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# SINGULARITIES OF PIECEWISE LINEAR SADDLE SPHERES ON $\mathbb{S}^3$

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Abstract. Segre's theorem asserts the following: *let a smooth closed simple curve $c \subset S^2$ have a non-empty intersection with any closed hemisphere. Then $c$ has at least 4 inflection points.*

In the paper, we prove two Segre-type theorems. The first one is a version of Segre's theorem for piecewise linear closed curves on $S^2$. Here we have *inflection edges* instead of inflection points.

Next, we go one dimension higher: we replace $S^2$ by $S^3$. Instead of simple curves, we treat immersed saddle surfaces which are homeomorphic to $S^2$ ("saddle spheres"). We prove that a piecewise linear saddle sphere $\Gamma \subset S^3$ necessarily has *inflection* or *reflex faces*. The latter replace inflection points and should be considered as singular phenomena.

As an application, we prove that a piecewise linear saddle surface cannot be altered in a neighborhood of its vertex maintaining its saddle property.

## 1. Introduction

Let us start with the following classical theorems.

**Theorem 1.1.** *Segre's theorem, see* [12], [17].
*Let a smooth closed simple (i.e., embedded) curve $c \subset S^2$ have a non-empty intersection with any closed hemisphere. Then $c$ has at least four inflection points.* $\square$

Here are its two famous corollaries:

**Theorem 1.2.** *V. Arnold's tennis ball theorem, see* [3], [12].
*Any smooth closed simple curve $c \subset S^2$ bisecting the area of the sphere has at least four inflection points.* $\square$

**Theorem 1.3.** *Möbius theorem, see* [12].
*A smooth closed simple non-contractible curve $c \subset \mathbb{R}P^2$ has at least three inflection points.* $\square$

Segre's theorem has various applications, generalizations and refinements. In the paper, we present one more Segre-type phenomenon. However, unlike the already existent ones, it deals with closed saddle surfaces on $S^3$ rather than closed curves. This object is not chosen just by chance: the study of closed saddle surfaces was originally motivated by A.D. Alexandrov's problem (see "Motivations" below).

*Key words and phrases.* Saddle surface, piecewise linear surface, inflection point, Segre's theorem.

**Definitions and the main result.** By $S^3 \subset \mathbb{R}^4$ we denote the unit sphere centered at the origin $O$. A *plane on the sphere* $S^3$ is a plane in the sense of spherical geometry, i.e., the intersection of $S^3$ with a Euclidean hyperplane passing through $O$.

**Definition 1.4.** A closed surface $\Gamma$ immersed in $S^3$ is called *saddle* if no (spherical) plane intersects $\Gamma$ locally at just one point.

**Definition 1.5.** A (spherical) *polygon* on the two-dimensional sphere $S^2$ is a part of $S^2$ bounded by a piecewise geodesic closed simple curve.

An angle of a polygon is called *convex* (respectively, *reflex*) if it is smaller (respectively, greater) than $\pi$.

A vertex of a polygon is called *convex* (respectively, *reflex*) if it is incident to a convex (respectively, *reflex*) angle.

**Definition 1.6.** A *piecewise linear saddle sphere* (a *PLS-sphere*, for short) on $S^3$ is an immersed piecewise linear saddle surface which is homeomorphic to $S^2$.

To avoid degeneracies and non-interesting exceptions, we assume in addition that all edges of a PLS-sphere are shorter than $\pi$, and that its vertex-edge graph is 3-connected.

Besides, we assume that the dihedral angle at each edge does not equal $\pi$, so the vertex-edge graph has no redundant edges.

Given an oriented PLS-sphere, we can speak of its *convex and concave* edges. In the sequel, we paint all the convex (respectively, concave) edges red (respectively, blue).

**Definition 1.7.** A PLS-sphere is called *elementary Barner* if there is a point $p \in S^3$ such that each great semicircle with endpoints at $p$ and at its antipode $-p$ hits the surface exactly once.

Equivalently, an elementary Barner PLS-sphere admits a bijective projection $\pi$ onto some equator $S^2 \subset S^3$, see Fig. 2.

Elementary Barner saddle spheres are of a particular interest because of a relationship to A.D. Alexandrov's problem (see "Motivations" below).

The interplay between PLS-spheres and smooth saddle spheres is not well understood yet. On the one hand, it seems plausible that a piecewise linear saddle sphere can be approximated by a smooth saddle sphere and vice versa. On the other hand, there is just one proven result (see [13]). It asserts that an elementary Barner PLS-sphere with a trivalent vertex-edge graph has a $C^\infty$-smooth saddle approximation.

By topological reasons, a smooth saddle sphere necessarily has flattening points. In some sense, the below defined inflection and reflex faces play the role of flattening phenomena of a piecewise linear saddle sphere.

**Definition 1.8.**
- A face $f$ of a PLS-sphere $\Gamma$ is *an inflection face* if
  (1) $f$ is bounded by two convex broken lines (say, by $L_1$ and $L_2$) such that the convexity directions look like in Fig. 1.
  (NB. A polygon with such convexity properties does not exist in Euclidean plane.)
  (2) All the edges of $L_1$ are convex, whereas all the edges of $L_2$ are concave.

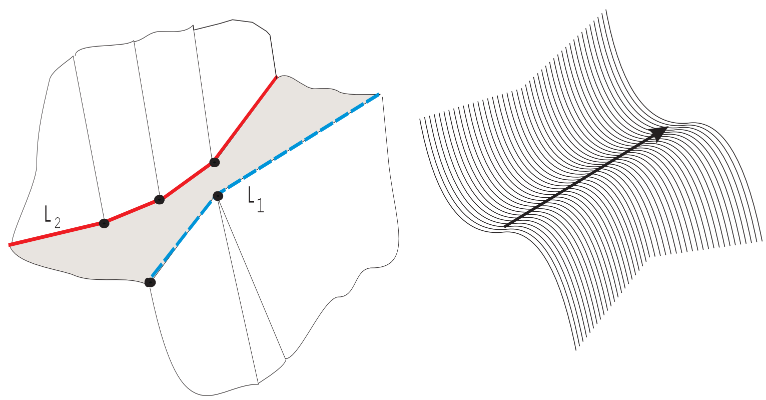


FIGURE 1. A fragment of an inflection face. An inflection arch of a smooth saddle surface

- A face $f$ of a PLS-sphere $\Gamma$ is called *a reflex face* if it contains a (two-dimensional) hemisphere.

Inflection faces as well as reflex faces represent a kind of singularity of the surface $\Gamma$: none of them fits in a hemisphere (see Lemma 5.1).

The main result of the paper describes singularities of a saddle sphere:

**Theorem 1.9.** (1) *Each saddle sphere $\Gamma \subset S^3$ belongs to one of the following disjoint classes:*
   (a) *$\Gamma$ has at least two reflex faces.*
   (b) *$\Gamma$ has exactly one reflex face and at least two inflection faces.*
   (c) *$\Gamma$ has no reflex faces and at least 4 inflection faces.*

(2) *There are saddle spheres with*
   (a) *exactly two reflex faces.*
   (b) *exactly one reflex face and exactly two inflection faces.*
   (c) *no reflex faces and any number of inflection faces greater than 4.*

(3) *There are no embedded PLS-spheres on $S^3$ of type (1a).*
(4) *There are no embedded PLS-spheres on $\mathbb{R}P^3$.*
(5) *There exist immersed PLS-spheres on $\mathbb{R}P^3$.*
(6) *There are no elementary Barner PLS-spheres of types (1a) and (1b).*
(7) *There exist elementary Barner PLS-spheres of type (1c) with any number of inflection faces greater than 4. Moreover, the set of elementary Barner PLS-spheres with a fixed number of inflection faces is disconnected.* □

**Outline of the proof.** Combinatorially, a PLS-sphere is a planar graph with additional equipment: its edges are colored and some of the angles (the reflex ones) are marked. This equipment necessarily has some properties which follow from the discrete Segre's theorem proven in Section 2.

This leads to a combinatorial notion of a saddle graph. Reflex and inflection faces are easily encoded in the combinatorial language, and we prove their existence using just combinatorics. Some similar phenomena are already discussed in [5] and [6]; our approach combines in a sense these ideas.

**Motivations.**

- The proof of the Theorem 1.9 is based on and generalizes the Segre's theorem. Here is one more link to the Segre's theorem: a surface $\Gamma$ is saddle if and only if its intersection with a small sphere centered at any of its vertices satisfies the condition of the discrete Segre's theorem.
- There exist embedded saddle tori on $\mathbb{R}P^3$. V. Arnold [2] formulated some conjectures about them (and about their higher dimensional versions). Some of the conjectures proved to be wrong [11], in partial cases some of them are true [7, 8], but two of them still stand open for $\mathbb{R}P^3$. In particular, Arnold conjectured that the set of all smooth saddle tori embedded in $\mathbb{R}P^3$ is connected (compare with Theorem 1.9, (7)). This paper sheds no light to Arnold's conjecture, but it treats some similar objects.
- Smooth elementary Barner saddle spheres arose originally in a relationship (see [10, 13]) to the following uniqueness conjecture proven for analytic surfaces by A. D. Alexandrov in [1]:

  *Let $K \subset \mathbb{R}^3$ be a smooth convex body. If for a constant $C$, at every point of $\partial K$, we have $R_1 \leq C \leq R_2$, then $K$ is a ball. ($R_1$ and $R_2$ stand for the principal curvature radii of $\partial K$).*

  Here is the link: let $K$ be a counterexample to the conjecture. Denote by $h_K$ its support function and denote by $h_C$ the support function of the ball of radius $C$. The graph $\gamma$ of the difference $h_K - h_C$ is a conical surface in $\mathbb{R}^4$ with the apex at the origin $O$. Its intersection with $S^3$ is an elementary Barner saddle sphere (see Fig. 2).

  Vice versa, a smooth elementary Barner saddle sphere yields a cone in $\mathbb{R}^4$ which can be interpreted as the graph of some positively homogeneous function $h$. For a sufficiently large $C$, the sum $h + h_C$ is a convex function. Then it is a support function of some convex body $K$ which is a counterexample to the conjecture.

  To summarize, each smooth elementary Barner saddle sphere yields a counterexample to the conjecture. An observation was made that all saddle spheres constructed in [10] and [13] have *inflection arches*. Later, the existence of at least four inflection arches for elementary Barner saddle spheres was proven in [14]. The above defined *inflection faces* represent a piecewise linear counterpart of inflection arches.
- We were also motivated by the following toy problem:

  *Given a piecewise linear saddle surface in $\mathbb{R}^3$, is it possible to alter it locally (i.e., in a neighborhood of a vertex), maintaining its saddle property?*

  In Section 5 we show that it is never possible.

**A convention about figures.** Fix a hyperplane $H \subset \mathbb{R}^4$ not passing through the origin $O$. The projection from the origin $pr : S^3 \to H$ maps bijectively some open hemisphere onto $H$. Spherical planes and lines are mapped to Euclidean planes and lines. Therefore, $pr$ preserves convexity and saddle property. By this reason, we will sometimes depict spherical objects as their images under $pr$ and refer to the convexity type of the image, as in Fig. 1, 3, 12.

Alternatively, if a spherical drawing does not fit in a hemisphere, it makes sense to depict it schematically, as in Fig. 6, 14.

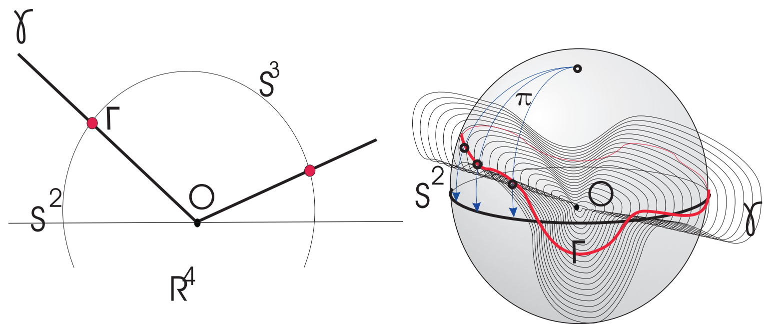


FIGURE 2. Elementary Barner sphere

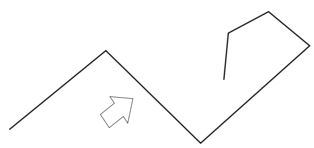

FIGURE 3. An inflection edge

## 2. Discrete Segre's theorem

We consider piecewise linear simple closed curves $c$ on the unit sphere $S^2$. If an edge of such a curve is shorter than $\pi$, it is called *short*. Otherwise, we call it *long*.

**Definition 2.1.** A closed simple (i.e., embedded) curve $c \subset S^2$ is *spanning* if it intersects each closed hemisphere.

A closed simple curve $c$ is *strongly spanning* if it intersects each open hemisphere.

**Definition 2.2.** Let $c \subset S^2$ be a piecewise linear simple closed curve. It splits $S^2$ into two (spherical) polygons. After fixing one of them, it makes sense to speak of convex and reflex angles of $c$.

An edge is called an *inflection edge* of $c$ (see Fig. 3) if it is incident to both convex and reflex angles.

**Theorem 2.3.** *(Discrete Segre's Theorem)*

(1) *A strongly spanning piecewise linear closed simple curve has at least 4 inflection edges.*
(2) *Let $c \subset S^2$ be a spanning piecewise linear closed simple curve. We assume that c has more than 2 vertices. Then one of the two (non-disjoint) assertions hold:*
   (a) *c has at least 4 inflection edges,*
   (b) *c has a long edge (say, e) and at least 2 inflection edges among the edges excluding e.*
(3) *Let $c = (P_1, ..., P_n) \subset S^2$ be a spanning piecewise linear closed simple curve with vertices $\{P_1, ..., P_n\}$. Assume that c has more than 2 vertices and at least two long edges. Then for any two long edges $P_iP_{i+1}$ and $P_jP_{j+1}$, there is at least one inflection edge among the edges lying between them (that is, among the edges $P_{i+1}P_{i+2}, P_{i+2}P_{i+3}, ..., P_{j-1}P_j$).*

Proof. The idea of the proof is to approximate $c$ by an appropriate smooth curve $c'$ and to apply then Segre's theorem. However, this needs some accuracy: if the curve $c$ is not strongly spanning, its smooth approximation $c'$ can be non-spanning.

(1) Suppose $c$ is strongly spanning. Then it can be approximated by a smooth curve $c'$ such that $c'$ has only isolated inflection points which are in a natural bijection with inflection edges of $c$. It remains to observe that a sufficiently close $c'$ is spanning, and to apply Segre's theorem to the curve $c'$.

(2) Suppose $c$ is spanning, but not strongly spanning. Then there exists a closed hemisphere $S^+$ containing $c$. Denote by $b$ its boundary ($b$ is a great circle). We may assume that $b \cap c$ is a union of some geodesic segments $e_{i_1}, ..., e_{i_m}$ of non-zero length (see Fig. 4). Two cases should be treated separately:

   (a) Suppose all the edges $e_{i_1}, ..., e_{i_m}$ are short.
   Note first that each semicircle $b^+ \subset b$ intersects the curve $c$. Take a smooth approximation $c'$ of the curve $c$ such that $c'$ has only isolated inflection points which are in a natural bijection with inflection edges of $c$ plus the following additional property: the curve $c'$ tangents each of the segments $e_{i_1}, ..., e_{i_m}$, and each semicircle $b^+$ contains at least one tangent point (see Fig. 4). This is always possible by Caratheodory theorem. This guarantees that $c'$ is spanning. It remains to apply Segre's theorem to the curve $c'$.

   (b) Suppose one of the edges (say, $e$) is long. We may assume that $|e| > \pi$. We approximate $c$ by a smooth curve $c'$ such that $c'$ has only isolated inflection points which are in a natural bijection with inflection edges of $c$ except for two extra inflection points lying on $e$ (see Fig. 5). It remains to apply Segre's theorem to the curve $c'$.
   We explore here the following phenomenon: suppose a (geodesic) segment in the plane is approximated by a smooth curve which tangents the segment at the endpoints. Then by Möbius Theorem, the curve has at least 2 inflection points (except for the endpoints). For a long segment on the sphere, such a curve can have no inflection points.

(3) The curve $c$ is strongly spanning and has therefore at least 4 inflection edges.

   Assume the contrary, i.e., that the chain $P_{i+1}, P_{i+2}, ..., P_j$ contains no inflection edges. The (non-closed) curve $P_{i+1}, P_{i+2}, ..., P_j$ is contained in the lune bounded by $P_iP_{i+1}$ and the extension of $P_{i+1}P_{i+2}$ (see Fig. 6).

   Indeed, if not, i.e., if $P_{i+1}, P_{i+2}, ..., P_j$ hits the extension of $P_{i+1}P_{i+2}$ (the dotted line) at a point $A$, then the curve $c''$ depicted in Fig. 6, 2 has at least two inflection edges. A contradiction.

   Now prove the theorem. We replace the curve $c$ by another curve $c'$ as is depicted in Fig. 6, 1. By the above proven, $c'$ is simple. Since the curve $c'$ is strongly spanning, it has at least 4 inflection edges. A contradiction. □

## 3. Saddle graphs

By a *graph* we mean a tuple $G = (V, E)$ where $V$ is a (finite) set of vertices and $E$ is the set of edges (unordered pairs of different vertices).

For $v \in V$, denote by $E(v)$ the set of edges incident to the vertex $v$.

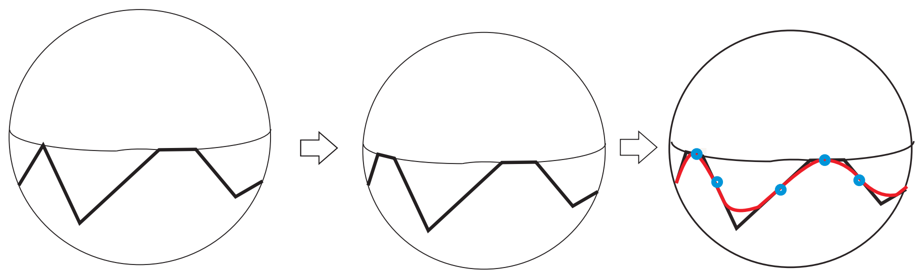

FIGURE 4. Smoothing of a curve without long edges

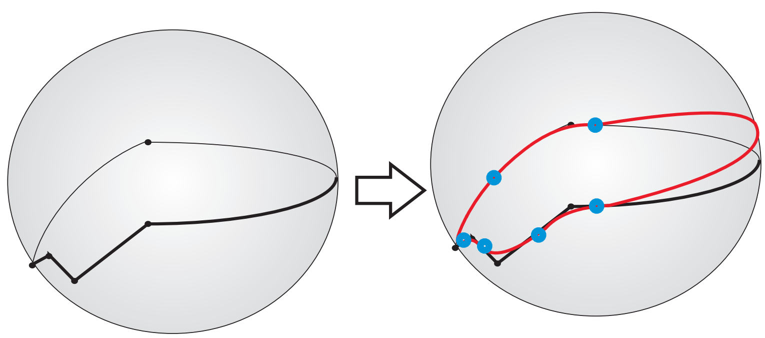

FIGURE 5. Smoothing of a non-strongly spanning curve with a long edge

Let $G = (V, E)$ be a 3-connected planar graph. All its embeddings in the sphere $S^2$ are known to have one and the same facial structure. Therefore, we have a natural notion of a *face* of the graph and a cyclic ordering on the set $E(V)$. Besides, we have a well-defined notion of angles:

**Definition 3.1.** An unordered pair of edges $(e_1, e_2)$ is called an *angle* of $G$ if the edges $e_1$ and $e_2$ are consecutive edges of a face of the graph $G$. The set of all angles we denote by $A(G)$. The set of all angles incident to a vertex $v$ we denote by $A(v)$.

The next idea is to add the so called saddle structure to a graph $G$. Namely, we paint convex edges red and we paint concave edges blue. Besides, we mark all the reflex angles.

Till now, a graph $G$ is just a combinatorial object, so in the below definition, the combinatorial convexity and concavity have no geometrical meaning. The saddle structure is defined axiomatically.

However, later we shall see that if a graph $G$ together with a coloring on its edges arise from some saddle sphere, then it satisfies the axioms from the below definition.

**Definition 3.2.** Let $G = (V, E)$ be a 3-connected planar graph.

Let $Col : E \to \{red, blue\}$ and $Refl : A(G) \to \{0, 1\}$ be some mappings.

Angles with $Refl(a) = 1$ we call (combinatorially) *reflex angles*.

We say that a triple $(G, Col, Refl)$ is a *graph equipped with a saddle structure* (*a saddle graph*, for short) if for any vertex $v$, we have the following (see Fig. 7):

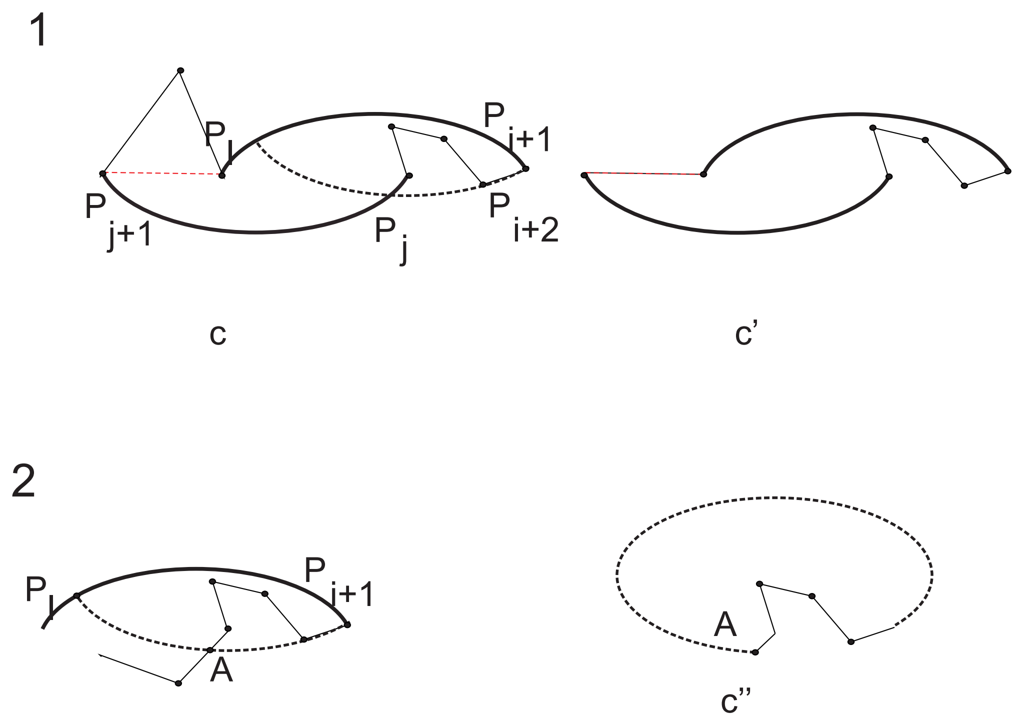


FIGURE 6.

(1) **"No reflex angles condition"** If $Refl$ is identically 0 on $A(v)$ (i.e., there are no reflex angles incident to $v$), then the number of changes of the function $Col$ when going around the vertex $v$ is greater or equal than 4.
(2) **"Exactly one reflex angle condition"** If there is exactly one reflex angle at $v$, (say, $Refl(e_i, e_j) = 1$), then the function $Col$ changes at least twice when going around the vertex $v$ from $e_i$ to $e_j$.
(3) **"More than one reflex angle condition"** If there are more than one reflex angle at $v$, then we claim two things: (1) that the total number of color changes when going around the vertex $v$ is greater or equal than 4 and (2) that the color changes at least once when going from one edge of a reflex angle to the edge of the next reflex angle.

**Definition 3.3.** For a face $f$ of a saddle graph, we algorithmically define its *index* $i(f)$, see Fig. 8:

(1) At the beginning, put $i(f) := 0$. Start going along the boundary of the face $f$.
(2) Once we pass by a vertex at which the color changes, put
$$i(f) := i(f) + 1.$$
(3) Once we pass by a vertex, if the color does not change and the angle we are passing by is reflex, we keep $i(f)$ unchanged.
(4) Once we pass by a vertex, if the color does not change and the angle we are passing by is not reflex, put $i(f) := i(f) + 2$.

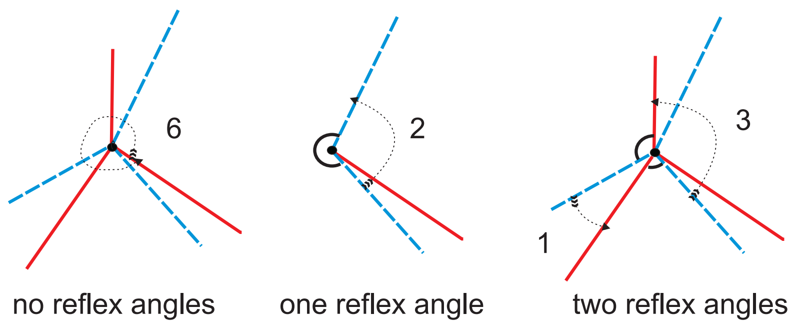


FIGURE 7. Counting color changes

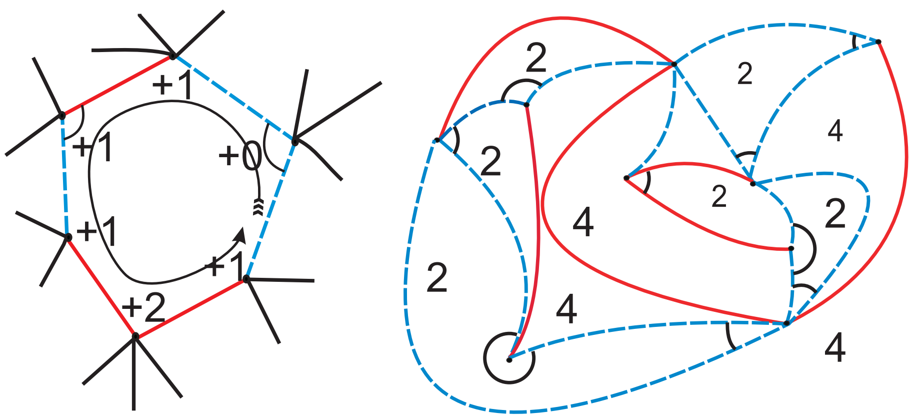


FIGURE 8. A saddle graph and the values of $i(f)$

**Definition 3.4.** Let $v$ be a vertex of a saddle graph. An edge $e$ incident to $v$ is called *superfluous with respect to the vertex* $v$ if its deletion maintains the properties (1)–(3) of the Definition 3.2 at the vertex $v$.

We describe below some local graph transformations, the *elementary splittings* of three types.

**Definition 3.5.** (1) For two neighbor edges of different colors, one of which is superfluous, the local graph transformation depicted in Fig. 9 is called the *first elementary splitting*.

Here are the formalities: if a blue edge $av$ is superfluous with respect to $v$ and a red edge $bv$ is neighbor to $av$ at the vertex $v$, then the first elementary splitting looks as follows:

(a) Remove from the graph the edges $av$ and $bv$
(b) Add a new vertex $d$, red edges $bd$ and $dv$, and a blue edge $ad$
(c) Mark the angle $bdv$ as reflex.

(2) Suppose that a vertex $v$ has no adjacent reflex angles and exactly 4 incident edges. The local graph transformation depicted in Fig. 10, 1 is called the *second elementary splitting*.

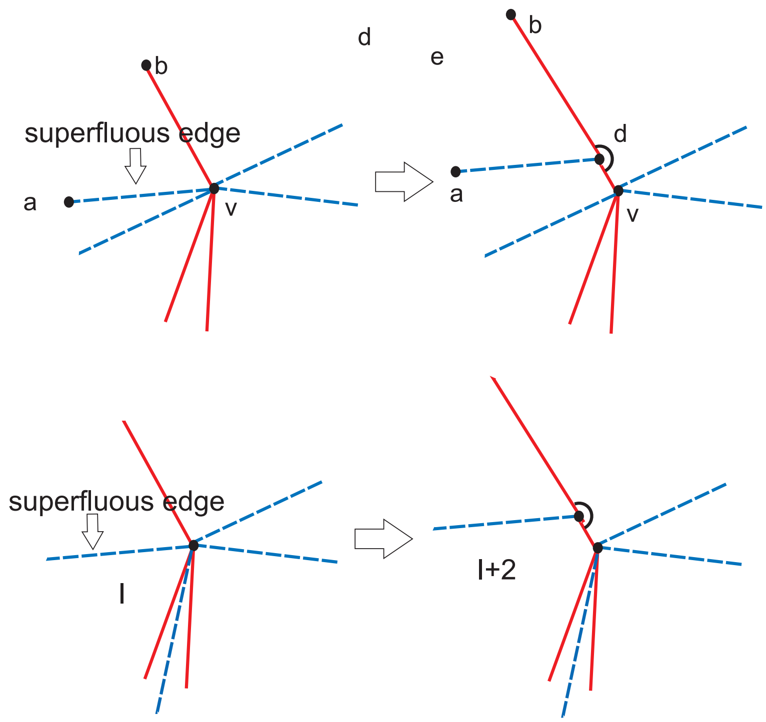


FIGURE 9. Two elementary splittings of the first type. For the first example, the index $i$ is maintained. For one face of the second example, it increases on 2.

More precisely, let $a, b, c$, and $d$ be vertices adjacent to $v$. Assume that the edges $av$ and $cv$ are red. We do the following:

(a) Remove from the graph the edges $cv$ and $bv$
(b) Add a new vertex $e$, red edges $ve$ and $ce$, and a blue edge $be$
(c) Mark the angles $ave$ and $vec$ as reflex.

**Definition 3.6.** Suppose a vertex $v$ is incident to more than one reflex angles. The following procedure describes the splitting which takes reflex angles apart.

(1) Choose two edges $e$ and $e'$ of one and the same color (say, red) incident to the vertex $v$ such that the edges $e$ and $e'$ are separated by reflex angles, see Fig. 11.
(2) Split the vertex $v$ into two vertices, split also the two edges $e$ and $e'$ and add one more edge of the other color (here it is blue) as is shown in Fig. 11. This local graph transformation is called the *third elementary splitting*.

More precisely, let $av$ and $bv$ be the edges $e$ and $e'$. Assume that they are red. We do the following:

(a) The set of all the edges incident to the vertex $v$ (except for the edges $e$ and $e'$) is divided by the broken line $avb$ into two parts $E_1$ and $E_2$.
(b) Add a new vertex $v'$, red edges $av'$ and $bv'$, and a blue edge $vv'$
(c) Each edge $xv$ from $E_2$ replace by the new edge $xv'$.

An easy check proves that:

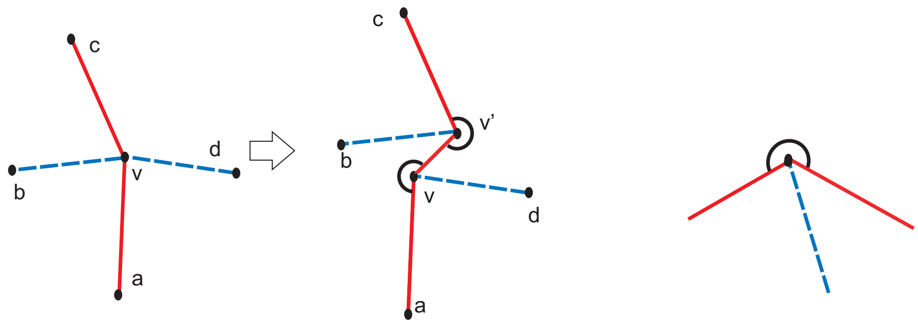


FIGURE 10. Second elementary splitting. All the indices are maintained.

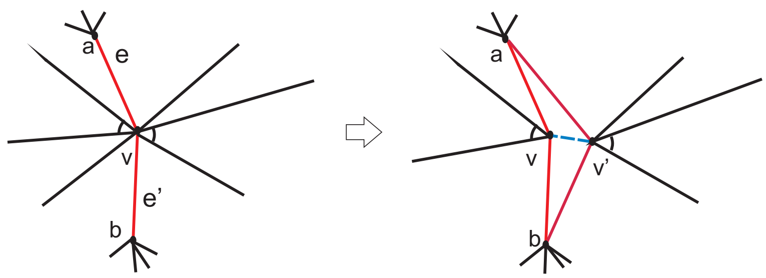


FIGURE 11. Third elementary splitting adds two faces with $i =$ 4. All the other indices are maintained.

**Lemma 3.7.** (1) *An elementary splitting of a saddle graph yields a saddle graph.*
(2) *For any first or second elementary splitting, the faces of the new graph are in a natural bijection with the faces of the original graph.*
(3) *A third elementary splitting adds two faces with $i = 4$.*
(4) *The index $i$ of a face does not decrease after any elementary splitting.* □

**Lemma 3.8.** *Each saddle graph is reducible to a trivalent saddle graph via a chain of elementary splittings.*

Proof. Third elementary splittings enable us to get a graph with at most one reflex angle at each vertex. Next, we treat all the vertices one by one. After fixing a vertex $v$, we first get rid of all superfluous edges incident to $v$. We arrive at one of the two possible cases depicted in Fig. 10. In the second case, we are done. In the first case it remains to apply the second splitting. □

The following theorem is a combinatorial version of Theorem 1.9, (1).

**Theorem 3.9.** *For each saddle graph, one of the following statements is valid:*
(1) *The graph has at least two faces with $i(f) = 0$.*
(2) *The graph has one face with $i(f) = 0$ and at least 2 faces with $i(f) = 2$.*
(3) *The graph has no faces with $i(f) = 0$ and at least 4 faces with $i(f) = 2$.*

Proof. Due to Lemma 3.8 and Lemma 3.7, we may assume that the graph is trivalent. At each of its vertex it looks like the graph in Fig. 10, 2 (up to color reverting). Count the total sum $\Sigma$ of indices $i(f)$ for all the faces $f$. The contribution of each vertex equals 2, therefore we have $\Sigma = 2 \mid V \mid$. The Euler formula for trivalent graphs $2 \mid F \mid = \mid V \mid +4$ implies that $\Sigma = 4 \mid F \mid -8$. Since the index $i(f)$ is always positive and even, we are done. (Here $\mid F \mid$ and $\mid V \mid$ denote the number of faces and vertices respectively.) $\square$

## Proof of Theorem 1.9, (1)

We associate a saddle graph $SG(\Gamma)$ to a saddle sphere $\Gamma$:

(1) Set the graph $G$ equal the vertex-edge graph of the surface $\Gamma$.
(2) Fix an orientation of $\Gamma$. Now it makes sense to speak of convex and concave edges. For an edge $e$, set
$$Col(e) = \begin{cases} \text{red}, & \text{if } e \text{ is convex;} \\ \text{blue}, & \text{otherwise.} \end{cases}$$
(3) For an angle $a$, set
$$Refl(a) = \begin{cases} 1, & a \text{ is a reflex angle on } \Gamma; \\ 0, & \text{otherwise.} \end{cases}$$

For a saddle sphere, $SG(\Gamma)$ is a saddle graph. Indeed, the properties from Definition 3.2 follow from the discrete Segre's theorem. Given a vertex $v$ of the surface $\Gamma$, take its Euclidean image and a small sphere $S_v$ centered at the point $v$. The intersection $\Gamma \cap S_v$ is a piecewise linear simple spanning curve with more than two vertices.

Next, we apply Theorem 3.9 to the saddle graph $SG(\Gamma)$. To conclude the proof, it remains to understand the geometrical meaning of the index $i(f)$.

**Lemma 3.10.** *For a a face $f$ of a saddle sphere $\Gamma$, we have:*

(1) $i(f) = 0$ *implies that $f$ is a reflex face.*
(2) $i(f) = 2$ *implies that $f$ is either a reflex face or an inflection face.*

Proof. (1). $i(f) = 0$ implies that the complement of $f$ is a (spherical) polygon with convex angles. Such polygons are known to lie in an open hemisphere.

(2). If $i(f) = 2$, three cases are possible:

(1) The face $f$ has no convex angles. Then its complement lies in an open hemisphere.
(2) The face $f$ has exactly one convex angle. This means that the boundary of $f$ has exactly 2 inflection edges (the ones adjacent to the only convex vertex). By Segre's Theorem, the boundary of $f$ is not a strongly spanning curve, and therefore, fits in an closed hemisphere.
(3) The face $f$ has two convex angles. This implies that the boundary of $f$ has both blue and red edges and the color changes at the convex vertices. This means by definition that $f$ is an inflection face.

$\square$

## 4. Proof of Theorem 1.9, (2-7)

**(2,a).** Here is the construction of a saddle sphere with two reflex faces (see Fig. 12): take two (spherical) planes an join them by a polytopal tube.

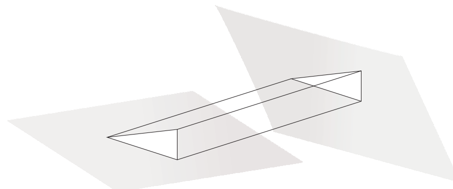

FIGURE 12. A saddle sphere with two reflex faces

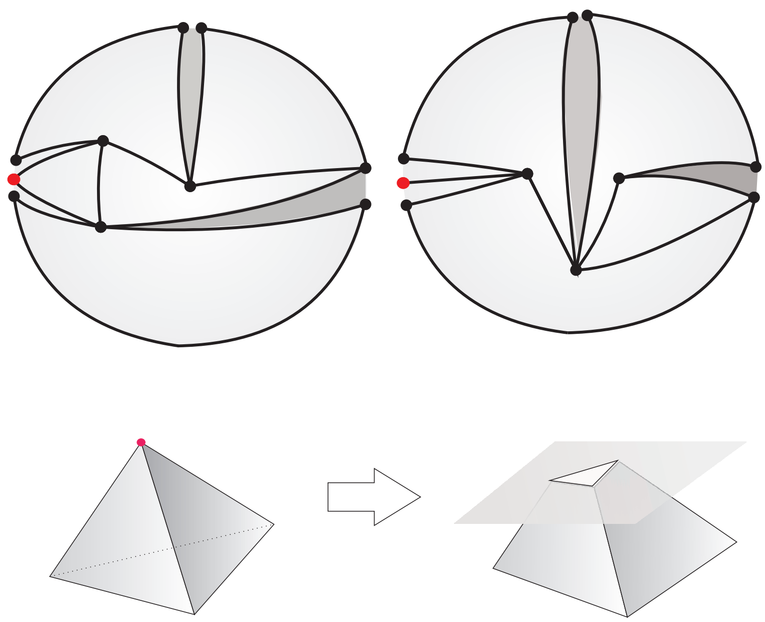

FIGURE 13. Saddle sphere with one reflex face and two inflection faces. The shadowed tiles correspond to inflection faces

**(2,b).** The construction of a saddle sphere with just one reflex face and two inflection faces is based on Maxwell-Cremona theorem and Laman theory for planar graphs embedded in $S^2$ (see details in [4] and [16]).

Figure 13 depicts a tiling of the sphere $S^2$ generated by an embedded graph. The graph is a *rigidity circuit*, therefore it has a 3D *lifting*, that is, there exists a piecewise linear surface $\Gamma$ embedded in $S^3$ whose bijective projection $\pi$ (see Fig. 2) onto $S^2$ yields this tiling. All the vertices of $\Gamma$ (except for a single one) have an incident reflex angle. Therefore, the surface $\Gamma$ is saddle everywhere except for just one vertex (marked red in Fig. 13). Next, we truncate $\Gamma$ at the convex vertex and patch a reflex face. The result is the desired surface.

**(2,c).** An example of an elementary Barner sphere with any number of inflection faces greater than 4 was constructed in [9].

**(3).** A saddle sphere with two reflex faces is never embedded since the reflex faces necessarily have an intersection.

**(4).** Suppose the contrary: there exists an embedded PLS-sphere $\Gamma \subset \mathbb{R}P^3$. Consider the standard covering $\varphi : S^3 \to \mathbb{R}P^3$. The preimage of $\Gamma$ is a union of two embedded saddle spheres on $S^3$. Each of them has either an inflection face or a reflex face $f$. But $\varphi$ is not injective on $f$.

**(5).** The mapping $\varphi$ maps an immersed saddle sphere to an immersed saddle sphere.

**(6).** Projections of two reflex faces (or a reflex face and an inflection face) on any (spherical) plane necessarily have an intersection. This is because each such projection necessarily contains a lune, see Lemma 5.1.

**(7).** The existence of an elementary Barner saddle sphere with any number of inflection faces greater than 3 was proven in [9]. The set of all elementary Barner saddle spheres with exactly 4 inflection faces is disconnected. This was proven in [14].

Furthermore, paper [15] gives a combinatorial classification of elementary Barner saddle sphere with any number of inflection faces greater than 3. Each elementary Barner saddle sphere $\Gamma \subset S^3$ generates an arrangement of (at least four) non-crossing oriented great semicircles on $S^2$. Namely, take the bijective projection of $\Gamma$ onto some equator $S^2$ (it exists by definition). The projection of each inflection face (see Lemma 5.1) contains a great semicircle which carries an orientation generated by red-blue sides of the projection. If we take one oriented great semicircle for each inflection face, we get an arrangement of non-crossing oriented great semicircles on $S^2$. In the paper [15] the converse is proven: each spanning arrangement of non-crossing oriented great semicircles is generated by an elementary Barner saddle sphere. Since there exist non-isotopic arrangements with one and the same number of great semicircles, the theorem is proven.

In particular, this means the diversity of saddle spheres on $S^3$. □

## 5. An application to saddle surfaces in Euclidean space

**Lemma 5.1.**
(1) *Two inflection faces of an elementary Barner saddle sphere cannot have a common convex vertex.*
(2) *For an inflection face $f$, let $s_1$ and $s_2$ be linear segments lying on $L_1$ and $L_2$ respectively (we use notation of Definition 1.8). Then the lune bounded by extended $s_1$ and $s_2$ lies in $f$.*
(3) *An inflection face contains a geodesic arc (a great semicircle) joining two antipodal points of $S^3$.*

Proof. (1). Indeed, in this case, projections of the faces to any spherical plane have a nonempty intersection. (2) follows from convexity properties of $L_1$ and $L_2$ and implies (3). □

Consider a piecewise linear saddle surface $M$ in $\mathbb{R}^3$ with the only vertex $O$ (i.e., $M$ is a conical surface, as in Fig. 15). Assume in addition that $M$ can be bijectively projected onto some plane $E$. A natural question which arose in attempts to develop a saddle approximation technique was the following:

Can we alter $M$ locally, maintaining its saddle properties?

The answer is "No":

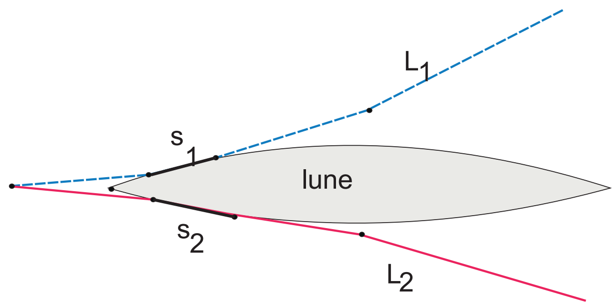


FIGURE 14. An inflection face (and its projection) contains a lune

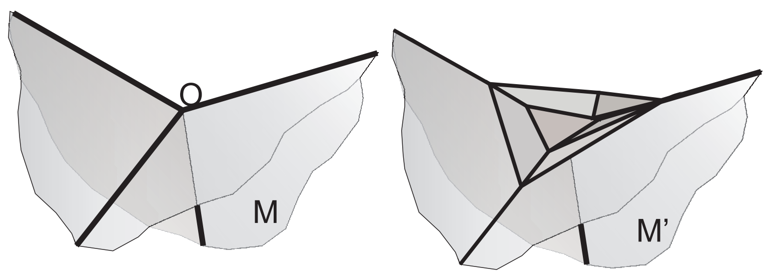


FIGURE 15.

**Proposition 5.2.** *In the above notation, suppose that for a piecewise linear saddle surface $M' \in R^3$ the following is true:*

- *$M'$ coincides with $M$ outside a ball centered at $O$;*
- *$M'$ can be bijectively projected onto the plane $E$.*

*Then $M = M'$.*

Proof. Assume that $M' \neq M$. We raise the surface $M'$ to the sphere $S^3$. Namely, we take the preimage $pr^{-1}(M')$ under the central projection $pr : S^3 \to \mathbb{R}^3$. The closure of the preimage is some elementary Barner saddle sphere $M'_{sph}$. By Theorem 1.9, surface $M'_{sph}$ necessarily has 4 inflection faces. The only candidates are those coming from unbounded faces of $M'$. But each two of them have an intersection, which is impossible. □

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