabrielov)**.** *If $E \subset M$ is subanalytic, then so is $M \setminus E$.*

Proposition 2.45. *Basic properties of a subanalytic set $E \subset M$:*

- *The closure and thus the interior (cf. Gabrielov's Theorem) of a subanalytic set are subanalytic.*
- *The connected components $C \in cc(E)$ are all subanalytic.*
- *The family $cc(E)$ is locally finite in M.*
- *If E is relatively compact, then $\#cc(E) < \infty$.*
- *E is locally connected.*
- *If $F \subset E$ is open-closed in E, then it is subanalytic.*
- *The Curve Selecting Lemma holds for subanalytic sets: if $a \in \overline{E \setminus \{a\}}$, then there is an analytic function $\gamma\colon (-1,1) \to M$ such that $\gamma(0) = a$ and $\gamma((0,1)) \subset E$. Moreover, γ is a homeomorphism on its image $\Gamma_{\gamma|_{[0,1)}}$ which is a semi-analytic arc of class $\mathscr{C}^1$.*

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[12] h comes from 'horizontally', while v stands for 'vertically', cf. one looks 'through' the graph.
[13] This is the case if e.g. f is analytic in M.
[14] This is the case if e.g. f is analytic in M and proper.

The proofs are based on the analoguous properties of semi-analytic sets and the Fibre-cutting Lemma (Lemmata A and B below).

Proposition 2.46. *Basic properties of subanalytic functions:*

- *The composition $g \circ f$ of subanalytic functions is subanalytic provided that either f is v-relatively compact, or g is h-relatively compact.*
- *If $f_i \dots, f_k \colon A \to N_i, i = 1, \dots, k$ are subanalytic, then the mapping*
$$(f_1, \dots, f_k) \colon A \to N_1 \times \dots \times N_k$$
is subanalytic, too.
- *The sum, the product and the quotient of real subanalytic functions defined on M is subanalytic, provided they are all locally bounded.*

Remark 2.47. Similar properties are satisfied by definable functions without extra assumptions. Note in particular that the composition of two subanalytic functions need not be subanalytic. The apparent analogy to semi-algebraic geometry or o-minimal structures is responsible for the fact that authors that use the subanalytic theory are often oblivious to that subtlety.

Definition 2.48. A *semi- or subanalytic leaf* in M is any analytic submanifold of M which is at the same time a semi- or, respectively, subanalytic set.

Example 2.49. The graph of $y = \sin 1/x$ is not subanalytic in the plane (note that the dimension of its frontier is again 1 which would be impossible for a subanalytic set, as Theorem 1.31 holds in the subanalytic category) although it is an analytic submanifold of it.

The following theorem of Łojasiewicz plays an important role in his theory of subanalytic sets without desingularization:

Theorem 2.50 (Łojasiewicz)**.** *Let Γ be a semi-analytic leaf in an affine space X. Denote by $G_k(X)$ the kth Grassmannian of X. Let $\tau \colon \Gamma \ni x \mapsto T_x\Gamma \in G_k(X)$, where $k = \dim \Gamma$, be the tangent mapping ($T_x\Gamma$ is the tangent space at x). Then for any semi-algebraic set $E \subset G_k(X)$, the pre-image $\tau^{-1}(E)$ is semi-analytic in X.*

For the subanalytic generalization see Theorem 4.23.

The key role in the subanalytic theory is played by the following lemmata suggested by René Thom (see [DLS]):

Lemma (A) (Decomposition)**.** *Let A be a semi-analytic, relatively compact subset of real, finite-dimensional vector space X. Assume that $X = U \oplus V$ is the direct sum of two vector spaces and let $\pi \colon X \to U$ be the projection parallel to V. Assume that $G_k(X)$ is decomposed into a finite number of open semi-algebraic sets: $G_k(X) = \bigcup G_i^{(k)}$. Then there exists a finite family of semi-analytic leaves $\{\Gamma_j\}$ such that $A = \bigcup \Gamma_j$ and*

(1) *the rank* $\mathrm{rk}\ \pi_{\Gamma_j}$ *is constant on each Γ_j,*
(2) *the Γ_j are members of some normal partitions,*
(3) *for any j there is an i such that $\tau(\Gamma_j) \subset G_i^{(k)}$ where $k = \dim \Gamma_j$.*

Lemma (B) (Replacement)**.** *Let A, X, U, V, π and $G_i^{(k)}$ be as in Lemma A. Then there is a finite family of semi-analytic leaves $\{\Gamma_j\}$ such that $\Gamma_j \subset A, \pi(A) = \pi(\bigcup \Gamma_j)$ and*

(1) *for any j, π_{Γ_j} is an immersion,*
(2) *the Γ_j are members of normal partitions,*
(3) *for any j there is an i such that $\tau(\Gamma_j) \subset G_i^{(k)}, k = \dim \Gamma_j$.*

Remark 2.51. If E is semi-analytic, then $\tau^{-1}(E)$ is only subanalytic (see 4.23), but in case where Γ is semi-algebraic, τ is semi-algebraic as well.

Hironaka started his theory with a different definition of subanalytic sets:

Definition 2.52. A set E is called *subanalytic* if for any point of M there is a neighbourhood V such that

$$E \cap V = \bigcup_{i=1}^{p} f_{i1}(A_{i1}) \setminus f_{i2}(A_{i2})$$

where f_{ij} are analytic and proper and A_{ij} are analytic sets.

The fourth definition of subanalytic sets is:

Definition 2.53. A subset $E \subset M$ is called *subanalytic in M* if for any point of M there is a neighbourhood V such that

$$E \cap V = \bigcup_{i=1}^{p} f_{i1}(M_{i1}) \setminus f_{i2}(M_{i2}),$$

with $f_{ij}\colon M_{ij} \to M$ analytic and proper, and this time M_{ij} analytic varieties.

Theorem 2.54 (see [DS1] for a proof)**.** *All four definitions of subanalytic sets are equivalent.*

Finally let us recall other important theorems:

Theorem 2.55 (Łojasiewicz)**.** *The Łojasiewicz inequality 2.14 and (#) as well as the regular separation 2.16 and Hölder continuity of functions hold for subanalytic sets.*

Theorem 2.56 (Gabrielov)**.** *Let $E \subset M \times N$ be a relatively compact subanalytic set, where M, N are analytic varieties. Then there is a constant N such that $\#cc(E_x) \le N$ for all $x \in M$.*

A deep result of W. Pawłucki below is a subanalytic version with parameter of the well-known complex Puiseux Theorem:

Theorem 2.57 ([P1])**.** *Let X, Y be two real, finite-dimensional vector spaces, Γ a subanalytic leaf relatively compact in X, $\Theta\colon \Gamma \times (0,1) \to Y$ an analytic map which is subanalytic in $X \times \mathbb{R} \times Y$ and bounded.*

Then there exists a closed subanalytic set $E \subset \Gamma$, $\dim E < \dim \Gamma$ and $k \in \mathbb{N}$ such that: for all $a \in \Gamma \setminus E$ the map $(x,t) \mapsto \Theta(x, t^k)$ has an analytic extension to a neighbourhood of $(a, 0)$ in $\Gamma \times \mathbb{R}$.

Using this K. Wachta obtained an important version of the Curve Selecting Lemma 2.28 for open subanalytic sets:

Theorem 2.58 (Wachta)**.** *Let $E \subset \mathbb{R}^n$ be an open subanalytic set and $a \in \overline{E}$. Then the arc from the Curve Selecting Lemma can be chosen semi-algebraic (i.e., Nash).*

Of course, the openness assumption is unavoidable due to the existence of transcendental curves.

At this point we stress again the fact that subanalytic sets do not form an o-minimal structure ([15]). They will, if we restrict ourselves to the so-called *globally (or totally) subanalytic sets*:

[15]The difference in behaviour of subanalytic and definable sets may be illustrated by the main result of [Di], see the last section.

Definition 2.59. A subanalytic set $E \subset \mathbb{R}^n$ is called *globally subanalytic* if its image by the semialgebraic homeomorphism $h(x) = x/(1 + ||x||)$ sending it to the unit Euclidean ball is subanalytic.

Theorem 2.60. *Globally subanalytic sets form an o-minimal structure which coincides with* $\mathbb{R}_{\mathrm{an}}$.

Remark 2.61. The same class of sets is obtained starting from functions *subanalytic at infinity* (see [T], see also [DS1]), i.e., such subanalytic functions $f\colon M \to \mathbb{R}$ which are subanalytic in $M \times \mathbb{S}^1$.

We end with a very useful lemma of K. Kurdyka, generalizing a result of M. Tamm (for $\mathscr{C}^k$), and its application:

Lemma 2.62 (Kurdyka)**.** *Let $f\colon U \to \mathbb{R}$ be a function subanalytic at infinity, $U \subset \mathbb{R}^n$ an open set. Then there is $k \in \mathbb{N}$ such that for any $x \in U$, if f is of Gâteaux class $\mathscr{G}^k$ ([16]) in a neighbourhood of x, then f is analytic at x.*

Remark 2.63. In connection with Remark 1.20 we may observe that this lemma readily implies that the structure $\mathbb{R}_{\mathrm{an}}$ admits analytic cell decomposition (compare Theorem 1.19).

This was used by Kurdyka to obtain a desingularization-free proof of the following:

Theorem 2.64 (Tamm [T])**.** *For any subanalytic set E the set of singular points $E \setminus \mathrm{Reg}E$ is subanalytic of dimension strictly smaller than* $\dim E$.

Remark 2.65. There is no direct counterpart of the subanalytic Puiseux Theorem or the lemma above in general o-minimal structures (a necessary condition would be their polynomial boundedness, cf. Definition 4.5). Tamm's Lemma can be extended to the structure $\mathbb{R}_{\mathrm{an},f_r,r\in\mathbb{R}}$ defined by the restricted analytic functions together with $f_r(t) = t^r$ for $t > 0$ and $f_r(t) = 0$ for $t \leq 0$. This implies analytic cell decomposition in the structure. See [vdDM] for details.

3. Pfaffian varieties and subpfaffian sets

This case is treated separately because it is much more recent than those dealt with in sections 1 and 2 and has an interesting history, often forgotten when Pfaffian sets are considered only on the ground of the model theory.

Subanalytic sets are insufficient for studying, for instance, the problems that arise in differential equations. Let us quote the following example from [MR]:

$$\begin{cases} \dot{x} = y, \\ \dot{y} = x^2. \end{cases}$$

The solutions of such a simple polynomial system are flat functions const.$\exp(-1/x)$ which are not subanalytic, but still quite regular, not to mention the fact that they arose from a simple polynomial dynamical system.

Outside France the history of Pfaffian varieties and the context in which they were born are totally unknown. And this despite the fact that [Ho2] contains a good historical introduction about how Pfaffian, semi-Pfaffian and sub-Pfaffian sets came into being. It all started with

[16] Recall that a function $f\colon U \to \mathbb{R}$ with $U \subset \mathbb{R}^n$ open is of class $\mathscr{G}^k$ in U if at any point $x \in U$ f possesses its kth Gâteaux derivative: for any $h \in \mathbb{R}^n$, the function $t \mapsto f(x + th)$ is k times differentiable at zero and the kth derivative is a homogeneous polynomial of degree k in h.

Hilbert XVIth problem and the works of Khovanskiĭ (see [Kh], [MR], [W2]). Hilbert XVIth problem deals with polynomial dynamical systems in the plane:

$$(PDS) \qquad \begin{cases} \dot{x} = P(x,y), \\ \dot{y} = Q(x,y), \end{cases}$$

and the question whether their *limit cycles* (closed trajectories that are isolated in the set of all closed trajectories of the system) can accumulate.

Extensive work was done on the subject in France and in Russia in the late 80's. Let us recall the names of Ilyashenko and Trifonov as well as those of Roussarie, Moussu, Ecalle and Ramis. Hilbert's question went further (Hilbert wanted to obtain a formula relating the maximal number of limit cycles to the degrees of P and Q above) but just their non-accumulation was a very difficult problem. As limit cycles can only accumulate on limit sets (cf. e.g. [DR]), it is possible to write a generalization of the classical Poincaré map, called the map of first return as it associates to the starting point (time t) the point of the first return to the curve we chose as transversal to the limit set, $\gamma(t)$. Back in 1988 R. Moussu started studying the properties of such mappings in order to show that $\gamma(t) - t$, even when it is not analytic, has no accumulation of zeroes. The map $\gamma(t)$ is seldom subanalytic — it often comes out infinitely flat. The idea was then to show it cannot oscillate.

Since solving (PDS) is equivalent to studying $\omega = 0$, where ω is the differential form (i.e., Pfaffian form) $\omega = -Q(x,y)dx + P(x,y)dy$, the notion of Pfaffian varieties was introduced by R. Moussu and C. Roche and studied, initially by Moussu, Roche and J.-M. Lion. There is a very good survey about that written by Moussu [M], based on [MR].

Definition 3.1. A *Pfaffian hypersurface* in $\mathbb{R}^n$ is a triplet (V, ω, M), where $M \subset \mathbb{R}^n$ is open and semi-analytic, ω is an analytic one-form defined on a neighbourhood of $\overline{M}$ and V is a maximal integral variety of $\omega = 0$ in M, smooth and of codimension 1 ([17]).

In other words we are given a codimension one foliation of a neighbourhood of $\overline{M}$ having V as one of its leaves and no singularities on M.

Definition 3.2. Let $X \subset \mathbb{R}^n$. (V, ω, M) is *of Rolle in X* (or just *of Rolle*, if $X = M$), if for any analytic $\gamma\colon [0,1] \to X \cap M$ there is a $t \in [0,1]$ such that $\gamma'(t) \in \mathrm{Ker}\omega(\gamma(t))$ ([18]).

In other words, any analytic path in $X \cap M$ connecting two points of V is tangent at some point to the field of hyperplanes defined by $\omega = 0$. In particular this excludes spiralling.

Definition 3.3. A Pfaffian hypersurface (V, ω, M) is *separating*, if the complement $M \setminus V$ has exactly two connected components whose common border in M is V.

By a theorem of Khovanskiĭ, a separating Pfaffian hypersurface is always of Rolle. The converse is not true as can be seen by considering $M = \mathbb{R}^2 \setminus \{0\}$ and $\omega = x^2dy - ydx$. Any integral curve of $\omega = 0$ is a Pfaffian hypersurface of Rolle and thus in particular the graphs of const.exp$(-1/x)$, $x > 0$. But their complement in M is connected. Besides, that example shows that in general V is just an analytic immersed submanifold which is not semi-analytic in $\mathbb{R}^n$.

Theorem 3.4 ([MR]). *Let $S(\omega) = \{x \mid \omega(x) = 0\}$ be the singular locus of ω. If $M \setminus S(\omega)$ is simply connected, then any Pfaffian hypersurface (V, ω, M) is of Rolle. If ω is integrable* ([19])*, then for any Pfaffian hypersurface (V, ω, M), V is a leaf of the foliation defined by ω.*

[17] That is to say: $\omega(x) \neq 0$ if $x \in V$, $\mathrm{Ker}\omega = T_xV$ and V is the maximal variety with this property among all the connected immersed subvarieties of M.

[18] In some sense that is an inverse approach to the classical Rolle Theorem: think of $\omega = dy$ in $\mathbb{R}^2$ and any differentiable function $y = \gamma(x)$ such that e.g. $\gamma(0) = \gamma(1) = 1$ — at some point t there is $\gamma'(t) = 0$.

[19] In the sense that $\omega \wedge d\omega = 0$ cf. the Frobenius Theorem.

Theorem 3.5 ([MR])**.** *Let $X \subset \mathbb{R}^n$ be semi-analytic and bounded and let $\omega_1, \dots, \omega_k$ be analytic one-forms in a neighbourhood of $\overline{M}$, where $M \subset \mathbb{R}^n$ is an open semi-analytic set. Then there exists a natural number $b = b(M, X, \omega_1, \dots, \omega_k)$ such that $\#cc(X \cap V_1 \cap \ldots \cap V_k) \le b$, where (V_i, ω_i, M) are Pfaffian hypersurface of Rolle.*

Remark 3.6. The last theorem implies Łojasiewicz's Theorem bounding the number of connected components of the sections of a semi-analytic set.

The interesting point here is that this is the only case of applications of Łojasiewicz's normal partitions outside Poland. Despite the fact that Łojasiewicz did this work in France (his preprint was published in 1965 by IHES), the normal partitions were almost exclusively used in Poland. Applying them to study the sets that appear as solutions of differential equations was, indeed, a very original, ingenuous and unexpected way to use them.

This happened before the o-minimal structures were introduced.

Lion and Rolin [LR] proved that relatively compact Rolle (i.e., non-spiralling) leaves of a real analytic foliations belong to a class of stratifiable subsets of $\mathbb{R}^n$ which is stable under intersection, union, set difference, linear projections and closure. That means that Rolle leaves belong to an o-minimal structure.

The basic properties of Pfaffian hypersurfaces are all gathered (with proofs) in the article of R. Moussu and C. Roche. Later, numerous other extremely useful properties of Pfaffian sets were proved. For instance Lion [L] showed, (with the use of Łojasiewicz's normal partitions) that there is a semi-analytic stratification of a neighbourhood of each point $a \in \mathbb{R}^n$, compatible with an analytic differential one-form ω and a semi-analytic open set M. This stratification allows a local decomposition of every integral hypersurface V of $\omega = 0$ into 'plaques' . Every leaf is the graph of an analytic function and if a is in the closure of a leaf, then a Pfaffian curve ending in a with a tangent lies in V. Lion and Roche obtained a Pfaffian Curve Selecting Lemma and then Lion proved a Pfaffian version of the Łojasiewicz inequality.

A natural thing is to construct *subpfaffian* sets starting from *semipfaffian* sets defined using intersections of leaves of Pfaffian foliations with the strata of Łojasiewicz's normal partitions (just like it was done for subanalytic sets). This way of proceeding originates in a question asked by R. Moussu and M. Shiota — what do we obtain by adding to the class of subanalytic sets the solutions of Pfaffian equations? And this is how the whole theory is presented in the interesting paper [Ho1]. (In what follows we can replace $\mathbb{R}^n$ by an analytic manifold N.) Semipfaffian geometry was suggested already by [L] or [MR]. In [Ho1] Z. Hajto proved a kind of analog of Gabrielov theorem on the complement 2.44. We present it hereafter.

Definition 3.7. A normal partition $\mathcal{N}$ is said to be *strongly adapted* to a finite family of Pfaffian hypersurfaces $\mathcal{V} := \{(V_i, \omega_i, M_i),\}_{i \in I}$ if it is adapted to $\{M_i\}_i$ and any subfamily of $\{\omega_i\}$ in the sense that for any leaf $\Gamma \in \mathcal{N}$ there are $\omega_{i_1}, \dots, \omega_{i_k}$ forming a base at each point $x \in \Gamma$ for the linear span (in $(\mathbb{R}^n)^*$) of $\{\omega_i(x)\}$.

Then by [L], for any leaf $\Gamma \in \mathcal{N}$ such that all the hypersurfaces from $\mathcal{V}$ is of Rolle for paths in Γ, the collection $\mathcal{V}_\Gamma := \{\bigcap_{i \in J} V_i \cap \Gamma\}_{J \subset I}$ is a finite family of analytic submanifolds with normal crossings in Γ; we call them *Pfaffian leaves*. These induce a stratification of Γ when we consider the connected components of $N_k \setminus N_{k-1}$ (with $N_{-1} = \varnothing$) where $N_k = \bigcup\{L \in \mathcal{V}_\Gamma \mid \dim L \le k\}$, $k = 0, \dots, \dim \Gamma$. These connected components are called *semi-pfaffian leaves*.

Definition 3.8. A subset $E \subset \mathbb{R}^n$ is *semi-pfaffian* (respectively: *basic semi-pfaffian*) if at every point $a \in \mathbb{R}^n$ there is a finite family of Pfaffian hypersurfaces $\mathcal{V}$ defined for neighbourhoods $M_i \ni a$ and a normal partition $\mathcal{N}$, defined in a normal neighbourhood $U \ni a$, strongly adapted to $\mathcal{V}$ and such that $E \cap U$ is a finite union of semipfaffian leaves. (respectively: of Pfaffian leaves defined by some strata of $\mathcal{N}$).

Locally finite unions and intersections and the Cartesian product of semipfaffian sets are semipfaffian. The family of connected components of a semipfaffian set is locally finite and the components are semipfaffian as well. However, there lacks the theorem on the closure of a semipfaffian set (and this is exactly a theorem one needs in the subanalytic category in order to prove the Gabrielov Theorem on the complement of a subanalytic set).

Definition 3.9. A subset $E \subset \mathbb{R}^n$ is *subpfaffian* if each point $a \in \mathbb{R}^n$ has a neighbourhood U such that $E \cap U = \pi(A)$ where $A \subset \mathbb{R}^n \times \mathbb{R}^k$ is a relatively compact *basic* semipfaffian set and $\pi\colon \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ is the natural projection.

Again locally finite unions and intersections remain in the category as well as the connected components which again form a locally finite family. Moreover, the projection on $\mathbb{R}^n$ of a $\mathbb{R}^k$-relatively compact subpfaffian set $E \subset \mathbb{R}^n \times \mathbb{R}^k$ is subpfaffian. In [Ho2] lemmata A and B are proved for subpfaffian sets. We remark that by a result of Cano, Lion and Moussu, the frontier of a Pfaffian hypersurface of Rolle is a subpfaffian set.

Definition 3.10. A semipfaffian set $E \subset \mathbb{R}^n$ is *subregular* if $\overline{E} \setminus E$ is contained in a closed subpfaffian set of dimension $< \dim E$ (the dimension being computed in the sense of Łojasiewicz 2.1).

Theorem 3.11 (Hajto)**.** *Any basic semipfaffian set is subregular.*

Remark 3.12. This theorem implies that the closure of any subpfaffian set is subpfaffian.

Theorem 3.13 (Hajto)**.** *The complement of a subpfaffian set is a subpfaffian set.*

All this is a good starting point for further study of the solutions of Pfaffian equations.

There is also another approach to *Pfaffian geometry* and we really do mean *another*, since until now nobody has compared $\mathbb{R}_{\mathrm{Pfaff}}$ with the following construction.

Definition 3.14. A $\mathscr{C}^1$ function $f\colon \mathbb{R}^n \to \mathbb{R}$ is called *Pfaffian* if there exist $\mathscr{C}^1$ functions $f_1, \ldots, f_k\colon \mathbb{R}^n \to \mathbb{R}$ with $f_k = f$, such that

$$\frac{\partial f_i}{\partial x_j}(x) = P_{ij}(x, f_1(x), \ldots, f_i(x)), \quad i = 1, \ldots, k, j = 1, \ldots, n,$$

for some polynomials P_{ij}.

The exponential function is clearly a Pfaffian function. Actually, any exponential polynomial

$$f(x_1, \ldots, x_n) := P(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n}),$$

where P is a polynomial in $2n$ variables, is a Pfaffian function. By a theorem of Khovanskiĭ [Kh], any set of the form $f^{-1}(0)$ where f is Pfaffian, has only finitely many connected components. Using these functions one constructs the structure $\mathbb{R}_{\mathrm{Pfaff}}$. It has remained for long an open question whether this structure is o-minimal. In 1991, A. J. Wilkie [W1] proved the theorem of the complement (an analogous to the Gabrielov theorem for subanalytic sets) for geometric cathegories that include functions of the form $P(x_1, ..., x_n, \log x_1, ..., \log x_n)$ or again $P(x_1, ..., x_n, \exp(x_1), ..., \exp(x_n))$.
Finally, in 1999 it was proved by Wilkie that:

Theorem 3.15 ([W2])**.** *The structure* $\mathbb{R}_{\mathrm{Pfaff}}$ *is o-minimal.*

4. Relations and differences between the classes of sets introduced so far

We start with observing that the following inclusions of the Boolean algebras we were talking about hold:

semi-algebraic sets $\subset$ locally semi-algebraic sets $\subset$ semi-analytic sets $\subset$ subanalytic sets.

In other words we have an increasing chain of classes used as a model for introducing o-minimal structures.

The simplest example of a semi-analytic set whose projection is no longer semi-analytic was given by Łojasiewicz using the Osgood transcendental function $f(x,y) = (xy, xe^y)$.

Example 4.1. Let $A := \{(x,y,xy,xe^y) \mid x,y \in (0,1)\}$ and consider $\pi(x,y,u,v) = (x,u,v)$. Then $\pi(A) = \{((x,y,xe^{y/x}) \mid 0 < y < x < 1\}$ and this set is not semi-analytic at $0 \in \overline{\pi(A)}$. If this were the case, there would be a description

$$\pi(A) \cap U = \bigcup_{i=1}^{p} \bigcap_{j=1}^{q} \{f_i(x,y,z) = 0, g_{ij}(x,y,z) > 0\}$$

with f_i, g_{ij} analytic in the neighbourhood U of zero. The set $\pi(A)$ is the graph of an analytic function and so it is not open. This implies that for some i there is $f_i \not\equiv 0$ and f_i vanishes on some open subset of $\pi(A)$. By the identity principle, $f_i \equiv 0$ on $\pi(A) \cap V$ with some neighbourhood $V \subset U$ of zero, i.e., $f(x, xy, xe^y) = 0$ for $x \in (0,\varepsilon)$, $y \in (0,1)$.

Expanding $f_i = \sum_{\nu \geq k} P_\nu$ into a series of homogeneous forms P_ν of degree ν, with $P_k \not\equiv 0$, yields then $P_\nu(1, y, e^y) \equiv 0$ for all ν and all $y \in (0,1)$, and thus for all $y \in \mathbb{R}$. Then

$$Q(y,z) := P_k(1,y,z)$$

is a non-zero polynomial vanishing on the graph of the exponential function which is a contradiction.

There are however two instances when the projection respects semi-analycity:

Theorem 4.2 ([Ł1])**.** *Let M, N be analytic varieties and $A \subset M \times N$ a semi-analytic set M-relatively compact. Let $\pi\colon M \times N \to N$ be the natural projection. If either $\dim A \leq 1$, or there is a semi-analytic set in N of dimension ≤ 2 containing $\pi(A)$, then $\pi(A)$ is semi-analytic. In particular, this is the case, if $\dim N \leq 2$.*

• Among the well-known and widely used results concerning subanalytic sets there is the fact that the Euclidean distance to a semi- or subanalytic set is subanalytic. As we have seen, this result is valid also in o-minimal structures: the distance to a definable set is definable. However, with subanalytic sets one has to be somewhat more cautious — the assertion stated above is not quite right (though one comes across it even in textbooks!).

Theorem 4.3 (Raby)**.** *Let E be subanalytic in an open set $U \subset \mathbb{R}^n$ and let $\delta(x) := \operatorname{dist}(x,E)$ denote the Euclidean distance. Then δ is subanalytic in some neighbourhood $V \subset U$ of E. Besides, if $U = \mathbb{R}^n$, then V can be taken to be $\mathbb{R}^n$, too.*

However, if $U \neq \mathbb{R}^n$, then in general $V \subsetneq U$ as is shown in the following example of Raby:

Example 4.4. The set $E = \{(1/n, 0) \mid n = 1,2,\dots\}$ is semi-analytic in $\mathbb{R}^2 \setminus \{0\}$. If δ were subanalytic in the whole of $\mathbb{R}^2 \setminus \{0\}$, one would have

$$\{x \in \mathbb{R}^2 \setminus \{0\} \mid \delta(x) = 1\} \cap (\mathbb{R} \times \{1\}) = \{(0,1), (1/n,1), n = 1,2,\dots\}$$

which clearly is not subanalytic, being discrete and accumulating in $\mathbb{R}^2 \setminus \{0\}$.

It is worth noting that for $\alpha \in \mathbb{R}$ the function t^α, $t > 0$ is subanalytic if and only if $\alpha \in \mathbb{Q}$. This is a consequence of Theorem 2.57. On the other hand, in the structure $\mathbb{R}_{\exp}$ any t^α is definable, because $\ln t$ is so (as the inverse of the exponential) and $t^\alpha = \exp(\alpha \ln t)$. Of course each t^α is definable in $\mathbb{R}_{\mathrm{an}, f_r, r\in\mathbb{R}}$.

On the other hand, as noted in Example 2.18, such nice properties as the Łojasiewicz inequalities do not hold in general o-minimal structures. They are satisfied in *polynomially bounded* o-minimal structures. These are defined by analogy to Lemma 2.13:

Definition 4.5. A structure is *polynomially bounded* if every function $f\colon \mathbb{R} \to \mathbb{R}$ definable in it satisfies for some N, $f = O(t^N)$ at infinity.

This property has a very nice characterization:

Theorem 4.6 (Miller [Mi])**.** *An o-minimal structure is not polynomially bounded iff the exponential function is definable in it.*

Theorem 4.7 (cf. [vdDM])**.** *In a polynomially bounded o-minimal structure, continuous definable functions on compact sets are Hölder continuous and they satisfy the Łojasiewicz inequality 2.14 (therefore also the property of regular separation 2.16 is satisfied in such structures).*

Nonetheless, there is a general definable counterpart of the Łojasiewicz inequality, namely:

Theorem 4.8 ([vdDM])**.** *If $f, g\colon A \to \mathbb{R}$ are continuous definable functions such that $f^{-1}(0) \subset g^{-1}(0)$ and $A \subset \mathbb{R}^n$ is compact, then there exists a $\mathscr{C}^p$ definable, strictly increasing bijection $\phi\colon \mathbb{R} \to \mathbb{R}$ which is p-flat at zero ([20]), such that $|\phi(g(t))| \leq |f(t)|$ on A.*

In [K1] Kurdyka showed in this spirit the general definable analogon of the Łojasiewicz gradient inequality, which is important due to its applications to the study of the gradient dynamics. We recall both versions:

Theorem 4.9. (Gradient inequality.)
(1) *Łojasiewicz's classical gradient inequality: Let $f\colon (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be an analytic germ ([21]). Then there exists $\theta \in (0, 1)$ such that in a neighbourhood of zero $||\mathrm{grad} f(x)|| \geq |f(x)|^\theta$.*
(2) *Kurdyka's definable version: Let $f\colon \Omega \to (0, +\infty)$ be a definable differentiable function on an open and bounded $\Omega \subset \mathbb{R}^n$. Then there exist positive constants $c, r > 0$ and a strictly increasing positive definable function $\phi\colon \mathbb{R}_+ \to \mathbb{R}$ of class $\mathscr{C}^1$ such that $||\mathrm{grad}(\phi \circ f)(x)|| \geq c$ whenever $f(x) \in (0, r)$.*

Remark 4.10. Of course the classical version cannot be applied to flat functions. Therefore it cannot hold e.g. in $\mathbb{R}_{\exp}$. Though it may not be apparent, Kurdyka's version is equivalent to the Kurdyka-Parusiński generalization of the classical Łojasiewicz's gradient inequality.

It may seem at first glance that the definable version consists only in avoiding the problem of possible existence of flat definable functions by composing f with a kind of 'desingularizing' function. However, even in this form the generalized gradient inequality has a great impact on the gradient dynamics (see [Ł3], [K1]):

Theorem 4.11. (1) *Łojasiewicz's gradient theorem: If $f\colon (0, \mathbb{R}^n) \to ([0, +\infty), 0)$ is analytic, then there is a neighbourhood U of zero such that each trajectory $y_x(t)$ of $x' = -\mathrm{grad} f(x)$ with $y_x(0) = x \in U$ satisfies:*

(1) *$y_x(t)$ is defined for all $t \geq 0$;*
(2) *the length $lg(y_x) = \int_0^{+\infty} ||y_x'(t)||\, dt$ is finite and uniformly bounded;*

[20] i.e., $\varphi^k(0) = 0, k = 0, \ldots, p$.
[21] It is still true for f just subanalytic $\mathscr{C}^1$.

(3) *there is an equilibrium point* $z \in \{\mathrm{grad} f = 0\}$ *for which there is* $\lim_{t\to+\infty} y_x(t) = z$ ([22]). *Moreover, the covergence is uniform with respect to* $x \in U$.

(2) *Kurdyka's gradient theorem: If* $f: U \to \mathbb{R}$ *is definable and* $\mathscr{C}^1$ *on a bounded open set* $U \subset \mathbb{R}^n$, *then:*

(1) *all the trajectories of* $-\mathrm{grad} f$ *have uniformly bounded length;*
(2) *the* ω*-limit set of any trajectory consists of only one point.*

Remark 4.12. For further information on applications in non-smooth analysis and optimization we refer the reader to [BDLM]. Note by the way, that the first applications in optimal control were done for subanalytic geometry, see e.g. [T], (or works of H. Sussmann, Łojasiewicz jr, Brunovsky in optimal control, some other applications by B. Teissier — cf. the most recent [BT] with J.-P. Brasselet — and J.-P. Françoise, Y. Yomdin, e.g. [FY] . . .).

Another kind of application of subanalytic geometry, this time in approximation theory, was performed by Pawłucki and Pleśniak who introduced in [PP] *uniformly polynomially cuspidal* sets in connection with the Markov inequality for bounded subanalytic sets (here the Wachta's Curve Selecting Lemma 2.58 is useful). Their result was then carried over to the definable setting (some o-minimal structures generated by quasi-analytic functions) by R. Pierzchała [Pr] — polynomial boundedness of the structure is needed.

Another result that found direct applications:

Theorem 4.13 (Denkowska-Wachta)**.** *Let* V *and* W *be two finite-dimensional real vector spaces and* $\pi: V \times W \to V$ *the natural projection. If* $E \subset V \times W$ *is subanalytic and* $F = \pi(E)$, *then there exists a subanalytic function* $\varphi: F \to W$ *such that* $\varphi \subset E$.

Here E can be seen as a subanalytic multifunction:

$$F \ni v \mapsto E_v \subset W$$

and φ is what is called *a selection* for this multifunction. The theorem above has applications in optimization where subanalytic multifunctions appear most naturally (cf. the works of R. J. Aumann, H. Halkin and E. C. Hendricks or, more recently M. Quincampoix) and the problem of finding a subanalytic selection is often crucial.

Remark 4.14. There exists a natural definable counterpart of this theorem, see e.g. [vdD]. It may be used to obtain the Curve Selecting Lemma.

• Some more metric properties:

Definition 4.15. A set $E \subset \mathbb{R}^n$ has *Whitney property* (in the class $\mathscr{C}$) if any two points $x, y \in E$ can be joined in E by a rectifiable arc (in the class $\mathscr{C}$) γ of length $lg(\gamma) \le c||x-y||^r$ for some $c, r > 0$.

The above notion is important. For instance if E is a fat set (i.e., $\overline{\mathrm{int} E} = E$) satisfying the Whitney property, then any $\mathscr{C}^\infty$ function in $\mathrm{int}E$ whose derivatives have continuous extensions onto E, has a $\mathscr{C}^\infty$ continuation to $\mathbb{R}^n \setminus E$.

Theorem 4.16 (Łojasiewicz-Stasica)**.** *The analytic Whitney property holds for semi- and subanalytic closed sets.*

Remark 4.17. Note that many properties of subanalytic sets hold in a 'parameter version', for instance regular separation (with a uniform exponent, Łojasiewicz-Wachta), Whitney property (uniform exponent, Denkowska), there is also a uniform bound on the lengths of arcs joining

[22] In other words, the ω-limit set of y_x consists of a single point.

points in the fibres of a bounded subanalytic set (Teissier and Denkowska-Kurdyka). See [DS1] for details.

On the other hand, Kurdyka in [K2] showed that any subanalytic set can be stratified into subanalytic leaves (regular in the sense of Mostowski-Parusiński) each of which satisfies the Whitney property with exponent 1. The same kind of result for definable sets, this time with parameter, has been obtained recently by B. Kocel-Cynk [KC].

The Whitney property is obviously involved in comparisons of the inner metric of a subanalytic or definable set ([23]) with the outer one and bi-Lipschitz equivalence problems. Here Kurdyka's *Pancake Lemma* from [K2] is the main ingredient: see the works of L. Birbrair and others e.g. [Bb]). We recall shortly the idea:
A definable or subanalytic set $X \subset \mathbb{R}^n$ is said to be *normally embedded*, if the identity map induces a bi-Lipschitz isomorphism between the metric spaces $(X; d_o)$ and $(X; d_i)$, d_o being the outer (Euclidean) metric, and d_i the inner one (this means precisely that the Łojasiewicz exponent of X is equal to 1).

Theorem 4.18 (Pancake Decomposition [K2])**.** *Let $X \subset \mathbb{R}^n$ be definable or subanalytic and bounded. Then there exists a finite collection of definable/subanalytic subsets $X_i \subset X$ such that*
1. $\bigcup X_i = X$;
2. *Each X_i is normally embedded in $\mathbb{R}^n$;*
3. $\forall i \neq j,\ \dim(X_i \cap X_j) < \min\{\dim X_i, \dim X_j\}$.

The collection $\{X_i\}$ is called Pancake Decomposition.

A nice decomposition of subanalytic sets, crucial from the point of view of the Whitney property (both in the subanalytic as in the definable setting):

Definition 4.19. An *(L)-analytic leaf* is a semi- or subanalytic subset of $\mathbb{R}^n$ which can be written in appropriate coordinates as the graph of a function $f \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$ with open domain and which is analytic with bounded differential.

Theorem 4.20 (Stasica)**.** *Any bounded subanalytic set in $\mathbb{R}^n$ is a finite union of (L)-analytic leaves.*

An analoguous theorem for semi-analytic sets is due to de Rham.

Just to stress once again the difference between the definable and subanalytic settings we quote part one of the results from [Di] where the following problem is considered. Let $M \subset \mathbb{R}^k_t \times \mathbb{R}^m_x$ be a set with closed t-sections M_t (not all empty) and let

$$m(t,x) = \{y \in M_t \mid ||x-y|| = \mathrm{dist}(x, M_t)\}.$$

Proposition 4.21 ([Di])**.** *If M is definable, then the set*

$$E := \{(t,x) \mid \#m(t,x) > 1\}$$

is definable, too.

Example 4.22. We have already observed that without an additional assumption (like that of M being x-relatively compact i.e., having proper projection onto $\mathbb{R}^k$) we cannot expect the function $(t,x) \mapsto \mathrm{dist}(x, M_t)$ to be subanalytic for a subanalytic M. Neither is the proposition true in the general subanalytic setting:

$$M = \{(x, 1/x) \mid x > 0\} \cup \bigcup_{n=1}^{+\infty} \{(1/n, -n)\} \subset \mathbb{R} \times \mathbb{R}$$

[23] i.e., the greatest lower bound of the lenghts of rectifiable curves joining two points in this set; triangulation theorems warrant this is well defined.

is subanalytic, but $E = \bigcup\{(1/n, 0)\}$ is not.

Nevertheless, the proposition above is true for subanalytic sets if we get rid of the parameter t. In this case we could be tempted to derive the proof from the definable case applied to the globally subanalytic sets $M_\nu = M \cap [-\nu, \nu]^n \subset \mathbb{R}^n$. However, the thing is more subtle than it seems and we do not have $E = \bigcup E_\nu$ where E_ν is constructed for M_ν. Indeed, take for instance M to be the union of semi-circles $\{x^2 + (y-\nu)^2 - (3/4)^2, y \leq \nu\}$. Then $(0, \nu) \in E_\nu \setminus E_{\nu+1}$ and in particular $(0, \nu) \notin E$.

• Many, though not all semi-analytic theorems have their subanalytic versions (cf. [DS1] for a thorough survey, e.g. each semi-analytic set germ admits an analytic germ of the same dimension as a superset which is no longer true for subanalytic germs cf. Example 4.1) and once again many, though not all, of these can be transposed to the definable setting. Here come some examples; first the theorem of the tangent mapping (compare Theorem 2.50):

Theorem 4.23. *Let $\Gamma \subset \mathbb{R}^n$ be a semi- or subanalytic leaf of dimension k. Then the tangent map $\tau\colon \Gamma \ni x \mapsto T_x\Gamma \in G_k(\mathbb{R}^n)$ is semi- or subanalytic (according to the case).*

Corollary 4.24. *If Γ is a subanalytic leaf, then for any subanalytic subset F of the Grassmannian $G_k(\mathbb{R}^n)$, $\tau^{-1}(E)$ is subanalytic, and for any bounded subanalytic set $E \subset \mathbb{R}^n$, $\tau(E)$ is subanalytic, too.*

Remark 4.25. The theorem above has a definable counterpart to be found in the articles by Ta Lê Loi.

The next result is a generalization of the Curve Selecting Lemma to higher dimensions:

Lemma 4.26 (Wings' Lemma)**.** *Let $\Gamma \subset M$ be a subanalytic leaf and $E \subset \overline{\Gamma} \setminus \Gamma$ a subanalytic set. Then there exists a subanalytic leaf Λ of dimension* $\dim E + 1$ *and such that $\Lambda \subset \Gamma$ and* $\dim \overline{\Lambda} \cap E = \dim E$.

A definable counterpart of the result above is given in [Loi] ([24]).

• Stratifications

Stratifications are an important tool and they are often asked to satisfy some additional properties — we shall discuss this briefly. Let V be a finite-dimensional real vector space and denote by J the family of pairs (V', V'') of subspaces of V satisfying $V' \subset V''$.
Let N_0, N be two differentiable subvarieties of V of dimension k and l respectively, with $k < l$.

Definition 4.27. We say that the pair (N_0, N) *satisfies Whitney's condition (a)* at $c \in N_0 \cap N$ if (T_cN_0, T_zN) tends to J in $G_k(V) \times G_l(V)$ when $z \in N$ tends to c.
We say that (N_0, N) *satisfies Whitney's condition (b)* at c if the pair $(\mathbb{R}\cdot(z - x), T_zN)$ tends to J in $G_1(V) \times G_l(V)$ when the point $(x, z) \in (N_0 \times N) \cap \{x \neq z\}$ tends to (c, c).

Remark 4.28. The convergence above is invariant with respect to diffeomorphisms, whence it can be formulated in the same way for a differentiable variety. Recall also that Whitney's condition (b) implies (a).

Theorem 4.29. *Let M be an affine space. Let $E_1, \ldots, E_r$ be subanalytic in M. Then there exists a stratification $\mathcal{N}$ of M into subanalytic leaves, compatible with $E_1, \ldots, E_r$ and such that for all pairs of strata $\Gamma_1, \Gamma_2 \in \mathcal{N}$ such that $\Gamma_1 \subset \overline{\Gamma_2} \setminus \Gamma_2$, the varieties Γ_1, Γ_2 satisfy Whitney's condition (b) at any point of Γ_1.*

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[24] We thank the referee for pointing this out.

Definition 4.30. Let $f\colon M \to N$ be an analytic map, $\mathcal{T}$ a stratification of the analytic variety M, $\mathcal{S}$ a stratification of another analytic variety N. The pair $\mathcal{T}, \mathcal{S}$ is said to be *compatible with* f if

(i) for all $T \in \mathcal{T}$, $f(T) \in \mathcal{S}$,
(ii) for all $T \in \mathcal{T}$, $\mathrm{rk} f|_T \equiv \dim f(T)$,
(iii) if $\mathrm{rk} f|_T = \dim T$, then $f|_T$ is injective.

Theorem 4.31 (Hardt)**.** *Let $f\colon M \to N$ be analytic. Given two locally finite families $\mathcal{M}, \mathcal{N}$ of subanalytic sets in M, N, respectively, and an open subanalytic set K such that $f|_{\overline{K}}$ is proper, there exists a stratification $\mathcal{S}$ compatible with $\mathcal{N}$ and a stratification $\mathcal{T}$ compatible with $\mathcal{M}$ together with K, such that the pair $(\mathcal{T}_K, \mathcal{S})$ is compatible with f_K, where $\mathcal{T}_K = \{T \in \mathcal{T}\colon T \subset K\}$.*

Let X be a finite-dimensional real vector space and U, V its linear subspaces. We define after T.-C. Kuo the function $\delta(U, V) := \sup\{d(x, V)\colon x \in U, |x| = 1\}$ where d is the Euclidean distance. There is $\delta(U, V) = 0$ if and only if $U \subset V$.

Definition 4.32. Let M, N be two $\mathscr{C}^\infty$ subvarieties of X such that $\overline{M} \cap N \neq \varnothing$. We say that the pair (M, N) *satisfies the Verdier condition (w) at a point* $a \in \overline{M} \cap N$ if there is a neighbourhood V of a in X and a constant $C > 0$ such that

$$\delta(T_xM, T_yN) \leq C||x - y||, \quad \text{for any } x \in V \cap M, y \in V \cap N.$$

We say that (M, N) *satisfies the condition (w)* if it satisfies this condition at all points $a \in \overline{M} \cap N$.

Remark 4.33. Kuo in [Kuo] showed that condition (w) implies Whitney's condition (b) in the semi-analytic case. Since the Curve Selecting Lemma and the Tangent Mapping Theorem hold also in the subanalytic case, the same kind of argument as that used by Kuo works also in the subanalytic case. Nonetheless, condition (w) in general is not stronger than condition (b) (see [Vd]). For more informations see [DSW], [DW], [KT], [OTr], [Tr].

Theorem 4.34 (Verdier [Vd] ([25]))**.** *Let $\{E_i\}$ be a locally finite family of subanalytic subsets of X. Then there is a subanalytic stratification of X compatible with that family and such that any pair of its strata satisfies the Verdier condition (w).*

It is worth adding a few words about Łojasiewicz's approach to stratifications. Needless to say, unlike e.g. Verdier, he made no use of Hironaka's desingularization. Instead, his idea was to start with the following key-lemma:

If M and N are subanalytic varieties in an affine space X and $N \subset \overline{M} \setminus M$, then the set $\{x \in N \mid (M, N)$ verifies condition $(\#)\}$ where $(\#)$ stands for one of the conditions introduced so far, is subanalytic in the space X and dense in N.

To prove *dense in N* (subanalytic is easy), we use Whitney's Wings' Lemma 4.26. See [D], [DW], [DS2].

• A natural question is whether subanalytic sets admit triangulation (cf. Theorem 1.30). The positive answer was given by Goresky [Go] as well as Verona [V]. Independently of the general result, H. Hironaka [H3] and R. Hardt [H2] gave both explicit methods of triangulation for subanalytic sets. Their constructions are natural and geometric. As noted by H. Hironaka the method is close to that used by S. Łojasiewicz for semi-algebraic sets, for both classes of sets — semi-algebraic and subanalytic — are closed with respect to projections.

[25] There are other proofs by Łojasiewicz-Stasica-Wachta, Coste-Roy and historically the first one by Denkowska-Wachta [DW] as an answer to a question of D. Trotman for a desingularization-free proof, presented in [D].

Theorem 4.35 (Hironaka)**.** *Let $\{X_\alpha\}_{\alpha\in A}$ be a locally finite family of subanalytic subsets of $\mathbb{R}^n$. Then there exists a simplicial decomposition of $\mathbb{R}^n = \bigcup \sigma_\mu$ into open simplices and a subanalytic homeomorphism $\theta\colon \mathbb{R}^n \to \mathbb{R}^n$ such that*

(i) *each X_α is a locally finite union of some of the images $\theta(\sigma_\mu)$,*
(ii) *for any μ, $\theta(\sigma_\mu)$ is an analytic subvariety of $\mathbb{R}^n$ and $\theta|_{\sigma_\mu}\colon \sigma_\mu \to \theta(\sigma_\mu)$ is an analytic isomorphism.*

Theorem 4.36 (Hardt)**.** *Let $\{X_\alpha\}_{\alpha\in A}$ be a locally finite family of subanalytic subsets of $\mathbb{R}^n$. Then there exists a simplicial decomposition Σ of $\mathbb{R}^n$ and a subanalytic map $f\colon [0,1]\times\mathbb{R}^n \to \mathbb{R}^n$ such that*

(i) *for each $t \in [0,1]$ the map $f_t(x) = f(t,x)$ is a homeomorphism,*
(ii) *$f_0 = id$,*
(iii) *for any $\alpha \in A$, $f_1^{-1}(X_\alpha)$ is a subcomplex of Σ.*

Remark 4.37. In the case of semi-analytic sets, a class of sets without the projection property, the construction of a semi-analytic triangulation is much more delicate (see S. Łojasiewicz [Ł4]).

Remark 4.38. Semi-algebraic, semi-analytic and subanalytic sets admit triangulation.
Quite recently, a student of W. Pawłucki, M. Czapla, proved in her Ph. D. Thesis (using a description of the Lipschitz structure of definable sets by G. Valette [Val]) that every definable set has a definable triangulation which is locally Lipschitz and weakly bi-Lipschitz on the natural stratification of a simplicial complex. She also proved that such a stratification may be obtained with Whitney's (b) condition or Verdier's condition.

On the other hand, it is well-known that subanalytic sets admit Lipschitz stratification (see [Pa]). A direct method of constructing a Lipschitz cell decomposition (which must involve some coordinate changes) has been produced recently by Pawłucki in [P2].

We started with semi-algebraic sets and we will end with them. The following theorem, proved using simple stratifications, show how ubiquous they are:

Theorem 4.39 ([DD])**.** *Let $E \subset \mathbb{R}^m$ be a compact subanalytic or definable set. Then there exists a sequence $\{A_\nu\}$ of semi-algebraic sets such that*

(1) *$E = \lim A_\nu$;*
(2) *For each $a \in E$ and any neighbourhood U of a one has for ν large enough,*

$$\dim U \cap E = \dim U \cap A_\nu.$$

Moreover, for each such a sequence $\{A_\nu\}$ one has the following: for any $S \in cc(E)$ there is a sequence $\{S_\nu\}$ such that each S_ν is the union of some connected components of A_ν and (1) *and* (2) *holds for S and the sequence $\{S_\nu\}$.*

Here the convergence is understood in the following sense (*Kuratowski convergence of closed sets*):
$A = \lim A_n$ *iff each point $a \in A$ is the limit of a sequence of points $a_n \in A_n$, $n \in \mathbb{N}$ and for each compact set K such that $K \cap A = \varnothing$ one has $K \cap A_n = \varnothing$ for almost all indices n.*

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