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KOENDERINK TYPE THEOREMS FOR FRONTS

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Dedicated to Professor Shyuichi Izumiya on the occasion of his 60*th birthday*

Abstract. We prove Koenderink type theorems with the terminology of the singular curvatures of cuspidal edges of wave fronts.

1. Introduction

In 1984 and 1990, J. J. Koenderink showed theorems that relate to how one actually sees a surface. Let $f : U \to \boldsymbol{R}^3$ be a non-singular smooth surface in $\boldsymbol{R}^3$ and $M = f(U)$. Let $\pi : \boldsymbol{R}^3 \to P$ be the orthogonal projection onto a plane $P \subset \boldsymbol{R}^3$ and $\pi_{\mathbf{0}} : \boldsymbol{R}^3 \to S^2$ the central projection onto a unit sphere S^2 of $\boldsymbol{R}^3$ centered at $\mathbf{0} \in \boldsymbol{R}^3$. We denote the singular set of a map g by $S(g)$. Koenderink showed the following:

Theorem. ([11, Appendix], [12, page 433]) *Suppose $p \in S(\pi \circ f)$, and $\pi \circ f(S(\pi \circ f))$ is a regular curve near p. Let κ_1 be the curvature of the plane curve $\pi \circ f(S(\pi \circ f)) \subset P$, and κ_2 the curvature of the normal section of M at p by the plane that contains the kernel of π. Then*

$$K = \kappa_1 \kappa_2$$

holds at p, where K is the Gaussian curvature of M.

Suppose $p \in S(\pi_{\mathbf{0}} \circ f)$, and $\pi_{\mathbf{0}} \circ f(S(\pi_{\mathbf{0}} \circ f))$ is a regular curve near p. Let κ_g be the geodesic curvature of the curve $\pi_{\mathbf{0}} \circ f(S(\pi_{\mathbf{0}} \circ f))$ and d be the distance of p from $\mathbf{0}$. Then $K = \kappa_g \kappa_2 / d$ holds at p.

See [15, p223] for further considerations of this type problem. See also [3, 2, 14, 8, 9, 10]. If f has a singular point, generically the Gaussian curvature is unbounded. Thus this theorem does not hold at the singular points of f. In [16], it was shown that if f is a front, then the Gaussian curvature form $K d\hat{A}$ is bounded, and introduced the singular curvature function on the singular set which consists of cuspidal edges. The singular curvature has a certain geometric property. So it is natural to expect a Koenderink type theorem of fronts using the Gaussian curvature form and the singular curvature. In this paper, we give Koenderink type theorems for cuspidal edges with the terminology of the Gaussian curvature form and the singular curvature. We also give the same type theorems for the cuspidal edges in the hyperbolic space.

2. Singular curvature and statement of results

Let $(U; u, v) \subset \boldsymbol{R}^2$ be a domain, N a three dimensional manifold, and W a five dimensional contact manifold with a Legendrian fibration $\mathrm{pr} : W \to N$. A smooth map $f : U \to N$ is called a *front* if there exists a Legendrian immersion lift $L_f : U \to W$ of f; that is, L is an immersion, the pull-buck of the contact form vanishes on U, and $\mathrm{pr} \circ L_f = f$ holds. We remark that a front in a two dimensional manifold can be defined in a similar manner by replacing U with an interval, N with a two dimensional manifold, and W with a three dimensional contact manifold respectively. Let us consider the case W is the unit tangent bundle $T_1 \boldsymbol{R}^3$ with the canonical contact form and pr is the Legendrian fibration $\mathrm{pr} : T_1\boldsymbol{R}^3 \to \boldsymbol{R}^3$. In this case, a smooth map $f : U \to \boldsymbol{R}^3$ is a front if there exists a unit vector field ν along f such that $L_f = (f, \nu) : U \to \boldsymbol{R}^3 \times S^2 = T_1\boldsymbol{R}^3$ is an immersion and the following orthogonality condition holds:

$$(df_p(X_p) \cdot \nu(p)) = 0 \quad (X \in TU, p \in U),$$

where $(\ \cdot\ )$ is the Euclidean inner product of $\boldsymbol{R}^3$. Let $f : U \to \boldsymbol{R}^3$ be a front. Set

$$\lambda(u, v) = \det\left(\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}, \nu\right)(u, v),$$

called the *signed area density function*. We also set

$$d\hat{A} = \lambda\, du \wedge dv, \tag{2.1}$$

called the *signed area form*. Suppose $p \in U$ is a singular point of f, then $\lambda(p) = 0$ holds. If $d\lambda(p) \neq 0$ holds, then there is a regular smooth curve $\gamma(t) : (-\varepsilon, \varepsilon) \to U$ $(\gamma(0) = p)$ such that the image of γ coincides with $S(f)$ near p. Furthermore, there exists a non-vanishing vector field η along γ satisfying

$$\langle \eta(t) \rangle_{\boldsymbol{R}} = \ker df_{\gamma(t)}.$$

We call γ the *singular curve* and η the *null vector field*.

It was shown in [13], if $\eta(0)$ transverse to $\gamma'(0)$, then the map germ f at p is $\mathcal{A}$-equivalent to a map germ $(u, v) \mapsto (u, v^2, v^3)$ at $\mathbf{0}$; that is, there exist diffeomorphic germs $\sigma : (\boldsymbol{R}^2, \mathbf{0}) \to (\boldsymbol{R}^2, p)$ and $\tau : (\boldsymbol{R}^3, f(p)) \to (\boldsymbol{R}^3, \mathbf{0})$ such that $\tau \circ f \circ \sigma(u, v) = (u, v^2, v^3)$ holds as map germs at $\mathbf{0}$. A singular point p of a front f is called a *cuspidal edge* if f at p is $\mathcal{A}$-equivalent to $(u, v) \mapsto (u, v^2, v^3)$.

Now we suppose that the singular curve γ of a front $f : U \to \boldsymbol{R}^3$ consists of cuspidal edges. Then we can choose the null vector field η such that $(\gamma'(t), \eta(t))$ is a positively oriented frame field along γ, where $' = d/dt$. We then define the *singular curvature* as follows ([16]):

$$\kappa_s(t) = \operatorname{sgn}(d\lambda(\eta)) \frac{\det(\hat{\boldsymbol{\gamma}}'(t), \hat{\boldsymbol{\gamma}}''(t), \nu \circ \gamma(t))}{|\hat{\boldsymbol{\gamma}}'(t)|^3},$$

where $\hat{\boldsymbol{\gamma}} = f \circ \gamma$. For the geometric meanings of the singular curvature, and further details, see [16, 17].

Now we consider the Gaussian curvature form of fronts.

Proposition 2.1 ([16]). *Let $f : U \to \boldsymbol{R}^3$ be a front, and K the Gaussian curvature of f which is defined on the set of regular points of f. Then $K\, d\hat{A}$ can be continuously extended as a globally defined 2-form on U, where $d\hat{A}$ is the signed area form as in (2.1).*

A similar proposition as above also holds for plane curves. Let $\boldsymbol{c} : I \to \boldsymbol{R}^2$ be a front, and κ the curvature of $\boldsymbol{c}$, defined on the set of regular points. By the same method, one can show that $\kappa\, ds$ can be continuously extended as a globally defined 1-form on I, where s is the arclength parameter of $\boldsymbol{c}$.

Let $f : U \to \boldsymbol{R}^3$ be a front and $p \in U$ a cuspidal edge. Then one can see that a section of $M = f(U)$ near $f(p)$ by a plane through $f(p)$ which transverse to $df_p(M)$ is a 3/2-cusp, in particular a front (see [13, Proposition 2.9], for example). Several curvatures of fronts in the plane are investigated in [18].

Using the notions of the curvature forms above, we state the Koenderink type theorems for fronts.

Theorem 2.2. *Let $f : U \to \boldsymbol{R}^3$ be a front, $p \in U$ a cuspidal edge, and γ the singular curve with $\gamma(0) = p$. Set $\hat{\boldsymbol{\gamma}} = f \circ \gamma$, $\boldsymbol{\xi}_p = \nu(p) \times \hat{\boldsymbol{\gamma}}'(p)/|\hat{\boldsymbol{\gamma}}'(p)|$ and $\boldsymbol{v}_\theta = \cos\theta \boldsymbol{\xi}_p + \sin\theta \nu(p)$. Let P_θ be a plane normal to $\boldsymbol{v}_\theta$ and π_θ the orthogonal projection $\pi_\theta : \boldsymbol{R}^3 \to P_\theta$ with respect to $\boldsymbol{v}_\theta$. Let $\kappa_1(t)$ be the curvature of the plane curve $\boldsymbol{\gamma}_1(t) := \pi_\theta \circ \hat{\boldsymbol{\gamma}}(t)$, and $\kappa_2(s)$ the curvature of the intersection* *curve $\boldsymbol{\gamma}_2$ of M at p by the plane $P := \langle \boldsymbol{\xi}_p, \nu(p) \rangle_{\boldsymbol{R}}$, where s is the arclength parameter of $\boldsymbol{\gamma}_2$. If $\theta \in (0, \pi/2)$ then*

$$K d\hat{A} = \frac{1}{\cos\theta} \left(\sin\theta \kappa_s - \kappa_1 \right) dt \wedge \kappa_2\, ds \tag{2.2}$$

holds at p, where κ_s is the singular curvature. Here, we give a orientation of $\boldsymbol{\gamma}_2(s)$ passing through p from the region $\{\lambda < 0\}$ to the region $\{\lambda > 0\}$. Also we give a orientation of P_θ such that $\{-\sin\theta \boldsymbol{\xi}_p + \cos\theta \nu(p),\ \boldsymbol{\gamma}_1'(0)\}$ forms a positive basis, and P such that $\{\boldsymbol{\xi}_p, \nu(p)\}$ forms a positive basis.

3. Proof of theorem 2.2

Let $f : U \to \boldsymbol{R}^3$ be a front and $p \in U$ a cuspidal edge. Then by [16, Lemma 3.2], we can take a coordinate system (u, v) near p satisfying

- (u, v) is compatible with the orientation of U,
- $p = \boldsymbol{0}$ and the u-axis is the singular curve,
- the null vector field is ∂_v on U,
- $\lambda_v(\boldsymbol{0}) > 0$, and
- $|f_u(u, 0)| = 1$.

We call such a coordinate system (u, v) *adapted coordinate system* with respect to p. In an adapted coordinate system (u, v), since $\lambda_v > 0$, it holds that

$$\kappa_s(u) = \det(f_u, f_{uu}, \nu)(u, 0) = \left(f_{uu} \cdot \nu \times f_u \right)(u, 0), \tag{3.1}$$

where $f_{uu} = \partial^2 f / \partial u^2$, for example.

Proof of theorem 2.2. We take an adapted coordinate system (u, v). Since $f_v(u, 0) = \boldsymbol{0}$ and $f_{vv}(0, 0) \neq \boldsymbol{0}$, there exists a smooth function φ satisfying $\varphi(\boldsymbol{0}) \neq 0$ and

$$f_v(u, v) = v \varphi(u, v). \tag{3.2}$$

In this setting, the Gaussian curvature form has the following expression on U:

$$K\, d\hat{A} = \frac{-\left(f_{uu} \cdot \nu \right) \left(\varphi \cdot \nu_v \right) - v \left(\varphi \cdot \nu_u \right)^2}{\left(\varphi \cdot \varphi \right) - \left(f_u \cdot \varphi \right)^2} \sqrt{\left(\varphi \cdot \varphi \right) - \left(f_v \cdot \varphi \right)^2}\, du \wedge dv.$$

This is equal to

$$-\frac{\left(f_{uu} \cdot \nu \right) \left(f_{vv} \cdot \nu_v \right)}{\sqrt{\left(f_{vv} \cdot f_{vv} \right) - \left(f_u \cdot f_{vv} \right)^2}}\, du \wedge dv \tag{3.3}$$

at p. On the other hand, we calculate the curvatures κ_1 and κ_2. Let $\gamma_1(u)$ be the plane curve $\pi_\theta \circ f(u, 0)$. Then the curvature κ_1 of γ_1 is

$$\kappa_1 = \left(-\cos\theta \left(f_{uu} \cdot \nu(p) \right) + \sin\theta \left(f_{uu} \cdot \boldsymbol{\xi}_p \right) \right). \tag{3.4}$$

Let γ_2 be the plane curve of the intersection of $f(M)$ at p by P and κ_2 its curvature. Since $(f_u(u, v) \cdot f_u(p)) \neq 0$, by the implicit function theorem, there exists a function $u = u(v)$ such that

$$\left(f(u(v), v) \cdot f_u(p) \right) = 0.$$

Hence γ_2 is expressed by

$$\gamma_2(v) = \left(\left(f(u(v), v) \cdot \boldsymbol{\xi}_p \right), \left(f(u(v), v) \cdot \nu(p) \right) \right).$$

Using (3.2), since $\nu = f_u \times \varphi / |f_u \times \varphi|$, one can compute $\kappa_2\, ds$ as follows

$$\kappa_2\, ds = \frac{\det(f_u,\ \varphi,\ \varphi_v)}{(\varphi \cdot \varphi) - (f_u \cdot \varphi)^2}\, dv = -\frac{(\nu_v \cdot f_{vv})}{\sqrt{(f_{vv} \cdot f_{vv}) - (f_u \cdot f_{vv})^2}}\, dv, \tag{3.5}$$

at p, where s is the arclength parameter of γ_2. By (3.4) and (3.5), we have (2.2). □

To get the spherical projection version of the theorem, we need the following lemma.

Lemma 3.1. *Let $\boldsymbol{\gamma} : I \to \boldsymbol{R}^3$ be a smooth curve and κ its curvature as a space curve. Take a point $p \in I$ satisfying that $\boldsymbol{\gamma}(p)$ and $\boldsymbol{\gamma}'(p)$ are linearly independent. Let $\pi_{\mathbf{0}} : \boldsymbol{R}^3 \to S^2$ be the central projection onto a unit sphere S^2 centered at $\mathbf{0}$ and κ_g be the geodesic curvature of $\pi_{\mathbf{0}} \circ \gamma$ as a spherical curve. Then*

$$\kappa(p) = \frac{\kappa_g(p)}{d} \tag{3.6}$$

holds, where d is the distance of p from $\mathbf{0}$.

Proof. Direct computations. □

By Lemma 3.1 and Theorem 2.2, we have the following:

Corollary 3.2. *In the same setting as in Theorem 2.2, suppose that $\hat{\boldsymbol{\gamma}}(0)$ and $\hat{\boldsymbol{\gamma}}'(0)$ are linearly independent, and $\hat{\boldsymbol{\gamma}}(0)$ and $\boldsymbol{v}_\theta$ are parallel. Let $\pi_{\mathbf{0}} : \boldsymbol{R}^3 \to S^2$ be the central projection onto a unit sphere S^2 centered at $\mathbf{0}$ and κ_g the geodesic curvature of $\pi_{\mathbf{0}} \circ \hat{\boldsymbol{\gamma}}$ as a spherical curve. If $\theta \in (0, \pi/2)$, then*

$$K d\hat{A} = \frac{1}{\cos\theta}\left(\sin\theta \kappa_s - \frac{\kappa_g}{d}\right) \kappa_2\, du \wedge dv \tag{3.7}$$

holds at p, where d is the distance of $f(p)$ from $\mathbf{0}$.

4. Horospherical Koenderink type theorem

Recently an extrinsic geometry on submanifolds in the hyperbolic space is discovered by Shyuichi Izumiya and investigated [5, 7]. See also [4, 6]. It is called *horospherical geometry.* In this section, we show a horospherical geometric Koenderink type theorem for cuspidal edges. It should be noted that horospherical geometric Koenderink type theorems for regular surfaces in the hyperbolic space are shown in [9]. See also [8, 10].

To state a Koenderink type theorem, we prepare some notion. Let $\boldsymbol{R}^4_1$ be the Minkowski 4-space with the inner product $\langle\ ,\ \rangle = (-,+,+,+)$. We denote by $H^3_+(-1)$, LC^*_+ and $S^3_1(1) \subset \boldsymbol{R}^4_1$ the *hyperbolic space*, the *lightcone* and the *de Sitter space* defined by

$$\begin{aligned} H^3_+(-1) &= \{\boldsymbol{x} \in \boldsymbol{R}^4_1 \mid \langle \boldsymbol{x}, \boldsymbol{x}\rangle = -1, u_0 > 0\}, \\ LC^*_+ &= \{\boldsymbol{x} \in \boldsymbol{R}^4_1 \mid \langle \boldsymbol{x}, \boldsymbol{x}\rangle = 0, u_0 > 0\}, \\ S^3_1(1) &= \{\boldsymbol{x} \in \boldsymbol{R}^4_1 \mid \langle \boldsymbol{x}, \boldsymbol{x}\rangle = 1\}. \end{aligned}$$

Let $(U; u, v) \subset \boldsymbol{R}^2$ be a domain and $f : U \to H^3_+(-1)$ a smooth regular surface. Define a vector

$$\boldsymbol{e}(u,v) = \frac{f_u \wedge f_v \wedge f}{|f_u \wedge f_v \wedge f|}(u,v),$$

where $f_u = \partial f/\partial u$, for example. Here for any $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3 \in \boldsymbol{R}^4_1$, the vector $\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3$ is defined as

$$\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3 = -\det\begin{pmatrix} x_1^1 & x_2^1 & x_3^1 \\ x_1^2 & x_2^2 & x_3^2 \\ x_1^3 & x_2^3 & x_3^3 \end{pmatrix}\boldsymbol{e}_0 - \det\begin{pmatrix} x_0^1 & x_2^1 & x_3^1 \\ x_0^2 & x_2^2 & x_3^2 \\ x_0^3 & x_2^3 & x_3^3 \end{pmatrix}\boldsymbol{e}_1$$
$$+\det\begin{pmatrix} x_0^1 & x_1^1 & x_3^1 \\ x_0^2 & x_1^2 & x_3^2 \\ x_0^3 & x_1^3 & x_3^3 \end{pmatrix}\boldsymbol{e}_2 - \det\begin{pmatrix} x_0^1 & x_1^1 & x_2^1 \\ x_0^2 & x_1^2 & x_2^2 \\ x_0^3 & x_1^3 & x_2^3 \end{pmatrix}\boldsymbol{e}_3$$

where $\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ is the canonical basis of $\boldsymbol{R}^4_1$ and $\boldsymbol{x}_i = (x_0^i, x_1^i, x_2^i, x_3^i)$ $(i = 1, 2, 3)$. We can easily show that $\langle \boldsymbol{x}, \boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3\rangle = \det(\boldsymbol{x}, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)$, so that $\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3$ is orthogonal to any $\boldsymbol{x}_i$ $(i = 1, 2, 3)$. Thus we have $\langle \boldsymbol{e}, f_u\rangle = \langle \boldsymbol{e}, f_v\rangle = \langle \boldsymbol{e}, f\rangle = 0$ and $\langle \boldsymbol{e}, \boldsymbol{e}\rangle = 1$. This map $\boldsymbol{e} : U \to S^3_1(1)$ is called the *de Sitter Gauss image*. We also define a map

$$\boldsymbol{l}^{\pm}(u, v) = f(u, v) \pm \boldsymbol{e}(u, v) : U \to LC^*_+,$$

which is called the *lightcone Gauss image*. We consider the lightcone Gauss image as a Gauss map. See [5] for details. With this notion, we consider fronts in the hyperbolic space as follows. Consider the following double fibration:

- $H^3_+(-1) \times LC^*_+ \supset \Delta_2 = \{(\boldsymbol{x}, \boldsymbol{y}) \mid \langle \boldsymbol{x}, \boldsymbol{y}\rangle = -1\}$,
- $\pi_{21} : \Delta_2 \to H^3_+(-1)$, $\pi_{22} : \Delta_2 \to LC^*_+$,
- $\theta_{21} = \langle d\boldsymbol{x}, \boldsymbol{y}\rangle\,|_{\Delta_2}, \theta_{22} = \langle \boldsymbol{x}, d\boldsymbol{y}\rangle\,|_{\Delta_2}$.

Here,

$$\pi_{21}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x},\ \pi_{22}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{y},\ \langle d\boldsymbol{x}, \boldsymbol{y}\rangle = -y_0\, dx_0 + \sum_{i=1}^3 y_i\, dx_i, \text{ and } \langle \boldsymbol{x}, d\boldsymbol{y}\rangle = -x_0\, dy_0 + \sum_{i=1}^3 x_i\, dy_i.$$

We remark that θ_{21} and θ_{22} define the same tangent hyperplane field over Δ_2 which is denoted by K_2. In [4], it has been shown that (Δ_2, K_2) is a contact manifold such that each fibration π_{2i} $(i = 1, 2)$ is a Legendrian fibration. See [4] for details.

As we have seen in Section 2, a smooth map $f : U \to H^3_+(-1)$ is a front if there exists a map $\boldsymbol{l} : U \to LC^*_+$ such that $(f, \boldsymbol{l}) : U \to \Delta_2$ is a Legendrian immersion with respect to K_2. The map $\boldsymbol{l}$ is called a Δ_2-*dual* of f. One can show that $-d_p\boldsymbol{l}$ is a linear transformation $-d_p\boldsymbol{l} : T_pU \to \big(\langle \boldsymbol{l}(p), f(p)\rangle_{\boldsymbol{R}}\big)^{\perp} \subset T_{f(p)}\boldsymbol{R}^4_1$, by an identification $T_{f(p)}\boldsymbol{R}^4_1 = \boldsymbol{R}^4_1$, where $\perp$ means the orthogonal complement. It is called the *hyperbolic shape operator*. The *hyperbolic Gaussian curvature* is defined as

$$K^h(p) = \det(-d_p\boldsymbol{l}),$$

and the *hyperbolic Gaussian curvature form* is defined as

$$K^h\, d\hat{A} = K^h\lambda^h\, du \wedge dv,$$

where λ^h is the the signed area density function $\lambda^h(u, v) = \det(f_u, f_v, \boldsymbol{l}, f)$. If K^h identically vanishes, then f is a one-parameter family of horocycles, more precisely, f is an envelope of a one-parameter family of horospheres and is a locus swept out by horocycles ([7]). It can be easily seen that if f is a front, then $K^h\, d\hat{A}$ can be continuously extended as a globally defined 2-form on U.

Let $f : U \to H^3_+(-1)$ be a front and $p \in U$ a cuspidal edge. We denote $\gamma(t) : I \to U$ by a parameterization of $S(f)$. Let $\boldsymbol{l}$ be a Δ_2-dual of f. We define the *hyperbolic singular curvature* κ^h_s as

$$\kappa^h_s(t) = \operatorname{sgn}(d\lambda(\eta))\frac{\det(\hat{\gamma}', \hat{\gamma}'', \boldsymbol{l} \circ \gamma, \hat{\gamma})}{|\hat{\gamma}'|^3}(t),$$

where $\hat{\boldsymbol{\gamma}}(t) = f\circ\gamma(t)$ and $\eta(t)$ is a null vector field, namely, non-zero vector field along γ satisfying $\langle \eta(t)\rangle_{\boldsymbol{R}} = \ker df_{\gamma(t)}$ and (γ', η) is positively oriented. Here, $' = d/dt$ and $\hat{\boldsymbol{\gamma}}''(t) = D_t\hat{\boldsymbol{\gamma}}'(t)$, where D is the Levi-Civita connection of $H^3_+(-1)$. The hyperbolic singular curvature has the same type geometric meaning as the Euclidean case. See Section 2 and [16, 17].

4.1. **Curves in hyperbolic space.** For a vector $\boldsymbol{v} \in S^3_1(1)$, define the hyperplane normal to $\boldsymbol{v}$ as $HP(\boldsymbol{v}, 0) = \{\boldsymbol{x} \in \boldsymbol{R}^4_1 \mid \langle \boldsymbol{x}, \boldsymbol{v}\rangle = 0\}$. It is well known that the set $H^2(\boldsymbol{v}) = HP(\boldsymbol{v}, 0) \cap H^3_+(-1)$ is a totally geodesic hyperbolic plane. Let $\boldsymbol{c}(s) : I \to H^2(\boldsymbol{v})$ be a regular curve and s an arclength parameter. Then since $T_pH^2(\boldsymbol{v}) = \big(\langle \boldsymbol{v}, p\rangle_{\boldsymbol{R}}\big)^{\perp}$ holds for $p \in H^2(\boldsymbol{v})$, the geodesic curvature of $\boldsymbol{c}$ is $\det(\boldsymbol{c}', \boldsymbol{c}'', \boldsymbol{v}, \boldsymbol{c})$ modulo a sign. Thus we define the *curvature in* $H^2(\boldsymbol{v})$ *of* $\boldsymbol{c}$ by $\kappa^h(s) = \det(\boldsymbol{c}', \boldsymbol{c}'', \boldsymbol{v}, \boldsymbol{c})(s)$. It can be easily seen that if a curve germ $\boldsymbol{c} : (I, 0) \to H^2(\boldsymbol{v})$ is a *cusp* ($\mathcal{A}$-equivalent to $t \mapsto (t^2, t^3)$ at 0), then $\kappa^h(s)\, ds$ can be continuously extended as a globally defined 1-form on I, where s is the arclength parameter of $\boldsymbol{c}$.

4.2. **Projections to planes.** To state Koenderink type theorems, we need orthogonal projections in $H^3_+(-1)$ to hyperbolic planes. Let us consider a hyperplane

$$HP(\boldsymbol{v}, 0) = \{\boldsymbol{x} \in \boldsymbol{R}^4_1 \mid \langle \boldsymbol{x}, \boldsymbol{v}\rangle = 0\}$$

for a vector $\boldsymbol{v} \in S^3_1(1)$. Given a point $\boldsymbol{q} \in H^3_+(-1)$, there is a unique geodesic in $H^3_+(-1)$ which intersects orthogonally the hyperbolic plane $H^2(\boldsymbol{v}) = HP(\boldsymbol{v}, 0) \cap H^3_+(-1)$ at some point $r(\boldsymbol{q}, \boldsymbol{v})$. We call the point $r(\boldsymbol{q}, \boldsymbol{v})$ the orthogonal projection of $\boldsymbol{q}$ in the direction $\boldsymbol{v}$ to $H^2(\boldsymbol{v})$. The point $r(\boldsymbol{q}, \boldsymbol{v})$ is given by

$$r(\boldsymbol{q}, \boldsymbol{v}) = \frac{1}{\sqrt{1 + \langle \boldsymbol{q}, \boldsymbol{v}\rangle^2}}\big(\boldsymbol{q} - \langle \boldsymbol{q}, \boldsymbol{v}\rangle\, \boldsymbol{v}\big).$$

See [9] for details.

4.3. **Koenderink type theorem.** In this section, we prove the following theorem:

Theorem 4.1. *Let $f : U \to H^3_+(-1)$ be a front, $p \in U$ a cuspidal edge, $M = f(U)$ and γ a singular curve with $\gamma(0) = p$. Set $\hat{\boldsymbol{\gamma}} = f \circ \gamma$,*

$$\boldsymbol{\xi}_p = \hat{\boldsymbol{\gamma}}'(p)/|\hat{\boldsymbol{\gamma}}'(p)| \wedge \boldsymbol{l}(p) \wedge f(p),$$

and $\boldsymbol{v}_\theta = \cos\theta\boldsymbol{\xi}_p + \sin\theta\boldsymbol{l}(p)$. Let r_θ the orthogonal projection $r_\theta : H^3_+(-1) \to H^2(\boldsymbol{v}_\theta)$ in the direction $\boldsymbol{v}_\theta$. Let $\kappa^h_1(t)$ be the curvature in $H^2(\boldsymbol{v}_\theta)$ of the curve $\boldsymbol{\gamma}_1(t) = r_\theta \circ \hat{\boldsymbol{\gamma}}(t)$, and $\kappa^h_2(s)$ the curvature in $H^2(\boldsymbol{l}(p) \wedge \boldsymbol{\xi}_p \wedge f(p))$ of the intersection curve $\boldsymbol{\gamma}_2$ of M at $f(p)$ by the hyperplane $HP(\boldsymbol{l}(p) \wedge \boldsymbol{\xi}_p \wedge f(p), 0)$, where s is the arclength parameter of $\boldsymbol{\gamma}_2$. If $\theta \in (0, \pi/2)$ then

$$K^h\, d\hat{A} = \frac{1}{\cos\theta}\big(-\cos\theta + \sin\theta\kappa^h_s - \kappa^h_1\big)\, dt \wedge \kappa^h_2\, ds$$

holds at p, where κ^h_s is the hyperbolic singular curvature. Here, we give a orientation of $\boldsymbol{\gamma}_2(s)$ passing through p from the region $\{\lambda^h < 0\}$ to the region $\{\lambda^h > 0\}$.

Proof. By changing coordinates on $(U; u, v)$, we may assume $p = \boldsymbol{0}$ and $S(f) = \{v = 0\}$. Also by isometries of $H^3_+(-1)$, we may assume

$$f(u, v) = \left(\sqrt{f_1(u, v)^2 + f_2(u, v)^2 + u^2 + 1}, f_1(u, v), f_2(u, v), u\right),$$

where $df_i = \boldsymbol{0}$ at $\boldsymbol{0}$ $(i = 1, 2)$. Then there exist functions $g_1(u)$, $g_2(u)$, $h_1(u, v)$, $h_2(u, v)$ such that $f_i(u, v) = u^2 g_i(u) + v h_i(u, v)$ $(i = 1, 2)$. Since $S(f) = \{v = 0\}$, it holds that $\partial h_i/\partial v(u, 0) = 0$ $(i = 1, 2)$. Thus there exist functions $\bar{h}_i(u, v)$ such that $h_i(u, v) = v\bar{h}_i(u, v)$ $(i = 1, 2)$. By a rotation of $H^3_+(-1)$, we may assume $\bar{h}_1(\boldsymbol{0}) = 0$. Thus we have $f_1(u, v) = u^2 a_1(u) + v^2 b_1(u, v)$

and $f_2(u,v) = u^2a_2(u) + uv^2a_3(u) + v^3b_2(u,v)$. where $a_1(u), a_2(u), a_3(u), b_1(u,v), b_2(u,v)$ are functions, and $b_1(\mathbf{0})b_2(\mathbf{0}) \neq 0$.

Then $\boldsymbol{l}(\mathbf{0}) = (0,0,1,0)$, $\boldsymbol{\xi_0} = (0,1,0,0)$ and $\boldsymbol{v}_\theta = (0,\cos\theta,\sin\theta,0)$ holds. By a direct calculation, we have

$$\kappa_s^h = -2a_1(0),\ \kappa_1^h = 2a_2(0)\cos\theta - 2a_1(0)\sin\theta,\ \kappa_2^h\, ds = -\frac{3b_2(\mathbf{0})}{2b_1(\mathbf{0})}\, ds$$

at $\mathbf{0}$ since one can consider $\hat{\boldsymbol{\gamma}}(t) = f(t,0)$ and $\boldsymbol{\gamma}_2(t) = f(0,t)$. On the other hand,

$$K^h\, du \wedge dv = \frac{3(1+2a_2(0))b_2(\mathbf{0})}{2b_1(\mathbf{0})}\, du \wedge dv$$

holds at $\mathbf{0}$. By these computations, we have the result. □

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