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QUASI CUSP SINGULARITIES

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Abstract. We obtain a list of all simple classes of singularities of function germs with respect to the quasi cusp equivalence relation. We discuss its connection with the singularities of Lagrangian projections in presence of a cuspidal edge. We also describe the bifurcation diagrams and caustics of simple quasi cusp singularities.

1. Introduction

In 2007, motivated by the needs of the theory of Lagrangian maps of singular varieties, Vladimir Zakalyukin classified function germs with respect to new non-standard equivalence relations (see [5, 6]) which he named quasi equivalences. The relations are aimed to control positions of only critical points of functions or maps and allow absolute freedom outside the critical locus.

Zakalyukin's quasi equivalences are a very natural and efficient tool for classification of Lagrangian maps of smooth manifolds containing distinguished singular hypersurfaces which are playing the role of boundaries or of initial conditions in the corresponding system of differential equations. For comparison, Arnold's classical approach to classification of boundary functions singularities and, in general, of functions in presence of possibly singular hypersurfaces [3] corresponds to consideration of a pair of Lagrangian submanifolds meeting along a codimension 1 intersection. Zakalyukin's method allows to keep information only about one of the submanifolds of a pair and of the intersection set. Thus, the quasi equivalences of functions are considerably rougher than equivalences of functions on manifolds with (singular) boundaries.

The quasi classifications of function singularities corresponding to smooth Lagrangian submanifolds with the boundary hypersurface being either smooth or a union of two smooth components meeting transversally have been considered in [6, 2]. In the present paper we study the case of the boundaries which are cylinders over generalized cuspidal curves $x_1^s = x_2^2$.

The paper is organized as follows. In Section 2 we introduce our main notions, of the pseudo and quasi border equivalence relations, and derive an expression for the tangent space to the quasi cusp equivalence class of a function. In Section 3 we obtain the classifications of simple quasi cusp singularities. In Section 4, the bifurcation diagrams and caustics of simple quasi cusp singularities are described. Finally, in Section 5 we discuss the singularities of Lagrangian projections in presence of a cuspidal edge.

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2. Pseudo and quasi border equivalence relations

Consider a coordinate space $\mathbb{R}^n$ with a hypersurface Γ in it. The hypersurface will usually be a cylinder, and we therefore split the coordinates on $\mathbb{R}^n$ into $y = (y_1, y_2, \dots, y_{n-m})$ which accommodate cylindrical directions for Γ and $x = (x_1, x_2, \dots, x_m)$ in which an equation $B(x) = 0$ of Γ is written. When this distinction between x and y is not crucial, we will be using the notation $w = (x, y)$ for the whole set of coordinates on $\mathbb{R}^n$.

In the current paper, we consider the following shapes of Γ.

1. The hypersurface is smooth, in which case we set $\Gamma = \Gamma_b = \{x_1 = 0\}$.
2. The hypersurface is a cylinder over a cusp: $\Gamma = \Gamma_{csp} = \{x_2^2 - x_1^s = 0 : s \geq 3\}$.

Remark 2.1. Notice that if $s = 2$ then the hypersurface $\{x_2^2 - x_1^2 = 0\}$ is diffeomorphic to the corner $\{x_1 x_2 = 0\}$ which was discussed in [2].

We consider germs of C^∞ functions $f : (\mathbb{R}^n, 0) \to \mathbb{R}$, in local coordinates w as above. We denote by $\mathbf{C}_w$ the ring of all such germs at the origin.

Definition 2.2. Two functions $f_0, f_1 : \mathbb{R}^n \to \mathbb{R}$ are called *pseudo border* equivalent if there exists a diffeomorphisim $\theta : \mathbb{R}^n \to \mathbb{R}^n$ such that $f_1 \circ \theta = f_0$, and if a critical point c of the function f_0 belongs to the border Γ then $\theta(c)$ also belongs to Γ and vice versa, if c is a critical point of f_1 and belongs to Γ then $\theta^{-1}(c)$ also belongs to Γ.

A similar definition can be introduced for germs of functions.

Remarks 2.3.

1. The general statements below are valid for reasonably good hypersurfaces. For rigorousness, we assume that the hypersurface Γ is a stratified set, and the stratification satisfies the Whitney condition A. Also, we assume in the definition 2.2 that if a critical point c belongs to some stratum of Γ then $\theta(c)$ belongs to the same stratum.
2. The pseudo border equivalence will be also called pseudo boundary or pseudo cusp for respective type of Γ.
3. The pseudo border equivalence is an equivalence relation: if $f_1 \sim f_2$ and $f_2 \sim f_3$ then $f_1 \sim f_3$. However, this relation is not a group action as the set of admissible diffeomorphisms depends on a function.

Definition 2.4. Let J be an ideal in $\mathbf{C}_w$, then we define the *radical* $Rad(J)$ of the ideal J as the set of all elements in $\mathbf{C}_w$, vanishing on the set of common zeros of germs from J:

$$Rad(J) = I(V(J)),$$

where

$$V(J) = \{w \in \mathbb{R}^n : h(w) = 0 \text{ for all } h \in J\},$$

and

$$I(V(J)) = \{\varphi \in \mathbf{C}_w : \varphi(w) = 0 \text{ for all } w \in V(J)\}.$$

A similar definition can be introduced when we replace $\mathbf{C}_w$ by the space $\mathbb{R}[w]$ of all real polynomials in the variables w.

In general, the radical of an ideal behaves badly when the ideal depends on a parameter.

Example 2.5. Consider the family of ideals $J_\varepsilon = w(w-\varepsilon)\mathbb{R}[w]$, $w \in \mathbb{R}$, depending on $\varepsilon \in \mathbb{R}$. Then,

$$Rad(J_\varepsilon) = \left\{ \begin{array}{ccc} J_\varepsilon & \text{if} & \varepsilon \neq 0 \\ w\mathbb{R}[w] & \text{if} & \varepsilon = 0. \end{array} \right.$$

Hence, the dimension of the quotient space $\mathbb{R}[w]/Rad(J_\varepsilon)$ varies with ε:

$$dim\,[\mathbb{R}[w]/Rad(J_\varepsilon)] = \left\{ \begin{array}{ccc} 2 & \text{if} & \varepsilon \neq 0 \\ 1 & \text{if} & \varepsilon = 0. \end{array} \right.$$

Recall that a vector field v preserves a hypersurface $\Gamma = \{B(x) = 0\}$ if the Lie derivative $L_v B$ belongs to the principal ideal generated by B. The module $\mathbf{S}_\Gamma$ of all germs of C^∞ vector fields preserving a hypersurface germ $(\Gamma, 0) \subset (\mathbb{R}^n, 0)$ is the Lie algebra of the group of diffeomorphims of $(\mathbb{R}^n, 0)$ preserving $(\Gamma, 0)$. The module $\mathbf{S}_\Gamma$ is called the *stationary algebra* of $(\Gamma, 0)$. Thus:

- If the border is smooth $\Gamma_b = \{x_1 = 0\}$, then
$$\mathbf{S}_{\Gamma_b} = \left\{ x_1 h \frac{\partial}{\partial x_1} + \sum_{i=1}^{n-1} k_i \frac{\partial}{\partial y_i}, \quad h, k_i \in \mathbf{C}_w \right\}.$$
Here $x = x_1 \in \mathbb{R}$ and $y = (y_1, y_2, \ldots, y_{n-1}) \in \mathbb{R}^{n-1}$.
- If the border is a cuspidal edge $\Gamma_{csp} = \{x_2^2 - x_1^s = 0 : s \geq 3\}$, then
$$\mathbf{S}_{\Gamma_{csp}} = \left\{ (\frac{x_1}{s} h_1 + 2x_2 h_2) \frac{\partial}{\partial x_1} + (\frac{x_2}{2} h_1 + s x_1^{s-1} h_2) \frac{\partial}{\partial x_2} + \sum_{i=1}^{n-2} k_i \frac{\partial}{\partial y_i}, \quad h_1, h_2, k_i \in \mathbf{C}_w \right\}.$$
Here $x = (x_1, x_2) \in \mathbb{R}^2$ and $y = (y_1, y_2, \ldots, y_{n-2}) \in \mathbb{R}^{n-2}$.

Suppose that all function germs in a smooth family f_t are pseudo border equivalent to the function germ f_0, $f_t \circ \theta_t = f_0$, $t \in [0, 1]$, with respect to a smooth family $\theta_t : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ of germs of diffeomorphisms such that $\theta_0 = id$ and $t \in [0, 1]$. Denote by $Rad\{I_t\}$ the radical of the gradient ideal I_t of the function f_t. Then we have the homological equation:

$$-\frac{\partial f_t}{\partial t} = \sum_{i=1}^{m} \frac{\partial f_t}{\partial x_i} \dot{X}_i(t) + \sum_{j=1}^{n-m} \frac{\partial f_t}{\partial y_j} \dot{Y}_j(t),$$

where the vector field

$$v_t = \sum_{i=1}^{m} \dot{X}_i(t) \frac{\partial}{\partial x_i} + \sum_{j=1}^{n-m} \dot{Y}_j(t) \frac{\partial}{\partial y_j}$$

generates the phase flow θ_t and its components satisfy the following:

- If the border is smooth then
$$\dot{X}_1(t) \in \{x_1 h + Rad\{I_t\}\} \quad \text{and} \quad \dot{Y}_i(t), h \in \mathbf{C}_w, \text{ for all } i. \text{ [6]}$$
- If the border is a cuspidal edge then
$$\dot{X}_1(t) \in \left\{ \frac{x_1}{s} h_1 + 2x_2 h_2 + Rad\{I_t\} \right\}, \quad \dot{X}_2(t) \in \left\{ \frac{x_2}{2} h_1 + s x_1^{s-1} h_2 + Rad\{I_t\} \right\}$$
and $\dot{Y}_i(t), h_1, h_2 \in \mathbf{C}_w$, for all i.

We modify the pseudo equivalence relation to have a better parameter dependence. Namely, we replace $Rad\{I_t\}$ by the ideal I_t itself in the equivalence definition.

Definition 2.6. Two functions $f_0, f_1 : \mathbb{R}^n \to \mathbb{R}$ are called *quasi border equivalent*, if they are pseudo border equivalent and there is a family of functions f_t which depends continuously on parameter $t \in [0,1]$ and a continuous piece-wise smooth family of diffeomorphisms $\theta_t : \mathbb{R}^n \to \mathbb{R}^n$ also depending on $t \in [0,1]$ such that: $f_t \circ \theta_t = f_0$, $\theta_0 = id$ and the components of the vector field v_t generating θ_t on each segment of smoothness satisfy the following:

- If the border is smooth, then
$$\dot{X}_1(t) \in \{x_1 h + I_t\} \quad \text{and} \quad \dot{Y}_i(t), h \in \mathbf{C}_w, \text{ for all } i. \quad [6]$$
- If the border is a cuspidal edge, then
$$\dot{X}_1(t) \in \left\{\frac{x_1}{s}h_1 + 2x_2h_2 + I_t\right\}, \quad \dot{X}_2(t) \in \left\{\frac{x_2}{2}h_1 + sx_1^{s-1}h_2 + I_t\right\}$$
and $\dot{Y}_i(t), h_1, h_2 \in \mathbf{C}_w$, for all i.

Remarks 2.7.

1. Such a family θ_t of diffeomorphisms generated by the vector field v_t as well as the vector field itself will be called *admissible* for the family f_t.
2. The tangent space $TQ\Gamma_f$ to the quasi border equivalence class of f has the following description:
   - If the border is smooth, then
$$TQ\Gamma_f := TQB_f = \left\{\frac{\partial f}{\partial x_1}\left(x_1 h + \frac{\partial f}{\partial x_1}A\right) + \sum_{i=1}^{n-1}\frac{\partial f}{\partial y_i}k_i, \quad h, A, k_i \in \mathbf{C}_w\right\}.$$
   - If the border is a cuspidal edge, then
$$\begin{aligned} TQ\Gamma_f := TQCU_f \quad &= \quad \left\{\frac{\partial f}{\partial x_1}\left(\frac{x_1}{s}h_1 + 2x_2h_2 + \frac{\partial f}{\partial x_1}A_1 + \frac{\partial f}{\partial x_2}A_2\right)\right. \\ &+ \quad \frac{\partial f}{\partial x_2}\left(\frac{x_2}{2}h_1 + sx_1^{s-1}h_2 + \frac{\partial f}{\partial x_1}B_1 + \frac{\partial f}{\partial x_2}B_2\right) \\ &+ \quad \left.\sum_{i=1}^{n-2}\frac{\partial f}{\partial y_i}C_i, \quad h_1, h_2, A_i, B_i, C_i \in \mathbf{C}_w\right\} \end{aligned}$$

Due to the inclusion $I_0^2 \subset TQ\Gamma_f \subset I_0$, where I_0 is the gradient ideal of f, we have

Proposition 2.8. *For any border, a function germ f has a finite codimension with respect to the quasi border equivalence if and only if f has a finite codimension with respect to the right equivalence.*

Definition 2.9. Two function germs are said to be *stably quasi border equivalent* if they become quasi border equivalent after the addition of non-degenerate quadratic forms in an appropriate number of extra cylindrical variables.

Following [4], we call a function germ *simple* if its sufficiently small neighbourhood in the space of all function germs contains only a finite number of quasi equivalence classes.

The quasi border classification of critical points outside the border Γ coincides with the standard right equivalence. Hence the standard classes A_k, D_k, E_6, E_7 and E_8 form the list of simple classes in this case. Also by definition, non-critical points are all equivalent wherever they are. Classification of critical points on a smooth border was done in [6]. So we classify in this paper only critical points on a cuspidal edge.

2.1. **Basic techniques of the classification and prenormal forms.** We will use Moser's homotopy method and the following technique which is similar to Lemma 8.1 [2] to establish quasi cusp equivalence between function germs.

Let us fix a convenient Newton diagram $\Delta \subset \mathbb{Z}^n_{\geq 0}$. The ideals S_γ of function germs of the Newton order at least γ, $\gamma \geq 0$, equip the ring $\mathbf{C}_w$ with the Newton filtration: $S_0 = \mathbf{C}_w$, $S_\delta \supset S_\gamma$ if $\delta < \gamma$ [4]. We assume here that the scaling factor for the orders is chosen so that functions with the Newton diagram Δ have order N.

Let $f = f_0 + f_*$ be a decomposition of a function germ f into its principal part f_0 of the Newton degree N and higher order terms f_*. We assume that f_0 has a finite codimension with respect to the right equivalence.

Lemma 2.10. *Consider a monomial basis of the linear space $\mathbf{C}_w/TQCU_{f_0}$. Let $e_1(w), e_2(w), \dots, e_s(w)$ be all its elements of Newton degrees higher than N.*

Suppose that for any $\varphi \in S_\gamma \setminus S_{>\gamma}$, $\gamma > N$:

1. There is an admissible vector field $\dot w = \sum \dot w_i \frac{\partial}{\partial w_i}$ where

$$\dot w = (\dot x_1, \dot x_2, \dot y_1, \dot y_2, \dots, \dot y_{n-2}),$$

$$\dot x_1 = \frac{x_1}{s} h_1 + 2x_2 h_2 + \sum_{i=1}^n A_{1,i}\frac{\partial f_0}{\partial w_i}, \quad \dot x_2 = \frac{x_2}{2} h_1 + s x_1^{s-1} h_2 + \sum_{i=1}^n A_{2,i}\frac{\partial f_0}{\partial w_i},$$

and

$$\dot y_1, \dots, \dot y_{n-2} \in \mathbf{C}_w,$$

with $h_1, h_2, A_{1,i}, A_{2,i} \in \mathbf{C}_w$, such that

$$\varphi = \sum_{i=1}^n \frac{\partial f_0}{\partial w_i} \dot w_i + \widehat{\varphi} + \sum_{i=1}^s c_i e_i(w),$$

where $\widehat{\varphi} \in S_{>\gamma}$ and $c_i \in \mathbb{R}$.

2. Moreover, for any δ, $N < \delta < \gamma$, and any $\psi \in S_\delta$ the expression

$$E(\psi,\varphi) = \sum_{i=1}^2 \frac{\partial \psi}{\partial x_i}\left[\dot x_i + \sum_{j=1}^n A_{i,j}\frac{\partial \psi}{\partial w_j}\right] + 2\sum_{i=1}^2 \frac{\partial f_0}{\partial x_i}\left[\sum_{j=1}^n A_{i,j}\frac{\partial \psi}{\partial w_j}\right] + \sum_{i=1}^{n-2}\frac{\partial \psi}{\partial y_i}\dot y_i$$

belongs to S_γ.

Then any germ $f = f_0 + f_*$ is quasi cusp equivalent to a germ $f_0 + \sum\limits_{i=1}^s a_i e_i$, where $a_i \in \mathbb{R}$.

We do not prove the Lemma here. The actual proof goes along the lines of Sections 12.5-12.17 of [4] by induction on increasing γ. Condition 1 of the Lemma allows us to move degree γ terms of a function f to higher degrees modulo a degree γ linear combination of the e_i. Condition 2 guarantees that the error term produced at such a move by the already normalised part (of degrees below γ) of the function does not affect this part.

Remark 2.11. A version of the Lemma is also valid for functions with the Newton principal part f_0 of infinite right equivalence codimension, which is the same as having

$$dim\,(\mathbf{C}_w/TQCU_{f_0}) = \infty.$$

Namely, still assuming the Newton degree of f_0 being N, let $e_1(w), \dots, e_s(w)$ be the degrees higher than N part of a monomial basis of $\mathbf{C}_w/\left(TQCU_{f_0} + S_M\right)$ for some $M > N$. Assume the conditions of Lemma 2.10 hold for all $\gamma < M$. Then, using the ideas hinted for the proof of the Lemma, one can show that any function with the Newton principal part f_0 is quasi cusp equivalent to

$$f_0 + \sum_{i=1}^{s} a_i e_i + \Psi, \quad \text{where} \quad \Psi \in S_M.$$

If M may be taken arbitrary here, this means in practice that, when classifying functions of finite quasi cusp codimension, we may consider functions with the principal part f_0 being reduced to the form $f_0 + \sum\limits_{i=1}^{s} a'_i e'_i$ where this time the sum is infinite: the e'_i are the degrees higher than N part of a monomial basis of a transversal to the quasi cusp orbit of f_0 in the ring of formal power series in w, and $a'_i \in \mathbb{R}$.

Lemma 2.10 and the above remark are essential for our normal form reduction in Section 3. It helps us in situations not covered by the technique of chapter 12 of [4], for example when f_0 is quasihomogeneous with respect to certain weights of the variables in which basic fields tangent to the border are not quasihomogeneous.

2.2. **Quadratic terms.** Now, let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a function germ of the form

$$f(x, y) = f_2(x, y) + f_3(x, y),$$

where f_2 is a quadratic form in x and y, and $f_3 \in \mathcal{M}^3_{x,y}$.

Lemma 2.12. *Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ be a function germ at the origin in local coordinates x_1 and x_2 only. If f_2 is a non-degenerate quadratic form then f is quasi cusp equivalent to $\pm x_1^2 \pm x_2^2$.*

Proof. In this case vector fields with components from the gradient ideal of a function with a non-degenerate quadratic part are all vector fields vanishing at the origin. Therefore any family of diffeomorphisms preserving the origin is admissible, and the Lemma follows from the standard Morse Lemma. □

Let $n \geq 2$ and set $f^*(y) = f|_{x=0}$. Denote by r^* the rank of the second differential $d_0^2 f^*$ at the origin. Set $c = n - 2 - r^*$. Denote by r the rank of the second differential $d_0^2 f$ at the origin.

Lemma 2.13. *(Stabilization) The function germ $f(x, y)$ is quasi cusp equivalent to a germ $\sum\limits_{i=1}^{r^*} \pm y_i^2 + g(x, \widetilde{y})$, where $\widetilde{y} \in \mathbb{R}^c$ and $g^* \in \mathcal{M}^3_{\widetilde{y}}$. For quasi cusp equivalent germs f, the respective reduced germs g are quasi cusp equivalent.*

Proof. Up to a linear transformation in y, we have

$$f = \sum_{i=1}^{r^*} \pm y_i^2 + \sum_{i=1}^{n-2} \sum_{j=1}^{2} a_{i,j} y_i x_j + Q_2(x) + f_3(x, y), \tag{1}$$

with $f_3 \in \mathcal{M}^3_{x,y}$ and Q_2 a quadratic form in x only.

Let $\hat{y} = (y_1, y_2, \ldots, y_{r^*})$ and $\widetilde{y} = (y_{r^*+1}, \ldots, y_{n-2}) \in \mathbb{R}^{n-2-r^*}$. Then, (1) can be written as

$$f_1 = \sum_{i=1}^{r^*} \pm y_i^2 + \varphi(x, \hat{y}, \widetilde{y}) + \widetilde{f}(x, \widetilde{y}),$$

where

$$\varphi = \sum_{i=1}^{r^*}\sum_{j=1}^{2} a_{i,j} y_i x_j + \sum_{l=1}^{r^*} y_l \widetilde{\varphi}_l(x,y) \quad \text{with} \quad \widetilde{\varphi}_l \in \mathcal{M}^2_{x,y},$$

and

$$\widetilde{f} = Q_2(x) + \sum_{i=r^*+1}^{n-2}\sum_{j=1}^{2} a_{i,j} y_i x_j + \widetilde{f}_3(x,\widetilde{y}) \quad \text{with} \quad \widetilde{f}_3 \in \mathcal{M}^3_{x,\widetilde{y}}.$$

We now aim to find a family

$$\theta_t : (x,y) \mapsto \big(x, \hat{Y}_t(x,\hat{y}), \widetilde{y}\big)$$

of diffeomorphisms which eliminates φ.

Take a family $f_t = \sum_{i=1}^{r^*} \pm y_i^2 + t\varphi(x,\hat{y},\widetilde{y}) + \widetilde{f}_t(x,\widetilde{y})$ which joins f_1 and $f_0 = \sum_{i=1}^{r^*} \pm y_i^2 + \widetilde{f}_0(x,\widetilde{y})$ with $t \in [0,1]$ and $\widetilde{f} = \widetilde{f}_1$. Here, $\widetilde{f}_t$ and $\widetilde{f}_0$ are unknown. So, we want to solve the homological equation for $\dot{y}$ and simultaneously for $\widetilde{f}_t$.

The homological equation takes the form

$$-\frac{\partial f_t}{\partial t} = \sum_{i=1}^{2} \frac{\partial f_t}{\partial x_i}\dot{x}_i + \sum_{i=1}^{r^*} \frac{\partial f_t}{\partial y_i}\dot{y}_i + \sum_{j=r^*+1}^{n-2} \frac{\partial f_t}{\partial y_j}\dot{y}_j. \tag{2}$$

Note that $\dot{x}_1 = \dot{x}_2 = \dot{y_j} = 0$, $j = r^*+1, \ldots, n-2$, as x and $\widetilde{y}$ do not change with t.

Thus, equation (2) can be written as

$$-(\varphi + \frac{\partial \widetilde{f}_t}{\partial t}) = \sum_{i=1}^{r^*} (\pm 2y_i + t\frac{\partial \varphi}{\partial y_i})\dot{y}_i. \tag{3}$$

Set $z_i = \pm 2y_i + t\frac{\partial\varphi}{\partial y_i}$, $i = 1, \ldots, r^*$, which are known functions. Note that the matrix $(\frac{\partial z_i}{\partial \hat{y}_j})$ has the maximal rank at the origin for any value of t. Hence we can take $z = (z_1, z_2, \ldots, z_{r^*})$ as new coordinates instead of $\hat{y}$. Thus, equation (3) takes the form

$$-(\varphi + \frac{\partial \widetilde{f}_t}{\partial t}) = \sum_{i=1}^{r^*} z_i \dot{y}_i$$

Using the Hadamard Lemma, we write this as

$$\sum_{i=1}^{r^*} z_i \psi_i(x,z,\widetilde{y},t) + \phi(x,\widetilde{y},t) + \frac{\partial \widetilde{f}_t}{\partial t} = \sum_{i=1}^{r^*} -z_i\dot{y}_i.$$

By taking $\psi_i = -\dot{y}_i$ and $\frac{\partial \widetilde{f}_t}{\partial t} = -\phi$, we show that the homological equation is solvable.

The last step is to find $\widetilde{f}_0$. This can be done using the relation

$$-\int_0^1 \phi dt = \int_0^1 \frac{\partial \widetilde{f}_t}{\partial t} dt = \widetilde{f}_1 - \widetilde{f}_0.$$

Note that the vector field $\dot{v} = \sum\limits_{i=1}^{r^*} \dot{y}_i \frac{\partial}{\partial y_i}$ is defined in some neighborhood of the segment $[0,1]$ of the t-axis in the space $\mathbb{R}^n \times \mathbb{R}_t$, which is due to the z_i vanishing on this segment.

Hence all the f_t are quasi cusp equivalent. In particular, the function germ f_1 is quasi cusp equivalent to f_0.

The second claim of the Lemma can be deduced directly as the family

$$\theta_t : (x, y) \mapsto \left(x, \hat{Y}_t(x, \hat{y}), \widetilde{y}\right)$$

preserves the projection $(x, \hat{y}, \widetilde{y}) \mapsto (x, \widetilde{y})$. $\square$

Lemma 2.14. *Let $f : (\mathbb{R}^n, 0) \to \mathbb{R}$ be a function germ with an isolated critical point at the origin, and I_0 its gradient ideal. Then f is quasi cusp equivalent for each $t \in [0, 1]$ to the function germ $g_t(w) = f(w) + th(w)$ with $h(w) \in I_0^2$, provided that the rank r of the second differential $d_0^2 g_t$ of g_t at the origin is constant.*

Proof. At first we claim that if the rank of $d_0^2 g_t$ is constant then for different t the gradient ideals I_t of g_t coincide. Since the claim does not depend on the choice of local coordinates, we may assume that the quadratic part of f at the origin has diagonal form $\sum_{i=1}^{r} \varepsilon_i w_i^2$, where $\varepsilon_i = \pm 1$ for $i = 1, \dots, r$. We also set $\varepsilon_i = 0$ for $i > r$.

The quadratic part of g_t at the origin is

$$\sum_{i,j=1}^{r} \left(\varepsilon_i \delta_{ij} w_i^2 + 4t h_{ij}(0) \varepsilon_i \varepsilon_j w_i w_j\right) = \sum_{i,j=1}^{r} D_{ij} w_i w_j$$

where the $h_{ij},\ i, j = 1, \dots, n$ are the coefficients of the decomposition

$$h(w) = \sum_{i,j=1}^{n} h_{ij}(w) \frac{\partial f}{\partial w_i} \frac{\partial f}{\partial w_j}$$

of the function h, δ_{ij} is the Kronecker symbol, and $D_{ij} = \varepsilon_i \delta_{ij} + 4t \varepsilon_i \varepsilon_j h_{ij}(0)$. We will assume here that $h_{ij} = h_{ji}$.

The $r \times r$ matrix with entries D_{ij} is invertible for any t since the rank of $d_0^2 g_t$ is r. Reversing signs of some of its rows, we see that the $n \times n$ matrix with the entries $\hat{D}_{ij} = \delta_{ij} + 4t \varepsilon_j h_{ij}(0)$, for $i, j = 1, \dots, r$ and $\hat{D}_{ij} = \delta_{ij}$ otherwise, is also invertible.

The differentiation

$$\frac{\partial g_t}{\partial w_i} = \frac{\partial f}{\partial w_i} + t \sum_{k,j=1}^{n} \left(2h_{kj} \frac{\partial^2 f}{\partial w_k \partial w_i} + \frac{\partial h_{kj}}{\partial w_i} \frac{\partial f}{\partial w_k}\right) \frac{\partial f}{\partial w_j}$$

implies that $I_t \subset I_0$. This derivative can also be written as

$$\frac{\partial g_t}{\partial w_i} = \sum_{j=1}^{n} \left(\delta_{ij} + 4t \varepsilon_i h_{ij}(0) + R_{ji}\right) \frac{\partial f}{\partial w_j} = \sum_{j=1}^{n} \left(\hat{D}_{ji} + R_{ji}\right) \frac{\partial f}{\partial w_j},$$

where the functions R_{ij} vanish at $w = 0$. So in some neighborhood of the interval $[0, 1]$ of the t-axis the matrix $\left(\hat{D}_{ji} + R_{ji}\right)$ is invertible. This implies that $I_0 \subset I_t$. Hence, $I_t = I_0$.

Now the homological equation $-\frac{\partial g_t}{\partial t} = \sum_{i=1}^{n} \frac{\partial g_t}{\partial w_i} V_i$ can be solved for the unknown functions V_i which belong to the gradient ideal I_t for any t, since the left hand side belongs to the square of this ideal. The phase flow of the vector field $\sum V_i \frac{\partial}{\partial w_i}$ leaves all critical points of g_t fixed. Hence all the germs g_t are quasi cusp equivalent. $\square$

Lemmas 2.13 and 2.14 imply the following improved stabilization splitting.

Lemma 2.15. *There is a non-negative integer $s \leq r - r^*$ such that the function germ $f(x, y)$ is quasi cusp equivalent to $\sum_{i=1}^{r^*+s} \pm y_i^2 + \widetilde{f}(x, \widetilde{y})$, where $\widetilde{y} \in \mathbb{R}^{c-s}$ and $\widetilde{f}$ is a sum of a function germ from $\mathcal{M}^3_{x,\widetilde{y}}$ and a quadratic form in x only. For quasi cusp equivalent germs f, the corresponding reduced germs $\widetilde{f}$ are quasi cusp equivalent.*

Proof. Due to Lemma 2.13, we can assume that the quadratic part of the function is $f_2 = \sum_{i=1}^{r^*} \pm y_i^2 + x_1 \sum_{i=r^*+1}^{n-2} \alpha_{1,i} y_i + x_2 \sum_{i=r^*+1}^{n-2} \alpha_{2,i} y_i + g_2(x)$ with constant coefficients $\alpha_{j,i}$ and the quadratic form g_2 in x only. Suppose that some of these coefficients, for example α_{1,r^*+1}, is non-zero. Then, summing up the function f with $\delta \left(\frac{\partial f}{\partial x_1}\right)^2$ for sufficiently small δ gives a new function g which (according to Lemma 2.14) is quasi cusp equivalent to f and contains the term $y^2_{r^*+1}$ with a non-zero coefficient. Therefore the rank of the restriction of g to the $x = 0$ subspace is larger than r^*. Repeating the procedure several times, if needed, we get a function germ with a larger value of r^* and without the $x_j y_{>r^*}$ terms. This is exactly the form required. □

3. Classification of simple functions

We start this section with recalling the classification of simple singularities with respect to the quasi boundary equivalence relation from [6]. After that we classify simple quasi cusp singularities, giving details of proofs of main results.

3.1. **Simple quasi boundary classes.** Classifications of simple quasi boundary singularities is as follows.

Theorem 3.1. [6] *On the boundary $x_1 = 0$, any simple quasi boundary singularity class is a class of stabilizations of one of the following two-variable germs:*

Notation	*Normal form*	*Restrictions*	*codimension*
B_k	$\pm y_1^2 \pm x_1^k$	$k \geq 2$	k
$F_{k,m}$	$\pm(x_1 \pm y_1^k)^2 \pm y_1^m$	$2 \leq k < m$	$k + m - 1$

Remarks 3.2.

1. Any germ f with the quadratic part of corank greater than 1 is non-simple.
2. The only fencing singularity is the uni-modal class $\widetilde{S}_5 : y_1^3 + x_1^3 + a x_1^2 y_1$, $a \in \mathbb{R} \setminus \{\frac{-3}{\sqrt[3]{4}}\}$, which is adjacent to $F_{2,3}$.
3. Any corank 1 germ is either simple (and hence quasi boundary equivalent to one of the germs in the above theorem) or belongs to a subset of infinite codimension in the space of all germs.
4. The graph of low codimension adjacencies is as follows:

$$\begin{array}{ccccccccccc}
B_2 & \leftarrow & B_3 & \leftarrow & B_4 & \leftarrow & B_5 & \leftarrow & B_6 & \leftarrow & \dots \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 & & F_{2,3} & \leftarrow & F_{2,4} & \leftarrow & F_{2,5} & \leftarrow & F_{2,6} & \leftarrow & \dots \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \widetilde{S}_5 & & F_{3,4} & \leftarrow & F_{3,5} & \leftarrow & F_{3,6} & \leftarrow & \dots \\
 & & & & & & \uparrow & & \uparrow & & \\
 & & & & & & \dots & & \dots & \dots &
\end{array}$$

3.2. **Simple quasi cusp classes.** We distinguish the following cases:

1. If the base point of a function germ is at a regular point of the border:
$$\Gamma_{csp} = \{x_2^2 - x_1^s = 0 : s \geq 3\},$$
then the quasi cusp equivalence coincides with the quasi boundary equivalence. Hence, the list of simple quasi cusp classes in this case is the same as that of simple quasi boundary classes.
2. The remaining case of a function germ having a critical base point on the cusp stratum is described by the following theorem.

Theorem 3.3. *Let a function germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$, be simple with respect to the quasi cusp equivalence. Then, either its quadratic part f_2 is non-degenerate and hence f is quasi cusp equivalent to $\mathcal{L}_2 : \pm x_1^2 \pm x_2^2 + \sum\limits_{i=1}^{n-2} \pm y_i^2$ or f_2 has corank 1 in which case f is stably quasi cusp equivalent to one of the following simple classes:*

Notation	*Normal form*	*Restrictions*	*Codimension*
$\mathcal{L}_k$	$\pm x_1^2 \pm x_2^k$	$k \geq 3$	$k+1$
$\mathcal{M}_k$	$\pm x_2^2 \pm x_1^k$	$s = 3,\ k \geq 3$	$k+2$
$\mathcal{M}_3$	$\pm x_2^2 + x_1^3$	$s \geq 4$	5
$\mathcal{N}_{2,2,k}$	$\pm(x_1 + y_1^2)^2 \pm (x_2 + y_1^2)^2 \pm y_1^k$	$s = 3,\ k \geq 3$	$k+3$
$\mathcal{N}_{2,2,3}$	$\pm(x_1 + y_1^2)^2 \pm (x_2 + y_1^2)^2 + y_1^3$	$s \geq 4$	6
$\mathcal{N}_{2,3,4}$	$\pm(x_1 + y_1^2)^2 \pm (x_2 + y_1^3)^2 \pm y_1^4$	$s = 3$	8
$\mathcal{N}_{3,3,4}$	$\pm(x_1 + y_1^3)^2 \pm (x_2 + y_1^3)^2 \pm y_1^4$	$s = 3$	9

Remarks 3.4.

1. Any germ f with the quadratic part of corank greater than 1 is non-simple.
2. Any germ of corank 1 is either simple (and hence quasi cusp equivalent to one of the germs in the above theorem) or belongs to a subset of infinite codimension in the space of all germs.
3. The fencing classes are stabilizations of the following:

Notation	Class	Restrictions	Codimension
$\widetilde{\mathcal{L}}$	$\pm x_1^3 + \beta x_1 x_2^2 + \gamma x_2^3$	$\beta, \gamma \in \mathbb{R}, 4\beta^3 \pm 27\gamma^2 \neq 0$	6
$\widetilde{\mathcal{M}}_4$	$\pm(x_2 + \delta x_1^2)^2 \pm x_1^4$	$s \geq 4,\ \delta \in \mathbb{R}$	6
$\mathcal{N}_{2,3,5}$	$\pm(x_1 + y_1^2)^2 \pm (x_2 + y_1^3)^2 + \alpha y_1^5$	$s = 3,\ \alpha \in \mathbb{R} \setminus \{0\}$	9

4. The graph of adjacencies in low codimension (when $s = 3$) is as follows:

$$\begin{array}{ccccccccccc}
\mathcal{L}_2 & \leftarrow & \mathcal{L}_3 & \leftarrow & \mathcal{L}_4 & \leftarrow & \mathcal{L}_5 & \leftarrow & \mathcal{L}_6 & \leftarrow & \dots \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \mathcal{M}_3 & \leftarrow & \mathcal{M}_4 & \leftarrow & \mathcal{M}_5 & \leftarrow & \mathcal{M}_6 & \leftarrow & \dots \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \mathcal{N}_{2,2,3} & \leftarrow & \mathcal{N}_{2,2,4} & \leftarrow & \mathcal{N}_{2,2,5} & \leftarrow & \mathcal{N}_{2,2,6} & \leftarrow & \dots \\
 & & & & \uparrow & & \uparrow & & & & \\
 & & & & \mathcal{N}_{2,3,4} & \leftarrow & \mathcal{N}_{2,3,5} & & & & \\
 & & & & \uparrow & & & & & & \\
 & & & & \mathcal{N}_{3,3,4} & & & & & &
\end{array}$$

Also, $\mathcal{M}_3 \leftarrow \widetilde{\mathcal{L}}$.

To prove Theorem 3.3, we need the following auxiliary results.

Lemma 3.5. *Let $\kappa = n - r$ be the corank of the second differential $d_0^2 f$ at the origin.*

1. *If $\kappa = 0$, then f is quasi cusp equivalent to $\sum\limits_{i=1}^{n-2} \pm y_i^2 + f_2(x_1, x_2) + f_3(x_1, x_2)$, where f_2 is a non-degenerate quadratic form and $f_3 \in \mathcal{M}^3_{x_1,x_2}$.*
2. *If $\kappa = 1$, then f is quasi cusp equivalent to either $\displaystyle\sum_{i=1}^{n-2} \pm y_i^2 + \widetilde{f}(x_1, x_2)$ with rank $\left(d_0^2 \widetilde{f}\right) = 1$ or to $\displaystyle\sum_{i=2}^{n-2} \pm y_i^2 \pm x_1^2 \pm x_2^2 + f_3(x_1, x_2, y_1)$ where $f_3(x_1, x_2, y_1) \in \mathcal{M}^3_{x_1,x_2,y_1}$.*
3. *If $\kappa \geq 2$, then f is non-simple.*

Proof. Lemmas 2.12 and 2.15 provide the first two parts of Lemma 3.5.

For part 3, suppose that $\kappa = 2$. Then Lemma 2.15 yields that any function germ $f(x_1, x_2, y)$ reduced to one of the following forms:

0. $F_0 = \displaystyle\sum_{i=1}^{n-2} \pm y_i^2 + f_3$ where $f_3 \in \mathcal{M}^3_{x_1,x_2}$, or

1. $F_1 = \displaystyle\sum_{i=2}^{n-2} \pm y_i^2 + f_2(x_1, x_2) + f_3$ where $f_3 \in \mathcal{M}^3_{x_1,x_2,\tilde{y}_1}$, $\tilde{y}_1 \in \mathbb{R}$ and f_2 is a quadratic form of rank one, or

2. $F_2 = \displaystyle\sum_{i=3}^{n-2} \pm y_i^2 + f_2(x_1, x_2) + f_3$ where $f_3 \in \mathcal{M}^3_{x_1,x_2,\tilde{y}}$, $\tilde{y} = (\tilde{y}_1, \tilde{y}_2)$ and f_2 is a non-degenerate quadratic form.

Consider the germ F_0. The tangent space to the quasi cusp orbit at the germ f_3 is

$$\begin{aligned} TQCU_{f_3} &= \frac{\partial f_3}{\partial x_1}\left\{\frac{x_1}{s}h_1 + 2x_2 h_2 + \frac{\partial f_3}{\partial x_1}A_1 + \frac{\partial f_3}{\partial x_2}A_2\right\} \\ &+ \frac{\partial f_3}{\partial x_2}\left\{\frac{x_2}{2}h_1 + s x_1^{s-1} h_2 + \frac{\partial f_3}{\partial x_1}B_1 + \frac{\partial f_3}{\partial x_2}B_2\right\}. \end{aligned}$$

The cubic terms in $TQCU_{f_3}$ are from

$$\left(\frac{x_1}{s}\frac{\partial f_3}{\partial x_1} + \frac{x_2}{2}\frac{\partial f_3}{\partial x_2}\right)h_1 \quad \text{and} \quad \left(2x_2\frac{\partial f_3}{\partial x_1} + s x_1^{s-1}\frac{\partial f_3}{\partial x_2}\right)h_2,$$

where $h_1, h_2 \in \mathbb{R}$. These terms form a subspace of dimension at most 2. The dimension of the space of all cubic forms in x_1 and x_2 is 4 which is greater than the subspace dimension. This means that the germ F_0 is non-simple.

In the next case we have $F_1 = \sum_{i=2}^{n-2} \pm y_i^2 \pm (ax_1 + bx_2)^2 + f_3$, where $f_3 \in \mathcal{M}_{x_1,x_2,\tilde{y}_1}$, and $a, b \in \mathbb{R}$ (a and b are not both zeros). Note that F_1 deforms to

$$\widetilde{F}_1 = \sum_{i=2}^{n-2} \pm y_i^2 \pm (ax_1 + bx_2 + \delta\tilde{y}_1)^2 + f_3, \quad \delta \neq 0.$$

According to Lemma 2.15, $\widetilde{F}_1$ is quasi cusp equivalent to $\sum_{i=1}^{n-2} \pm y_i^2 + \widetilde{f}$ with $\widetilde{f} \in \mathcal{M}^3_{x_1,x_2}$, which we have already shown to be non-simple. Similar argument yields that F_2 is adjacent to F_1 and the result follows. $\square$

Lemma 3.6. *Let $f:(\mathbb{R}^2,0)\mapsto(\mathbb{R},0)$ be a function germ in local coordinates x_1 and x_2, and with a critical point at the origin. If the quadratic form f_2 of f has rank 1 then f is quasi cusp equivalent to either $\pm x_1^2+\varphi_1(x_2)$ where $\varphi_1\in\mathcal{M}^3_{x_2}$ or $\pm x_2^2+\varphi_2(x_1,x_2)$ where $\varphi_2\in\mathcal{M}^3_{x_1,x_2}$. Moreover, if $s=3$ then $\pm x_2^2+\varphi_2(x_1,x_2)$ is quasi cusp equivalent to $\pm x_2^2+\varphi_3(x_1)$ where $\varphi_3\in\mathcal{M}^3_{x_1}$.*

Proof. We have $f=\pm(ax_1+bx_2)^2+f_3(x_1,x_2)$, where $f_3\in\mathcal{M}^3_{x_1,x_2}$ and $a,b\in\mathbb{R}$ (a and b are not both zeros). Consider $Q_1=\pm(ax_1+bx_2)^2$ and suppose that $a\neq 0$. Take the homotopy $Q_t=\pm(ax_1+tbx_2)^2$ where $t\in[0,1]$. The corresponding homological equation is

$$\begin{aligned}\pm 2bx_2(ax_1+tbx_2) \quad &= \quad \pm 2a(ax_1+tbx_2)\left\{\frac{x_1}{s}h_1+2x_2h_2+(ax_1+tbx_2)A\right\}\\ &\quad\quad \pm 2tb(ax_1+tbx_2)\left\{\frac{x_2}{2}h_1+sx_1^{s-1}h_2+(ax_1+tbx_2)B\right\}.\end{aligned}$$

This is equivalent to

$$bx_2=a\left\{\frac{x_1}{s}h_1+2x_2h_2+(ax_1+tbx_2)A\right\}+tb\left\{\frac{x_2}{2}h_1+sx_1^{s-1}h_2+(ax_1+tbx_2)B\right\}.$$

The homological equation is solvable by setting $h_1=B=0$ and taking constants A and h_2 such that

$$a^2A+tbsx_1^{s-2}h_2=0 \quad \text{and} \quad 2ah_2+atbA=b.$$

These two equations in the variables A and h_2 are linearly independent. Thus, all the $Q_t, t\in[0,1]$, are quasi cusp equivalent. In particular, $Q_1=\pm(ax_1+bx_2)^2$ is quasi cusp equivalent to $\pm x_1^2$.

Now, consider the germ $F=\pm x_1^2+f_3(x_1,x_2)$, $f_3\in\mathcal{M}^3_{x_1,x_2}$. Let $F_0=\pm x_1^2$. The quasi cusp tangent space at F_0 is

$$TQCU_{F_0}=\pm 2x_1\left\{\frac{x_1}{s}h_1+2x_2h_2+x_1A_1\right\}.$$

Thus, we get *mod* $TQCU_{F_0}$: $x_1^2\equiv 0$ and $x_1x_2\equiv 0$. Hence, $\mathbf{C}_{x_1,x_2}/TQCU_{F_0}\simeq\mathbf{C}_{x_2}+\mathbb{R}x_1$. According to Remark 2.11, the germ F is quasi cusp equivalent to $\pm x_1^2+\varphi_1(x_2)$ with $\varphi_1\in\mathcal{M}^3_{x_2}$.

If $a=0$ and $b\neq 0$ then f reduces to $\pm x_2^2+\varphi_2(x_1,x_2)$, $\varphi_2\in\mathcal{M}^3_{x_1,x_2}$.

Suppose that $s=3$. Consider the germ $f_0=\pm x_2^2$. Similar to the argument above, the germ f is quasi cusp equivalent to $\pm x_2^2+\varphi_3(x_1)$, where $\varphi_3\in\mathcal{M}^3_{x_1}$. □

Lemma 3.7. *A function germ $f(x_1,x_2,y_1)=\pm x_1^2\pm x_2^2+f_3(x_1,x_2,y_1)$, $f_3\in\mathcal{M}^3_{x_1,x_2,y_1}$, is quasi cusp equivalent to $\widetilde{f}(x_1,x_2,y_1)=\pm x_1^2\pm x_2^2+x_1\phi_1(y_1)+x_2\phi_2(y_1)+\phi_3(y_1)$ where $\phi_1,\phi_2\in\mathcal{M}^2_{y_1}$ and $\phi_3\in\mathcal{M}^3_{y_1}$.*

Proof. Consider the principal part $f_0=\pm x_1^2\pm x_2^2$. The quasi cusp tangent space to the orbit at f_0 is

$$TQCU_{f_0}=\pm 2x_1\left\{\frac{x_1}{s}h_1+2x_2h_2+x_1A_1+x_2A_2\right\}\pm 2x_2\left\{\frac{x_2}{2}h_1+sx_1^{s-1}h_2+x_1B_1+x_2B_2\right\}.$$

Thus, we get *mod* $TQCU_{f_0}$: $x_1^2\equiv 0$, $x_2^2\equiv 0$ and $x_1x_2\equiv 0$. Hence, we have $\mathbf{C}_{x_1,x_2,y_1}/TQCU_{f_0}\simeq\{x_1\varphi_1(y_1)+x_2\varphi_2(y_1)+\varphi_3(y_1):\varphi_1,\varphi_2,\varphi_3\in\mathbf{C}_{y_1}\}$. Due to the constraint in the lemma on the term f_3, the claim of the lemma follows. □

3.2.1. *Proof of the main Theorem 3.3.* Lemmas 3.5, 3.6 and 3.7 yield that all simple quasi cusp singularities are among the following germs:

1. $G_1 = \pm x_1^2 + \varphi_1(x_2)$, where $\varphi_1 \in \mathcal{M}_{x_2}^3$.
2. $G_2 = \pm x_2^2 + \varphi_2(x_1, x_2)$, where $\varphi_2 \in \mathcal{M}_{x_1,x_2}^3$.
3. $G_3 = \pm x_1^2 \pm x_2^2 + x_1\phi_1(y_1) + x_2\phi_2(y_1) + \phi_3(y_1)$, where $\phi_1, \phi_2 \in \mathcal{M}_{y_1}^2$ and $\phi_3 \in \mathcal{M}_{y_1}^3$.

Using Lemma 2.10, one can easily prove the results below.

Consider the germ G_1. Let $\varphi_1(x_2) = a_k x_2^k + \widetilde{\varphi}(x_2)$ where $a_k \neq 0$, $k \geq 3$ and $\widetilde{\varphi} \in \mathcal{M}_{x_2}^{k+1}$. Then, G_1 is quasi cusp equivalent to the germ $\mathcal{L}_k : \pm x_1^2 \pm x_2^k$.

Next, consider the germ G_2.

Let $s = 3$. Then, by Lemma 3.6, G_2 is quasi cusp equivalent to $\widetilde{G} = \pm x_2^2 + \varphi_3(x_1)$, where $\varphi_3 \in \mathcal{M}_{x_1}^3$. In this case $\widetilde{G}$ can be reduced to one of the functions $\mathcal{M}_k : \pm x_2^2 \pm x_1^k$, $k \geq 3$.

Let $s \geq 4$. If φ_2 contains a term ax_1^3, $a \neq 0$, then G_2 is equivalent to $\mathcal{M}_3 : \pm x_2^2 + x_1^3$. Otherwise, in the most general case, G_2 is equivalent to a non-simple germ $\widetilde{\mathcal{M}}_4 : \pm(x_2 + \delta x_1^2) \pm x_1^4$, $\delta \in \mathbb{R}$.

Finally, consider the germ G_3:

$$G_3 = \pm x_1^2 \pm x_2^2 + x_1(a_2y_1^2 + a_3y_1^3 + a_4y_1^4 + \dots) + x_2(b_2y_1^2 + b_3y_1^3 + b_4y_1^4 + \dots) + c_3y_1^3 + c_4y_1^4 + c_5y_1^5 + \dots$$

Suppose $s = 3$.

- If $c_3 \neq 0$, then G_3 is quasi cusp equivalent to $\pm x_1^2 \pm x_2^2 + y_1^3$, which can be written equivalently as $\mathcal{N}_{2,2,3} : \pm(x_1 + y_1^2)^2 \pm (x_2 + y_1^2)^2 + y_1^3$.
- If $c_3 = 0$, $b_2 \neq 0$ and $c_4 \neq \pm\frac{b_2^2}{4}$, then G_3 is quasi cusp equivalent to $\pm x_1^2 \pm x_2^2 + x_2y_1^2 \pm y_1^4$, which is also quasi cusp equivalent to $\mathcal{N}_{2,2,4} : \pm(x_1 + y_1^2)^2 \pm (x_2 + y_1^2)^2 \pm y_1^4$.
- If $c_3 = 0$, $b_2 \neq 0$ and $c_4 = \pm\frac{b_2^2}{4}$, then we get the classes $\mathcal{N}_{2,2,k} : \pm(x_1+y_1^2)^2 \pm (x_2+y_1^2)^2 \pm y_1^k$, where $k \geq 5$ (one may omit y_1^2 in the first bracket here).
- If $c_3 = b_2 = 0$, $a_2 \neq 0$ and $c_4 \neq \pm\frac{a_2^2}{4}$, then G_3 is quasi cusp equivalent to $\pm x_1^2 \pm x_2^2 \pm x_1y_1^2 \pm y_1^4$, which is equivalent to $\mathcal{N}_{2,3,4} : \pm(x_1 + y_1^2)^2 \pm (x_2 + y_1^3)^2 \pm y_1^4$.
- If $c_3 = a_2 = b_2 = 0$, and $c_4 \neq 0$, then we get the class $\pm x_1^2 \pm x_2^2 \pm y_1^4$ or, equivalently, the $\mathcal{N}_{3,3,4}$ class: $\pm(x_1 + y_1^3)^2 \pm (x_2 + y_1^3)^2 \pm y_1^4$.
- If $c_3 = b_2 = 0$, $a_2 \neq 0$ and $c_4 = \pm\frac{a_2^2}{4}$, then we get a non-simple class

$$\mathcal{N}_{2,3,5} : \pm(x_1 + y_1^2)^2 \pm (x_2 + y_1^3)^2 + \alpha y_1^5$$

  with $\alpha \in \mathbb{R} \setminus \{0\}$.

Now let $s \geq 4$. Suppose $c_3 \neq 0$. Then, similar to the $s = 3$ case, the germ G_3 is quasi cusp equivalent to $\pm x_1^2 \pm x_2^2 \pm y_1^3$, and hence to $\mathcal{N}_{2,2,3} : \pm(x_1 + y_1^2)^2 \pm (x_2 + y_1^2)^2 + y_1^3$. If $c_3 = 0$, then G_3 is adjacent to the non-simple germ $\widetilde{\mathcal{M}}_4$. This finishes the proof of the theorem.

4. Bifurcation diagrams and caustics of simple quasi cusp singularities

A quasi cusp miniversal deformation of a function germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ may be constructed in the standard way as

$$F(x, y, \lambda) = f(x, y) + \sum_{i=0}^{\tau-1} \lambda_i e_i(x, y)\,, \tag{4}$$

where $e_0, \ldots, e_{\tau-1} \in \mathbf{C}_{x,y}$ project to a basis of $\mathbf{C}_{x,y}/TQCU_f$. We will use the notation F_λ for $F|_{\lambda=const}$, so that $F_0 = f$.

Definition 4.1. The *quasi cusp bifurcation diagram* of a function germ f is the set of all points λ in the base $\mathbb{R}^\tau$ of its quasi cusp miniversal deformation for which

- either the set $\{F_\lambda = 0\} \subset \mathbb{R}^n$ is singular,
- or a singularity of $\{F_\lambda = 0\}$ is on the border $x_2^2 - x_1^s = 0$.

Respectively, this diagram consists of two components, W_1 and W_2: $W_2 \subset W_1$, $\dim W_j = \tau - j$.

Now assume that $e_0 = 1$ in (4), and all the other e_i are from $\mathcal{M}_{x,y}$. Following the standard approach, we call the space $\mathbb{R}^{\tau-1}$ of the parameter $\lambda_1, \ldots, \lambda_{\tau-1}$ the base of a *truncated quasi cusp miniversal deformation* of f.

Definition 4.2. Consider the projection map $\Pi : \mathbb{R}^\tau \to \mathbb{R}^{\tau-1}$ between the two bases, forgetting λ_0. The *quasi cusp caustic* of a function f is a hypersurface in the base $\mathbb{R}^{\tau-1}$ which is a union of the Π-image Σ_1 of the cuspidal edge of the set $W_1 \subset \mathbb{R}^\tau$, and of the set $\Sigma_2 = \Pi(W_2)$.

Remark 4.3. In terms of Section 5 below, the component W_1 is the critical value set of the Lagrangian map of the manifold L, and W_2 is the image of the border Γ.

All simple quasi cusp singularities are the A_k singularities with respect to the standard right equivalence. So, the first component of the quasi cusp bifurcation diagram of a simple quasi cusp function is a product of a generalized swallowtail and $\mathbb{R}^{\tau-k}$. A similar observation is valid for the first components of the caustics.

The versal deformations listed below provide an explicit description of the bifurcation diagrams and caustics of simple quasi cusp singularities.

Proposition 4.4. *Quasi cusp miniversal deformations of simple quasi cusp classes are as follows:*

Singularity	Miniversal deformation	Restrictions
$\mathcal{L}_k$	$\pm x_1^2 \pm x_2^k + \sum_{i=0}^{k-1} \lambda_i x_2^i + \lambda_k x_1$	$k \geq 2$
$\mathcal{M}_k$	$\pm x_2^2 \pm x_1^k + \sum_{i=0}^{k-1} \lambda_i x_1^i + \lambda_k x_2 + \lambda_{k+1} x_1 x_2$	$s = 3, k \geq 3$
$\mathcal{M}_3$	$\pm x_2^2 + x_1^3 + \lambda_0 + \lambda_1 x_1 + \lambda_2 x_1^2 + \lambda_3 x_2 + \lambda_4 x_1 x_2$	$s \geq 4$
$\mathcal{N}_{2,2,k}$	$\pm(x_1 + y_1^2)^2 \pm (x_2 + y_1^2)^2 \pm y_1^k + \sum_{i=0}^{k-2} \lambda_i y_1^i$ $+\lambda_{k-1} x_1 + \lambda_k x_2 + \lambda_{k+1} x_1 y_1 + \lambda_{k+2} x_2 y_1$	$s = 3,\ k \geq 3$
$\mathcal{N}_{2,2,3}$	$\pm(x_1 + y_1^2)^2 \pm (x_2 + y_1^2)^2 + y_1^3 + \lambda_0 + \lambda_1 y_1 + \lambda_2 x_1$ $+\lambda_3 x_2 + \lambda_4 x_1 y_1 + \lambda_5 x_2 y_1$	$s \geq 4$
$\mathcal{N}_{2,3,4}$	$\pm(x_1 + y_1^2)^2 \pm (x_2 + y_1^3)^2 \pm y_1^4 + \lambda_0 + \lambda_1 x_1 + \lambda_2 x_2$ $+\lambda_3 x_1 y_1 + \lambda_4 x_2 y_1 + \lambda_5 x_2 y_1^2 + \lambda_6 y_1 + \lambda_7 y_1^2$	$s = 3$
$\mathcal{N}_{3,3,4}$	$\pm(x_1 + y_1^3)^2 \pm (x_2 + y_1^3)^2 \pm y_1^4 + \lambda_0 + \lambda_1 x_1 + \lambda_2 x_2$ $+\lambda_3 x_1 y_1 + \lambda_4 x_2 y_1 + \lambda_5 x_1 y_1^2 + \lambda_6 x_2 y_1^2 + \lambda_7 y_1 + \lambda_8 y_1^2$	$s = 3$

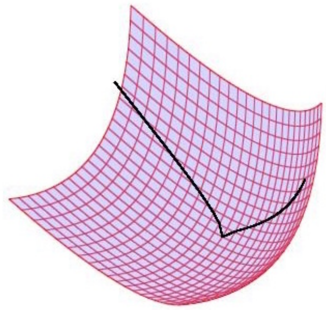
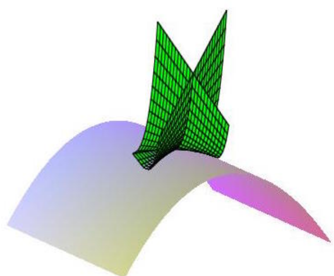
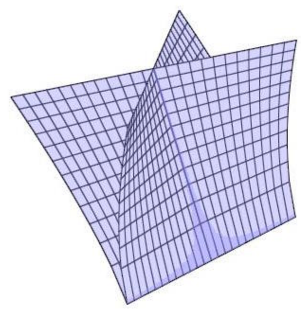


FIGURE 1. The bifurcation diagram of $\mathcal{L}_2$.



(A) The two components of the $\mathcal{L}_3$ caustic.



(B) Folded Umbrella: $\{a^2 + c^3 b^2 = 0 \subset \mathbb{R}^3\}$.

FIGURE 2. The $\mathcal{L}_3$ caustics.

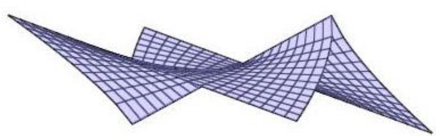


FIGURE 3. Overlapping cuspidal edges: $\{a^3 + bc^2 = 0 \subset \mathbb{R}^3\}$.

We have for the $\mathcal{L}$ and $\mathcal{M}$ series of singularities:

1. The bifurcation diagram of $\mathcal{L}_2$ is a smooth surface and a cuspidal curve on it (see Figure 1).
2. The bifurcation diagram of $\mathcal{L}_3$ in $\mathbb{R}^4$ is a product of a cuspidal curve and a plane, and a folded umbrella in this product.
3. The caustic of the $\mathcal{L}_k$ singularity is a union of a cylinder over a generalized swallowtail and a cylinder over a folded umbrella. In particular, the $\mathcal{L}_3$ caustic is a union of a folded umbrella and a smooth surface tangent to it (see Figure 2).
4. For $s = 3$, the caustic of the $\mathcal{M}_3$ singularity in $\mathbb{R}^4$ is a union of a cylinder of a smooth surface and a cylinder over a union of two overlapping cuspidal edges (see Figure 3).

5. Application to Lagrangian border singularities

Standard notions and basic definitions concerning Lagrangian singularities can be found in [4].

Singularities of Lagrangian projections (mappings) are essentially the singularities of their generating families treated as families of functions depending on parameters and considered up to the right equivalence depending on parameters and addition of functions in parameters. In particular, the caustic $\Sigma(L)$ of a Lagrangian projection of a Lagrangian submanifold L coincides with the set of values of the parameters λ of the generating family $F(w,\lambda)$ for which the family member F_λ has a non-Morse critical point.

Stability of a Lagrangian projection with respect to symplectomorphisms preserving the fibration structure corresponds to the versality of the generating family with respect to the $\mathcal{R}_+$-equivalence group.

Consider the standard symplectic space $M = T^*\mathbb{R}^n$ with coordinates q on the base $\mathbb{R}^n$ and dual coordinates p on the fibers of the Lagrangian projection $\pi : T^*\mathbb{R}^n \to \mathbb{R}^n$.

Locally any Lagrangian submanifold L^n in an ambient symplectic space M is determined by a generating family of functions $F(w,q)$ in variables $w \in \mathbb{R}^m$ and parameters $q \in \mathbb{R}^n$ according to the standard formula:

$$L = \left\{ (p,q) \in \mathbb{R}^n \times \mathbb{R}^n : \exists w \in \mathbb{R}^m, \frac{\partial F}{\partial w_i} = 0, p = \frac{\partial F}{\partial q} \right\},$$

provided that the Morse non-degeneracy condition (the matrix $\begin{pmatrix} \frac{\partial^2 F}{\partial w_i \partial w_j} & \frac{\partial^2 F}{\partial w_i \partial q_j} \end{pmatrix}$ has rank n) holds. The condition guarantees L being a smooth manifold.

Definition 5.1. [4] Two family germs $F_i(w,q)$, $w \in \mathbb{R}^m$, $q \in \mathbb{R}^n$, $i = 1,2$, at the origin are called *$\mathcal{R}_+$-equivalent* if there exists a diffeomorphism $\Phi : (w,q) \mapsto (W(w,q), Q(q))$ and a smooth function Θ of the parameters q such that $F_2(w,q) = (F_1 \circ \Phi)(w,q) + \Theta(q)$.

Following the application of the quasi corner equivalence relation considered in [2], we introduce

Definition 5.2. A pair (L,Γ) consisting of a Lagrangian submanifold L^n in an ambient symplectic space M and an $(n-1)$-dimensional isotropic variety $\Gamma \subset L$ is called *a Lagrangian submanifold with a border* Γ.

Definition 5.3. Lagrangian projections of two Lagrangian submanifolds with borders (L_i, Γ_i), $i = 1,2$, are *Lagrange equivalent* if there exists a symplectomorphism of the ambient space M which preserves the π-bundle structure and sends one pair to the other.

The notions of stability and simplicity of Lagrangian submanifolds with borders with respect to this Lagrangian equivalence are straightforward.

Up to a Lagrange equivalence we may assume that in a vicinity of a base point the tangent space to L has a regular projection onto the fiber of π and the coordinates p can be taken as coordinates w on the fibers of the source space of the generating family.

Generating family is defined up to $\mathcal{R}_+$-equivalence. So having two Lagrange equivalent pairs (L_i, Γ_i) we can choose a generating family for one of them in coordinates p,q and the generating family for the second pair in transformed coordinates $\widetilde{P}(p)$ so that the projection of Γ_1 to p-coordinate subspace coincides with the projection of Γ_2 to the $\widetilde{P}$-coordinate subspace.

Assume that the Γ_i are borders, $i = 1, 2$. Rename the coordinates p by w and q by λ. Let $g_i(w) = 0$ be the equation of the border Γ_i, $i = 1, 2$.

Now we have generating families $F_i(w, \lambda)$ for both submanifolds such that the critical points of F_i with respect to variables w at the set $\{g_i(w) = 0\}$ correspond to the border Γ_i.

Hence, the Lagrange equivalence of pairs (L_i, Γ_i), $\quad i = 1, 2$, gives rise to an equivalence of the generating families F_i which is a pseudo border equivalence and addition with a function in parameters.

Moreover the following holds.

Proposition 5.4. *Let* (L_t, Γ_t), $t \in [0, 1]$*, be a family of equivalent pairs of Lagrangian submanifolds with cuspidal edges. Then the respective generating families are* **quasi cusp** *equivalent up to addition of functions depending on parameters.*

The above equivalence of generating families will be called the *quasi cusp* $+$*-equivalence.*

The last proposition and the classification of simple quasi cusp singularities imply the following theorem.

Theorem 5.5. *1. A germ* (L, Γ) *is stable if and only if its arbitrary generating family is quasi border* $+$*-versal, that is, versal with respect to the quasi border equivalence and addition of functions in parameters.*

2. Any stable and simple projection of a Lagrangian submanifold with a cuspidal border is symplectically equivalent to the projection determined by a generating family which is a quasi cusp $+$*-versal deformation of one of the classes from Theorem 3.3.*

Proof. Suppose that a germ (L_0, Γ_0) is stable. Then any germ $(\widetilde{L}, \widetilde{\Gamma})$ close (L_0, Γ_0) is Lagrange equivalent to it.

Assume we have a family (L_t, Γ_t) of deformations of (L_0, Γ_0), with $t \in [0, 1]$. Also assume that there is a family of diffeomorphism $\theta_t : T^*\mathbb{R}^n \to T^*\mathbb{R}^n$ which preserves Lagrange fibration $\pi : T^*\mathbb{R}^n \to \mathbb{R}^n, (p, q) \mapsto q$ and the standard symplectic form ω, and maps (L_t, Γ_t) to (L_0, Γ_0).

Consider families depending on t of respective generating families $G_t(w, q)$ of (L_t, Γ_t) with $t \in [0, 1]$ and G_0 being a generating family of the pair (L_0, Γ_0). By Proposition 5.4, all the G_t are quasi cusp $+$-equivalent. Thus, there exist a family of diffeomorphisms

$$\Phi_t : (w, q) \mapsto (\widetilde{w}_t(w, q), Q_t(q))$$

and a family Ψ_t of smooth functions of the parameters q such that: $G_t \circ \Phi_t = G_0 + \Psi_t$, and the critical points sets $\left\{\frac{\partial G_t}{\partial w} = 0\right\}$ correspond to the Lagrangian submanifolds L_t. This yields, in particular, that G_0 is versal with respect to quasi border $+$-equivalence.

By reversing the previous argument we prove the converse claim.

The second part of the theorem is a consequence of the classification of function germs with respect to the quasi cusp equivalence. $\square$

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