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Families of Gauss indicatrices on smooth surfaces in pseudo-spheres in the Minkowski 4-space

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Abstract

We study families of Gauss indicatrices on surfaces in pseudo-spheres in the Minkowski 4-space and obtain the generic local models of the configurations of the foliations determined by the fibres of their principal curvatures functions.

1 Introduction

In [9], Izumiya-Pei-Sano defined the hyperbolic Gauss indicatrix of a hypersurface in the Minkowski space model of the hyperbolic space. The work in [9] set the foundations of applications of singularity theory to the extrinsic geometry of submanifolds in the hyperbolic space. Given a point p on a hypersurface M in the hyperbolic space $H^n_+(-1)$, there is a well defined (at least locally) unit normal vector $\boldsymbol{e}(p)$ to M at p; see §2. The vector $\boldsymbol{e}(p)$ is in the de Sitter space S^n_1 and defines the de Sitter Gauss indicatrix

$$\begin{array}{rccc} \mathbb{E}: & M & \to & S^n_1 \\ & p & \to & \boldsymbol{e}(p) \end{array}$$

The de Sitter Gauss-Kronecker curvature at p is $K_e(p) := \det(-(d\mathbb{E})_p)$ and the totally umbilic hypersurfaces with $K_e \equiv 0$ are the hyperplanes in $H^n_+(-1)$. The de Sitter Gauss indicatrix on M is related to the contact of M with hyperplanes ([9]).

Another Gauss indicatrix on M is introduced in [9] and is called the hyperbolic or lightcone Gauss indicatrix; see §2. The vector $p \pm \boldsymbol{e}(p)$ is lightlike (i.e., belongs to the lightcone LC^*) and defines the hyperbolic Gauss indicatrices

$$\begin{array}{rccc} \mathbb{L}^{\pm}: & M & \to & LC^* \\ & p & \to & p \pm \boldsymbol{e}(p) \end{array}$$

The hyperbolic Gauss-Kronecker curvature at p is $K_h(p) := \det(-(d\mathbb{L}^{\pm})_p)$ and the totally umbilic hypersurfaces with $K_h \equiv 0$ are the hyperhorospheres in $H^n_+(-1)$. The hyperbolic Gauss indicatrix on M is related to the contact of M with hyperhorospheres ([9]).

In [1] is constructed a 1-parameter family of Gauss indicatrices which links $\mathbb{E}$ and $\mathbb{L}^{\pm}$. The family is given by $N_\theta(p) = \cos\theta p \pm \boldsymbol{e}(p) \in S^n(\sin^2(\theta))$, $\theta \in [0, \pi/2]$, and is called the Slant Gauss indicatrix. Observe that $N_\theta(p)$ is always spacelike for $\theta \neq 0$. The above family links the geometry of M related to hyperplanes to that related to hyperhorospheres. See also [11] for slant geometry in the de Sitter space and in the lightcone.

The work in this paper is inspired by that in [1, 11]. A hypersurface M in $H^n_+(-1)$ can be viewed as a codimension 2 spacelike submanifold in $\mathbb{R}^{n+1}_1$. It has then a timelike normal plane in $\mathbb{R}^{n+1}_1$ at

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any of its points. For this reason, we consider normal vector fields (Gauss indicatrix) on M which are not necessarily spacelike. We define two families of Gauss indicatrices on M. One is spacelike and is given by $N^s_\theta = \tanh(\theta)p + \boldsymbol{e}(p)$ and the other is timelike and is given by $N^t_\theta = \tanh(\theta)^{-1}p + \boldsymbol{e}(p)$ (we use hyperbolic angles here, see [17] for definition and properties). The families N^w_θ, $w = s, t$ tend to $\mathbb{L}^\pm$ as θ tends to $\pm\infty$. We define the θ^w-Gauss-Kronecker curvature by $K^w_\theta(p) := \det(-(dN^w_\theta)_p)$.

We give in §3 general results about the families N^w_θ on hypersurfaces in $H^n_+(-1)$ and deal in more details with surfaces in $H^3_+(-1)$ in §3.1. We denote by κ_1 and κ_2 the eigenvalues of the de Sitter shape operator and call them the de Sitter principal curvatures. It turns out that the θ^w-parabolic sets (points where K^w_θ vanishes) are given by $\kappa_i = constant$. The θ^s-parabolic sets foliate the region in M where $|\kappa_i| < 1$ and the θ^t-parabolic sets foliate the region in M where $|\kappa_i| > 1$; see Theorem 3.2. (One motivation behind considering the timelike Gauss indicatrices is that the θ^s-parabolic sets do not cover the whole surface. The other is that N^t_θ gives information about the contact of M with hyperspheres.) Note that the parabolic sets of the limiting families $N^w_{\pm\infty} = \mathbb{L}^\pm$ are the horospherical parabolic sets given by $\kappa_i = \pm 1$. We obtain the generic local configurations of the foliations $\kappa_i = constant$, $i = 1, 2$ (Theorem 3.7), and characterise geometrically their singularities (Theorems 3.4, 3.5, 3.8).

One can view $M \subset H^3_+(-1)$ as a surface in $\mathbb{R}^4_1$. Asymptotic directions are defined via the contact of M with lines. They are metric independent and we have thus well defined asymptotic curves on M given by a quadratic binary differential equation (BDE for short). We show that these asymptotic curves are in fact the lines of the de Sitter principal curvature. This is true for any spacelike or timelike surface in a pseudo-sphere in the Minkowski 4-space (Theorem 3.9).

We consider in §4 families of Gauss indicatrices on timelike hypersurfaces in the de Sitter space S^n_1, with emphasis on timelike surfaces in S^3_1. The foliations $\kappa_i = constant$, $i = 1, 2$, behave differently from those on spacelike surfaces (Theorem 4.1). We recall in the Appendix §5 the classification of codimension ≤ 1 singularities of BDEs.

2 Preliminaries

We start by recalling some basic concepts in hyperbolic geometry (see for example [16, 19]). The *Minkowski* $(n+1)$*-space* $(\mathbb{R}^{n+1}_1, \langle , \rangle)$ is the $(n+1)$-dimensional vector space $\mathbb{R}^{n+1}$ endowed with the *pseudo scalar product* $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = -u_0v_0 + \sum_{i=1}^n u_iv_i$, for any $\boldsymbol{u} = (u_0, \dots, u_n)$ and $\boldsymbol{v} = (v_0, \dots, v_n)$ in $\mathbb{R}^{n+1}_1$. We say that a vector $\boldsymbol{u}$ in $\mathbb{R}^{n+1}_1 \setminus \{\boldsymbol{0}\}$ is *spacelike*, *lightlike* or *timelike* if $\langle \boldsymbol{u}, \boldsymbol{u} \rangle > 0, = 0$ or < 0 respectively. The norm of a vector $\boldsymbol{u} \in \mathbb{R}^{n+1}_1$ is defined by $\|\boldsymbol{u}\| = \sqrt{|\langle \boldsymbol{u}, \boldsymbol{u} \rangle|}$. Given a vector $\boldsymbol{v} \in \mathbb{R}^{n+1}_1$ and a real number c, a hyperplane with pseudo normal $\boldsymbol{v}$ is defined by

$$HP(\boldsymbol{v}, c) = \{\boldsymbol{u} \in \mathbb{R}^{n+1}_1 \mid \langle \boldsymbol{u}, \boldsymbol{v} \rangle = c\}.$$

We say that $HP(\boldsymbol{v}, c)$ is a *spacelike*, *timelike* or *lightlike hyperplane* if $\boldsymbol{v}$ is timelike, spacelike or lightlike respectively. We have the following pseudo-spheres in $\mathbb{R}^{n+1}_1$ with centre $p \in \mathbb{R}^{n+1}_1$ and radius $r > 0$,

$$\begin{array}{rcl} H^n(p, -r) & = & \{\boldsymbol{u} \in \mathbb{R}^{n+1}_1 \mid \langle \boldsymbol{u} - p, \boldsymbol{u} - p \rangle = -r^2\}, \\ S^n(p, r) & = & \{\boldsymbol{u} \in \mathbb{R}^{n+1}_1 \mid \langle \boldsymbol{u} - p, \boldsymbol{u} - p \rangle = r^2\}, \\ LC^*(p) & = & \{\boldsymbol{u} \in \mathbb{R}^{n+1}_1 \mid \langle \boldsymbol{u} - p, \boldsymbol{u} - p \rangle = 0\}. \end{array}$$

We denote by $H^n(-r)$ and $S^n(r)$ the pseudo-spheres centred at the origin in $\mathbb{R}^{n+1}_1$. The pseudo sphere $H^n(-r)$ has two connected components. The hyperbolic space $H^n_+(-1)$ is the connected component of $H^n(-1)$ whose points $\boldsymbol{u}$ have positive coordinate u_0. The de Sitter space is $S^n_1 = S^n(1)$ and the lightcone is $LC^* = LC^*(\boldsymbol{0})$.

A hypersurface given by the intersection of $H^n_+(-1)$ with a spacelike, timelike or lightlike hyperplane is called respectively *hypersphere*, *equidistant hypersurface* or *hyperhorosphere*. The intersection of a hypersurface with a timelike hyperplane through the origin is called simply a *hyperplane*.

The study of the extrinsic geometry of hypersurfaces in the hyperbolic space from the viewpoint of Legendrian singularities was initiated in [9]. Let $\boldsymbol{x} : U \to H^n_+(-1)$ be a local parametrisation of a hypersurface M embedded in $H^n_+(-1)$, where U is an open subset of $\mathbb{R}^{n-1}$. We write $M = \boldsymbol{x}(U)$. Since $\langle \boldsymbol{x}, \boldsymbol{x} \rangle \equiv -1$, we have $\langle \boldsymbol{x}_{u_i}, \boldsymbol{x} \rangle \equiv 0$, for $i = 1, \ldots, n-1$, where $\underline{u} = (u_1, \ldots, u_{n-1}) \in U$. The spacelike unit normal vector $\boldsymbol{e}(\underline{u})$ to M at $\boldsymbol{x}(\underline{u})$ is defined by

$$\boldsymbol{e}(\underline{u}) = \frac{\boldsymbol{x}(\underline{u}) \wedge \boldsymbol{x}_{u_1}(\underline{u}) \wedge \ldots \wedge \boldsymbol{x}_{u_{n-1}}(\underline{u})}{\|\boldsymbol{x}(\underline{u}) \wedge \boldsymbol{x}_{u_1}(\underline{u}) \wedge \ldots \wedge \boldsymbol{x}_{u_{n-1}}(\underline{u})\|}.$$

It follows that $\boldsymbol{x}(\underline{u}) \pm \boldsymbol{e}(\underline{u})$ is a lightlike vector for all $\underline{u} \in U$. The de Sitter and hyperbolic Gauss indicatrices $\mathbb{E}$ and $\mathbb{L}^{\pm}$ respectively are defined in the introduction. The linear transformation $-(d\mathbb{E})_p$ at $p = \boldsymbol{x}(\underline{u})$ is called the *de Sitter shape operator*. Its eigenvalues κ_i, $i = 1, \ldots, n-1$, are called the *de Sitter principal curvatures* and the corresponding eigenvectors $\boldsymbol{p}_i$, $i = 1, \ldots, n-1$, are called the *de Sitter principal directions*. The linear transformation $-(d\mathbb{L}^{\pm})_p$ is labelled the *hyperbolic shape operator* of M at p. It has the same eigenvectors as $-(d\mathbb{E})_p$ but has distinct eigenvalues. In fact the eigenvalues $\bar{\kappa}_i^{\pm}$ of $-(d\mathbb{L}^{\pm})_p$ satisfy $\bar{\kappa}_i^{\pm} = -1 \pm \kappa_i$, $i = 1, \ldots, n-1$.

A smooth submanifold M of the Minkowski space is said to be *spacelike* (resp. *timelike*) if the induced metric on M is Riemannian (resp. Lorentzian, i.e., of signature 1). For a spacelike (resp. timelike) hypersurface in the de Sitter space S^n_1, the vector $\boldsymbol{e}(\underline{u})$ is timelike (resp. spacelike) and defines a Gauss indicatrix with values in the hyperbolic (resp. de Sitter) space.

3 Hypersurfaces in $H^n_+(-1)$

We start with some general results on hypersurfaces M in $H^n_+(-1)$. Let $\boldsymbol{x} : U \to M$ be a local parametrisation of M. At each point $\boldsymbol{x}(\underline{u})$, the normal plane $N_{\boldsymbol{x}(\underline{u})}M$ to M in $\mathbb{R}^{n+1}$ is timelike and is generated by $\boldsymbol{e}(\underline{u})$ and $\boldsymbol{x}(\underline{u})$. Any choice of a normal vector in $N_{\boldsymbol{x}(\underline{u})}M$ generates a Gauss indicatrix. For instance, the hyperbolic Gauss indicatrix $\mathbb{L}^{\pm}$ is given by $\boldsymbol{x}(\underline{u}) \pm \boldsymbol{e}(\underline{u})$. We can parametrise a circle of vectors in $N_{\boldsymbol{x}(\underline{u})}M$ by $\cos(\theta)\boldsymbol{x}(\underline{u}) + \sin(\theta)\boldsymbol{e}(\underline{u})$ and get a family of Gauss indicatrices. However, we would like the parameter to have some geometric meaning and also to distinguish between the timelike and spacelike normal vectors as these lead to the contact of M with different models of hypersurfaces. The differential of the Gauss indicatrix given by the vector $\boldsymbol{x}(\underline{u})$ is the identity map so does not give any geometric information. For these reasons, we define the family of spacelike Gauss indicatrices by

$$\begin{array}{rccl} N^s_\theta : & U & \to & S^n(\cosh(\theta)^{-2}) \\ & \underline{u} & \mapsto & \tanh(\theta)\boldsymbol{x}(\underline{u}) + \boldsymbol{e}(\underline{u}) \end{array}$$

where $\theta \in \mathbb{R}$ is the hyperbolic angle between $N^s_\theta(\underline{u})$ and $\boldsymbol{x}(\underline{u})$. If we take $\sinh(\theta)\boldsymbol{x}(\underline{u}) + \cosh(\theta)\boldsymbol{e}(\underline{u}) \in S^n_1$ as a unit normal spacelike vector we will not get the desired limit $N^s_\theta \to \mathbb{L}^{\pm}$ when $\theta \to \pm\infty$. We define the family of timelike Gauss indicatrices by

$$\begin{array}{rccl} N^t_\theta : & U & \to & H^n(-\sinh(\theta)^{-2}) \\ & \underline{u} & \mapsto & \tanh(\theta)^{-1}\boldsymbol{x}(\underline{u}) + \boldsymbol{e}(\underline{u}) \end{array}$$

where $\theta \in \mathbb{R} \setminus \{0\}$ is the hyperbolic angle between $N^t_\theta(\underline{u})$ and $\boldsymbol{x}(\underline{u})$. Again, if we take $\cosh(\theta)\boldsymbol{x}(\underline{u}) + \sinh(\theta)\boldsymbol{e}(\underline{u}) \in H^n(-1)$ as a unit normal timelike vector we will not get the desired limit $N^t_\theta \to \mathbb{L}^{\pm}$ when $\theta \to \pm\infty$. (Observe that $\boldsymbol{x}$ is not a member of the family N^t_θ.)

We have the following result which follows from the definitions of N^w_θ, $w = s, t$.

Theorem 3.1 *The differential map $-(dN^s_\theta)_p = -\tanh(\theta)I_p - (d\mathbb{E})_p$ is a self-adjoint operator on T_pM. Its eigenvalues are $\kappa^s_{\theta i} = -\tanh(\theta) + \kappa_i$, where κ_i are the de Sitter principal curvatures. The eigenvectors of $-(dN^s_\theta)_p$, for any $\theta \in \mathbb{R}$, coincide with those of the de Sitter shape operator $-(d\mathbb{E})_p$.*

Similarly, the differential map $-(dN^t_\theta)_p = -\tanh(\theta)^{-1}I_p - (d\mathbb{E})_p$ is a self-adjoint operator on T_pM. Its eigenvalues are $\kappa^t_{\theta i} = -\tanh(\theta)^{-1} + \kappa_i$. The eigenvectors of $-(dN^t_\theta)_p$, for any $\theta \in \mathbb{R}\setminus\{0\}$, also coincide with those of the de Sitter shape operator.

We call $\kappa^w_{\theta i}$, $w = s,t$, the θ^w-principal curvatures and call $K^w_\theta(p) = det(-(dN^w_\theta)_p) = \prod_{i=1}^{n-1}\kappa^w_{\theta i}$ the θ^w-Gauss-Kronecker curvature of M at $p = \boldsymbol{x}(\underline{u})$. A point p on M is called (spacelike) θ^w-umbilic ($w = s$ or t) if $\kappa^w_{\theta i} = \kappa^w_{\theta j}$ for all i,j at p. It is called θ^w-parabolic if $K^w_\theta(p) = 0$.

We are interested in hypersurfaces whose points are all θ^w-umbilics, which we label totally θ^w-umbilic hypersurfaces. These will form the "model" hypersurfaces in the hyperbolic space. One can characterised θ^w-umbilic hypersurfaces in the same way as in Proposition 2.3 in [9]. For instance, if a hypersurface $M \subset H^n_+(-1)$ is totally θ^w-umbilic, then $\kappa^w_{\theta i}$ are all equal to the same constant, say κ^w_θ, on M. Then M is a subset of the intersection of $H^n_+(-1)$ with a hyperplane and the type of the hyperplane is determined by the value of the constant κ^w_θ.

We consider the contact of M with model hypersurfaces in $H^n_+(-1)$. We define the family of spacelike height functions by

$$\begin{array}{cccc} H^s_\theta & U\times S^n(\cosh(\theta)^{-2}) & \to & \mathbb{R} \\ & (\underline{u},\boldsymbol{v}) & \mapsto & \langle \boldsymbol{x}(\underline{u}),\boldsymbol{v}\rangle + \tanh(\theta) \end{array}$$

This measures the contact of M with the equidistant hypersurfaces $HP(\boldsymbol{v}, -\tanh(\theta)) \cap H^n_+(-1)$. We have $H^s_\theta = \partial H^s_\theta/\partial u_i = 0$ if and only if $\boldsymbol{v} = N^s_\theta(\underline{u})$. A point $p = \boldsymbol{x}(\underline{u})$ is a θ^s-parabolic point if and only if the Hessian of $H^s_\theta(-,\boldsymbol{v})$, with $\boldsymbol{v} = N^s_\theta(\underline{u})$, is singular. This means that the θ^s-parabolic set is the set of points on M which correspond to the singular points of the discriminant of the family H^s_θ. (One can show, using the same arguments in the proof of Proposition 4.2 in [9] that H^s_θ is a Morse family. This yields a Legendrian immersion whose generating family is H^s_θ. The wavefront of the Legendrian immersion is the Gauss indicatrix N^s_θ.)

We also define the family of timelike height functions

$$\begin{array}{cccc} H^t_\theta & U\times H^n(-\sinh(\theta)^{-2}) & \to & \mathbb{R} \\ & (\underline{u},\boldsymbol{v}) & \mapsto & \langle \boldsymbol{x}(\underline{u}),\boldsymbol{v}\rangle + \tanh(\theta)^{-1} \end{array}$$

which measures the contact of M with the hyperspheres $HP(\boldsymbol{v}, -\tanh^{-1}(\theta)) \cap H^n_+(-1)$. We have similar results to those for the family H^s_θ.

3.1 Surfaces in $H^3_+(-1)$

We obtain in this section geometric information about the foliations determined by $\kappa^w_{\theta i} = constant$, $i = 1,2$, $w = s,t$. As the θ^w-principal curvatures define the same foliations, we work with the de Sitter curvatures κ_1 and κ_2. Let $\boldsymbol{x} : U \subset \mathbb{R}^2 \to M \subset H^3_+(-1)$ be a local parametrisation of M and denote by (u,v) the coordinates in U. In this paper, subscripts involving the parameters u,v refer to partial differentiation with respect to these parameters. The coefficients of the first fundamental form with respect to $\boldsymbol{x}$ are denoted by

$$E = \langle \boldsymbol{x}_u, \boldsymbol{x}_u\rangle, \quad F = \langle \boldsymbol{x}_u, \boldsymbol{x}_v\rangle, \quad G = \langle \boldsymbol{x}_v, \boldsymbol{x}_v\rangle.$$

The θ^w-second fundamental form ($w = s,t$) at $p = \boldsymbol{x}(u,v)$, with associated shape operator $-(dN^w_\theta)_p$, is given by $\mathrm{II}^w_\theta(\boldsymbol{u},\boldsymbol{v}) = \langle -(dN^w_\theta)_p(\boldsymbol{u}),\boldsymbol{v}\rangle$, for $\boldsymbol{u},\boldsymbol{v} \in T_pM$. We denote by

$$\begin{array}{rcccl} l^w_\theta & = & \langle -(dN^w_\theta)_p(\boldsymbol{x}_u),\boldsymbol{x}_u\rangle & = & \langle N^w_\theta, \boldsymbol{x}_{uu}\rangle, \\ m^w_\theta & = & \langle -(dN^w_\theta)_p(\boldsymbol{x}_u),\boldsymbol{x}_v\rangle & = & \langle N^w_\theta, \boldsymbol{x}_{uv}\rangle, \\ n^w_\theta & = & \langle -(dN^w_\theta)_p(\boldsymbol{x}_v),\boldsymbol{x}_v\rangle & = & \langle N^w_\theta, \boldsymbol{x}_{vv}\rangle \end{array}$$

its coefficients with respect to the basis $\{\boldsymbol{x}_u, \boldsymbol{x}_v\}$. We have

$$\begin{array}{rcl} l_\theta^w & = & -\tanh(\theta)^\epsilon E + l, \\ m_\theta^w & = & -\tanh(\theta)^\epsilon F + m, \\ n_\theta^w & = & -\tanh(\theta)^\epsilon G + n, \end{array}$$

with $\epsilon = 1$ if $w = s$ and $\epsilon = -1$ if $w = t$ and where l, m, n denote the coefficients of the second fundamental form associated to the de Sitter shape operator $-d\mathbb{E}$. Because the induced metric on M is Riemannian, $-(dN_\theta^w)_p$ has always two real eigenvalues. The θ^w-lines of principal curvature are the same for all θ and coincide with the de Sitter lines of principal curvature. These are given by a BDE that can be represented in the following determinant form

$$\begin{vmatrix} dv^2 & -dudv & du^2 \\ E & F & G \\ l & m & n \end{vmatrix} = 0. \tag{1}$$

For a generic surface, the discriminant of equation (1) (which is the set of points on the surface where the equation determines a unique direction, see §5 for details) consists of the isolated umbilic points. We write

$$\begin{array}{l} K_e = \kappa_1\kappa_2 = \dfrac{ln - m^2}{EG - F^2}, \\ H_e = \dfrac{1}{2}(\kappa_1 + \kappa_2) = \dfrac{lG - 2mF + nE}{2(EG - F^2)}, \end{array}$$

for the de Sitter Gauss-Kronecker curvature and the de Sitter mean curvature, respectively. We have the following result.

Theorem 3.2 (1) *The θ^s-parabolic set is given by*

$$\tanh^2(\theta) - 2H_e \tanh(\theta) + K_e = 0.$$

It consists of the curves (which could be empty) $\kappa_i = \tanh(\theta)$, $i = 1, 2$. Each family of these curves foliate the region of M where $|\kappa_i| < 1$ as θ varies in $\mathbb{R}$. The leaves of the foliations tend to the horospherical parabolic set $|\kappa_i| = 1$ as θ tends to $\pm\infty$.

(2) *The θ^t-parabolic set is given by*

$$\tanh^2(\theta) K_e - 2H_e \tanh(\theta) + 1 = 0.$$

It consists of the curves (which could be empty) $\kappa_i = \tanh(\theta)^{-1}$, $i = 1, 2$. Each family of these curves foliate the region of M where $|\kappa_i| > 1$ as θ varies in $\mathbb{R} \setminus \{0\}$. The leaves of the foliations tend to the horospherical parabolic set as θ tends to $\pm\infty$.

Proof The θ^w-Gauss-Kronecker curvature is given by

$$K_\theta^w = \det(-(dN_\theta^w)_p) = \frac{l_\theta^w n_\theta^w - (m_\theta^w)^2}{EG - F^2} = \kappa_{\theta 1}^w \kappa_{\theta 2}^w.$$

The equations for the θ^w-parabolic sets follow from the fact that $\kappa_{\theta i}^w = -\tanh(\theta)^\epsilon + \kappa_i$ with $\epsilon = 1$ if $w = s$ and $\epsilon = -1$ if $w = t$ and observing that $K_e = \kappa_1\kappa_2$ and $2H_e = \kappa_1 + \kappa_2$. If we take, for example $w = s$, it follows that the θ^s-parabolic sets consists of the curves $\kappa_i = \tanh(\theta)$, $i = 1, 2$. As $|\tanh(\theta)| < 1$, these curves foliate the regions where $|\kappa_i| < 1$ as θ varies in $\mathbb{R}$. The case $w = t$ follows similarly.

□

Remark 3.3 It follows from Theorem 3.2 that the θ^s-parabolic sets do not cover the whole surface M. This is one of the reasons why we need to consider the family N_θ^t of timelike Gauss indicatrices.

A direction $\boldsymbol{u} \in T_pM$ is said to be θ^w*-asymptotic*, $w = s, t$, if $\langle (dN_\theta^w)_p(\boldsymbol{u}), \boldsymbol{u}\rangle = 0$. The integral curves of the θ^w-asymptotic directions are called the θ^w*-asymptotic curves.* It is not hard to show that the θ^w-asymptotic curves are the solutions of the binary differential equation (BDE)

$$(A_\theta^w):\ n_\theta^w dv^2 + 2m_\theta^w dudv + l_\theta^w du^2 = 0. \tag{2}$$

Equation (2) determines two θ^w-asymptotic directions in the region where $\delta_\theta^w = l_\theta^w n_\theta^w - (m_\theta^w)^2 > 0$, none where $\delta_\theta^w < 0$, and a unique (double) θ^w-asymptotic direction on the θ^w-parabolic set $\delta_\theta^w = 0$. See Appendix (§5) for topological models of the solutions of a BDE.

We show below that the singularities of the foliations $\kappa_i = constant$, $i = 1, 2$, are picked up by the families of height functions and by the BDE (2). This will allow us to determine their configurations at their singular points. We start with the families of height functions. The contact group is denoted by $\mathcal{K}$, the $\mathcal{K}$-singularities A_k are modelled by $u^2 \pm v^{k+1}$ and the $\mathcal{K}$-singularities D_k by $u^2v \pm v^{k-1}$.

Theorem 3.4 *Away from a discrete set of values of $\theta \in \mathbb{R}$, the height function $H_\theta^w(-, \boldsymbol{v})$, $w = s, t$, along $\boldsymbol{v} = N_\theta^w(p)$, has generically $\mathcal{K}$-singularities of type A_1, A_2 and A_3 at p. These are characterised geometrically as follows:*

A_1 : *p is not a θ^w-parabolic point.*

A_2 : *p is a θ^w-parabolic point and the unique θ^w-asymptotic direction at p is transverse to the θ^w-parabolic set.*

A_3 : *p is a θ^w-parabolic point and the unique θ^w-asymptotic direction at p is tangent to the θ^w-parabolic set.*

Proof The height function $H_\theta^w(-, \boldsymbol{v})$ is singular at (u_0, v_0) if $\boldsymbol{v} = N_\theta^w(u_0, v_0)$. (In fact it is singular at (u_0, v_0) if and only if $\boldsymbol{v}$ is a normal vector to M at $\boldsymbol{x}(u_0, v_0)$.) We suppose that (u_0, v_0) is a singularity of $H_\theta^w(-, \boldsymbol{v})$ with $\boldsymbol{v} = N_\theta^w(u_0, v_0)$, and write H_θ^w for $H_\theta^w(-, \boldsymbol{v})$.

At (u_0, v_0), $(H_\theta^w)_{uu} = l_\theta^w = -\tanh(\theta)^\epsilon E + l$, $(H_\theta^w)_{uv} = m_\theta^w = -\tanh(\theta)^\epsilon F + m$, and $(H_\theta^w)_{vv} = n_\theta^w = -\tanh(\theta)^\epsilon G + n$. Thus the Hessian of H_θ^w at (u_0, v_0) is singular if and only if $\boldsymbol{x}(u_0, v_0)$ is a θ^w-parabolic point. The singularity is of type A_2 if the cubic part of the Taylor expansion of H_θ^w at (u_0, v_0) does not divide Q, where Q^2 is its quadratic part. To make the conditions more apparent, we choose a special local parametrisation of M where the coordinate curves are the de Sitter lines of principal curvature. (We can do this away from the de Sitter umbilic points and we can assume this to be the case at $\boldsymbol{x}(u_0, v_0)$.) Then $F \equiv 0$, $m \equiv 0$ and (u_0, v_0) is a singularity of H_θ^w if and only if $(H_\theta^w)_{uu}(u_0, v_0) = 0$ or $(H_\theta^w)_{vv}(u_0, v_0) = 0$. If both are zero we get a D_4-singularity and this is dealt with in Theorem 3.5. Suppose that $(H_\theta^w)_{uu}(u_0, v_0) = 0$ and $(H_\theta^w)_{vv}(u_0, v_0) \neq 0$. Then the singularity is of type A_2 if and only if $(H_\theta^w)_{uuu}(u_0, v_0) \neq 0$. We have $H_{uu} = \langle \boldsymbol{x}_{uu}, \boldsymbol{v}\rangle$, so at (u_0, v_0)

$$\begin{aligned} H_{uuu} &= \langle \boldsymbol{x}_{uuu}, \boldsymbol{v}\rangle \\ &= \langle \boldsymbol{x}_{uuu}, \tanh(\theta)^\epsilon \boldsymbol{x} + \boldsymbol{e}\rangle \\ &= \tanh(\theta)^\epsilon \langle \boldsymbol{x}_{uuu}, \boldsymbol{x}\rangle + \langle \boldsymbol{x}_{uuu}, \boldsymbol{e}\rangle. \end{aligned}$$

By differentiating twice the identity $\langle \boldsymbol{x}, \boldsymbol{x}_u\rangle = 0$ we get

$$\langle \boldsymbol{x}_{uuu}, \boldsymbol{x}\rangle = -3\langle \boldsymbol{x}_u, \boldsymbol{x}_{uu}\rangle = -\frac{3}{2}E_u.$$

We have $\langle \boldsymbol{x}_{uu}, \boldsymbol{e}\rangle = l$, so $\langle \boldsymbol{x}_{uuu}, \boldsymbol{e}\rangle + \langle \boldsymbol{x}_{uu}, \boldsymbol{e}_u\rangle = l_u$. However,

$$\langle \boldsymbol{x}_{uu}, \boldsymbol{e}_u\rangle = \langle \boldsymbol{x}_{uu}, -\kappa_1 \boldsymbol{x}_u\rangle = -\frac{1}{2}\kappa_1 E_u.$$

Thus

$$\langle \boldsymbol{x}_{uuu}, \boldsymbol{e} \rangle = l_u + \frac{1}{2}\kappa_1 E_u.$$

We have $\kappa_1 = \tanh(\theta)^\epsilon$ at (u_0, v_0), so at this point

$$\begin{array}{rcl} H_{uuu} & = & \tanh(\theta)^\epsilon \langle \boldsymbol{x}_{uuu}, \boldsymbol{x} \rangle + \langle \boldsymbol{x}_{uuu}, \boldsymbol{e} \rangle \\ & = & -\frac{3}{2}\tanh(\theta)^\epsilon E_u + l_u + \frac{1}{2}\tanh(\theta)^\epsilon E_u \\ & = & -\tanh(\theta)^\epsilon E_u + l_u. \end{array}$$

Now the discriminant of the asymptotic curves (the θ^w-parabolic set) is given by $l_\theta^w = -\tanh^\epsilon(\theta)E + l = 0$ and the unique asymptotic direction at (u_0, v_0) is along $(1, 0)$. The direction $(1, 0)$ is transverse the the θ^w-parabolic set at (u_0, v_0) if and only if $(-\tanh^\epsilon(\theta)E_u + l_u)(u_0, v_0) \neq 0$, that is, if and only if $(H_\theta^w)_{uuu}(u_0, v_0) \neq 0$. When $(H_\theta^w)_{uu} = (H_\theta^w)_{uuu} = 0$ at (u_0, v_0), we get and A_3-singularity for generic θ.

For θ fixed, the family H_θ^w is a 3-parameter family. Therefore, for a generic embedding of M in $H_+^3(-1)$, only singularities of $\mathcal{K}$-codimension ≤ 3 can occur. (See for example [14]. We are interested in the discriminant of the family H_θ^w, this is why we consider the $\mathcal{K}$-codimension and not the $\mathcal{K}_e$-codimension.) These are the A_1, A_2 and A_3-singularities. If we let θ vary, we get generically singularities of $\mathcal{K}$-codimension 4 at isolated points, which can occur for a discrete set of values of θ. $\Box$

Denote by $\mathcal{S} = \{(\theta, \boldsymbol{v}) \in \mathbb{R} \times S^3(\cosh(\theta)^{-2})\}$ and $\mathcal{T} = \{(\theta, \boldsymbol{v}) \in \mathbb{R} \setminus \{0\} \times H^3(-\sinh(\theta)^{-2})\}$. We consider the "big" families of height functions given by

$$\begin{array}{cccc} H^s & U \times \mathcal{S} & \to & \mathbb{R} \\ & ((u, v), (\theta, \boldsymbol{v})) & \mapsto & \langle \boldsymbol{x}(u, v), \boldsymbol{v} \rangle + \tanh(\theta) \end{array}$$

and

$$\begin{array}{cccc} H^t & U \times \mathcal{T} & \to & \mathbb{R} \\ & ((u, v), (\theta, \boldsymbol{v})) & \mapsto & \langle \boldsymbol{x}(u, v), \boldsymbol{v} \rangle + \tanh(\theta)^{-1} \end{array}$$

For a generic embedding of the surface the big family H^w, $w = s, t$, along $N_\theta^w(p)$ can have the following local catastrophic events at p:

(i) an A_3-singularity which is not $\mathcal{K}$-versally unfolded by the family H_θ^w.

(ii) an A_4-singularity of H_θ^w.

(iii) a D_4-singularity of H_θ^w; this occurs at an umbilic point with $\kappa_1 = \kappa_2 = \tanh(\theta)^\epsilon$.

Theorem 3.5 (1) *The family H_θ^w, $w = s, t$, for θ fixed, is always a $\mathcal{K}$-versal unfolding of the A_1 and A_2 singularities of the height function at p along $\boldsymbol{v} = N_\theta^w(p)$. It fails to be a $\mathcal{K}$-versal unfolding of an A_3-singularity if and only if the θ^w-parabolic set is singular.*

(2) *The big family H^w is always a $\mathcal{K}$-versal unfolding of a non-versal A_3-singularity of H_θ^w along $\boldsymbol{v} = N_\theta^w(p)$.*

(3) *For a generic surface, the big family H^w is a $\mathcal{K}$-versal unfolding of an A_4-singularity of H_θ^w at p along $\boldsymbol{v} = N_\theta^w(p)$.*

(4) *For a generic surface, the big family H^w is a $\mathcal{K}$-versal unfolding of a D_4-singularity of H_θ^w at p along $\boldsymbol{v} = N_\theta^w(p)$.*

Proof The proof is similar to those given in [3] for families of height functions on surfaces in $\mathbb{R}^3$. We deal here with the D_4-singularity case and with $w = s$. This occurs when $\kappa_1 = \kappa_1 = \tanh(\theta_0)$, say at $(u_0, v_0) = (0, 0)$. Every direction in $T_{\boldsymbol{x}(0,0)}M$ is a de Sitter principal direction, so we cannot take a parametrisation with $F \equiv 0, m \equiv 0$. We take without loss of generality, $j^1\boldsymbol{x}(u, v) = (1, u, v, 0)$, $\boldsymbol{e}(0, 0) = (0, 0, 0, 1)$ and $\boldsymbol{v}_0 = (\tanh(\theta_0), 0, 0, 1)$. We write $\boldsymbol{x} = (x_0, x_1, x_2, x_3)$. For

$\boldsymbol{v} = (v_0, v_1, v_2, v_3) \in S^3(\cosh(\theta)^{-2})$ near $\boldsymbol{v}_0$, we can write $v_3 = \sqrt{\cosh^{-2}(\theta) + v_0^2 - v_1^2 - v_2^2}$. Then the family H^s is a $\mathcal{K}$-versal deformation of the D_4-singularity of $H^s_{\theta_0}$ at $(0,0)$ if and only if

$$\mathcal{E}_2 \left\langle \frac{\partial H^s_{\theta_0}}{\partial u}, \frac{\partial H^s_{\theta_0}}{\partial v}, H^s_{\theta_0} \right\rangle + \mathbb{R}.\left\{ \frac{\partial H^s}{\partial v_0}, \frac{\partial H^s}{\partial v_1}, \frac{\partial H^s}{\partial v_2}, \frac{\partial H^s}{\partial \theta} \right\} = \mathcal{E}_2 \tag{3}$$

where $H^s_{\theta_0}$, $\partial H^s/\partial v_i$, $i = 1, 2, 3$, $\partial H^s/\partial \theta$ are evaluated at $(u, v, \theta_0, \boldsymbol{v}_0)$, and $\mathcal{E}_2$ denotes the ring of germs of smooth functions at $(0,0)$.

The 2-jet of $H^s_{\theta_0}$ is identically zero and its 3-jet is a non-degenerate cubic (the singularity is of type D_4). Therefore, it is 3-$\mathcal{K}$-determined. We can then work in the 3-jet space and show that all degree 3 monomials in u and v are in the left hand side of (3). For degree ≤ 2 we proceed as follows. We have

$$\boldsymbol{x}(u, v) = (1, u, v, 0) + \frac{1}{2}(\boldsymbol{x}_{uu}(0,0)u^2 + 2\boldsymbol{x}_{uv}(0,0)uv + 2\boldsymbol{x}_{vv}(0,0)v^2).$$

One can show that

$$\begin{array}{rcl}
\boldsymbol{x}_{uu}(0,0) & = & (-E, \frac{1}{2}E_u, F_v - \frac{1}{2}E_v, \tanh(\theta_0)E), \\
\boldsymbol{x}_{uv}(0,0) & = & (-F, \frac{1}{2}E_v, \frac{1}{2}G_u, \tanh(\theta_0)F), \\
\boldsymbol{x}_{vv}(0,0) & = & (-G, F_v - \frac{1}{2}G_u, \frac{1}{2}G_v, \tanh(\theta_0)G).
\end{array}$$

We have $\partial H^s/\partial\theta((u, v), (\theta_0, \boldsymbol{v}_0)) = \cosh(\theta_0)^{-2}$, so the constant terms are in the left hand side of (3) and we can work modulo these terms. We have

$$\begin{array}{rcl}
j^2 \frac{\partial H^s}{\partial v_0}((u, v), (\theta_0, \boldsymbol{v}_0)) - 1 & = & j^2(-x_0(u, v) + \tanh(\theta)x_3(u, v)) - 1 \\
& = & \frac{1}{2}(1 + \tanh^2(\theta_0))(Eu^2 + 2Fuv + Gv^2).
\end{array}$$

Also, by similar calculations to those in the proof of Theorem 3.4,

$$\begin{array}{rl}
j^2(H^s_{\theta_0})_u(u, v) & = \frac{1}{2}((H^s_{\theta_0})_{uuu}u^2 + 2(H^s_{\theta_0})_{uuv}uv + (H^s_{\theta_0})_{uvv}v^2) \\
& = \frac{1}{2}((-\tanh(\theta_0)E_u + l_u)u^2 + 2(-\tanh(\theta_0)F_u + m_u)uv + (-\tanh(\theta_0)G_u + n_u)v^2), \\
j^2(H^s_{\theta_0})_v(u, v) & = \frac{1}{2}((H^s_{\theta_0})_{uuv}u^2 + 2(H^s_{\theta_0})_{uvv}uv + (H^s_{\theta_0})_{vvv}v^2) \\
& = \frac{1}{2}((-\tanh(\theta_0)E_v + l_v)u^2 + 2(-\tanh(\theta_0)F_v + m_v)uv + (-\tanh(\theta_0)G_v + n_v)v^2).
\end{array}$$

We put a multiple of the above three vectors in the following matrix form

$$\begin{array}{cccc}
 & u^2 & uv & v^2 \\
\frac{2}{1+\tanh^2(\theta_0)} j^2 \frac{\partial H^s}{\partial v_0} & E & 2F & G \\
2j^2(H^s_{\theta_0})_u & -\tanh(\theta_0)E_u + l_u & 2(-\tanh(\theta_0)F_u + m_u) & -\tanh(\theta_0)G_u + n_u \\
2j^2(H^s_{\theta_0})_v & -\tanh(\theta_0)E_v + l_v & 2(-\tanh(\theta_0)F_v + m_v) & -\tanh(\theta_0)G_v + n_v
\end{array} \tag{4}$$

The determinant of the above matrix is not zero at a generic umbilic point. Therefore, $u^2, uv, , v^2$ are in the left hand side of (3). We can work now on the 1-jet level and obtain u, v using

$$j^1 \frac{\partial H^s}{\partial v_1}((u, v), (\theta_0, \boldsymbol{v}_0)) = j^1(x_1(u, v)) = u \text{ and } j^1 \frac{\partial H^s}{\partial v_2}((u, v), (\theta_0, \boldsymbol{v}_0)) = j^1 x_2(u, v) = v.$$

□

Remark 3.6 It follows from Theorem 3.5 that the de Sitter parabolic set can have singularities if it is considered as a member of the θ-parabolic sets. This means that there is nothing special about the de Sitter Gauss map $\mathbb{E}$ when considered as a member of the family N^s_θ.

Theorem 3.7 *The curves $\kappa_i =$ constant, $i = 1, 2$, undergo Morse transitions at a non-versal A_3-singularity of the height function (Figure 1, first two figures) and remain smooth at an A_4-singularity. At a D_4^+ (resp. D_4^-)-singularity (i.e., at an umbilic point) the generic configuration is as in the third (resp. fourth) figure in Figure 1.*

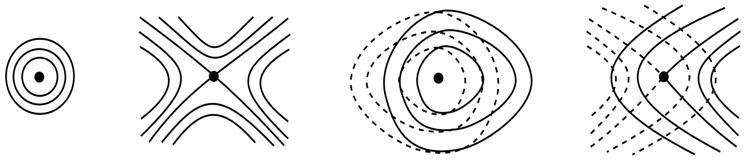

Figure 1: The foliation $\kappa_i = constant$ ($i = 1$ or 2) at a non-versal A_3-singularity (first two figures). The third (resp. fourth) figure is the generic configuration of the foliations $\kappa_i = constant, i = 1, 2$ at a D_4^+ (resp. D_4^-)-singularity, continuous lines for κ_i and dashed for $\kappa_j, j \neq i$.

Proof The first two statements are a consequence of Theorem 3.5. At an umbilic point (u_0, v_0) with $\tanh(\theta_0)^\epsilon = \kappa_1 = \kappa_2$, the foliations $\kappa_i = constant$ are given by $\tanh(\alpha)^{2\epsilon} - 2H_e \tanh(\alpha)^\epsilon + K_e = 0$ and $\tanh(\alpha)^\epsilon = constant$. The first equation determines a surface S in the (θ, u, v)-space which has a cone singularity at $q_0 = (\theta_0, u_0, v_0)$. The projection $\pi : S \to U$ maps diffeomorphically each connected component of $S \setminus \{q_0\}$ to $U \setminus \{(u_0, v_0)\}$. The foliations $\kappa_i = constant$ are the images by π of the traces of the planes $\theta = constant$ on S. The traces of these planes on one component on $S \setminus \{q_0\}$ project to $\kappa_1 = constant$ and those on the other component project to $\kappa_2 = constant$. The plane $\theta = \theta_0$ is generically not tangent to the cone, so we have two possible configurations for its trace on the cone: it is either an isolated point (this is the case when the height function $H^w_{\theta_0}$ along $N^w_{\theta_0}(p)$ has a D_4^+-singularity) or it is a pair of crossing curves (this is the case when the height function $H^w_{\theta_0}$ along $N^w_{\theta_0}(p)$ has a D_4^--singularity). As θ varies near θ_0 we obtain generic cone sections. If the cone sections are closed curves (resp. hyperbole), the configuration of their projections to the (u, v)-plane is as in Figure 2, third (resp. fourth) figure. □

We turn now the θ^w-asymptotic curves and their singularities (see Appendix for notation).

Theorem 3.8 *For a generic surface M in $H^3_+(-1)$, the BDE (A^w_θ) of the θ^w-asymptotic curves can have singularities of codimension ≤ 1.*

(1) *The BDE (A^w_θ) has a folded singularity (or worse) at p if and only if H^w_θ along $N^w_\theta(p)$ has an A_3-singularity (or worse) at p. The three types of the folded singularities of BDEs can occur in (A^w_θ) (Figure 3).*

(2) *The BDE (A^w_θ) has a folded saddle-node singularity at p for some $\theta = \theta_0$ if and only if $H^w_{\theta_0}$ has an A_4-singularity at p. The family (A^w_θ), as θ varies near θ_0, is generic if and only if the big family H^w is a versal unfolding of the A_4-singularity of $H^w_{\theta_0}$ (Figure 4, left).*

(3) *The BDE (A^w_θ) can have a node-focus change at p for some $\theta = \theta_0$. This is not detected by the family H^w_θ. The family (A^w_θ), as θ varies near θ_0, is generic for generic surfaces in $H^3_+(-1)$ (Figure 4, right).*

(4) *The BDE (A^w_θ) has a Morse Type 1 singularity at p for some $\theta = \theta_0$ if and only if $H^w_{\theta_0}$ has a non-versal A_3-singularity at p. The family (A^w_θ), as θ varies near θ_0, is always a generic family (Figure 5).*

(5) *At an umbilic point the BDE $A^w_{\theta_0}$ has a Morse Type 2 singularity with discriminant of type A_1^+ (Figure 6) or A_1^- (Figure 7). The family (A^w_θ) as θ varies near θ_0 is a generic family if and only if the family H^w is a versal unfolding of the D_4-singularity of $H^w_{\theta_0}$.*

Proof The proofs here are also similar to those for surfaces in $\mathbb{R}^3$ ([2, 6]). For the case (5), the condition for the family (A_θ^s) to be a generic family at an umbilic point is

$$\begin{vmatrix} a_\theta & b_\theta & c_\theta \\ a_u & b_u & c_u \\ a_v & b_v & c_v \end{vmatrix} \neq 0,$$

where $a, 2b, c$ are the coefficients of (A_θ^s) (see [6]). The above determinant is, up to a scalar multiple, the determinant of the matrix (4) in the proof of Theorem 3.5. $\square$

3.2 Surfaces in $H_+^3(-1)$ viewed as surfaces in $\mathbb{R}_1^4$

In §3.1 we defined a θ^w-asymptotic direction $\boldsymbol{u} \in T_pM$ by $\langle (dN_\theta^w)_p(\boldsymbol{u}), \boldsymbol{u}\rangle = 0$. This notion depends on the shape operator $-dN_\theta^w$. For surfaces in $\mathbb{R}^4$, there is another notion of asymptotic directions which is defined in terms of the contact of the surface with lines and hyperplanes ([4, 13]; see also [12] for their definition in terms of the curvature ellipse). For this reason, these asymptotic directions and their integral curves (the asymptotic curves) are affine properties of the surface, i.e., they do not depend on the metric on $\mathbb{R}^4$ and can be defined in the same way on a surface in $\mathbb{R}_1^4$.

Let $\mathbf{r} : U \subset \mathbb{R}^2 \to M \subset \mathbb{R}_1^4$ be a local parametrisation of a spacelike or timelike surface M. We have a well defined second fundamental form on M using the Levi-Civita connection on $\mathbb{R}_1^4$ (see for example [16]). Let $\{e_3, e_4\}$ be a frame in the normal plane N_pM. Then the coefficient of this second fundamental form are given by

$$a_i = \langle e_i, \mathbf{r}_{uu}\rangle, \quad b_i = \langle e_i, \mathbf{r}_{uv}\rangle, \quad c_i = \langle e_i, \mathbf{r}_{vv}\rangle, \quad i = 3, 4.$$

Given any normal vector field μ, with coordinates (α, β) in the normal space N_pM, the shape operator $S_\mu : T_pM \to T_pM$ along μ is represented, with respect to the basis $\{\mathbf{r}_u, \mathbf{r}_v\}$, by the matrix

$$S_\mu = \frac{1}{EG - F^2} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix} \begin{pmatrix} \alpha a_3 + \beta a_4 & \alpha b_3 + \beta b_4 \\ \alpha b_3 + \beta b_4 & \alpha c_3 + \beta c_4 \end{pmatrix}.$$

We denote by

$$[S_\mu] = \begin{pmatrix} \alpha a_3 + \beta a_4 & \alpha b_3 + \beta b_4 \\ \alpha b_3 + \beta b_4 & \alpha c_3 + \beta c_4 \end{pmatrix}$$

the symmetric matrix associated to S_μ (it completely determines S_μ). We call the eigenvectors of S_μ (when they exist) the μ-principal directions and call their integral curves the μ-principal curves. These are given by the binary differential equation

$$\begin{vmatrix} dv^2 & -dudv & du^2 \\ E & F & G \\ \alpha a_3 + \beta a_4 & \alpha b_3 + \beta b_4 & \alpha c_3 + \beta c_4 \end{vmatrix} = 0. \tag{5}$$

Following [4], we say that a direction $\boldsymbol{u} \in T_pM$ is asymptotic if the projection of M along $\boldsymbol{u}$ to a transverse hyperplane has an $\mathcal{A}$-singularity more degenerate than a cross-cap at p. It is not difficult to show that the asymptotic curves on $M \subset \mathbb{R}_1^4$ are given by a BDE which has the same form as that of a surface in $\mathbb{R}^4$, namely

$$(A) : \, (b_3c_4 - b_4c_3)dv^2 + (a_3c_4 - a_4c_3)dudv + (a_3b_4 - a_4b_3)du^2 = 0 \tag{6}$$

where a_i, b_i, c_i, $i = 3, 4$, are the coefficients of the second fundamental form at (u, v). This equation can also be written in a determinant form

$$\begin{vmatrix} dv^2 & -dudv & du^2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix} = 0. \tag{7}$$

We follow the notation for surfaces in $\mathbb{R}^4$ and label the discriminant of equation (6) by Δ. Points where Δ is singular (generically a Morse singularity $A_1^{\pm}$) are labelled *inflection points*. The generic configurations of the asymptotic curves at inflection points are the same as those for surfaces in $\mathbb{R}^4$, top figures in Figure 6 and Figure 7 (see [5, 7]).

Theorem 3.9 *Let M be a spacelike or timelike surface contained in a pseudo-sphere in $\mathbb{R}^4_1$. Then the μ-principal curves coincide for all normal vector fields μ on M and are precisely the asymptotic curves of M when viewed as a surface in $\mathbb{R}^4_1$.*

Proof Let $\boldsymbol{x} : U \to M$ be a local parametrisation of M. Because the metric on M is not degenerate, $\{\boldsymbol{x}(u,v), \boldsymbol{e}(u,v)\}$ is a basis of the normal plane N_pM at all points $p = \boldsymbol{x}(u,v)$. The coefficients of the second fundamental form (with respect to $\{\boldsymbol{x}, \boldsymbol{e}\}$) are given by

$$\begin{array}{ll} a_3 = \langle \boldsymbol{x}, \boldsymbol{x}_{uu}\rangle = -\langle \boldsymbol{x}_u, \boldsymbol{x}_u\rangle = -E, & a_4 = \langle \boldsymbol{e}, \boldsymbol{x}_{uu}\rangle = l, \\ b_3 = \langle \boldsymbol{x}, \boldsymbol{x}_{uv}\rangle = -\langle \boldsymbol{x}_u, \boldsymbol{x}_v\rangle = -F, & b_4 = \langle \boldsymbol{e}, \boldsymbol{x}_{uv}\rangle = m, \\ c_3 = \langle \boldsymbol{x}, \boldsymbol{x}_{vv}\rangle = -\langle \boldsymbol{x}_v, \boldsymbol{x}_v\rangle = -G, & c_4 = \langle \boldsymbol{e}, \boldsymbol{x}_{vv}\rangle = n. \end{array}$$

Let $\mu = \alpha\boldsymbol{x} + \beta\boldsymbol{e}$ be a normal vector field to M (we assume that $\beta \neq 0$). Then the equation of the μ-lines of principal curvature is given by

$$\begin{vmatrix} dv^2 & -dudv & du^2 \\ E & F & G \\ -\alpha E + \beta l & -\alpha F + \beta m & -\alpha G + \beta n \end{vmatrix} = 0 = \begin{vmatrix} dv^2 & -dudv & du^2 \\ E & F & G \\ l & m & n \end{vmatrix}.$$

The last determinant above is equation (7) of the asymptotic curves of M when viewed as a surface in $\mathbb{R}^4_1$. $\square$

Remark 3.10 The proof of Theorem 3.9 is an alternative to that in [18] for surfaces in the Euclidean 4-space and for spacelike surfaces in the Minkowski 4-space [8].

We shall not distinguish between a general BDE (9) (see Appendix) and its non-zero multiples, so at each point $(u,v) \in U$ we can view the BDE as a quadratic form $a\beta^2 + 2b\beta\gamma + c\gamma^2 = 0$ ($\beta = dv$ and $\gamma = du$) and represent it by the point $Q = (a : 2b : c)$ in the projective plane $\mathbb{R}P^2$. In $\mathbb{R}P^2$ there is a conic $\Gamma = \{Q : b^2 - ac = 0\}$ of singular quadratic forms. These can be put in the form $(a_1\beta + b_1\gamma)^2$.

The *polar line* $\widehat{Q}$ of a point Q (with respect to the conic Γ) is the line that contains all points O such that Q and O are harmonic conjugate points with respect to the intersection points R_1 and R_2 of the conic Γ and a variable line through Q. Geometrically, if the polar line $\widehat{Q}$ meets Γ, then the tangents to Γ at the points of intersection meet at Q.

The symmetric matrix $[S_\mu]$ associated to the shape operator S_μ can be represented by a point $S_\mu = (\alpha a_3 + \beta a_4 : \alpha b_3 + \beta b_4 : \alpha c_3 + \beta c_4) \in \mathbb{R}P^2$. Then these points trace at each point $p \in M$ a pencil in $\mathbb{R}P^2$ (by varying α, β). This pencil is precisely the polar line $\hat{A}$ of the asymptotic BDE (6), [15, 21]. We also represent the metric $Gdv^2 + 2Fdudv + Edu^2$ by the point $L = (G : F : E)$.

Corollary 3.11 *Let M be a surface in $H^3_+(-1)$. The families of shape operators $-dN^w_\theta$, $\theta \in \mathbb{R}$, trace the polar line of the de Sitter lines of principal curvature with the points $\mathbb{L}^{\pm}$ and L removed. The family $-dN^s_\theta$ (resp. $-dN^t_\theta$) trace the part of the polar line corresponding to spacelike (resp. timelike) shape operators. The hyperbolic shape operators $\mathbb{L}^+$ and $\mathbb{L}^-$ form an obstruction for joining spacelike and timelike shape operators.*

4 Timelike hypersurfaces in S^n_1

Let M be a hypersurface in the de Sitter space S^n_1. If M is spacelike, then its normal plane in $\mathbb{R}^{n+1}_1$ is timelike and we have similar results to those in §3 for a hypersurface in the hyperbolic space. We deal here with the case when M is timelike. Then the normal plane N_pM in $\mathbb{R}^{n+1}_1$ is spacelike for all $p \in M$. The vectors $\boldsymbol{x}(\underline{u})$ and $\boldsymbol{e}(\underline{u})$ form an orthonormal basis of N_pM. Therefore, we can parametrise the unit normal vectors in N_pM by $\sin(\alpha)\boldsymbol{x}(\underline{u}) + \cos(\alpha)\boldsymbol{e}(\underline{u})$. However, the derivative of the Gauss indicatrix $\boldsymbol{x}(\underline{u})$ is the identity map on $T_{\boldsymbol{x}(\underline{u})}M$, so all points on M are umbilic points with respect to this Gauss indicatrix. This is why we define the family of (spacelike) Gauss indicatrices by

$$\begin{array}{rccc} N_\alpha : & U & \to & S^n_1(\cos(\alpha)^{-2}) \\ & \underline{u} & \mapsto & \tan(\alpha)\boldsymbol{x}(\underline{u}) + \boldsymbol{e}(\underline{u}) \end{array}$$

where $\alpha \in (-\pi/2, \pi/2)$ is the angle between $N_\alpha(\underline{u})$ and $\boldsymbol{e}(\underline{u})$. The family N_α does not contain the normal vector $\boldsymbol{x}$. We associate the same notions to $-(dN_\alpha)_p$ as those associated to $-(dN^w_\theta)_p$ in §3. We have, for instance, the α-principal curvatures given by $\kappa_{\alpha i} = -\tan(\alpha) + \kappa_i$. The α-principal directions do not depend on α.

We define the family of height functions

$$\begin{array}{rccc} H_\alpha & U \times S^n_1(\cos(\alpha)^{-2}) & \to & \mathbb{R} \\ & (\underline{u}, \boldsymbol{v}) & \mapsto & \langle \boldsymbol{x}(\underline{u}), \boldsymbol{v} \rangle - \tan(\alpha) \end{array}$$

We have similar results to those in §3 concerning the families N_α and H_α. In this section we deal mainly with timelike surfaces in S^3_1 and give only the results that are distinct from those in §3.1.

4.1 Surfaces in S^3_1

Let $\boldsymbol{x} : U \subset \mathbb{R}^2 \to M \subset S^3_1$ be a local parametrisation of M and let

$$\begin{array}{rclcl} l_\alpha & = & \langle -(dN_\alpha)_p(\boldsymbol{x}_u), \boldsymbol{x}_u \rangle & = & \langle N_\alpha, \boldsymbol{x}_{uu} \rangle, \\ m_\alpha & = & \langle -(dN_\alpha)_p(\boldsymbol{x}_u), \boldsymbol{x}_v \rangle & = & \langle N_\alpha, \boldsymbol{x}_{uv} \rangle, \\ n_\alpha & = & \langle -(dN_\alpha)_p(\boldsymbol{x}_v), \boldsymbol{x}_v \rangle & = & \langle N_\alpha, \boldsymbol{x}_{uv} \rangle, \end{array}$$

denote the coefficients of the α-second fundamental form at $p = \boldsymbol{x}(u, v)$ associated to the shape operator $-(dN_\alpha)_p$. We have

$$l_\alpha = -\tan(\alpha)E + l, \ m_\alpha = -\tan(\alpha)F + m, \ n_\alpha = -\tan(\alpha)G + n,$$

where l, m, n denote the coefficients of second fundamental form associated to the de Sitter shape operator $-d\mathbb{E}$. We denote, as in §3.1, by K_e and H_e the de Sitter Gauss-Kronecker curvature and the de Sitter mean curvature, respectively.

The (de Sitter) lines of principal curvature are given by the same equation as for the case of a surface in $H^3_+(-1)$ (i.e., equation (1)). The difference here is that the induced metric on the surface M is Lorentzian, so $-dN_\alpha$ does not always have two real eigenvalues. For a generic surface, the discriminant of the lines of principal curvature is a smooth curve except possibly at isolated points where it has Morse singularities of type A^-_1 (node) ([10]). This discriminant is denoted by the LPL in [10] (Lightlike Principal Locus) and consists of points where the two principal directions coincide and become lightlike. The singular points of the LPL are labelled *timelike umbilic points*. In view of Theorem 3.9, the LPL is precisely the Δ-set of M as a surface in $\mathbb{R}^4_1$.

Theorem 4.1 *The α-parabolic set, $\alpha \in (-\pi/2, \pi/2)$, is given by*

$$\tan^2(\alpha) - 2H_e \tan(\alpha) + K_e = 0.$$

It consists of the curves $\kappa_i = \tan(\alpha)$, $i = 1, 2$. Each of these curves foliate, as α varies in $(-\pi/2, \pi/2)$, the region of M where there are two principal directions.

Proof The proof is similar to that of Theorem 3.2. Here the de Sitter principal curvatures κ_1 and κ_2 may be complex conjugate but $K_e = \kappa_1\kappa_2$ and $H_e = (\kappa_1 + \kappa_2)/2$ are always real numbers. $\square$

The α-asymptotic curves (which we define following §3.1) are given by

$$(A_\alpha): \; n_\alpha dv^2 + 2m_\alpha du dv + l_\alpha du^2 = 0. \tag{8}$$

The α-parabolic set is the discriminant of equation (8). Away from the LPL, the α-parabolic sets behave as the θ-parabolic sets in §3.1 (we have similar results to those in Theorems 3.4, 3.5, 3.8). We shall consider their behaviour at points on the LPL. We observe that the generic configurations of the lines of principal curvature at points on the LPL are obtained in [10].

Theorem 4.2 *Let M be a timelike surface in S_1^3 and p a point on the LPL of M.*

At most points on the LPL the height function H_α along the normal direction N_α has an A_2-singularity.

The singularity is of type A_3 if and only if p is a folded singularity of the de Sitter lines of curvature (and hence of all α-lines of curvature) and of an α-asymptotic curves.

The singularity is of type D_4 if and only if p is a timelike umbilic point (i.e., a singularity of the LPL) and $\tan(\alpha) = \kappa_1 = \kappa_2$*. At such point, the de Sitter lines of curvature has a Morse Type 2 singularity with a discriminant having a singularity of type A_1^- (Figure 7, top figures). The α-asymptotic curves have a Morse Type 2 singularity with the discriminant of type A_1^+ (Figure 6, top figures) or A_1^- (Figure 7, top figures).*

Proof We take a special parametrisation of the surface where the coordinate curves coincide with the lightlike curves, so $E \equiv 0$, $F \equiv 0$. The equation of the de Sitter lines of curvature becomes

$$ndv^2 - ldu^2 = 0,$$

and its discriminant (the LPL) is given by $ln = 0$. Suppose that p is a smooth point on the LPL, and assume that $l = 0$ and $n \neq 0$. Then the de Sitter lines of curvature have (generically) a folded singularity if and only if $l_u = 0$.

At a singular point of the LPL ($l = n = 0$) both coefficients of the de Sitter lines of curvature vanish. Thus, the de Sitter lines of curvature have generically a Morse Type 2 singularity with a discriminant ($ln = 0$) having a singularity of type A_1^-. The five generic configurations in Figure 7 (top figures) can occur.

The α-asymptotic curves are given by

$$ndv^2 + 2(-\tan(\alpha)F + m)dudv + ldu^2 = 0,$$

and the α-parabolic set (its discriminant) is given by $(-\tan(\alpha)F + m)^2 - ln = 0$. With the same setting as above, a smooth point $\boldsymbol{x}(u_0, v_0)$ on the α-parabolic set is also on the LPL if $\tan(\alpha) = (m/F)(u_0, v_0)$. Then the α-asymptotic curves, with $\tan(\alpha) = (m/F)(u_0, v_0)$, have (generically) a folded singularity if and only if $l_u = 0$.

At a singular point of the LPL, all the coefficients of the α-asymptotic curves BDE, with $\tan(\alpha) = (m/F)(u_0, v_0)$, vanish. The discriminant can have either an A_1^+ or an A_1^- singularity, so the α-asymptotic curves have generically a Morse Type 2 singularity with both discriminant types. All the generic configurations of Morse Type 2 singularities can occur (Figures 6, 7, top figures).

The height function $H_\alpha(-, \boldsymbol{v})$ is singular at (u_0, v_0) if $\boldsymbol{v} = N_\alpha(u_0, v_0)$. We write H_α for $H_\alpha(-, \boldsymbol{v})$. We have at (u_0, v_0), $(H_\alpha)_{uu} = l$, $(H_\alpha)_{uv} = -\tan(\alpha)F + m$, and $(H_\alpha)_{vv} = n$, so on the LPL (and with the setting above), $(H_\alpha)_{uu} = 0$ and the Hessian of H_α is degenerate if and only if $(H_\alpha)_{uv} = 0$, that is, $\tan(\alpha) = (m/F)(u_0, v_0)$. Calculations similar to those in the proof of Theorem 3.4 show that the singularity is of type A_2 if and only if $(H_\theta^w)_{uuu}(u_0, v_0) = l_u(u_0, v_0) \neq 0$. When $l_u = 0$, we get generically an A_3-singularity. At a timelike umbilic point $l = n = 0$, and with $\tan(\alpha) = (m/F)(u_0, v_0)$, the 2-jet of H vanishes, so the singularity is generically of type D_4. $\square$

Theorem 4.3 *Let M be a timelike surface in S^3_1 and p a point on the LPL of M.*

(1) *At most points on the LPL the foliations $\kappa_i = constant$, $i = 1, 2$ are as in Figure 2, top left. The leaves of $\kappa_1 = constant$ join those of $\kappa_2 = constant$ on the LPL and form smooth curves which have ordinary tangency with the LPL. At isolated points on the smooth part of the LPL the foliation $\kappa_i = constant$, $i = 1, 2$ are as in Figure 2, top right. These points are generically distinct from the folded singularities of the de Sitter lines of principal curvature.*

(3) *There are generically three configurations of the foliations $\kappa_i = constant$, $i = 1, 2$ at a timelike umbilic point. These are as in Figure 2, bottom figures.*

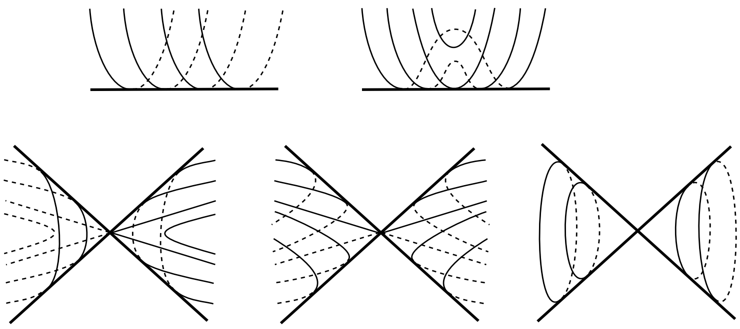

Figure 2: Generic configurations of the foliations $\kappa_i = constant, i = 1, 2$ at points on the LPL (continuous lines for κ_i and dashed for $\kappa_j, j \neq i$).

Proof The α-parabolic sets, which give the foliations $\kappa_i = constant$, are given by $\tan(\alpha)^2 - 2H_e \tan(\alpha) + K_e = 0$. In a local chart with $E \equiv 0$ and $F \equiv 0$, this becomes $(-\tan(\alpha)F + m)^2 - ln = 0$. To simplify notation, we denote by

$$\phi(u, v, \lambda) = (-\lambda F + m)^2 - ln,$$

where $\lambda = \tan(\alpha)$. The surface $\phi^{-1}(0)$ is smooth at (u, v, λ) if and only if $p = \boldsymbol{x}(u, v)$ is not a timelike umbilic point. At a timelike umbilic point with $\lambda = m/F$, $\phi^{-1}(0)$ is generically diffeomorphic to a cone. The projection $\pi : \phi^{-1}(0) \to U$ is a fold map at (u, v, λ) when $p = \boldsymbol{x}(u, v) \in LPL$ and is not a timelike umbilic point. The discriminant of π is the LPL. We call *criminant* the critical set of π.

Suppose that $p \in LPL$ is not a timelike umbilic point. The α-parabolic sets are the images by π of the intersection of $\phi^{-1}(0)$ with the planes $\lambda = constant$. These planes are transverse to $\phi^{-1}(0)$. Therefore their traces on $\phi^{-1}(0)$ is a family of smooth curves. We have two possible generic configurations of their projections to the (u, v)-plane (i.e., of the α-parabolic sets) depending on whether the criminant is transverse to the plane $\lambda = constant$ (Figure 2, top left) or tangent to it (Figure 2, top right). A condition for tangency is $\phi_{u\lambda}\phi_v - \phi_{v\lambda}\phi_u = 0$ (the tangency is ordinary in general) and is distinct from that for having a folded singularity of the de Sitter lines of curvature. The criminant splits $\phi^{-1}(0)$ locally into two components. The projections of the traces of $\lambda = constant$ in one component give the foliation $\kappa_1 = constant$ and those in the other component give the foliation $\kappa_2 = constant$.

We consider now the case when $p = \boldsymbol{x}(u_0, v_0)$ is a timelike umbilic point with $\lambda_0 = \tan(\alpha_0) = (m/F)(u_0, v_0)$. Then $\phi^{-1}(0)$ is a cone at (u_0, v_0, λ_0). The plane $\lambda = \lambda_0$ is not tangent to the cone, so we have two possible configurations for its trace on the cone: it is either an isolated point (this is the case when the α-parabolic set has a singularity of type A_1^+) or it is a pair of crossing curves (this is the case when the α-parabolic set has a singularity of type A_1^-). As λ varies near λ_0 we obtain generic cone sections. The LPL lifts to two smooth curves on $\phi^{-1}(0)$. If the cone sections are closed curves, we have one possible configuration for their projections to the (u, v)-plane (Figure 2, last bottom figure). If the cone sections are hyperbole, then we have two possible configurations depending on the position of the lift of the LPL with respect to the plane $\lambda = \lambda_0$. If both components of the LPL in a connected component of the cone with the singularity removed are on one side of the plane $\lambda = \lambda_0$, then the projections to the (u, v)-plane of the $\lambda = constant$ sections are as in Figure 2, first figure of the bottom row. Otherwise they are as in Figure 2, middle figure of the bottom row. If we take the special parametrisation $E \equiv 0, G \equiv 0$, the last two types of configurations are distinguished by the sign of

$$((m_u l_v - m_v l_u)F - (F_u l_v - F_v l_u)m)\,((m_u n_v - m_v n_u)F - (F_u n_v - F_v n_u)m)$$

at (u_0, v_0), positive for the first case and negative for the second. $\square$

5 Appendix: singularities of BDEs

We give a brief summary of results concerning the singularities of quadratic Binary Differential Equations (BDEs) and their bifurcations (see [20] for a survey article and references). A BDE is given in the form

$$a(u, v)dv^2 + 2b(u, v)dudv + c(u, v)du^2 = 0, \tag{9}$$

with $(u, v) \in U \subset \mathbb{R}^2$. It determines a pair of transverse foliations away from the discriminant curve, which is the set of points where the function $\delta = b^2 - ac$ vanishes. The pair of foliations together with the discriminant curve are called the *configuration* of the solutions of the BDE. In all the figures, we draw one foliation in continuous line and the other in dashed line. The discriminant curve is drawn in thick black.

We consider here topological equivalence among BDEs and say that two BDEs are topologically equivalent if there is a local homeomorphism in the plane taking the configuration of one equation to the configuration of the other. We suppose the point of interest to be the origin. There are two cases to consider depending on whether all the coefficients of the BDE vanish or not at the origin.

When the coefficients do not all vanish at the origin, the stable configurations are as shown in Figure 3. The last three figures are called folded saddle, folded node and folded focus in that order. Folded singularities occur when the unique direction determined by the BDE on the discriminant is tangent to the discriminant.

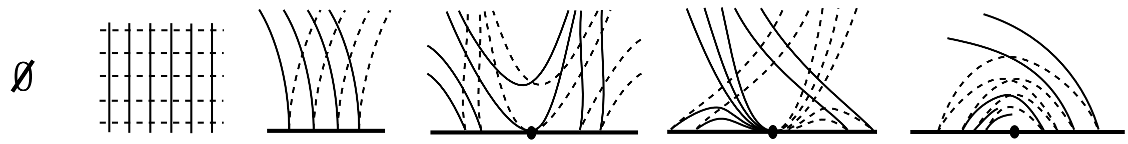

Figure 3: Stable configurations of BDEs: last three figures are the folded saddle, folded node and folded focus respectively.

Codimension 1 singularities can occur in three ways: (i) a folded saddle and a folded node coming together and disappearing (folded saddle-node singularity) on a smooth discriminant (Figure 4, left); (ii) a change from a folded saddle to a folded node on a smooth discriminant (Figure 4, right); (iii) the discriminant undergoes a Morse transition of type A_1^+ or A_1^-. For each type we have two

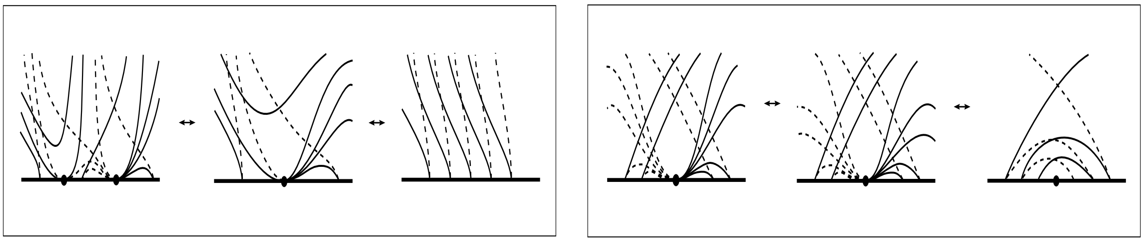

Figure 4: Folded saddle-node bifurcations (left) and a folded node-focus change (right).

cases depending on whether two folded saddles or two folded foci appear in the bifurcations. These singularities are label Morse Type 1 (Figure 5).

When the coefficients of the BDE all vanish at the origin, the singularities are automatically of codimension ≥ 1. If the discriminant has a Morse singularity, then we label the singularities of the BDE Morse Type 2 singularities. We have three generic configurations when the singularity of the discriminant is of type A_1^+ (Figure 6) and five (one case splits into two sub-cases when deformed) when it is of type A_1^- (Figure 7). In Figures 6 and 7 only one side of the transition is drawn the other side is symmetrical.

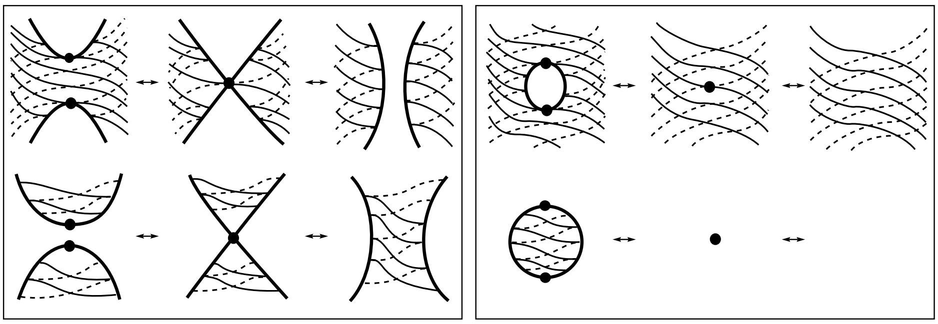

Figure 5: Bifurcations at a Morse Type 1 singularity: A_1^- left and A_1^+ right.

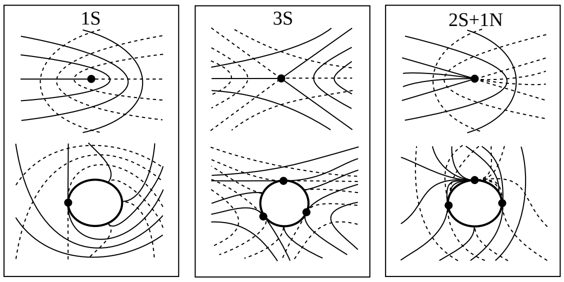

Figure 6: Bifurcations at a Morse Type 2 singularity (A_1^+).

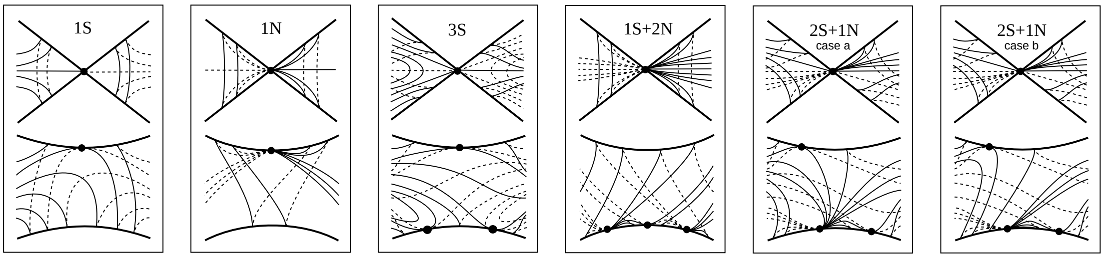

Figure 7: Bifurcations at a Morse Type 2 singularity (A_1^-).

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