EXTREMAL CONFIGURATIONS OF ROBOT ARMS IN THREE DIMENSIONS

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ABSTRACT. We define a volume function for a robot arm in $\mathbb{R}^3$ and give geometric conditions for its critical points.

1. INTRODUCTION

Linkages are flexible 1-dimensional structures, where edges are straight intervals of a fixed length, where flexes are allowed at vertices. For general properties of linkages we refer to [1],[2] and [3].

Recently G. Khimshiashvili, G. Panina, their co-workers and the author investigated various extremal problems on the moduli spaces of linkages. An important part of that studies considers cyclic configurations of planar polygonal linkages and open robot arms as critical points of the oriented area function [4], [5] , [7], [8] and [12].

The aim of the current paper is to generalize these statements to the 3-dimensional case. We will give a geometric description of the critical configurations in the case of oriented volume in 3D. The extremal arms consist of planar circular contributions combined with zigzags (theorem 4.5). For computational reasons we consider the signed volume function on a parameter space and not on the moduli space. The isotropy groups of oriented isometries acting on this parameter space are not constant. We study this effect for the 3-arm and show in that case:

The oriented moduli space of 3-arms in $\mathbb{R}^3$ is a 3-sphere. The Volume function is an exact topological Morse function on this space with precisely two Morse critical points.

2. PRELIMINARIES AND NOTATION

An *n-linkage* is a sequence of positive numbers $l_1, \ldots, l_n$. It should be interpreted as a collection of rigid bars of lengths l_i joined consecutively by revolving joints in a chain, either open or closed. Open linkages are sometimes called *robot arms*. We study the flexes of the both types of chain with allowed self-intersections. This is formalized in the following definitions.

Definition 2.1. For an open linkage L, *a configuration* in the Euclidean space $\mathbb{R}^d$ is a sequence of points $R = (p_1, \ldots, p_{n+1})$, $p_i \in \mathbb{R}^d$ with $l_i = |p_i, p_{i+1}|$ modulo the action of orientation preserving isometries. We also call R *an open chain.*

The set $M_d^\circ(L)$ of all such configurations is *the moduli space, or the configuration space of the robot arm L.*

For a closed polygonal linkage, we claim in addition that the last point coincides with the first point: a configuration of the linkage L in the Euclidean space $\mathbb{R}^d$ is a sequence of points $P = (p_1, \ldots, p_n)$, $p_i \in \mathbb{R}^d$ with $l_i = |p_i, p_{i+1}|$ for $i = 1, .., n-1$ and $l_n = |p_n, p_1|$. As above, the action of orientation preserving isometries is factored out. We also call P *a closed chain* or a *polygon.*

The set $M_d(L)$ of all such configurations is *the moduli space, or the configuration space of the polygonal linkage L.*

In [5] and [8] the 2-dimensional case was treated with the signed area function on the configuration space. We recall some definitions and results.

Definition 2.2. The *signed area* of a polygon P with the vertices $p_i = (x_i, y_i)$ is defined by

$$2A(P) = (x_1 y_2 - x_2 y_1) + \ldots + (x_n y_1 - x_1 y_n).$$

The *signed area* of an open chain with the vertices $p_i = (x_i, y_i)$ is defined by

$$2A(P) = (x_1 y_2 - x_2 y_1) + \ldots + (x_n y_{n+1} - x_{n+1} y_n) + (x_{n+1} y_1 - x_1 y_{n+1}).$$

In other words, we add one more edge that turns an open chain to a closed polygon and take the signed area of the polygon.

Definition 2.3. A polygon P is called *cyclic* if all its vertices p_i lie on a circle.

A robot arm R is called *diacyclic* if all its vertices p_i lie on a circle, and $p_1 p_{n+1}$ is the diameter of the circle.

Cyclic polygons and cyclic open chains arise as critical points of the signed area:

Theorem 2.4. *([5], [8])*

Generically, a polygon P is a critical point of the signed area function A iff P is a cyclic configuration.

Generically, an open robot arm R is a critical point of the signed area function A iff R is a diacyclic configuration. □

3. About 3-arm in $\mathbb{R}^3$

Before we treat in the next section open linkages with n arms in $\mathbb{R}^3$, we study here 3-arms in $\mathbb{R}^3$.

Let us fix some notation. The arm vectors are: $a = (1, 0, 0)$, b and c of length $|a|, |b|, |c|$.

A spatial arm is constructed as follows: we take the segments from O to the end points A, B, C of a, $a + b$, $a + b + c$. This yields a tetrahedron $OABC$.

Definition 3.1. We define the *signed volume* V of the 3-arm as the triple vector product:

$$V = [a, a + b, a + b + c] = [a, b, c].$$

We intend to study V on several parameter spaces:
- On $S^2 \times S^2$,
- On $S^1 \times S^2$, where we fix the vector b to lie in the xy plane,
- On the moduli space M_3^o (mod the $SO(3)$ action).

In each of these cases critical points may be different. We intend to compare the critical points and the Morse theory for the three cases.

3.1. **On $S^2 \times S^2$.** Before starting we define some special positions of the 3-arm:

- *Tri-orthogonal*: The vectors a, b, c are tri-orthogonal; equivalently: the sphere with diameter OC contains also the points A and B,
- *Degenerate*: The arm lies in a two-dimensional subspace,
- *Aligned*: The arm is contained in a line.

Proposition 3.2. *The signed area $V : S^2 \times S^2 \to \mathbb{R}$ has the following critical points:*

- *Tri-orthogonal arms (maximum, resp minimum). These are Bott-Morse critical points with transversal index 3 and critical value $\pm|a||b||c|$.*
- *Isolated points, corresponding to the aligned configurations. Here V has Morse index 2 and the critical value 0.*

Proof. We use coordinate systems on the spheres; we take partial derivatives with respect to all coordinates. We denote the partial derivatives of b by $\delta_1 b$ and $\delta_2 b$. Both are non-zero and orthogonal to b. We take partial derivatives of $V = [a, b, c]$ in the $(\delta_1 b, \delta_2 b)$ directions: $[a, \delta_1 b, c] = 0$ and $[a, \delta_2 b, c] = 0$.

We will shorten this to $[a, \dot{b}, c] = 0$ meaning that the equation holds for all vectors in the tangent space of b (which is orthogonal to b and spanned by $\delta_1 b$ and $\delta_2 b$). In this way we get:

$$[a, \dot{b}, c] = 0, \quad [a, b, \dot{c}] = 0.$$

For both equations we will consider two cases:

equation	ortho condition	parallel condition
$[a, \dot{b}, c] = 0$	$a \times c \neq o$ equivalent to $b \perp a$ and $b \perp c$	$a \times c = o$ equivalent to $a \parallel c$
$[a, b, \dot{c}] = 0$	$a \times b \neq o$ equivalent to $c \perp a$ and $c \perp b$	$a \times b = o$ equivalent to $a \parallel b$

The combination of the two ortho conditions gives the tri-orthogonal case of the proposition; combining the two parallel conditions is the aligned case. Combining one ortho condition with the other parallel condition gives a contradiction. □

Next we describe the type of the critical points. For the positively oriented tri-orthogonal case we get a maximum. Due to the remaining SO-action the singular set is an S^1, and its transversal Morse index is 3. The other orientation gives a minimum on S^1 with the transversal Morse index 0. The aligned configurations (4 cases) occur in isolated points. In all these cases we have index 2. We check the Bott-Morse formula:

$$\sum t^{\lambda(C)} P(C) - P(M) = (1 + t)R(t)$$

where $R(t)$ must have non-negative coefficients. In our case we have

$$t^3(1 + t) + (1 + t) + (1 + t) + 4t^2 - (t^4 + 2t^2 + 1) = t^3 + 2t^2 + t = (1 + t)(t^2 + t),$$

so this is OK. □

3.2. **On $S^1 \times S^2$.** After a rotation we can always assume that b lies in the xy-plane. We consider SO-action, that fixes this plane.

Proposition 3.3. *The signed volume $V : S^1 \times S^2 \to \mathbb{R}$ has the following critical points:*

- *4 points, corresponding to tri-orthogonal arms (2 maxima, respectively 2 minima). At these points V has critical value 0.*
- *Two circles corresponding to degenerate configurations. where a and b are aligned and c is free to move in the xy-plane. At these points V has Bott-Morse critical points with transversal index 1.*

The proof is a straight forward computation [6].

We check the result with Bott-Morse formula:

$$2t^3 + 2 + 2t(1+t) - (t^3 + t^2 + t + 1) = t^3 + t^2 + t + 1 = (t+1)(t^2+1) \ .$$

Note the difference between the situation on $S^2 \times S^2$ and on $S^1 \times S^2$.

3.3. **On the moduli space M_3^o.** This moduli space is homeomorphic to S^3. This is shown in [9]. We return to this later in this paper. An outline is as follows: First construct the non oriented moduli space and show that this is a topological 3-ball. The sphere S^3 appears as a gluing of two such balls along their common boundary. This boundary consists of degenerate arms (those who are not the maximal dimension).

The function V will be studied separately on the two hemispheres, each of whom has exactly one Morse point. Near the common boundary one can show that V glues to a topologically regular function. In Section 6 we give details and prove the following:
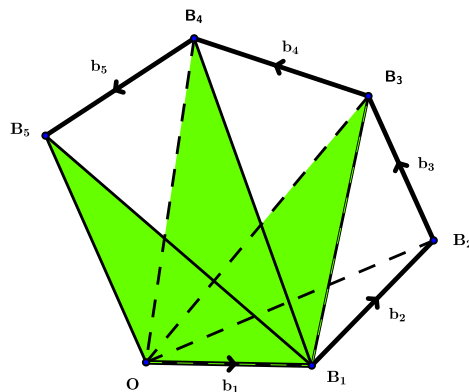
Theorem 3.4. *The oriented moduli space of 3-arms in $\mathbb{R}^3$ is a 3-sphere. V is an exact topological Morse function on this space with precisely two Morse critical points.* $\qquad\square$

Note that the critical points with $V = 0$, which we got before in the cases with parametrization $S^2 \times S^2$ or $S^1 \times S^2$, are no longer (topological) critical on the moduli space.

4. ABOUT n-ARMS IN $\mathbb{R}^3$

There is no unique way to attach a volume to a polygonal chain. We take one special situation as starting point for our definition of (signed) volume in case of a n-arm in $\mathbb{R}^3$. The following picture where all simplices contain $a = b_1$ illustrates this definition.

The relation with the volume of the convex hull can be lost, especially when the combinatorics of the convex hull changes.

Definition 4.1. Let an n-arm be given by the vectors $b_1, \cdots, b_n$. The vertices are $O, B_1, \cdots, B_n$. We fix $b_1 = a$ (as before). We denote $c_k = \sum_{i=1}^{k} b_i$ (the endpoint of this vector is B_k). The *signed volume function* is defined as

$$V = \sum_{k=1}^{n-1} [b_1, c_k, c_{k+1}],$$

which can be rewritten as:

$$V = [b_1, b_2, b_3] + [b_1, b_2 + b_3, b_4] + [b_1, b_2 + b_3 + b_4, b_5] + \cdots [b_1, b_2 + \cdots b_{n-1}, b_n].$$

N.B. Note that this signed volume is essentially the signed area of the projection onto the plane orthogonal to b_1.

Lemma 4.2. (Mirror lemma) *Let two arms differ on a permutation of the arms $2, \ldots, n$. Then there exists a bijection (by 'mirror-symmetry') between their "moduli spaces" which preserves the signed volume function. Consequently this bijection preserves critical points and their local (Morse) types.*

Proof. As in the planar case [7]. □

The conditions for critical points are:

$$\forall \dot{b_2} \perp b_2 : [b_1, \dot{b_2}, b_3] + [b_1, \dot{b_2}, b_4] + \cdots + [b_1, \dot{b_2}, b_n] = [b_1, \dot{b_2}, b_3 + \cdots + b_n] = 0,$$

$$\forall \dot{b_3} \perp b_3 : [b_1, b_2, \dot{b_3}] + [b_1, \dot{b_3}, b_4] + \cdots [b_1, \dot{b_3}, b_n] = [b_1, b_2 - (b_4 + \cdots + b_n), \dot{b_3}] = 0.$$

The rth -derivative gives the following:

$$\forall \dot{b_r} \perp b_r : [b_1, b_2 + \cdots + b_{r-1}, \dot{b_r}] + [b_1, \dot{b_r}, b_{r+1}] + \cdots + [b_1, \dot{b_r}, b_n] =$$

$$= [b_1, b_2 + \cdots + b_{r-1} - (b_{r+1} + \cdots + b_n), \dot{b_r}] = 0.$$

There are two cases for any $2 \leq r \leq n$ (which we call *ortho* and *parallel*):

- case O_r:

$$b_1 \times ((b_2 + \cdots + b_{r-1}) - (b_{r+1} + \cdots + b_n)) \neq 0.$$

Hence we have the following orthogonalities

$$b_r \perp b_1 \ \wedge \ b_r \perp (b_2 + \cdots + b_{r-1}) - (b_{r+1} + \cdots + b_n).$$

- case P_r:

$$b_1 \times ((b_2 + \cdots + b_{r-1}) - (b_{r+1} + \cdots + b_n)) = 0,$$

which means that $(b_2 + \cdots + b_{r-1}) - (b_{r+1} + \cdots + b_n) \in \mathbb{R}b_1$.

Next we decompose vectors into their $\mathbb{R}b_1$-component and its orthogonal complement:

$$b_r = b_r' + b_r^{\perp}$$

Lemma 4.3. *For all $r = 2, \cdots, n$:*

$$b_r^{\perp} \perp (b_2^{\perp} + \cdots + b_{r-1}^{\perp}) - (b_{r+1}^{\perp} + \cdots + b_n^{\perp})$$

and also

$$(b_2^{\perp} + \cdots + b_{r-1}^{\perp}) \perp (b_r^{\perp} + \cdots b_n^{\perp}) \quad (*)$$

For any critical point of the signed volume function on n-arms in $\mathbb{R}^3$ one can consider the projection of the arm onto the hyperplane orthogonal to b_1.

Proposition 4.4. *The vertices of this planar $(n-1)$-arm $b_2^\perp, \ldots, b_n^\perp$ lie on a circle with diameter the interval $B_1 B_n^\perp$ from the start point to the end point of this arm. This configuration corresponds to a critical point of such arms (but with fixed lengths) under the signed area function.* □

Note that in general we don't have fixed lengths of the projections and that projections can be "degenerate".

We next treat several cases of the spatial situations and after that state the general result in Theorem 4.5.

4.1. **Full ortho case:** O_r for all $r = 2, \ldots, n$.
Now $b_r = b_r^\perp$. So we have:

Statement 1. The critical points of the signed volume function on n-arms in $\mathbb{R}^3$ are exactly those configurations, where all vertices (including the first O and the last B_r) are on a sphere with diameter OB_r; the first arm is perpendicular to the all other arms. Delete the first arm: the vertices of this planar $(n-1)$-arm lie on a circle with $B_1 B_r$ as the diameter. This configuration corresponds precisely to a critical point of such arms under the signed area function. Moreover,

$$V = |b_1| \cdot sA.$$

4.2. **Full parallel case:** P_r for all $r = 2, \ldots, n$.

If n is odd we find $b_r \in \mathbb{R}b_1$ $(r = 2, \ldots, n)$.
If n is even we find $b_r + b_{r+1} \in \mathbb{R}b_1$ $(r = 2, \ldots, n-1)$.

Statement 2. Critical points of V are aligned configurations if n is odd and 1-parameter families of zigzags if n is even. Zigzags are arms, which project all to the same interval (see Fig. 1, right).

Zigzags also contain the aligned configuration. In a zigzag the lengths of the projections can vary the from 0 to the minimum lengths of $b_2, \ldots, b_r$.
Both full cases (see Fig. 1) have the property that solutions exists for all length vectors.
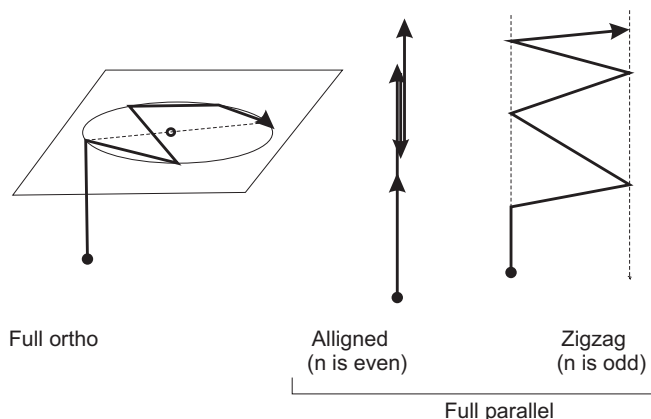

FIGURE 1.

4.3. **General case: $n-k$ parallel conditions, and $k-1$ ortho conditions.** We can assume (due to the mirror lemma) that the last $n-k$ conditions are parallel. That is, we have

$$b_2 + \cdots + b_k + b_{k+1}^{\perp} + \cdots + b_{n-1}^{\perp} = 0$$

together with

$$b_{k+1} + b_{k+2} \in \mathbb{R}b_1, \cdots, b_{n-1} + b_n \in \mathbb{R}b_1.$$

So

$$b_{k+1}^{\perp} + b_{k+2}^{\perp} = 0, \cdots, b_{n-1}^{\perp} + b_n^{\perp} = 0.$$

This has the following consequences:

- The $b_{k+1}^{\perp}, \cdots, b_n^{\perp}$ are diameters of the critical circle,
- If $n-k$ is even, then $b_2 + \cdots + b_k + b_{k+1}^{\perp} = 0$.
  The $(k-1)$-arm $b_2, \cdots, b_k$ is an open planar diacyclic chain (*diameter condition*).
- If $n-k$ is odd, then $b_2 + \cdots + b_k = 0$. The $(k-1)$-arm $b_2, \cdots, b_{n-k-1}$ is a closed planar cyclic polygon (*closing condition*).

In both cases (odd and even) the projections of the vertices lie on a circle (see Fig. 2). There are only finite number of these circles possible. For a realization it is necessary that $|b_i| \geq R$ (radius of circle) if $k+1 \leq i \leq n$.
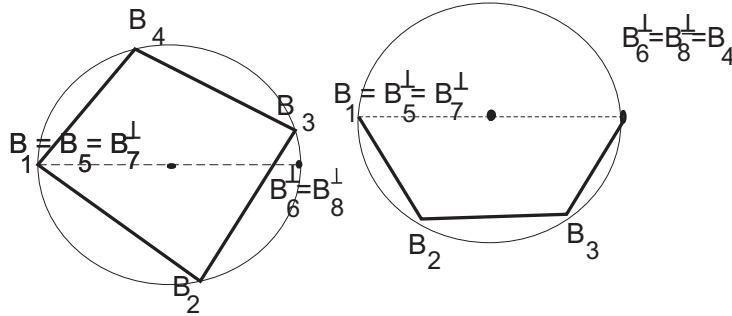

FIGURE 2. Projected vertices are on a circle.

The above discussion shows the following:

Theorem 4.5. *The critical points of V up to "mirror-symmetry" are as follows (see Fig. 3): There exists a division of the n-arm into a sub-arm b_1, a sub-arm $b_2, \ldots, b_k$ and a sub-arm $b_{k+1}, \ldots, b_n$ such that:*

- *b_1 is orthogonal to each of $b_2, \ldots, b_k$ (which lie in a plane $\mathbb{R}b_1^{\perp}$).*
- *The vertices of the arm $b_2, \ldots, b_k$ lie on a circle, satisfying*
  - *the closing condition if $n-k=$ odd,*
  - *the diameter condition if $n-k=$ even.*
- *The arm $b_{k+1}, \ldots, b_n$ is a zigzag, which projects orthogonally to the diameter of the circle.*   □
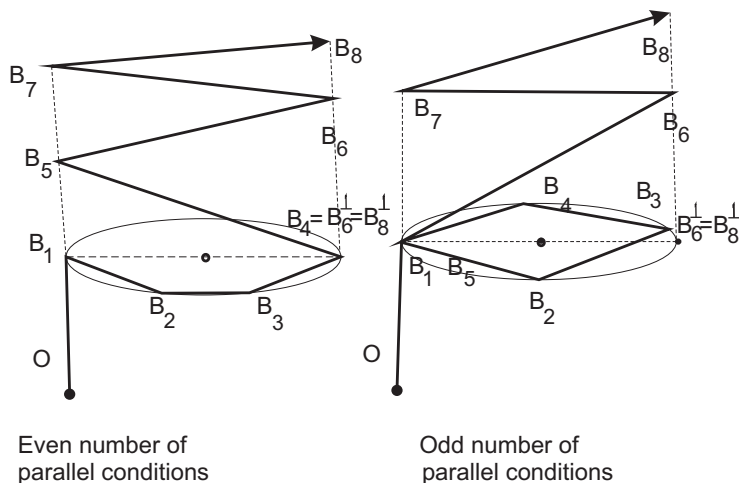
FIGURE 3.   Solutions in the general case.

5. ABOUT N-ARMS IN $\mathbb{R}^3$; PROJECTION ON PLANES

As mentioned befor the signed volume is essentially the signed area of the projection onto the plane orthogonal to b_1. The same reasoning can be applied to more general projections. We consider in $\mathbb{R}^3$ a vector p, which is the direction of the orthogonal projection on a plane $\mathbb{R}p^\perp$.

Let the n-arm be given by the vectors $b_1, \cdots, b_n$. The vertices are $O, B_1, \cdots, B_n$.

Define the signed Projected Area function as follows:

$$PA = [p, b_1, b_2] + [p, b_1 + b_2, b_3] + [p, b_1 + b_2 + b_3, b_4] +$$

$$[p, b_1 + b_2 + b_3 + b_4, b_5] + \cdots + [p, b_1 + \cdots + b_{n-1}, b_n].$$

We fix first both the positions of p and b_1!.

We assume that $p \times b_1 \neq 0$.

Theorem 5.1. (Projection with fixed p and b_1) *The critical points of PA up to "mirror-symmetry" are as follows:*

There exists a division of the n-arm into two sub-arms $b_1, \ldots, b_k$ and $b_{k+1}, \ldots, b_n$, such that:

- *The vertices of the arm $b_1^\perp, b_2, \ldots, b_k$ lie on a circle in the projection plane, satisfying*
  - *the closing condition if $n - k = $ odd,*
  - *the diameter condition if $n - k = $ even.*
- *The arm $b_{k+1}, \ldots, b_n$ is a zigzag, which projects orthogonally to the diameter of the circle.*

Proof. As in the signed volume case, see Theorem 4.5. $\square$

Remark 1. The special case that p is orthogonal to b_1 is included. In this case we obviously have $b_1^\perp = b_1$.

If p is parallel to b_1 we are in the case of signed volume studied before.

Remark 2. If we fix only p and not b_1 the study of the signed projected area of the n-arm $b_1, \ldots, b_n$ is equivalent to that of the signed volume of the $(n+1)$-arm $p, b_1, \ldots, b_n$. We state this:

Theorem 5.2. *(**General projection on plane**) The critical points of PA up to "mirror-symmetry" are as follows:*
There exists a division of the n-arm into two sub-arms $b_1, \ldots, b_k$ and $b_{k+1}, \ldots, b_n$, such that:

- *The vertices of the arm $b_1, b_2, \ldots, b_k$ lie on a circle in the projection plane, satisfying*
  - *the closing condition if $n - k = $ odd,*
  - *the diameter condition if $n - k = $ even.*
- *The arm $b_{k+1}, \ldots, b_n$ is a zigzag, which projects orthogonally to the diameter of the circle.*   □

6. GRAM MATRICES AND MODULI SPACE

One way to study the moduli space of n-arms in $\mathbb{R}^n$ is to use the Gram matrix. This has an advantage that there is a direct relation with the volume.

Given a set of vectors, the Gram matrix G is the matrix of all possible inner products. Let B be the matrix whose columns are the arm vectors $b_1, \ldots, b_n$. Then the Gram matrix is $G = B^t B$. Its determinant is the square of the volume of the simplex spanned by these vectors:

$$\det G = (V)^2.$$

The Gram matrix is always a positive semi definite symmetric matrix and any positive semi definite symmetric matrix is the Gram matrix of some B. If G is positive definite it determines B up to isometry.

In our case of n-arm in $\mathbb{R}^n$ the inner products $(b_i.b_i)$ are the fixed numbers b_i^2. The other entries of the Gram matrix we consider as variables x_{ij}. Its determinant is:

$$
\begin{vmatrix}
b_1^2 & x_{12} & x_{13} & & & x_{1n} \\
x_{12} & b_2^2 & x_{23} & & & x_{2n} \\
x_{13} & x_{23} & b_3^2 & & & x_{3n} \\
& & & & x_{ij} & \\
& & & x_{ij} & & \\
x_{1n} & & & & & b_n^2
\end{vmatrix}
$$

For a given n-arm, Gram matrix is contained in a subspace of dimension $\frac{n(n-1)}{2}$.

Remark. Note that the equivalence is only up to isometry and not with respect to orientation. On the set GRAM of all Gram matrices we will consider $|V|$. In order to treat the oriented version we have to take two copies of GRAM and to glue it on the common boundary. The set GRAM is contained in the product of intervals $-b_i b_j \leq x_{ij} \leq b_i b_j$.

In [9] diagonals are used as coordinates of the moduli space. GRAM is related to that description by the cosine rule:

$$d_{ij} = b_i^2 + b_j^2 - 2x_{ij}.$$

Note that G is differentiable on the entire space $\mathbb{R}^{n(n-1)/2}$. In turn, $|V|$ is defined on GRAM, but is only differentiable on the interior $\{|V| > 0\}$. What happens on the boundary?

We consider next the 3 dimensional case and use the notations from section 3.

$$\det G = \begin{vmatrix} a^2 & z & y \\ z & b^2 & x \\ y & x & c^2 \end{vmatrix} = 2xyz - a^2x^2 - b^2y^2 - c^2z^2 + a^2b^2c^2 = 0$$

In Figure 4 this equation is visualized. Note that GRAM is equal to the intersection $\{\det G \geq 0\}$ with the box defined by $\{|x| < bc, \ |y| < ac, \ |z| < ab\}$. The boundary of the box intersects $\det G = 0$ only in four points.

The critical points of $\det G$ are given by the conditions
$\partial \det G / \partial x = 2(yz - a^2x) = 0$,
$\partial \det G / \partial y = 2(xz - b^2y) = 0$,
$\partial \det G / \partial z = 2(xy - c^2z) = 0.$

We find the following critical points of $\det G$:

- $(x, y, z) = (0, 0, 0)$: maximum $a^2b^2c^2$ (index 3)
- $(x, y, z) = (bc, ac, ab), (-bc, ac, -ab), (-bc, -ac, ab)$ or $(bc, -ac, -ab)$ (just the four intersection points mentioned above).

  The critical value is equal to 0. What are the types of these 4 critical points? We compute the Hessian matrix and its determinant:

$$\det H = \begin{vmatrix} -a^2 & z & y \\ z & -b^2 & x \\ y & x & -c^2 \end{vmatrix}$$

Note that $\det H(x, y, z) = -\det G(-x, -y, -z)$.
Each of our 4 critical points is non-degenerate; since $\det H \neq 0$. The Morse index is 2. Note also that they are related to aligned situations.

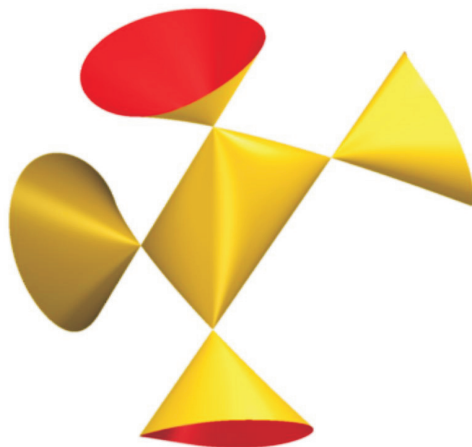
FIGURE 4. Zero locus of the determinant of G. The compact region corresponds to the set of Gram matrices. (The figure is produced by SINGULAR software.)

The local behavior of the level surfaces near the critical level can be studied with the local formula:

$$\det G = -\zeta_1^2 - \zeta_2^2 + \zeta_3^2.$$

Its zero level is a quadratic cone. We restrict ourselves by points inside the box. Near the singular points we have a homeomorphism:

$$(\det G)^{-1}[0, \epsilon] = (\det G)^{-1}[\epsilon] \times [0, \epsilon]$$

For the non-critical points this is is guaranteed by the regular interval theorem; so the product structure is global. We have shown the following:

Proposition 6.1. *(Fig. [4]) The closure of the component of $G^{-1}(0, a^2b^2c^2)$, which contains $(0, 0, 0)$ is a topological 3-ball. Its boundary is a topological 2-sphere (differentiable outside 4 critical points).* □

This component is exactly the set GRAM. Moreover, in this 3-dimensional case GRAM is equivalent (up to isometry) to the set of triples of arm vectors.

Since we have $\det G = |V|^2$, the both functions have the same level curves on the domain of common definition. So the above proposition tell us that the (unoriented) moduli space of 3-arm is a topological disc. By gluing two copies of GRAM along the common boundary we get:

Theorem 6.2. *The oriented moduli space of 3-arms in $\mathbb{R}^3$ is a 3-sphere. V is an exact topological Morse function on this space with precisely two Morse critical points.* □

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