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LIPSCHITZ GEOMETRY OF COMPLEX CURVES

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Abstract. We describe the Lipschitz geometry of complex curves. To a large part this is well known material, but we give a stronger version even of known results. In particular, we give a quick proof, without any analytic restrictions, that the outer Lipschitz geometry of a germ of a complex plane curve determines and is determined by its embedded topology. This was first proved by Pham and Teissier, but in an analytic category. We also show the embedded topology of a plane curve determines its ambient Lipschitz geometry.

1. Introduction

The germ of a complex set $(X,0) \subset (\mathbb{C}^N,0)$ has two metrics induced from the standard hermitian metric on $\mathbb{C}^N$: the *outer metric* given by distance in $\mathbb{C}^N$ and the *inner metric* given by arc-length of curves on X. Both are well defined up to bilipschitz equivalence, *i.e.*, they only depend on the analytic type of the germ $(X,0)$ and not on the embedding $(X,0) \subset (\mathbb{C}^N,0)$. Studies of what information can be extracted from this metric structure have generally worked under analytic restrictions, e.g., that equivalences be restricted to be analytic or semi-algebraic or similar. In this note we prove the metric classification of germs of complex plane curves, but without any analytic restrictions (equivalence of item (1) of the following theorem with the other items):

Theorem 1.1. *Let $(C_1,0) \subset (\mathbb{C}^2,0)$ and $(C_2,0) \subset (\mathbb{C}^2,0)$ be two germs of complex curves. The following are equivalent:*

(1) *$(C_1,0)$ and $(C_2,0)$ have same Lipschitz geometry, i.e., there is a homeomorphism of germs $\phi\colon (C_1,0) \to (C_2,0)$ which is bilipschitz for the outer metric;*
(2) *there is a homeomorphism of germs $\phi\colon (C_1,0) \to (C_2,0)$, holomorphic except at 0, which is bilipschitz for the outer metric;*
(3) *$(C_1,0)$ and $(C_2,0)$ have the same embedded topology, i.e., there is a homeomorphism of germs $h\colon (\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$ such that $h(C_1) = C_2$;*
(4) *there is a bilipschitz homeomorphism of germs $h\colon (\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$ with $h(C_1) = C_2$.*

The equivalence of (1), (3) and (4) is our new contribution. The equivalence of (2) and (3) was first proved by Pham and Teissier [7]. By Teissier [8, Remarque, p.354] (see also Fernandes [5]) it then also follows that the outer bilipschitz geometry of any curve germ $(X,0) \subset (\mathbb{C}^N,0)$ determines the embedded topology of its general plane projection (Corollary 5.2).

For completeness we give quick proofs of all the equivalences. We start with the result for inner geometry, which will be used in examining outer geometry.

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2. Inner geometry

An algebraic germ $(X,0)\subset(\mathbb{C}^N,0)$ is homeomorphic to the cone on its link $X\cap S_\epsilon$, where S_ϵ is the sphere of radius ϵ about the origin with ϵ sufficiently small. If it is endowed with a metric, it is *metrically conical* if it is bilipschitz equivalent to the metric cone on its link. This basically means that the metric tells one no more than the topology (and is therefore uninteresting).

Proposition 2.1. *Any space curve germ $(C,0)\subset(\mathbb{C}^N,0)$ is metrically conical for the inner geometry.*

Proof. Take a linear projection $p\colon\mathbb{C}^N\to\mathbb{C}$ which is generic for the curve $(C,0)$ (*i.e.*, its kernel contains no tangent line of C at 0) and let $\pi:=p|_C$, which is a branched cover of germs. Let $D_\epsilon=\{z\in\mathbb{C}:|z|\le\epsilon\}$ with ϵ small, and let C_ϵ be the part of C which branched covers D_ϵ. Since π is holomorphic away from 0 we have a local Lipschitz constant $K(x)$ at each point $x\in C\setminus\{0\}$ given by absolute value of the derivative map of π at x. On each branch of C this $K(x)$ extends continuously over 0, so the infimum and supremum K^- and K^+ of $K(x)$ on $C_\epsilon\setminus\{0\}$ are defined and positive. For any arc γ in C_ϵ which is smooth except where it passes through 0 we have $K^-\ell(\gamma)\le\ell'(\gamma)\le K^+\ell(\gamma)$, where ℓ respectively ℓ' represent arc length using inner metric on $\mathbb{C}_\epsilon$ respectively the metric lifted from B_ϵ. Since C_ϵ with the latter metric is strictly conical, we are done. $\square$

3. Outer geometry determines embedded topological type

In this section, we prove (1) $\Rightarrow$ (3) of Theorem 1.1, *i.e.*, that the embedded topological type of a plane curve germ $(C,0)\subset(\mathbb{C}^2,0)$ is determined by the outer Lipschitz geometry of $(C,0)$.

We first prove this using the analytic structure and the outer metric on $(C,0)$. The proof is close to Fernandes' approach in [5]. We then modify the proof to make it purely topological and to allow a bilipschitz change of the metric.

The tangent space to C at 0 is a union of lines $L^{(j)}$, $j=1,\dots,m$, and by choosing our coordinates we can assume they are all transverse to the y-axis.

There is $\epsilon_0>0$ such that for any $\epsilon\le\epsilon_0$ the curve C meets transversely the set

$$T_\epsilon:=\left\{(x,y)\in\mathbb{C}^2:|x|=\epsilon\right\}.$$

Let μ be the multiplicity of C. The lines $x=t$ for $t\in(0,\epsilon_0]$ intersect C in μ points $p_1(t),\dots,p_\mu(t)$ which depend continuously on t. Denote by $[\mu]$ the set $\{1,2,\dots,\mu\}$. For each $j,k\in[\mu]$ with $j<k$, the distance $d(p_j(t),p_k(t))$ has the form $O(t^{q(j,k)})$, where $q(j,k)=q(k,j)$ is either a characteristic Puiseux exponent for a branch of the plane curve C or a coincidence exponent between two branches of C in the sense of e.g., [1, Chapitre 1, p. 12]. We call such exponents *essential*. For $j\in[\mu]$ define $q(j,j)=\infty$.

Lemma 3.1. *The map $q\colon[\mu]\times[\mu]\to\mathbb{Q}\cup\{\infty\}$, $(j,k)\mapsto q(j,k)$, determines the embedded topology of C.*

Proof. There are many combinatorial objects that encode the embedded topology of C, for example the Eisenbud-Neumann splice diagram [4] of the curve or the Eggers tree [3] (both are described, with the relationship between them, in C.T.C. Wall's book [9]). The "carrousel tree" described below is closely related (first described in [6]). All three are rooted trees with edges or vertices decorated with numeric labels.

To prove the lemma we will construct the carrousel tree from q. We also describe how one derives the splice diagram from it.

The $q(j,k)$ have the property that $q(j,l)\ge min(q(j,k),q(k,l))$ for any triple j,k,l. So for any $q\in\mathbb{Q}\cup\{\infty\},q>0$, the relation on the set $[\mu]$ given by $j\sim_q k\Leftrightarrow q(j,k)\ge q$ is an equivalence relation.

Name the elements of the set $q([\mu] \times [\mu]) \cup \{1\}$ in decreasing order of size:

$$\infty = q_0 > q_1 > q_2 > \cdots > q_s = 1.$$

For each $i = 0, \ldots, s$ let $G_{i,1}, \ldots, G_{i,\mu_i}$ be the equivalence classes for the relation $\sim_{q_i}$. So $\mu_0 = \mu$ and the sets $G_{0,j}$ are singletons while $\mu_s = 1$ and $G_{s,1} = [\mu]$. We form a tree with these equivalence classes $G_{i,j}$ as vertices, and edges given by inclusion relations: the singleton sets $G_{0,j}$ are the leaves and there is an edge between $G_{i,j}$ and $G_{i+1,k}$ if $G_{i,j} \subseteq G_{i+1,k}$. The vertex $G_{s,1}$ is the root of this tree. We weight each vertex with its corresponding q_i.

The *carrousel tree* is the tree obtained from this tree by suppressing valence 2 vertices: we remove each such vertex and amalgamate its two adjacent edges into one edge. We will describe how one gets from this to the splice diagram, but we first give an illustrative example.

We will use the plane curve C with two branches given by

$$y = x^{3/2} + x^{13/6}, \quad y = x^{7/3}.$$

Fig. 1 gives pictures of sections of C with complex lines $x = 0.1$, 0.05, 0.025 and 0. The central three-points set corresponds to the branch $y = x^{7/3}$ while the two lateral three-points sets correspond to the other branch.

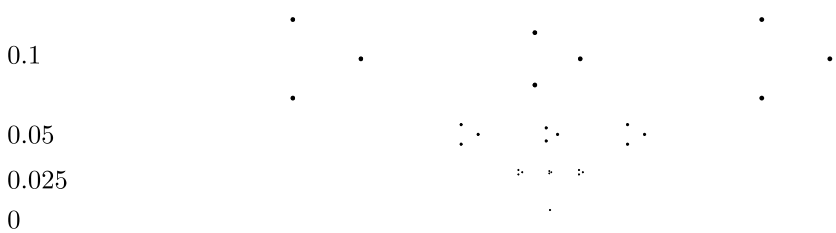

FIGURE 1. Sections of C

The carrousel tree for this example is the tree on the left in Fig. 2 and the procedure we will describe for getting from it to the splice diagram is then illustrated in the middle and right trees. We will follow the computer science convention of drawing the tree with its root vertex at the top, descending to its leaves at the bottom. At any non-leaf vertex v of the carrousel tree we

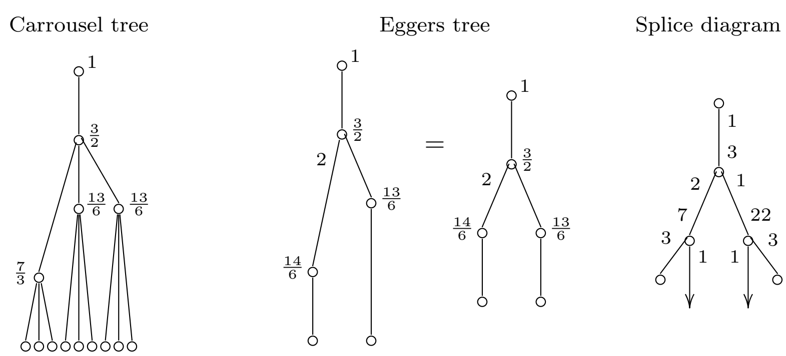

FIGURE 2. Carrousel tree to splice diagram

have a weight q_v, $1 \le q_v \le q_1$, which is one of the q_i's. We write it as m_v/n_v, where n_v is the lcm of the denominators of the q-weights at the vertices on the path from v up to the root vertex. If

v' is the adjacent vertex above v along this path, we put $r_v = n_v/n_{v'}$ and $s_v = n_v(q_v - q_{v'})$. At each vertex v the subtrees cut off below v consist of groups of r_v isomorphic trees, with possibly one additional tree. We label the top of the edge connecting to this additional tree at v, if it exists, with the number r_v, and then delete all but one from each group of r_v isomorphic trees below v. We do this for each non-leaf vertex of the carrousel tree. The resulting tree, with the q_v labels at vertices and the extra label on a downward edge at some vertices is easily recognized as a mild modification of the Eggers tree.

We construct the splice diagram starting from this tree. We first replace every leaf by an arrowhead. Then at each vertex v which did not have a downward edge with an r_v label we add such an edge (ending in a new leaf which is not an arrowhead). Each still unlabeled top end of an edge is then given the label 1. Finally, starting from the top of the tree we move down the tree adding a label to the bottom end of each edge ending in a vertex v which is not a leaf as follows. If v is directly below the root the label is $m'_v := m_v$. For a vertex v directly below a vertex v' other than the root the label is $m'_v := s_v + r_v r_{v'} m'_{v'}$ if $r_{v'}$ does not label the edge $v'v$ and $m'_v := (s_v + r_v m'_{v'})/r_{v'}$ if it does (see [4, Prop. 1A.1]). □

As already noted, this discovery of the embedded topology involved the complex structure and outer metric. We must show we can discover it without use of the complex structure, even after applying a bilipschitz change to the outer metric.

Recall that the tangent space of C is a union of lines $L^{(j)}$. We denote by $C^{(j)}$ the part of C tangent to the line $L^{(j)}$. It suffices to discover the topology of each $C^{(j)}$ independently, since the $C^{(j)}$'s are distinguished by the fact that the distance between any two of them outside a ball of radius ϵ around 0 is $O(\epsilon)$, even after bilipschitz change to the metric. We therefore assume from now on that the tangent to C is a single complex line.

The points $p_1(t), \dots, p_\mu(t)$ we used to find the numbers $q(j,k)$ were obtained by intersecting C with the line $x = t$. The arc $p_1(t)$, $t \in [0, \epsilon_0]$ satisfies $d(0, p_1(t)) = O(t)$. Moreover, the other points $p_2(t), \dots, p_\mu(t)$ are in the transverse disk of radius rt centered at $p_1(t)$ in the plane $x = t$. Here r can be as small as we like, so long as ϵ_0 is then chosen sufficiently small.

Instead of a transverse disk of radius rt, we can use a ball $B(p_1(t), rt)$ of radius rt centered at $p_1(t)$. This $B(p_1(t), rt)$ intersects C in μ disks $D_1(t), \dots, D_\mu(t)$, and we have $d(D_j(t), D_k(t)) = O(t^{q(j,k)})$, so we still recover the numbers $q(j,k)$. In fact, the ball in the outer metric on C of radius rt around $p_1(t)$ is $B_C(p_1(t), rt) := C \cap B(p_1(t), rt)$, which consists of these μ disks $D_1(t), \dots, D_\mu(t)$.

We now replace the arc $p_1(t)$ by any continuous arc $p'_1(t)$ on C with the property that $d(0, p'_1(t)) = O(t)$, and if r is sufficiently small it is still true that $B_C(p'_1(t), rt)$ consists of μ disks $D'_1(t), \dots, D'_\mu(t)$ with $d\big(D'_j(t), D'_k(t)\big) = O(t^{q(j,k)})$. So at this point, we have gotten rid of the dependence on analytic structure in discovering the topology, but not yet dependence on the outer geometry.

A K-bilipschitz change to the metric may make the components of $B_C(p'_1(t), rt)$ disintegrate into many pieces, so we can no longer simply use distance between pieces. To resolve this, we consider both $B'_C(p'_1(t), rt)$ and $B'_C(p'_1(t), \frac{r}{K^4}t)$ where B' means we are using the modified metric. Then only μ components of $B'_C(p_1(t), rt)$ will intersect $B'_C(p_1(t), \frac{r}{K^4}t)$. Naming these components $D'_1(t), \dots, D'_\mu(t)$ again, we still have $d(D'_j(t), D'_k(t)) = O(t^{q(j,k)})$ so the $q(j,k)$ are determined as before. □

4. Embedded topological type determines outer geometry

In this section, we prove (3) ⇒ (2) of Theorem 1.1. The implication (2) ⇒ (1) is trivial, so we then have the equivalence of the first three items of Theorem 1.1.

We will use the following lemma:

Lemma 4.1. *Let $(C,0) \subset (\mathbb{C}^N,0)$ be a germ of complex plane curve and let $p\colon \mathbb{C}^N \to \mathbb{C}$ be a linear projection whose kernel does not contain any tangent line to C. Then there exists a neighborhood U of 0 in C and a constant $M>1$ such that for each $u,u' \in U \smallsetminus \{0\}$, there is an arc $\tilde\alpha$ in C joining u to a point u'' with $p(u'')=p(u')$ and*

$$d(u,u') \le L(\tilde\alpha) + d(u'',u') \le M d(u,u')$$

where $L(\tilde\alpha)$ denotes the length of $\tilde\alpha$.

Proof. There exists a neighbourhood U of 0 in C such that the restriction $p|_C$ is a bilipschitz local homeomorphism for the inner metric on $U \smallsetminus \{0\}$ (see proof of Proposition 2.1). Choose any $\delta > 1$. If 0 is not in the segment $[p(u),p(u')]$, we set $\alpha = [p(u),p(u')]$. If $0 \in [p(u),p(u')]$, we modify this segment to a curve α avoiding 0 which has length at most δ times the length of $[p(u),p(u')]$. Consider the lifting $\tilde\alpha$ of α by $p|_C$ with origin u and let u'' be its extremity. We obviously have:

$$d(u,u') \le L(\tilde\alpha) + d(u',u'')\,.$$

On the other hand, $L(\tilde\alpha) \le K_0 L(\alpha) \le \delta K_0 d(p(u),p(u'))$, where K_0 is a bound for the local inner bilipschitz constant of p on $U \smallsetminus \{0\}$. As $d(p(u),p(u')) \le d(u,u')$, we then obtain:

$$L(\tilde\alpha) \le \delta K_0 d(u,u').$$

If we join the segment $[u,u']$ to $\tilde\alpha$ at u we get a curve from u' to u'', so

$$d(u',u'') \le (1+\delta K_0) d(u,u').$$

We then obtain:

$$L(\tilde\alpha) + d(u',u'') \le (1+2\delta K_0) d(u,u'),$$

and $M = 1+2\delta K_0$ is the desired constant. ☐

Proof of (3) ⇒ (2) *of Theorem 1.1.* Let $(C_1,0) \subset (\mathbb{C}^2,0)$ be an irreducible plane curve which is not tangent to the y-axis. Then there exists a minimal integer $n>0$ such that $(C_1,0)$ has Puiseux parametrization

$$\gamma_1(w) = \Big(w^n, \sum_{i\ge n} a_i w^i\Big)\,.$$

Denote $A := \{i : a_i \ne 0\}$. Recall that the embedded topology of C_1 is determined by n and the essential integer exponents in the sum $\sum_{i\ge n} a_i w^i$, where an $i \in A \smallsetminus \{n\}$ is an *essential integer exponent* if and only if $\gcd\{j \in \{n\}\cup A : j \le i\} < \gcd\{j \in \{n\}\cup A : j < i\}$ (equivalently $\frac{i}{n}$ is a characteristic exponent). Denote by A_e the subset of A consisting of the essential integer exponents.

Now let $(C_2,0) \subset (\mathbb{C}^2,0)$, given by

$$\gamma_2(w) = \Big(w^n, \sum_{i\ge n} b_i w^i\Big)\,,$$

be a second plane curve with the same embedded topology as C_1, so that the set of essential integer exponents $B_e \subset B := \{i : b_i \ne 0\}$ is equal to A_e.

We will prove that the homeomorphism $\Phi\colon C_1 \to C_2$ defined by $\Phi(\gamma_1(w)) = \gamma_2(w)$ is bilipschitz on small neighborhoods of the origin.

We first prove that there exists $K>0$ and a neighborhood U of 0 in $\mathbb{C}$ such that for each pair (w,w') with $w \in U$, $w \ne w'$ and $w^n = (w')^n$, we have

$$d\big(\gamma_1(w),\gamma_1(w')\big) \le K d\big(\gamma_2(w),\gamma_2(w')\big)$$

For (w,w') as above, consider the two real arcs $s\in[0,1]\mapsto\gamma_1(sw)$ and $s\mapsto\gamma_1(sw')$ and their images by Φ. Then we have

$$d\big(\gamma_1(ws),\gamma_1(w's)\big)=s^n\Big|\sum_{i>n}a_is^{i-n}\big(w^i-(w')^i\big)\Big|$$

and

$$d\Big(\Phi\big(\gamma_1(ws)\big),\Phi\big(\gamma_1(w's)\big)\Big)=s^n\Big|\sum_{i>n}b_js^{i-n}\big(w^i-(w')^i\big)\Big|$$

Let i_0 be the minimal element of $\{i\in A; w^i\neq(w')^i\}$. Then i_0 is an essential integer exponent, so a_{i_0} and b_{i_0} are non-zero. Moreover, as s tends to 0 we have

$$d\big(\gamma_1(ws),\gamma_1(w's)\big)\sim s^{i_0}|w^{i_0}-(w')^{i_0}||a_{i_0}|$$

and $d\big(\Phi\big(\gamma_1(ws)\big),\Phi\big(\gamma_1(w's)\big)\big)\sim s^{i_0}|w^{i_0}-(w')^{i_0}||b_{i_0}|$ and hence the ratio

$$d\Big(\gamma_1(ws),\gamma_1(w's)\Big)\Big/ d\Big(\Phi\big(\gamma_1(ws)\big),\Phi\big(\gamma_1(w's)\big)\Big)\qquad(*)$$

tends to the non zero constant $c_{i_0}=\frac{|a_{i_0}|}{|b_{i_0}|}$.

Notice that the integer i_0 depends on the pair of points (w,w'). But i_0 is either n or an essential integer exponent for γ_1. Therefore there are a finite number of values for i_0 and c_{i_0}. Moreover, the set of pairs (w,w') such that $w^n=(w')^n$ consists of a disjoint union of n lines. So there exists $s_0>0$ such that for each such (w,w') with $|w|=1$ and each $s\leq s_0$, the quotient $(*)$ belongs to $[1/K,K]$ where $K>0$. Then $U=\{w:|w|\leq s_0\}$ is the desired neighbourhood of 0.

We now prove that Φ is bilipschitz on $\gamma_1(U)$. Consider the projection $p\colon\mathbb{C}^2\to\mathbb{C}$ given by $p(x,y)=x$. Let w and w' be any two complex numbers in U. Let α be the segment in $\mathbb{C}$ joining w^n to $(w')^n$ and let $\tilde\alpha_1$ (resp. $\tilde\alpha_2$) be the lifting of α by the restriction $p|_{C_1}$ (resp. $p|_{C_2}$) with origin $\gamma_1(w)$ (resp. $\gamma_2(w)$). Consider the unique $w''\in\mathbb{C}$ such that $\gamma_1(w'')$ is the extremity of $\tilde\alpha_1$. Notice that $\gamma_2(w'')$ is the extremity of $\tilde\alpha_2$. We have

$$d\big(\gamma_1(w),\gamma_1(w')\big)\leq L(\tilde\alpha_1)+d\big(\gamma_1(w''),\gamma_1(w')\big).$$

According to Section 2, $p|_{C_1}$ (resp. $p|_{C_2}$) is an inner bilipschitz homeomorphism with bilipschitz constant say K_1 (resp. K_2). We then have $L(\tilde\alpha_1)\leq K_1K_2L(\tilde\alpha_2)$. Therefore setting $C=\max(K_1K_2,K)$, we obtain:

$$d\big(\gamma_1(w),\gamma_1(w')\big)\leq C\Big(L(\tilde\alpha_2)+d\big(\gamma_2(w''),\gamma_2(w')\big)\Big)\qquad(**)$$

Applying Lemma 4.1 to the restriction $p|_{C_2}$ with $u=\gamma_2(w)$ and $u'=\gamma_2(w')$, we then obtain:

$$d\big(\gamma_1(w),\gamma_1(w')\big)\leq CMd\big(\gamma_2(w),\gamma_2(w')\big)$$

This proves Φ is Lipschitz. It is then bilipschitz by symmetry of the roles.

In the general case where C_1 and C_2 are not necessarily irreducible, the same arguments work taking into account a Puiseux parametrization for each branch and the fact that the sets of characteristic exponents and coincidence exponents between branches coincide. □

5. Outer geometry of space curves

Before proving the final equivalence of Theorem 1.1 we give a quick proof, based on the preceding proof, of the following result of Teissier [8, pp. 352–354].

Theorem 5.1. *For a complex curve germ $(C,0)\subset(\mathbb{C}^N,0)$ the restriction to C of a generic linear projection $\ell\colon\mathbb{C}^N\to\mathbb{C}^2$ is bilipschitz for the outer geometry.*

Our notion of *generic linear projection* to $\mathbb{C}^2$, defined in the proof below, is equivalent to Teissier's, which says that the kernel of the projection should contain no limit of secant lines to the curve.

Proof of Theorem 5.1. We have to prove that the restriction $\ell|_C\colon C \to \ell(C)$ is bilipschitz for the outer metric. We choose coordinates (x,y) in $\mathbb{C}^2$ so $\ell(C)$ is transverse to the y-axis at 0 and coordinates $(z_1,\dots,z_n)$ in $\mathbb{C}^n$ with $z_1 = x\circ\ell$. So ℓ has the form $(z_1,\dots,z_N)\mapsto(z_1,\sum_1^N b_jz_j)$ and any component of C has a Puiseux expansion of the form (n is the multiplicity of the component):

$$\gamma(w) = \Big(w^n, \sum_{i\ge n} a_{2i}w^i, \dots, \sum_{i\ge n} a_{Ni}w^i\Big)\,.$$

We first assume $(C,0)$ is irreducible. We again denote $A := \{i : \exists j, a_{ji}\ne 0\}$ and call an exponent $i\in A\smallsetminus\{n\}$ an *essential integer exponent* if and only if

$$\gcd\{j\in\{n\}\cup A : j\le i\} < \gcd\{j\in\{n\}\cup A : j< i\}.$$

Define $a_{1n}=1$ and $a_{1i}=0$ for $i>n$. We say ℓ is *generic* if $\sum_{j=1}^N b_ja_{ji}\ne 0$ for each essential integer exponent i. We now assume ℓ is generic.

As in the proof of the second part of Theorem 1.1 there then exists $K>0$ and a neighborhood U of 0 in $\mathbb{C}$ such that for each pair (w,w') with $w\in U$ and $w^n=(w')^n$ we have

$$\frac{1}{K}d\big(\ell\gamma(w),\ell\gamma(w')\big)\le d\big(\gamma(w),\gamma(w')\big)\le Kd\big(\ell\gamma(w),\ell\gamma(w')\big)\,.$$

Lemma 4.1 then completes the proof, as before.

The proof when C is reducible is essentially the same, but the genericity condition must take both characteristic and coincidence exponents into consideration. Namely, ℓ should be generic as above for each individual branch of C; and for any two branches, given by (with n now the lcm of their multiplicities)

$$\gamma(w) = \Big(w^n, \sum_{i\ge n} a_{2i}w^i, \dots, \sum_{i\ge n} a_{Ni}w^i\Big), \quad \gamma'(w) = \Big(w^n, \sum_{i\ge n} a'_{2i}w^i, \dots, \sum_{i\ge n} a'_{Ni}w^i\Big),$$

we require $\sum_{j=1}^N b_j(a_{ji}-\lambda^ia'_{ji})\ne 0$ for each n-th root of unity λ, where i is the smallest exponent for which some $a_{ji}-a'_{ji}$ is non-zero. □

Corollary 5.2. *Let $(C_1,0)\subset(\mathbb{C}^{N_1},0)$ and $(C_2,0)\subset(\mathbb{C}^{N_2},0)$ be two germs of complex curves. The following are equivalent:*

(1) *$(C_1,0)$ and $(C_2,0)$ have same Lipschitz geometry i.e., there is a homeomorphism of germs $\phi\colon(C_1,0)\to(C_2,0)$ which is bilipschitz for the outer metric;*
(2) *there is a homeomorphism of germs $\phi\colon(C_1,0)\to(C_2,0)$, holomorphic except at 0, which is bilipschitz for the outer metric;*
(3) *the generic plane projections of $(C_1,0)$ and $(C_2,0)$ have the same embedded topology.* □

6. Ambient geometry of plane curves

To complete the proof of Theorem 1.1 we must show the implication (3) ⇒ (4) of that theorem, since (4) ⇒ (3) is trivial. We will use a *carrousel decomposition* of $(\mathbb{C}^2,0)$ with respect to a plane curve, so we first describe this (it is essentially the one described in [2]).

The tangent space to C at 0 is a union $\bigcup_{j=1}^m L^{(j)}$ of lines. For each j we denote the union of components of C which are tangent to $L^{(j)}$ by $C^{(j)}$. We can assume our coordinates (x,y) in $\mathbb{C}^2$ are chosen so that no $L^{(j)}$ is tangent to an axis. Then $L^{(j)}$ is given by an equation $y=a_1^{(j)}x$ with $a_1^{(j)}\ne 0$.

We choose $\epsilon_0 > 0$ sufficiently small that the set $\{(x,y) : |x| = \epsilon\}$ is transverse to C for all $\epsilon \le \epsilon_0$. We define conical sets $V^{(j)}$ of the form

$$V^{(j)} := \big\{(x,y) : |y - a_1^{(j)}x| \le \eta|x|, |x| \le \epsilon_0\big\} \subset \mathbb{C}^2\,,$$

where the equation of the line $L^{(j)}$ is $y = a_1^{(j)}x$ and $\eta > 0$ is small enough that the cones are disjoint except at 0. If ϵ_0 is small enough $C^{(j)} \cap \{|x| \le \epsilon_0\}$ will lie completely in $V^{(j)}$.

There is then an $R > 0$ such that for any $\epsilon \le \epsilon_0$ the sets $V^{(j)}$ meet the boundary of the "square ball"

$$B_\epsilon := \big\{(x,y) \in \mathbb{C}^2 : |x| \le \epsilon, |y| \le R\epsilon\big\}$$

only in the part $|x| = \epsilon$ of the boundary. We will use these balls as a system of Milnor balls.

We now describe our carrousel decomposition for each $V^{(j)}$, so we will fix j for the moment.

We first truncate the Puiseux series for each component of $C^{(j)}$ at a point where truncation does not affect the topology of $C^{(j)}$. Then for each pair $\kappa = (f, p_k)$ consisting of a Puiseux polynomial $f = \sum_{i=1}^{k-1} a_i^{(j)} x^{p_i^{(j)}}$ and an exponent $p_k^{(j)}$ for which there is a Puiseux series

$$y = \sum_{i=1}^{k} a_i^{(j)} x^{p_i^{(j)}} + \dots$$

describing some component of $C^{(j)}$, we consider all components of $C^{(j)}$ which fit this data. If $a_{k1}^{(j)}, \dots, a_{km_\kappa}^{(j)}$ are the coefficients of $x^{p_k^{(j)}}$ which occur in these Puiseux polynomials we define

$$B_\kappa := \Big\{(x,y) : \alpha_\kappa|x^{p_k^{(j)}}| \le \Big|y - \sum_{i=1}^{k-1} a_i^{(j)} x^{p_i^{(j)}}\Big| \le \beta_\kappa|x^{p_k^{(j)}}|$$
$$\Big|y - (\sum_{i=1}^{k-1} a_i^{(j)} x^{p_i^{(j)}} + a_{kj}^{(j)} x^{p_k^{(j)}})\Big| \ge \gamma_\kappa|x^{p_k^{(j)}}| \text{ for } j = 1, \dots, m_\kappa\Big\}\,.$$

Here $\alpha_\kappa, \beta_\kappa, \gamma_\kappa$ are chosen so that $\alpha_\kappa < |a_{k\nu}^{(j)}| - \gamma_\kappa < |a_{k\nu}^{(j)}| + \gamma_\kappa < \beta_\kappa$ for each $\nu = 1, \dots, m_\kappa$. If ϵ is small enough, the sets B_κ will be disjoint for different κ.

The intersection $B_\kappa \cap \{x = t\}$ is a finite collection of disks with smaller disks removed. We call B_κ a *B-piece*. The closure of the complement in $V^{(j)}$ of the union of the B_κ's is a union of pieces, each of which has link either a solid torus or a "toral annulus" (annulus $\times\, \mathbb{S}^1$). We call the latter *annular pieces* or *A-pieces* and the ones with solid torus link *D-pieces* (a B-piece corresponding to an inessential exponent has the same topology as an A-piece, but we do not call it annular).

This is our carrousel decomposition of $V = V^{(j)}$. We call $\overline{B_\epsilon \smallsetminus \bigcup V^{(j)}}$ a $B(1)$ piece (even though it may have A- or D-topology). It is metrically conical, and together with the carrousel decompositions of the $V^{(j)}$'s we get a carrousel decomposition of the whole of B_ϵ.

Proof of (3) $\Rightarrow$ (4) *of Theorem 1.1.* Let $(C_1, 0) \subset (\mathbb{C}^2, 0)$ and $(C_2, 0) \subset (\mathbb{C}^2, 0)$ have the same embedded topological type. Consider two carrousel decompositions of $(\mathbb{C}^2, 0)$: one with respect to C_1 and the other with respect to C_2, constructed as above. The proof consists of constructing a bilipschitz map of germs $h\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ which sends the carrousel decomposition for C_1 to the one for C_2 (being careful to include matching pieces for inessential exponents which occur in just one of C_1 and C_2). We first construct it to respect the carrousels, but not necessarily map C_1 to C_2. Once this is done, we adjust it so that C_1 is mapped to C_2.

Let $L_1^{(j)}$ and $L_2^{(j)}$, $j = 1, \dots, m$, be the tangent lines to C_1 and C_2 and $C_1^{(j)}$ resp. $C_2^{(j)}$ the union of components of C_1 resp. C_2 which are tangent to $L_1^{(j)}$ resp. $L_2^{(j)}$. We may assume we have numbered them so $C_1^{(j)}$ and $C_2^{(j)}$ have matching embedded topology. Let $V_1^{(j)}$ and $V_2^{(j)}$, $j = 1, \dots, m$, be the conical sets around the tangent lines as defined earlier.

The $B(1)$ pieces of the carrousel decompositions for C_1 and C_2 are metrically conical with the same topology, so there is a conical bilipschitz diffeomorphism between them. We can arrange that it is a translation on each $x = t$ section of each $\partial V_1^{(j)}$. We will extend it over the cones $V_1^{(j)}$ and $V_2^{(j)}$ using the carrousels.

Consider the Puiseux series $y = \sum_{i=1}^k a_i^{(j)} x^{p_i^{(j)}} + \dots$ describing some component of $C_1^{(j)}$ and the Puiseux series $y = \sum_{i=1}^k b_i^{(j)} x^{p_i^{(j)}} + \dots$ describing the corresponding component of $C_2^{(j)}$. If a term with inessential exponent appears in one of the series, we include it also in the other, even if its coefficient there is zero. This way, when we construct the carrousel as above we have corresponding B-pieces for the two carrousels. Moreover, we can choose the constants $\alpha_\kappa, \beta_\kappa, \gamma_\kappa$ used to construct these corresponding B-pieces to be the same for both. The $\{x = t\}$ sections of a pair of corresponding A-pieces will then be congruent, so we can map the one A-piece to the other by preserving x coordinate and using translation on each $x = t$ section. The same holds for D-pieces. It then remains to extend to the B-pieces.

A B-piece B_{κ_1} in the decomposition for C_1 is determined by some $\kappa_1 = (f_1, p_k)$ with

$$f_1 = \sum_{i=1}^{k-1} a_i x^{p_i},$$

and is foliated by curves of the form $y = f_1 + \xi x^{p_k}$ for varying ξ (we call p_k the *rate* of B_κ). The corresponding piece B_{κ_2} for C_2 is similarly determined by some $\kappa_2 = (f_2, p_k)$ with

$$f_2 = \sum_{i=1}^{k-1} b_i x^{p_i}$$

and is foliated by curves $y = f_2 + \xi x^{p_k}$. The $x = \epsilon_0$ section of B_{κ_1} has a free cyclic group action generated by the first return map of the foliation, and the same is true for B_{κ_2}. We choose a smooth map $(B_{\kappa_1} \cap \{x = \epsilon_0\}) \to (B_{\kappa_2} \cap \{x = \epsilon_0\})$ which is equivariant for this action and on the boundary matches the maps, coming from A- and D-pieces, already chosen. This map extends to the whole of B_{κ_1} by requiring it to preserve the foliation and x-coordinate.

By construction, the resulting map of germs $\phi\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ is an isometry on the A- and D-pieces and bilipschitz on the $B(1)$ piece. We must check that it is bilipschitz on the B-pieces of type B_κ. Pick such a B and suppose the rate of B is r. The Lipschitz constant of ϕ is bounded in a neighborhood of the link $B^{(\epsilon)} := B \cap \{|x| = \epsilon\}$ of B by compactness. For $0 < \epsilon' < \epsilon$, if we move points inwards x-distance $\epsilon - \epsilon'$ along the leaves of the foliation of B, each section at $x = t$ with $|t| = \epsilon$ moves to the section at $x = \frac{\epsilon'}{\epsilon} t$ while scaling by a factor of $(\epsilon'/\epsilon)^r$. The same holds for the images of these sections in the carrousel for C_2. So to high order the Lipschitz constant of ϕ at a point of the $x = t$ section equals the Lipschitz constant at the corresponding point of the $x = \frac{\epsilon'}{\epsilon} t$ section. It follows that the local Lipschitz constant is bounded on the whole of B, so ϕ is bilipschitz.

However, ϕ maps C_1 not to C_2, but to a small deformation of it, since we constructed the carrousels by first truncating our Puiseux series beyond any terms which contributed to the topology. But it is not hard to see that, by a small change of the constructed map inside the D-pieces which intersect C_1, one can change ϕ so it maps C_1 to C_2 while changing the bilipschitz coefficient by an amount which approaches zero as one approaches the origin. Namely, let D_1 be such a piece and $D_2 = \phi(D_1)$ the corresponding piece for the curve C_2. In each $x = t$ slice $D_1(t) := D_1 \cap \{x = t\}$ we take the map $D_1(t) \to D_1(t)$ which moves the point $p_1(t) := D_1(t) \cap C_1$ to $p_2(t) := \phi^{-1}(D_2(t) \cap C_2)$ and maps each ray from $p_1(t)$ to a point $p \in \partial D_1(t)$ linearly to the ray from $p_2(t)$ to p. This gives a map $\psi\colon D_1 \to D_1$ whose bilipschitz constant rapidly approaches 1 as $t \to 0$ and $\phi \circ \psi$ does what is required on this piece. $\square$

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