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FAST LOOPS ON SEMI-WEIGHTED HOMOGENEOUS HYPERSURFACE SINGULARITIES

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Abstract. We show the existence of essential fast loops on semi-weighted homogeneous hypersurface singularities with weights $w_1 \geq w_2 > w_3$. In particular we show that semi-weighted homogeneous hypersurface singularities are metrically conical only if their two lowest weights are equal.

1. Introduction

Let $X \subset \mathbb{R}^n$ be a subanalytic set with a singularity at x. It is well-known for small real numbers $\epsilon > 0$ that there exists a homeomorphism from the Euclidean ball $B(x,\epsilon)$ to itself which maps $X \cap B(x,\epsilon)$ onto the straight cone over $X \cap S(x,\epsilon)$ with vertex at x. The homeomorphism h is called a *topologically conical structure* of X at x and, since John Milnor proved the existence of topologically conical structure for algebraic complex hypersurfaces with an isolated singularity [9], some authors say ϵ is a *Milnor radius* of X at x. Some developments of the Lipschitz geometry of complex algebraic singularities come from the following question: given an algebraic subset $X \subset \mathbb{C}^n$ with an isolated singularity at x, is there $\epsilon > 0$ such that $X \cap B(x,\epsilon)$ is bi-Lipschitz homeomorphic to the cone over $X \cap S(x,\epsilon)$ with vertex at x? When we have a positive answer for this question we say that (X,x) *is metrically conical*. Some motivations for this question were given in [3], [6] and, in the same papers, the above question was answered negatively. The strategy used in [6] to show that some examples of complex weighted homogeneous surface singularities (X,x) are not metrically conical was to exhibit nontrivial loops on $X \cap S(x,\epsilon)$ which the diameter goes to 0 faster than linearly as $\epsilon \to 0$. In this paper we analyze semi-weighted homogeneous hypersurface singularities under the same

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point of view above and, in particular, we show that semi-weighted homogeneous hypersurface singularities are metrically conical only if its two lowest weights are equal.

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2. Preliminaries

2.1. **Inner metric.** Given an arc $\gamma\colon [0,1] \to \mathbb{R}^n$, we remember that the *length of* γ is defined by

$$l(\gamma) = \inf\{\sum_{i=1}^{m} |\gamma(t_i) - \gamma(t_{i-1})| \ : \ 0 = t_0 < t_1 < \cdots < t_{m-1} < t_m = 1\}.$$

Let $X \subset \mathbb{R}^n$ be a subanalytic connected subset. It is well-know that the function

$$d_X\colon X \times X \to [0, +\infty)$$

defined by

$$d_X(x,y) = \inf\{l(\gamma) \ : \ \gamma\colon [0,1] \to X;\ \gamma(0) = x,\ \gamma(1) = y\}$$

is a metric on X, so-called *inner metric* on X.

Theorem 2.1 (Pancake Decomposition [8])**.** *Let $X \subset \mathbb{R}^n$ be a subanalytic connected subset. Then, there exist $\lambda > 0$ and $X_1, \ldots, X_m$ subanalytic subsets such that:*

a. $X = \bigcup_{i=1}^m X_i$,
b. $d_X(x,y) \leq \lambda |x-y|$ *for any* $x, y \in X_i$, $i = 1, \ldots, m$.

2.2. **Horn exponents.** Let $\beta \geq 1$ be a rational number. The germ of

$$H_\beta = \{(x,y,z) \in \mathbb{R}^3 \ : \ x^2 + y^2 = z^\beta,\ z \geq 0\}$$

at $0 \in \mathbb{R}^3$ is called a *β-horn*.

By results of [1], we know that a β_1-horn is bi-Lipschitz equivalent, with respect to the inner metric, to a β_2-horn if, and only if $\beta_1 = \beta_2$. Let $\Omega \subset \mathbb{R}^n$ be a 2-dimensional subanalytic set. Let $x_0 \in \Omega$ be a point such that Ω is a topological 2-dimensional manifold without boundary near x_0.

Theorem 2.2. [1] *There exists a unique rational number $\beta \geq 1$ such that the germ of Ω at x_0 is bi-Lipschitz equivalent, with respect to the inner metric, to a β-horn.*

The number β is called *the horn exponent of* Ω *at* x_0. We use the notation $\beta(\Omega, x_0)$. By Theorem 2.2, $\beta(\Omega, x_0)$ is a complete intrinsic bi-Lipschitz invariant of germs of subanalytic sets which are topological 2-dimensional manifold without boundary. In the following, we show a way to compute horn exponents.

According to [2], $\beta(\Omega, x_0) + 1$ is the volume growth number of Ω at x_0, i. e.

$$\beta(\Omega, x_0) + 1 = \lim_{r \to 0+} \frac{\log \mathcal{H}^2[\Omega \cap B(x_0, r)]}{\log r}$$

where $\mathcal{H}^2$ denotes the 2-dimensional Hausdorff measure with respect to Euclidean metric on $\mathbb{R}^n$.

2.3. **Order of contact of arcs.** Let $\gamma_1 \colon [0, \epsilon) \to \Omega$ and $\gamma_2 \colon [0, \epsilon) \to \Omega$ be two continuous semianalytic arcs with $\gamma_1(0) = \gamma_2(0) = x_0$ and not identically equal to x_0. We suppose that the arcs are parameterized in the following way:

$$\|\gamma_i(t) - x_0\| = t, \ i = 1, 2.$$

Let $\rho(t)$ be a function defined as follows: $\rho(t) = \|\gamma_1(t) - \gamma_2(t)\|$. Since ρ is a subanalytic function there exist numbers $\lambda \in \mathbb{Q}$ and $a \in \mathbb{R}$, $a \neq 0$, such that

$$\rho(t) = at^\lambda + o(t^\lambda).$$

The number λ is called *an order of contact of* γ_1 *and* γ_2. We use the notation $\lambda(\gamma_1, \gamma_2)$ (see [4]).

Let K be the field of germs of subanalytic functions $f \colon (0, \epsilon) \to \mathbb{R}$. Let $\nu \colon K \to \mathbb{R}$ be a canonical valuation on K. Namely, if $f(t) = \alpha t^\beta + o(t^\beta)$ with $\alpha \neq 0$ we put $ord_t(f(t)) = \beta$.

Lemma 2.3. *Let* γ_1, γ_2 *be a pair of semianalytic continuous arcs such that* $\gamma_1(0) = \gamma_2(0) = x_0$ *and* $\gamma_i \neq x_0$ *(*$i = 1, 2$*). Let* $\tilde{\gamma_1}(\tau)$ *and* $\tilde{\gamma_2}(\tau)$ *be semianalytic parameterizations of* γ_1 *and* γ_2 *such that* $\|\tilde{\gamma_i}(\tau) - x_0\| = \tau + o_i(\tau)$, $i = 1, 2$. *Then* $ord_\tau \|\tilde{\gamma_1}(\tau) - \tilde{\gamma_2}(\tau)\| \leq \lambda(\gamma_1, \gamma_2)$.

The following result is an alternative way to compute horn exponents of germs of subanalytic sets which are topological 2-dimensional manifold without boundary.

Theorem 2.4. *Let $\Omega \subset \mathbb{R}^n$ be a 2-dimensional subanalytic set. Let $x_0 \in \Omega$ be a point such that Ω is a topological 2-dimensional manifold without boundary near x_0. Then $\beta(\Omega, x_0) = \min\{\lambda(\gamma_1, \gamma_2) \ : \ \gamma_1, \gamma_2$ are semianalytic arcs on Ω with $\gamma_1(0) = \gamma_2(0) = x_0\}$.*

Lemma 2.3 and Theorem 2.4 were proved in [5].

3. Fast loops

Let $X \subset \mathbb{R}^n$ be a subanalytic set with a singularity at x. Let $\epsilon > 0$ be a Milnor radius of X at x and let us denote by X^* the set $X \cap B(x, \epsilon) \setminus \{x\}$. Given a positive real number α, a continuous map $\gamma \colon S^1 \to X^*$ is called an α-fast loop if there exists a homotopy $H \colon S^1 \times [0,1] \to X \cap B(x, \epsilon)$ such that

(1) $H(\theta, 0) = x$ and $H(\theta, 1) = \gamma(\theta)$, $\forall\ \theta \in S^1$,
(2) $\displaystyle\lim_{r \to 0^+} \frac{1}{r^a} \mathcal{H}^2(Im(H) \cap B(x, r)) = 0$ for each $0 < a < \alpha$,

where $Im(H)$ denotes the image of H.

Given a subanalytic set X and a singular point $x \in X$, according to [2], there exists a positive number c such that any α-fast loop $\gamma \colon S^1 \to X^*$ with $\alpha > c$ is necessarily homotopically trivial. Such a number c is called *distinguished for* (X, x). We define the υ invariant in the following way:

$$\upsilon(X, x) = \inf\{c \ : \ c \text{ is distinguished for } (X, x)\}.$$

The number $\upsilon(X, x)$ defined above is inspired by the first characteristic exponent for the local metric homology presented in [2].

Example 3.1. Let $K \subset \mathbb{R}^n$ be a straight cone over a Nash submanifold $N \subset \mathbb{R}^n$, with vertex at p. Then every loop $\gamma \colon S^1 \to K^*$ is a 2-fast loop. Moreover, if $\alpha > 2$, then each α-fast loop $\gamma \colon S^1 \to K^*$ is homotopically trivial. We can sum up it saying $\upsilon(K, p) = 2$.

Proposition 3.2. *Let (X, x) and (Y, y) be subanalytic germs. If there exists a germ of a bi-Lipschitz homeomorphism, with respect to inner metric, between (X, x) and (Y, y), then $\upsilon(X, x) = \upsilon(Y, y)$.*

Proof. Let $f\colon (X,x) \to (Y,y)$ be a bi-Lipschitz homeomorphism, with respect to the inner metric. Given $A \subset X$, let us denote $\tilde{A} = f(A)$. In this case, $A = f^{-1}(\tilde{A})$, where f^{-1} denotes the inverse map of $f\colon (X,x) \to (Y,y)$.

`Claim.` There are positive constants $k_1, k_2, \lambda_1, \lambda_2$ such that

$$\frac{1}{k_1}\mathcal{H}^2(\tilde{A} \cap B(y, \frac{r}{\lambda_2})) \leq \mathcal{H}^2(A \cap B(x,r)) \leq k_2\mathcal{H}^2(\tilde{A} \cap B(y, \lambda_1 r)).$$

In fact, using Pancake Decomposition Theorem (see Subsection 2.1) and using that f and f^{-1} are Lipschitz maps, we obtain positive constants λ_1, λ_2 such that

$$f(A \cap B(x,r)) \subset (\tilde{A} \cap B(y\lambda_1 r)) \text{ and } f(\tilde{A} \cap B(y,r)) \subset (A \cap B(x\lambda_2 r))$$

and we also obtain positive constants k_1, k_2 such that

$$\mathcal{H}^2(f(A \cap B(x,r))) \leq k_1\mathcal{H}^2(A \cap B(x,r)) \text{ and } \mathcal{H}^2(f^{-1}(\tilde{A} \cap B(y,r))) \leq k_2\mathcal{H}^2(\tilde{A} \cap B(y,r)).$$

Our claim follows from these two inequalities and the two inclusions above.

Now, we use this claim to show that given $\alpha > 0$, a loop $\gamma\colon S^1 \to X \setminus \{x\}$ is an α-fast loop if, and only if, $f \circ \gamma\colon S^1 \to Y \setminus \{y\}$ is an α-fast loop. In fact, let $\gamma\colon S^1 \to X \setminus \{x\}$ be a loop and $H\colon S^1 \times [0,1] \to X$ a homotopy such that $H(\theta, 0) = x$ and $H(\theta, 1) = \gamma(\theta)$, $\forall\ \theta \in S^1$. Thus, $f \circ \gamma\colon S^1 \to Y \setminus \{y\}$ is a loop and $f \circ H\colon S^1 \times [0,1] \to Y$ is a homotopy such that $f \circ H(\theta, 0) = x$ and $f \circ H(\theta, 1) = f \circ \gamma(\theta)$, $\forall\ \theta \in S^1$. Let us denote $A = Im(H)$ and $\tilde{A} = Im(f \circ H)$, i. e., $\tilde{A} = f(A)$. Given $0 < a < \alpha$, by the above claim, we have that

$$\lim_{r \to 0^+} \frac{1}{r^a}\mathcal{H}^2(A \cap B(x,r)) = 0$$

if, and only if,

$$\lim_{r \to 0^+} \frac{1}{r^a}\mathcal{H}^2(\tilde{A} \cap B(y,r)) = 0.$$

In other words, it was shown that $\gamma\colon S^1 \to X \setminus \{x\}$ is an α-fast loop if, and only if, $f \circ \gamma\colon S^1 \to Y \setminus \{y\}$ is an α-fast loop, hence $\upsilon(X,x) = \upsilon(Y,y)$. □

Corollary 3.3. *Let $X \subset \mathbb{R}^n$ be a subanalytic set and $x \in X$ an isolated singular point. If $\upsilon(X,x) > 2$, then (X,x) is not metrically conical.*

Proof. Let N be the intersection $X \cap S(x,\epsilon)$ where $\epsilon > 0$ is chosen sufficiently small. Since x is an isolated singular point of X, we have $N \subset \mathbb{R}^n$ is a Nash submanifold. If (X,x) is metrically conical, $X \cap B(x,\epsilon)$ must be bi-Lipschitz homeomorphic (with respect to the inner metric) to the straight cone over N with vertex at x. Thus, it follows from Proposition 3.2 that $\upsilon(X,x) = 2$. □

4. SEMI-WEIGHTED HOMOGENEOUS HYPERSURFACE SINGULARITIES

Remind that a polynomial function $f\colon \mathbb{C}^3 \to \mathbb{C}$ is called *semi-weighted homogeneous* of degree $d \in \mathbb{N}$ with respect to the weights $w_1, w_2, w_3 \in \mathbb{N}$ if f can be presented in the following form: $f = h + \theta$ where h is a weighted homogeneous polynomial of degree d with respect to the weights w_1, w_2, w_3, the origin is an isolated singularity of h and θ contains only monomials $x_1^{m_1} x_2^{m_2} x_3^{m_3}$ such that $w_1 m_1 + w_2 m_2 + w_3 m_3 > d$.

An algebraic surface $S \subset \mathbb{C}^3$ is called *semi-weighted homogeneous* if there exists a semi-weighted homogeneous polynomial $f = h + \theta$ such that

$$S = \{(x_1, x_2, x_3) \in \mathbb{C}^3 \ : \ f(x_1, x_2, x_3) = 0\}.$$

The set

$$S_0 = \{(x_1, x_2, x_3) \in \mathbb{C}^3 \ : \ f(x_1, x_2, x_3) = 0\}$$

is called a *weighted approximation* of S.

Theorem 4.1. *Let $S \subset \mathbb{C}^3$ be a semi-weighted homogeneous algebraic surface with an isolated singularity at origin $0 \in \mathbb{C}^3$. If the weights of S satisfy $w_1 \geq w_2 > w_3$, then $\upsilon(S,0) > 2$. In particular, $(S,0)$ is not metrically conical.*

Proof. Let $S \subset \mathbb{C}^3$ be defined by the semi-weighted polynomial $f = h + \theta$ of degree d and let S_0 be the following weighted homogeneous approximation of S:

$$S_0 = \{(x_1, x_2, x_3) \in \mathbb{C}^3 \ : \ h(x_1, x_2, x_3) = 0\}.$$

Let us consider a family of functions defined as follows:

$$F(X,u) = h(X) + u\theta(X),$$

where $u \in [0,1]$, $X = (x_1, x_2, x_3)$. Let $V(X,u)$ be the vector field defined by:

$$V(X,u) = -\sum_{i=1}^{3} \frac{Q_i(X,u)}{P(X,u)} \frac{\partial}{\partial x_i} + \frac{\partial}{\partial u}$$

where

$$P(X,u) = \sum_{i=1}^{3} |\frac{\partial F}{\partial x_i}(X,u)|^{2\alpha_i} \text{ and } Q_i(X,u) = \theta(X) |\frac{\partial F}{\partial x_i}(X,u)|^{2\alpha_i - 2} \overline{\frac{\partial F}{\partial x_i}}(X,u)$$

and $\alpha_i = \frac{(d-w_1)(d-w_2)(d-w_3)}{d-w_i}$, $i = 1, 2, 3$.

It was shown, by L. Fukui and L. Paunescu (see [7] p. 445), that the flow of this vector field gives a modified analytic trivialization [7] of the family $F^{-1}(0)$. In particular, we obtain a homeomorphism $\Phi\colon (S_0, 0) \to (S, 0)$ which defines a correspondence of subanalytic continuous arcs. Moreover, Φ satisfies the following equation

$$\Phi(X) = X + \int_0^1 W(\Phi(X), u)du \tag{4.1}$$

where $W(X, u) = V(X, u) - \frac{\partial}{\partial u}$.

Proposition 4.2. *Let $\gamma(t) = (t^{w_1}x_1(t), t^{w_2}x_2(t), t^{w_3}x_3(t))$ be such that $x_1(t), x_2(t)$ and $x_3(t)$ are subanalytic continuous functions, $0 \le t < \epsilon$, with $(x_1(0), x_2(0), x_3(0)) \neq (0, 0, 0)$. If*

$$\eta(t) = \int_0^1 W(\gamma(t), u)du$$

with $\eta(t) = (\eta_1(t), \eta_2(t), \eta_3(t))$, then $ord_t|\eta_i(t)| > w_i$ for all $i = 1, 2, 3$.

Proof of the proposition. Let $m = (d - w_1)(d - w_2)(d - w_3)$. Since h has isolated singularity at $0 \in \mathbb{C}^3$, $\exists\ \lambda_1 > 0$ such that

$$P(\gamma(t), u) \ge \lambda_1 \sum_{i=1}^{3} |\frac{\partial h}{\partial x_i}(\gamma(t))|^{2\alpha_i}.$$

Moreover, since each $\dfrac{\partial h}{\partial x_i}$ is weighted homogeneous of degree $d - w_i$, $\exists\ \lambda_2 > 0$ such that

$$|\frac{\partial h}{\partial x_i}(\gamma(t))|^{2\alpha_i} \ge \lambda_2 t^{2m}.$$

Hence, $ord_t P(\gamma(t), u) \le 2m$. By hypothesis,

$$ord_t|\theta(\gamma(t))| > d \quad \text{and} \quad ord_t|\frac{\partial F}{\partial x_i}(\gamma(t), u)|^{2\alpha_i - 1} \ge (2\alpha_i - 1)(d - w_i).$$

Now, we can conclude that ord_t of $\dfrac{\theta}{P}|\dfrac{\partial F}{\partial x_i}|^{2\alpha_i - 1}$ on $\gamma(t)$ is bigger than

$$d + (2\alpha_i - 1)(d - w_i) - 2m = w_i.$$

Finally, since

$$\eta_i(t) = \int_0^1 \frac{\theta(\gamma(t))}{P(\gamma(t), u)} |\frac{\partial F}{\partial x_i}(\gamma(t), u)|^{2\alpha_i - 2} \overline{\frac{\partial F}{\partial x_i}}(\gamma(t), u)du,$$

we have proved the proposition. □

According to Lemma 1 of [6], we can take an essential loop Γ from S^1 to the link of the weighted homogeneous approximation S_0 of S of the form:

$$\Gamma(\theta) = (x_1(\theta), x_2(\theta), 1).$$

Let $H_0\colon [0,1] \times S^1 \to S_0$ be defined by

$$H_0(r,\theta) = (r^{\frac{w_1}{w_3}} x_1(\theta), r^{\frac{w_2}{w_3}} x_2(\theta), r).$$

Then, $H\colon [0,1] \times S^1 \to S$ defined by

$$H(r,\theta) = \Phi \circ H_0(r,\theta)$$

is a subanalytic homotopy satisfying: $H(0,\theta) = x$ and $H(1,\theta) = \Phi \circ \Gamma(\theta)$. We are going to show the image of H $(Im(H) = \Omega)$ has volume growth number at origin bigger than 2. Actually, since the volume growth number of Ω at 0 is $1 + \beta(\Omega, 0)$, we are going to show that $\beta(\Omega, 0)$ is bigger than 1. So, let us consider two arcs γ_1 and γ_2 on $(\Omega, 0)$.

`Claim.` Each γ_i can be parameterized in the following form:

$$\gamma_i(s) = (s^{\frac{w_1}{w_3}} x_{i1}(s), s^{\frac{w_2}{w_3}} x_{i2}(s), s x_{i3}(s))$$

where $x_{i1}(s), x_{i2}(s)$ and $x_{i3}(s)$ are subanalytic continuous functions and $x_{i3}(0) = 1$, $(i = 1, 2)$.

In fact, first of all, let us fix i and denote $\gamma = \gamma_i$. For each $s > 0$, let $\gamma(s)$ be the point on the arc γ such that $\rho(\gamma(s)) = s^{\frac{1}{w_3}}$, where

$$\rho(x_1, x_2, x_3) := [|x_1|^{w_2 w_3} + |x_2|^{w_1 w_3} + |x_3|^{w_1 w_2}]^{\frac{1}{w_1 w_2 w_3}}.$$

In particular, $\gamma(s) = (s^{\frac{w_1}{w_3}} x_1(s), s^{\frac{w_2}{w_3}} x_2(s), s x_3(s))$ where $x_1(s), x_2(s)$ and $x_3(s)$ are subanalytic continuous functions, with $(x_1(0), x_2(0), x_3(0)) \neq (0,0,0)$. For each $s > 0$, let $\xi(s)$ be the point on the image $Im(H_0) \subset S_0$ such that $\Phi(\xi(s)) = \gamma(s)$. It follows from eq. (4.1) that

$$\gamma(s) = \xi(s) + \eta(s)$$

where $\eta(s) = \int_0^1 W(\gamma(s), u) du$. By Proposition 4.2 (taking $s = t^{w_3}$), it follows that

$$\xi(s) = (s^{\frac{w_1}{w_3}} z_1(s), s^{\frac{w_2}{w_3}} z_2(s), s z_3(s)) \quad \text{with} \quad (z_1(0), z_2(0), z_3(0)) = (x_1(0), x_2(0), x_3(0)).$$

Since the image $Im(H_0)$ is invariant by the $\mathbb{R}_+$-action

$$s \cdot (x_1, x_2, x_3) = (s^{\frac{w_1}{w_3}} x_1, s^{\frac{w_2}{w_3}} x_2, s x_3)$$

and $\xi(s) \in Im(H_0)$ for all $s > 0$, we have that $(z_1(0), z_2(0), z_3(0)) \in Im(H_0)$, hence $z_3(0) = x_3(0) = a$ is a positive real number. Finally, via the simple change $s \mapsto a^{-1}s$, we show what was claimed above.

In order to finalize the proof of Theorem 4.1, by Lemma 2.3, we have

$$\lambda(\gamma_1, \gamma_2) \geq ord_s(|\gamma_1(s) - \gamma_2(s)|)$$

and, since

$$ord_s(|\gamma_1(s) - \gamma_2(s)|) > 1,$$

$\lambda(\gamma_1, \gamma_2) > 1$. Therefore, we can use Theorem 2.4 to get $\beta(\Omega, 0) > 1$. □

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