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SINGULARITIES FOR NORMAL HYPERSURFACES OF DE SITTER TIMELIKE CURVES IN MINKOWSKI 4-SPACE

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Abstract. In this paper, we consider the normal hypersurfaces associated with timelike curves in Minkowski 4-space which are confined in de Sitter 3-space. We classify the generic singularities of the normal hypersurfaces, which are cuspidal edges, swallowtails and butterflies. And reveal the relationships between these singularities and the Lorentzian invariants of timelike curves by applying the singularity theory.

1. Introduction

Since the second half of the 20th century, singularity theory and semi-Riemannian geometry have been active areas of research in differential geometry.

In [5], the second author et al. used Montaldi's characterization of submanifold contacts in terms of $\mathcal{K}$-equivalent functions, which provided a technical linkage to Lagrangian singularity theory. They presented the classification of singularities of de Sitter Gauss map of timelike hypersurfaces which were based on the Lagrangian singularity theory.

In [8], Z. Wang et al. investigated singularities of the focal surfaces and the binormal indicatrix associated with a null Cartan curve. The relationships were revealed between singularities of the above two subjects and differential geometric invariants of null Cartan curves. L. Chen defined the timelike Anti de Sitter Gauss images and timelike Anti de Sitter height functions on spacelike surfaces in [3], he investigated the geometric meanings of singularities of these mappings. The authors of these papers investigated the singularities of some geometrical objects by using the theory of singularities of differential mappings.

In Minkowski 4-space, T. Fusho and S. Izumiya [4] discussed the the generic singularities of lightlike surface which is generated by a spacelike curve in de Sitter 3-space. De Sitter 3-space is an important cosmological model for the physical universe. The spacelike curve had a degenerate contact with a lightcone at the singularities of the lightlike surface. The study on the contact of lightlike curves with lightcones is an interesting case. The lightcone is an important model in physics too. T. Fusho and S. Izumiya [4] had classified the singularities of the lightlike surface of spacelike curve, in addition to investigating the geometric meanings of the singularities of such surfaces in de Sitter 3-space.

In [9], the null developables of timelike curves that lie on the nullcone in 3-dimensional semi-Euclidean space with index 2 were investigated by the second author, Z. Wang and X. Fan. They also classified the singularities of the null developables of timelike curves.

However, to the best of the authors' knowledge, no literature exists regarding the singularities of surfaces and curves as they relate to timelike curves in de Sitter 3-space. Thus, the current

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study hopes to serve such a need and it is inspired by the reports of T. Fusho and S. Izumiya [4].

This paper is supplementary for [4]. We consider the timelike curve in de Sitter 3-space, then we define a normal hypersurface associated to the timelike curve. The normal hypersurface is different from the lightlike surface which is in [4]. T. Fusho and S. Izumiya [4] considered the lightlike surface in de Sitter 3-space while the normal hypersurface is in Minkowski 4-space. Therefore we stick to the hypersurface in this paper. We get an invariant σ of timelike curve which describes the contact between a given model and the timelike curve. A kind of height function has been constructed which is related to the timelike curve, as it will be quite useful to study the singularities of hypersurface. Our main results are stated in Theorem 2.1. By these results, we give a classification of the singularities of the normal hypersurfaces in Minkowski 4-space and get some geometric properties of the singularities.

We shall assume throughout the whole paper that all manifolds and maps are C^∞ unless the contrary is explicitly stated.

2. Basic notions and results

In this section we give the basic notions and the main results. For the basic results in the Lorentzian geometry see [7]. Let $\mathbb{R}^4$ be a 4-dimensional vector space. For any two vectors $\boldsymbol{x} = (x_1, x_2, x_3, x_4)$, $\boldsymbol{y} = (y_1, y_2, y_3, y_4)$ in $\mathbb{R}^4$, their pseudo scalar product is defined by

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4.$$

The pair $(\mathbb{R}^4, \langle , \rangle)$ is called *Minkowski 4-space*. We denote it as $\mathbb{R}^4_1$.

For any three vectors $\boldsymbol{x} = (x_1, x_2, x_3, x_4)$, $\boldsymbol{y} = (y_1, y_2, y_3, y_4)$, $\boldsymbol{z} = (z_1, z_2, z_3, z_4) \in \mathbb{R}^4_1$, we define a vector $\boldsymbol{x} \wedge \boldsymbol{y} \wedge \boldsymbol{z}$ by

$$\boldsymbol{x} \wedge \boldsymbol{y} \wedge \boldsymbol{z} = \begin{vmatrix} -\boldsymbol{e_1} & \boldsymbol{e_2} & \boldsymbol{e_3} & \boldsymbol{e_4} \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix},$$

where $\{\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3}, \boldsymbol{e_4}\}$ is the canonical basis of $\mathbb{R}^4_1$. We have $\langle \boldsymbol{x}_0, \boldsymbol{x} \wedge \boldsymbol{y} \wedge \boldsymbol{z} \rangle = \det(\boldsymbol{x}_0, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$, so $\boldsymbol{x} \wedge \boldsymbol{y} \wedge \boldsymbol{z}$ is pseudo orthogonal to $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$. A non-zero vector $\boldsymbol{x} \in \mathbb{R}^4_1$ is called *spacelike, lightlike* or *timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0$, $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$ or $\langle \boldsymbol{x}, \boldsymbol{x} \rangle < 0$, respectively. The norm of $\boldsymbol{x} \in \mathbb{R}^4_1$ is defined by $\|\boldsymbol{x}\| = (\mathrm{sign}(\boldsymbol{x})\langle \boldsymbol{x}, \boldsymbol{x} \rangle)^{1/2}$, where $\mathrm{sign}(\boldsymbol{x})$ denotes the signature of $\boldsymbol{x}$ which is given by $\mathrm{sign}(\boldsymbol{x}) = 1, 0$ or-1 when $\boldsymbol{x}$ is a spacelike, lightlike or timelike vector, respectively.

Let $\boldsymbol{\gamma} : I \to \mathbb{R}^4_1$ be a regular curve in $\mathbb{R}^4_1$ (i.e.,$\dot{\boldsymbol{\gamma}}(t) = d\boldsymbol{\gamma}/dt \neq 0$), where I is an open interval. For any $t \in I$, the curve $\boldsymbol{\gamma}$ is called *spacelike, lightlike or timelike* if $\langle \dot{\boldsymbol{\gamma}}(t), \dot{\boldsymbol{\gamma}}(t) \rangle > 0$, $\langle \dot{\boldsymbol{\gamma}}(t), \dot{\boldsymbol{\gamma}}(t) \rangle = 0$ or $\langle \dot{\boldsymbol{\gamma}}(t), \dot{\boldsymbol{\gamma}}(t) \rangle < 0$ respectively. We call $\boldsymbol{\gamma}$ a *nonlightlike curve* if $\boldsymbol{\gamma}$ is a spacelike or timelike curve. The acr-length of a nonlightlike curve $\boldsymbol{\gamma}$ measured from $\boldsymbol{\gamma}(t_0)(t_0 \in I)$ is $s(t) = \int_{t_0}^{t} \|\dot{\boldsymbol{\gamma}}(t)\| dt$.

The parameter s is determined by $\|\boldsymbol{\gamma}'(s)\|$=1 for the nonlightlike curve, where $\boldsymbol{\gamma}'(s) = d\boldsymbol{\gamma}/ds$ is the unit tangent vector of $\boldsymbol{\gamma}$ at s. The *de Sitter 3-space* is defined by

$$S^3_1 = \{\boldsymbol{x} \in \mathbb{R}^4_1 | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1\}.$$

Let $\boldsymbol{\gamma} : I \to S^3_1$ be a timelike regular curve $(\langle \dot{\boldsymbol{\gamma}}(t), \dot{\boldsymbol{\gamma}}(t) \rangle < 0, t \in I)$. Since the curve $\boldsymbol{\gamma}$ is timelike, we can reparametrize it by the acr-length s. Then we have the tangent vector $\boldsymbol{t}(s) = \boldsymbol{\gamma}'(s)$, obviously $\|\boldsymbol{t}(s)\| = 1$. When $\langle \boldsymbol{t}'(s), \boldsymbol{t}'(s) \rangle \neq 1$, we define a unit vector

$$\boldsymbol{n}(s) = \big(\boldsymbol{t}'(s) - \boldsymbol{\gamma}(s)\big)/\|\boldsymbol{t}'(s) - \boldsymbol{\gamma}(s)\|,$$

let $\boldsymbol{e}(s) = \boldsymbol{\gamma}(s) \wedge \boldsymbol{t}(s) \wedge \boldsymbol{n}(s)$. Then we have a pseudo orthonormal frame $\{\boldsymbol{\gamma}(s), \boldsymbol{t}(s), \boldsymbol{n}(s), \boldsymbol{e}(s)\}$ of $\mathbb{R}^4_1$ along $\boldsymbol{\gamma}$. By directly calculating, the following Frenet-Serret type is displayed, under the assumption that $\langle \boldsymbol{t}'(s), \boldsymbol{t}'(s)\rangle \neq 1$.

$$\begin{cases} \boldsymbol{\gamma}'(s) = \boldsymbol{t}(s) \\ \boldsymbol{t}'(s) = \boldsymbol{\gamma}(s) + \kappa_g(s)\boldsymbol{n}(s) \\ \boldsymbol{n}'(s) = \kappa_g(s)\boldsymbol{t}(s) - \tau_g(s)\boldsymbol{e}(s) \\ \boldsymbol{e}'(s) = \tau_g(s)\boldsymbol{n}(s). \end{cases}$$

Here, $\kappa_g(s) = \|\boldsymbol{t}'(s) - \boldsymbol{\gamma}(s)\|$ is the *geodesic curvature*, $\tau_g(s) = -\kappa_g^{-2}(s)\det\big(\boldsymbol{\gamma}(s), \boldsymbol{\gamma}'(s), \boldsymbol{\gamma}''(s), \boldsymbol{\gamma}'''(s)\big)$ is the *geodesic torsion*.

We now define a normal hypersurface associate to a timelike curve. Let $\boldsymbol{\gamma} : I \to S^3_1$ be a unit speed timelike curve, we define $NHS : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^4_1$ by

$$NHS(s, u, w) = \boldsymbol{\gamma}(s) + u\boldsymbol{n}(s) + w\boldsymbol{e}(s).$$

We call $NHS(s, u, w)$ the *normal hypersurface* of $\boldsymbol{\gamma}$. We also define the following model surface. For any $\boldsymbol{v}_0 \in NHS(s, u, w)$, $S^2_1(\boldsymbol{v}_0) = \{\boldsymbol{x} \in S^3_1 \mid \langle \boldsymbol{x}, \boldsymbol{v}_0\rangle - 1 = 0\}$, where

$$\langle \boldsymbol{v}_0, \boldsymbol{v}_0\rangle = 1 + u^2 + w^2 \geq 1.$$

In this paper, the major purpose is to study the Lorentzian geometric meanings of the singularities of the normal hypersurface. We get σ equivalent to the conformal torsion in [2],

$$\sigma(s) = \kappa_g^2(s)\tau_g^3(s) - \kappa_g(s)\kappa_g''(s)\tau_g(s) + 2(\kappa_g'(s))^2\tau_g(s) + \kappa_g(s)\kappa_g'(s)\tau_g'(s).$$

On the other hand, let $F : S^3_1 \to \mathbb{R}$ be a submersion and $\boldsymbol{\gamma} : I \to S^3_1$ be a timelike curve. We say that $\boldsymbol{\gamma}$ and $F^{-1}(0)$ have k-point contact for $t = t_0$ if the function $g(t) = F \circ \boldsymbol{\gamma}(t)$ satisfies $g(t_0) = g'(t_0) = \cdots = g^{(k-1)}(t_0) = 0$, $g^{(k)}(t_0) \neq 0$. We also have that $\boldsymbol{\gamma}$ and $F^{-1}(0)$ have at least k-point contact for $t = t_0$ if the function $g(t) = F \circ \boldsymbol{\gamma}(t)$ satisfies

$$g(t_0) = g'(t_0) = \cdots = g^{(k-1)}(t_0) = 0.$$

We now consider the following conditions:

(A1) The number of points p of $\boldsymbol{\gamma}(s)$ where the $S^2_1(\boldsymbol{v}_0)$ at p having five-point contact with the curve $\boldsymbol{\gamma}$ is finite.

(A2) There is no point p of $\boldsymbol{\gamma}(s)$ where the $S^2_1(\boldsymbol{v}_0)$ at p having greater than or equal to six-point contact with the curve $\boldsymbol{\gamma}$.

Our main results is as follows.

Theorem 2.1. *Let $\boldsymbol{\gamma} : I \to S^3_1$ be a unit regular timelike curve with $\langle \boldsymbol{t}'(s), \boldsymbol{t}'(s)\rangle \neq 1$, $\tau_g(s) \neq 0$, $\boldsymbol{v}_0 = NHS(s_0, u_0, w_0)$ and $S^2_1(\boldsymbol{v}_0) = \{\mathbf{u} \in S^3_1 \mid \langle \mathbf{u}, \boldsymbol{v}_0\rangle - 1 = 0\}$, we can state the following facts.*

(1) $S^2_1(\boldsymbol{v}_0)$ and $\boldsymbol{\gamma}$ have at least 2-point contact at s_0.

(2) $S^2_1(\boldsymbol{v}_0)$ and $\boldsymbol{\gamma}$ have 3-point contact at $\mathbf{s}_0$ if and only if

$$\boldsymbol{v}_0 = \boldsymbol{\gamma}(s_0) - (1/\kappa_g(s_0))\boldsymbol{n}(s_0) + u_0\boldsymbol{e}(s_0)$$

and $u_0 \neq -\kappa_g'(s_0)/\kappa_g^2(s_0)\tau_g(s_0)$, under this condition the germ of image NHS at $NHS(s_0, u_0, w_0)$ is diffeomorphic to the cuspidal edge $C \times \mathbb{R}^2$.

(3) $S^2_1(\boldsymbol{v}_0)$ and $\boldsymbol{\gamma}$ have 4-point contact at $\mathbf{s}_0$ if and only if

$$\boldsymbol{v}_0 = \boldsymbol{\gamma}(s_0) - (1/\kappa_g(s_0))\boldsymbol{n}(s_0) - \big(\kappa_g'(s_0)/\kappa_g^2(s_0)\tau_g(s_0)\big)\boldsymbol{e}(s_0) \text{ and } \sigma(s_0) \neq 0,$$

under this condition the germ of image NHS at $NHS(s_0, u_0, w_0)$ is diffeomorphic to the swallowtail $SW \times \mathbb{R}$.

(4) $S^2_1(\boldsymbol{v}_0)$ and $\boldsymbol{\gamma}$ have 5-point contact at $\mathbf{s}_0$ if and only if

$$\boldsymbol{v}_0 = \boldsymbol{\gamma}(s_0) - (1/\kappa_g(s_0))\boldsymbol{n}(s_0) - \big(\kappa_g'(s_0)/\kappa_g^2(s_0)\tau_g(s_0)\big)\boldsymbol{e}(s_0),\ \sigma(s_0) = 0 \text{ and } \sigma'(s_0) \neq 0,$$

under this condition the germ of image NHS at $NHS(s_0, u_0, w_0)$ is diffeomorphic to the BF.

Here, $SW \times \mathbb{R} = \{(x_1, x_2, x_3) \mid x_1 = 3u^4 + u^2 v, x_2 = 4u^3 + 2uv, x_3 = v\} \times \mathbb{R}$ is the swallowtail, $BF = \{(x_1, x_2, x_3, x_4) \mid x_1 = 4u^5 + 2u^3 v + u^2 w, x_2 = -5u^4 - 3u^2 v - 2uw, x_3 = v, x_4 = w\}$ is the butterfly and $C \times \mathbb{R}^2 = \{(x_1, x_2) \mid x_1^2 = x_2^3\} \times \mathbb{R}^2$ is the cuspidal edge.

We will give the proof of Theorem 2.1 in §4.

3. Timelike height functions and the singularities of normal hypersurfaces

In this section we discuss a kind of Lorentzian invariant function on a timelike curve in $\mathbb{R}^4_1$. It is useful to study the normal hypersurface of the timelike curve. Let $\boldsymbol{\gamma} : I \to S^3_1$ be a unit timilike curve and $\langle \boldsymbol{t}'(s), \boldsymbol{t}'(s)\rangle \neq 1$. We now define a function

$$H : I \times \mathbb{R}^4_1 \to \mathbb{R}$$

by $H(s, \boldsymbol{v}) = \langle \boldsymbol{\gamma}(s), \boldsymbol{v}\rangle - 1$, we call H a *timelike height function* on the timelike curve $\boldsymbol{\gamma}$. We denote that $h_{\boldsymbol{v}}(s) = H(s, \boldsymbol{v})$, for any fixed $\boldsymbol{v} \in \mathbb{R}^4_1$. Then, we have the following Proposition.

Proposition 3.1. *Let $\boldsymbol{\gamma} : I \to S^3_1$ be a unit timelike curve with $\langle \boldsymbol{t}'(s), \boldsymbol{t}'(s)\rangle \neq 1$ and $\tau(s) \neq 0$, then we have the following.*

(1) $h_{\boldsymbol{v}}(s) = 0$ if and only if there exist $b, c, d \in \mathbb{R}$ such that $\boldsymbol{v} = \boldsymbol{\gamma}(s) + b\boldsymbol{t}(s) + c\boldsymbol{n}(s) + d\boldsymbol{e}(s)$.

(2) $h_{\boldsymbol{v}}(s) = h'_{\boldsymbol{v}}(s) = 0$ if and only if there exist $c, d \in \mathbb{R}$ such that $\boldsymbol{v} = \boldsymbol{\gamma}(s) + c\boldsymbol{n}(s) + d\boldsymbol{e}(s)$.

(3) $h_{\boldsymbol{v}}(s) = h'_{\boldsymbol{v}}(s) = h''_{\boldsymbol{v}}(s) = 0$ if and only if there exists $d \in \mathbb{R}$ such that

$$\boldsymbol{v} = \boldsymbol{\gamma}(s) - (1/\kappa_g(s))\boldsymbol{n}(s) + d\boldsymbol{e}(s).$$

(4) $h_{\boldsymbol{v}}(s) = h'_{\boldsymbol{v}}(s) = h''_{\boldsymbol{v}}(s) = h'''_{\boldsymbol{v}}(s) = 0$ if and only if

$$\boldsymbol{v} = \boldsymbol{\gamma}(s) - (1/\kappa_g(s))\boldsymbol{n}(s) - \big(\kappa'_g(s)/\kappa^2_g(s)\tau_g(s)\big)\boldsymbol{e}(s).$$

(5) $h_{\boldsymbol{v}}(s) = h'_{\boldsymbol{v}}(s) = h''_{\boldsymbol{v}}(s) = h'''_{\boldsymbol{v}}(s) = h^{(4)}_{\boldsymbol{v}}(s) = 0$ if and only if

$$\boldsymbol{v} = \boldsymbol{\gamma}(s) - (1/\kappa_g(s))\boldsymbol{n}(s) - \big(\kappa'_g(s)/\kappa^2_g(s)\tau_g(s)\big)\boldsymbol{e}(s) \text{ and } \sigma(s) = 0.$$

Proof. (1) Since $\boldsymbol{v} \in \mathbb{R}^4_1$, we can find $a, b, c, d \in \mathbb{R}$ such that $\boldsymbol{v} = a\boldsymbol{\gamma}(s) + b\boldsymbol{t}(s) + c\boldsymbol{n}(s) + d\boldsymbol{e}(s)$. Because $h_{\boldsymbol{v}}(s) = \langle \boldsymbol{\gamma}(s), \boldsymbol{v}\rangle - 1 = 0$, we can get $a = 1$, then $\boldsymbol{v} = \boldsymbol{\gamma}(s) + b\boldsymbol{t}(s) + c\boldsymbol{n}(s) + d\boldsymbol{e}(s)$, the converse direction also holds.

(2) By (1), an easy computation shows that $\langle \boldsymbol{t}(s), \boldsymbol{v}\rangle - 1 = 0$, we get $b = 0$, therefore

$$\boldsymbol{v} = \boldsymbol{\gamma}(s) + c\boldsymbol{n}(s) + d\boldsymbol{e}(s).$$

(3) Under the assumption that $h_{\boldsymbol{v}}(s) = h'_{\boldsymbol{v}}(s) = 0$,

$$h''_{\boldsymbol{v}}(s) = \langle \boldsymbol{\gamma}(s) + \kappa_g(s)\boldsymbol{n}(s), \boldsymbol{\gamma}(s) + c\boldsymbol{n}(s) + d\boldsymbol{e}(s)\rangle,$$

we can get $\kappa_g(s)c+1 = 0$, it is that $c = -1/\kappa_g(s)$, then we have $\boldsymbol{v} = \boldsymbol{\gamma}(s) - (1/\kappa_g(s))\boldsymbol{n}(s) + d\boldsymbol{e}(s)$.

(4) Based on the assumption that $h_{\boldsymbol{v}}(s) = h'_{\boldsymbol{v}}(s) = h''_{\boldsymbol{v}}(s) = 0$, the relation

$$h'''_{\boldsymbol{v}}(s) = \langle (1 + \kappa^2_g(s))\boldsymbol{t}(s) + \kappa'_g(s)\boldsymbol{n}(s) - \kappa_g(s)\tau_g(s)\boldsymbol{e}(s), \boldsymbol{v}\rangle,$$

it follows that $h'''_{\boldsymbol{v}}(s) = 0$ is equivalent to $(-\kappa'_g(s)/\kappa_g(s)) - \kappa_g(s)\tau_g(s)d = 0$, so

$$d = -\kappa'_g(s)/\kappa^2_g(s)\tau_g(s).$$

This proves assertion (4).

(5) When $h_{\boldsymbol{v}}(s) = h'_{\boldsymbol{v}}(s) = h''_{\boldsymbol{v}}(s) = h'''_{\boldsymbol{v}}(s) = 0$, the fourth derivative

$$\begin{aligned} h^{(4)}_{\boldsymbol{v}}(s) =& \langle (1 + \kappa^2_g(s))\boldsymbol{\gamma}(s) + (\kappa_g(s) + \kappa^3_g(s) + \kappa''_g(s)\kappa'_g(s) - \kappa_g(s)\tau^2_g(s))\boldsymbol{n}(s) \\ & - (2\kappa'_g(s)\tau(s) + \kappa_g(s)\tau'_g(s))\boldsymbol{e}(s) + 3\kappa_g(s)\kappa'_g(s)\boldsymbol{t}(s), \boldsymbol{v}\rangle, \end{aligned}$$

by directly calculating we have $\sigma(s)/\kappa_g^2(s)\tau_g(s) = 0$, where

$$\sigma(s) = \kappa_g^2(s)\tau_g^3(s) - \kappa_g(s)\kappa_g''(s)\tau_g(s) + 2(\kappa_g'(s))^2\tau_g(s) + \kappa_g(s)\kappa_g'(s)\tau_g'(s),$$

therefore $\sigma(s) = 0$.

Now, we research some properties of the normal hypersurface of the timelike curve in $\mathbb{R}^4_1$. As we can know the functions $\kappa_g(s)$, $\tau_g(s)$ and $\sigma(s)$ have particular meanings. Here, we consider the case when the normal hypersurface has the most degenerate singularities. We have the following proposition.

Proposition 3.2. *Let $\boldsymbol{\gamma} : I \to S^3_1$ be a unit timelike curve with $\langle \boldsymbol{t}'(s), \boldsymbol{t}'(s)\rangle \neq 1$ and $\tau_g(s) \neq 0$, then we have the following.*

(1) The set $\{(s, u, w) \mid u = -1/\kappa_g(s), s \in I\}$ is the singularities of normal hypersurface NHS.

(2) If $\boldsymbol{v}_0 = NHS(s, -1/\kappa_g(s), -\kappa_g'(s)/\kappa_g^2(s)\tau_g(s))$ is s constant vector, we have $\boldsymbol{\gamma}(s) \in S^2_1(\boldsymbol{v}_0)$ for any $s \in I$ at the same time $\sigma(s) = 0$.

Proof. By calculations we have

$$\frac{\partial NHS}{\partial u} = \boldsymbol{n}(s),\ \frac{\partial NHS}{\partial w} = \boldsymbol{e}(s),$$
$$\frac{\partial NHS}{\partial s} = (1 + u\kappa_g(s))\boldsymbol{t}(s) - u\tau_g(s)\boldsymbol{e}(s) + w\tau_g(s)\boldsymbol{n}(s).$$

(1) If the above three vectors are linearly dependent, we can get the singularities of NHS if and only if $1 + u\kappa_g(s) = 0$, $u = -1/\kappa_g(s)$.

(2) If $f(s) = \boldsymbol{\gamma}(s) + u(s)\boldsymbol{n}(s) + w(s)\boldsymbol{e}(s)$ is a constant, then

$$\frac{df}{ds} = (1 + u(s)\kappa_g(s))\boldsymbol{t}(s) + (u'(s) + w(s)\tau_g(s))\boldsymbol{n}(s) + (w'(s) - u(s)\tau_g(s))\boldsymbol{e}(s) = 0.$$

Since

$$u(s) = -\frac{1}{\kappa_g(s)},\ w(s) = -\frac{\kappa_g'(s)}{\kappa_g^2(s)\tau_g(s)},$$

then

$$w'(s) - u(s)\tau_g(s) = 0.$$

We have

$$\frac{2(\kappa_g'(s))^2\tau_g(s) + \kappa_g(s)\kappa_g'(s)\tau_g'(s) - \kappa_g''(s)\kappa_g(s)\tau_g(s)}{\kappa_g^3(s)\tau_g^2(s)} = -\frac{\tau_g(s)}{\kappa_g(s)},$$

$$\sigma(s) = 0,$$

therefore

$$\left\langle \boldsymbol{\gamma}(s), \boldsymbol{\gamma}(s) - \frac{1}{\kappa_g(s)}\boldsymbol{n}(s) - \frac{\kappa_g'(s)}{\kappa_g^2(s)\tau_g(s)}\boldsymbol{e}(s)\right\rangle - 1 = 0.$$

This completes the proof.

4. Unfoldings of height function

In this section we classify singularities of the normal hypersurface along $\boldsymbol{\gamma}$ as an application of the unfolding theory of functions. Let $F : (\mathbb{R}\times\mathbb{R}^r, (s_0, \boldsymbol{x}_0)) \to \mathbb{R}$ be a function germ, $f(s) = F_{\boldsymbol{x}_0}(s, \boldsymbol{x}_0)$. We call F an *r-parameter unfolding* of f. If $f^{(p)}(s_0) = 0$ for all $1 \le p \le k$ and $f^{(k+1)}(s_0) \neq 0$, we say f has A_k-singularity at s_0. We also say f has $A_{\ge k}$-singularity at s_0 if $f^{(p)}(s_0) = 0$ for all $1 \le p \le k$. Let F be a r-parameter unfolding of f and f has A_k-singularity $(k \ge 1)$ at s_0, we define the $(k-1)$-jet of the partial derivative $\partial F/\partial x_i$ at s_0 as

$$j^{(k-1)}\Big(\frac{\partial F}{\partial x_i}(s, \boldsymbol{x}_0)\Big)(s_0) = \sum_{j=1}^{k+1} \alpha_{ji}(s-s_0)^j,\ (i = 1, \dots, r).$$

If the rank of $k \times r$ matrix $(\alpha_{0i}, \alpha_{ji})$ is k $(k \le r)$, then F is called a *versal unfolding* of f, where $\alpha_{0i} = \partial F/\partial x_i(s_0, \boldsymbol{x}_0)$. The *discriminant set* of F is defined by

$$D_F = \{\boldsymbol{x} \in \mathbb{R}^r \mid \exists\ s \in \mathbb{R}, F(s, \boldsymbol{x}) = \frac{\partial F}{\partial s}(s, \boldsymbol{x}) = 0\}.$$

There have been the following famous result (Theorem 6.14 on page 150 in [1]).

Theorem 4.1. [1] *Let $F : (\mathbb{R}\times\mathbb{R}^r, (s_0, \boldsymbol{x}_0)) \to \mathbb{R}$ be an r-parameter unfolding of $f(s)$ which has A_k-singularity at s_0, suppose F is a versal unfolding of f, then we have the following.*
(a) If $k = 1$, then D_F is locally diffeomorphic to $\{0\} \times \mathbb{R}^{r-1}$.
(b) If $k = 2$, then D_F is locally diffeomorphic to $C \times \mathbb{R}^{r-2}$.
(c) If $k = 3$, then D_F is locally diffeomorphic to $SW \times \mathbb{R}^{r-3}$.
(d) If $k = 4$, then D_F is locally diffeomorphic to $BF \times \mathbb{R}^{r-4}$.

By Proposition 3.1, the discriminant set of the timelike height function $H(s, \boldsymbol{v})$ is given by

$$D_H = \{\boldsymbol{\gamma}(s) + c\boldsymbol{n}(s) + d\boldsymbol{e}(s) \mid s \in I,\ c, d \in \mathbb{R}\}.$$

Proposition 4.2. *If $h_{\boldsymbol{v}}$ has A_k-singularity at s $(k = 1, 2, 3, 4)$, then H is a versal unfolding of $h_{\boldsymbol{v}}$.*

Proof. We notice that $\boldsymbol{\gamma}(s) \in \mathbb{R}^4_1$.

$$\text{Let } \boldsymbol{\gamma}(s) = (x_1(s), x_2(s), x_3(s), x_4(s)),\ \boldsymbol{v} = (v_1, v_2, v_3, v_4),$$

we have

$$H(s, \boldsymbol{v}) = -x_1v_1 + x_2v_2 + x_3v_3 + x_4v_4 - 1,$$
$$\frac{\partial H(s, \boldsymbol{v})}{\partial v_1} = -x_1(s),\ \frac{\partial}{\partial s}\frac{\partial H(s, \boldsymbol{v})}{\partial v_1} = -x_1'(s),$$
$$\frac{\partial^2}{\partial s^2}\frac{\partial H(s, \boldsymbol{v})}{\partial v_1} = -x_1''(s),\ \frac{\partial^3}{\partial s^3}\frac{\partial H(s, \boldsymbol{v})}{\partial v_1} = -x_1'''(s).$$

We also have

$$\frac{\partial H(s, \boldsymbol{v})}{\partial v_i} = x_i(s),\ \frac{\partial}{\partial s}\frac{\partial H(s, \boldsymbol{v})}{\partial v_i} = x_i'(s),$$
$$\frac{\partial^2}{\partial s^2}\frac{\partial H(s, \boldsymbol{v})}{\partial v_i} = x_i''(s),\ \frac{\partial^3}{\partial s^3}\frac{\partial H(s, \boldsymbol{v})}{\partial v_i} = x_i'''(s),\ (i = 2, 3, 4).$$

The 3-*jet* of $\dfrac{\partial H(s,\boldsymbol{v})}{\partial v_i}$, $(i = 1, 2, 3, 4)$ at s_0 is given by

$$\frac{\partial H(s,\boldsymbol{v})}{\partial v_i} =$$

$$\frac{\partial H(s_0,\boldsymbol{v})}{\partial v_i} + \frac{\partial}{\partial s}\frac{\partial H(s_0,\boldsymbol{v})}{\partial v_i}(s-s_0) + \frac{1}{2}\frac{\partial^2}{\partial s^2}\frac{\partial H(s_0,\boldsymbol{v})}{\partial v_i}(s-s_0)^2 + \frac{1}{6}\frac{\partial^3}{\partial s^3}\frac{\partial H(s_0,\boldsymbol{v})}{\partial v_i}(s-s_0)^3 =$$

$$\alpha_{0,i} + \alpha_{1,i}(s-s_0) + \frac{1}{2}(s-s_0)^2 + \frac{1}{6}\alpha_{3,i}(s-s_0)^3.$$

By Proposition 3.1, h has the $A_{\geq 1}$-singularity at s_0 if and only if $\boldsymbol{v} = \boldsymbol{\gamma}(s) + c\boldsymbol{n}(s) + d\boldsymbol{e}(s)$. Since the curve $\boldsymbol{\gamma}(s)$ is regular, the rank of $(-x_1(s)\ x_2(s)\ x_3(s)\ x_4(s))$ is 1. We can get that h has the $A_{\geq 2}$-singularity at s_0 if and only if $\boldsymbol{v} = \boldsymbol{\gamma}(s) - (1/\kappa_g(s))\boldsymbol{n}(s) + d\boldsymbol{e}(s)$. When h has the $A_{\geq 2}$-singularity at s_0, we require the 2×4 matrix

$$\begin{pmatrix} -x_1(s) & x_2(s) & x_3(s) & x_4(s) \\ -x_1'(s) & x_2'(s) & x_3'(s) & x_4'(s) \end{pmatrix}$$

to have rank 2, which it always does since $\boldsymbol{\gamma}(s)$ in de Sitter 3-space.

It also follows from Proposition 3.1 that h has the $A_{\geq 3}$-singularity at s_0 if and only if

$$\boldsymbol{v} = \boldsymbol{\gamma}(s) - (1/\kappa_g(s))\boldsymbol{n}(s) - \big(\kappa_g'(s)/\kappa_g^2(s)\tau_g(s)\big)\boldsymbol{e}(s).$$

We require the 3×4 matrix

$$\begin{pmatrix} -x_1(s) & x_2(s) & x_3(s) & x_4(s) \\ -x_1'(s) & x_2'(s) & x_3'(s) & x_4'(s) \\ -x_1''(s) & x_2''(s) & x_3''(s) & x_4''(s) \end{pmatrix}$$

to have rank 3, which follows from the proof of the next case.

By Proposition 3.1, h has the $A_{\geq 4}$-singularity at s_0 if and only if

$$\boldsymbol{v} = \boldsymbol{\gamma}(s) - (1/\kappa_g(s))\boldsymbol{n}(s) - \big(\kappa_g'(s)/\kappa_g^2(s)\tau_g(s)\big)\boldsymbol{e}(s) \text{ and } \sigma(s) = 0.$$

We require 4×4 matrix

$$\begin{pmatrix} -x_1(s) & x_2(s) & x_3(s) & x_4(s) \\ -x_1'(s) & x_2'(s) & x_3'(s) & x_4'(s) \\ -x_1''(s) & x_2''(s) & x_3''(s) & x_4''(s) \\ -x_1'''(s) & x_2'''(s) & x_3'''(s) & x_4'''(s) \end{pmatrix}$$

to have rank 4. In fact

$$\begin{aligned} &\begin{vmatrix} -x_1(s) & x_2(s) & x_3(s) & x_4(s) \\ -x_1'(s) & x_2'(s) & x_3'(s) & x_4'(s) \\ -x_1''(s) & x_2''(s) & x_3''(s) & x_4''(s) \\ -x_1'''(s) & x_2'''(s) & x_3'''(s) & x_4'''(s) \end{vmatrix} \\ &= -\begin{vmatrix} x_1(s) & x_2(s) & x_3(s) & x_4(s) \\ x_1'(s) & x_2'(s) & x_3'(s) & x_4'(s) \\ x_1''(s) & x_2''(s) & x_3''(s) & x_4''(s) \\ x_1'''(s) & x_2'''(s) & x_3'''(s) & x_4'''(s) \end{vmatrix} \\ &= -\langle \boldsymbol{\gamma}, \boldsymbol{\gamma}'(s) \wedge \boldsymbol{\gamma}''(s) \wedge \boldsymbol{\gamma}'''(s)\rangle \\ &= \kappa_g^2(s)\tau_g(s) \neq 0. \end{aligned}$$

In summary, H is a versal unfolding of $h_{\boldsymbol{v}}$, this completes the proof.

The proof of Theorem 2.1. Let $\boldsymbol{\gamma} : I \to S_1^3$ be a timelike regular curve and $\langle \boldsymbol{t}'(s), \boldsymbol{t}'(s)\rangle \neq 1$ and $\tau_g(s) \neq 0$. As $\boldsymbol{v}_0 = NHS(s_0, u_0, w_0)$, we give a function $H : S_1^3 \to \mathbb{R}$, by $H(u) = \langle u, v_0\rangle - 1$, then we assume that $h_{\boldsymbol{v}_0}(s) = H(\boldsymbol{\gamma}(s))$. Because $H^{-1}(0) = S^2_1(v_0)$ and 0 is a regular value of H, $\boldsymbol{\gamma}$ and $S^2_1(v_0)$ have $(k+1)$-point contact for s_0 if and only if $h_{\boldsymbol{v}_0}(s)$ has the A_k-singularity at s_0. By Proposition 3.1, Theorem 4.1, and Proposition 4.2 the proven of Theorem 2.1 is obvious.

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